Swinging Sticks Derivations

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1 Problem Set Up

In order to find the equations of motion (EOM) of the proposed system, it can be very elegant to solve the Euler-Lagrange equations. To this end, it is useful to define the frames and state vector of the problem and identify the transformations needed to describe the system.

Frames:

w: world frame
b: base frame at first pendulum axle
$p1$: frame aligned with first pendulum
$p2$: frame at second pendulum axle aligned with second pendulum
x: cart COM directly below base frame

Parameters:

x: x position of first pendulum axle
θ_1: angle of first pendulum s horizontal
θ_2: angle of second pendulum vs horizontal
l: pendulum length
h: first pendulum axle height

State Vector:

$$q = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Transforms:

$$T_{wb} = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{bp1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{bp2} = T_{bp1} * \begin{bmatrix} 1 & 0 & 0 & l/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{bx} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now that we have transforms from the base frame to each body in the system, we will now finish describing the system positionally by calculating the transform with respect to the world frame.

$$T_{wp1} = T_{wb} * T_{bp1}$$
$$T_{wp2} = T_{wb} * T_{bp2}$$
$$T_{wx} = T_{wb} * T_{bx}$$

At this point we're ready to assemble the Lagrangian:

$$L = KE_{tot} - PE_{tot}$$

Find the kinetic energy of each body:

$$KE_{tot} = KE_x + KE_{p1} + KE_{p2}$$
$$KE_x = \frac{1}{2} * V_x^T * I_{cart} * V_x$$
$$KE_{p1} = \frac{1}{2} * V_{p1}^T * I_{pend1} * V_{p1}$$
$$KE_{p2} = \frac{1}{2} * V_{p2}^T * I_{pend2} * V_{p2}$$

Where:
V: is a 6 vector twist
I: is a spacial inertial matrix
m: is the mass of the respective body

J: is the moment of inertia about a principal axis

$$V_a = T_{wa}^{-1} * \dot{T}_{wa}$$

$$I_a = \begin{bmatrix} m_a & 0 & 0 & 0 & 0 & 0 \\ 0 & m_a & 0 & 0 & 0 & 0 \\ 0 & 0 & m_a & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{xx} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{yy} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{zz} \end{bmatrix}$$

Find the potential energy of each body. Note that we will be taking the derivative of the Lagrangian and thus care only about the change in potential energy. Therefore, shifting the potential energy by a constant has no effect:

$$PE_{tot} = PE_x + PE_{p1} + PE_{p2}$$
$$PE_x = 0$$
$$PE_{p1} = 0$$
$$PE_{p2} = m_{p2} * g * \frac{l}{2} * \sin\theta_1$$

Last, define the external force vector:

$$F_{ext} = \begin{bmatrix} F_x \\ F_{\theta_1} \\ F_{\theta_2} \end{bmatrix}$$

$$F_x = 5 * \frac{\dot{x}}{|\dot{x}|}$$

$$F_\theta = 5 * \frac{\dot{\theta}_1}{\dot{\theta}_1} * e^{(\theta_1 + \frac{\pi}{2})\%2\pi}$$

$$F_{\theta_2} = 0$$

Finally, solve the system of equations for $\ddot{q}$:

$$\frac{dL}{d\dot{q}} - \frac{dL}{dq} = F_{ext}$$

Now we'll solve for the impact update laws. First, we must calculate the generalized momentum and conserved Hamiltonian.

$$p = \frac{dL}{d\dot{q}}$$

$$H = p_1 * \dot{q}_1 + p_2 * \dot{q}_2 + \cdots + p_n * \dot{q}_n - L$$

Next we set up our impact condition, ϕ.

$$\phi = x - 6$$

Finally, solve the system of equations for the updated $\dot{q}$ and λ:

$$p|_{\tau^-}^{\tau^+} = \lambda \frac{\partial \phi}{\partial q}$$

$$H|_{\tau^-}^{\tau^+} = 0$$

Where:

τ^+: is the time directly after impact
τ^- is the time directly before impact

Note that the impact law will depend upon the direction from which the system is approaching the impact barrier. Thus each element for which you are solving will have two solutions.