Final Exam

Instructions

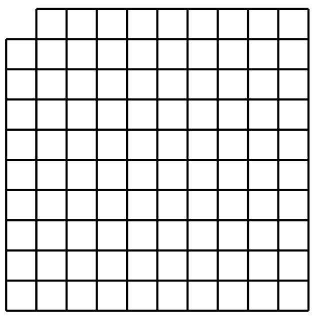
Complete the **specified number** of problems in each part. You should provide detailed solutions with sufficient justification on your own paper to the problems you choose to complete. This exam is worth a total of 34 points and 20% of your overall grade. Good luck and have fun!

Part A

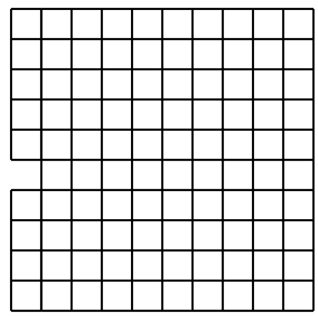
Complete **two** of the following problems. (4 points each)

A1. Tile each of the grids below with trominoes that consist of 3 squares in a line. If a tiling is not possible, explain why.

(a)



(b)



A2. Suppose you randomly cut a stick into 3 pieces. What is the probability that you can form a triangle out of these 3 pieces?

A3. Recent archaeological work on Mars discovered a site containing a pile of white spheres, each about the size of a tennis ball. A plaque near the mound states that each sphere contains a jewel that come in many different colors while strictly more than half of the spheres contain jewels of the same color. When two spheres are brought together, they both glow white if their internal jewels are the same color; otherwise, no glow. Argue that you can find a sphere that you are certain holds a jewel of the majority color in at most 3 tests if total the number of spheres is 6.

A4. Consider the regular hexagon $ABCDEF$. Let X be the midpoint of CD and let Y be the midpoint of DE. Let Z be the common point of AX and BY. Which polygon has larger area, ABZ or $DXZY$?

A5. Show that in any group of 6 students there are 3 students who know each other or 3 students who do not know each other.

A6. In the senate of the Klingon home world no senator has more than three enemies. Show that the senate can be separated into two houses so that nobody has more than one enemy in the same house.

Part B

Complete $\boxed{\textbf{one}}$ of the following problems. (4 points)

B1. An overfull prison has decided to terminate some prisoners. The jailer comes up with a game for selecting who gets terminated. Here is his scheme. 10 prisoners are to be lined up all facing the same direction. On the back of each prisoner's head, the jailer places either a black or a red dot. Each prisoner can only see the color of the dot for all of the prisoners in front of them and the prisoners do not know how many of each color there are. The jailer may use all black dots, or perhaps he uses 3 red and 7 black, but the prisoners do not know. The jailer tells the prisoners that if a prisoner can guess the color of the dot on the back of their head, they will live, but if they guess incorrectly, they will be terminated. The jailer will call on them in order starting at the back of the line. Before lining up the prisoners and placing the dots, the jailer allows the prisoners 5 minutes to come up with a plan that will maximize their survival. What plan can the prisoners devise that will maximize the number of prisoners that survive? Some more info: each prisoner can hear the answer of the prisoner behind them and they will know whether the prisoner behind them has lived or died. Also, each prisoner can only respond with the word "black" or "red.'

B2. A certain store sells a product called widgets in boxes of $7, 9$, and 11. A number n is called *widgetable* if one can buy exactly n widgets by buying some number of boxes. What is the largest non-widgetable number?

B3. Alice and Brenda both ran in a 100-meter race. When Alice crossed the finish line, Brenda was 10 meters behind her. Assuming the girls run the same rate, how many meters behind Brenda should Alice start in order for them to finish in a tie?

B4. A town of Smurfs consists of 24 blue, 8 pink, and 16 purple individuals. When two Smurfs of different colors shake hands, they both change their colors to the third color. Is it possible that all Smurfs in the town eventually have the same color? If so, describe a pattern of handshakes that will convert all the Smurfs to the same color. If this is not possible, provide a detailed argument as to why.

B5. Two prisoners are locked away in two separate towers, say North Tower and South Tower, and each tower has its own prison guard. Each morning, the respective guards toss a fair coin and then radio the guard in the other tower and report the outcome (heads or tails) of their coin toss. The guard then shows the prisoner in his/her respective tower the outcome of the coin toss in the opposite tower. At this point, each prisoner must guess the outcome of the coin toss that occurred in his/her tower. If at least one of the prisoners guesses correctly, then the prisoners survive another day. If both guess incorrectly, then both will be executed. Is there a strategy that the prisoners can implement that will ensure their survival (until they die of old age in prison) or are they doomed to eventually guess incorrectly and perish? You may assume that prior to being permanently locked up, the prisoners had a few minutes to concoct a plan.

Part C

Complete $\boxed{\textbf{one}}$ of the following problems. (4 points)

C1. Suppose we draw n lines in the plane that have the maximum number of unique intersections. This partitions the plane into disjoint regions (some of which are polygons with finite area and some are not). Suppose we color each of the regions so that no two adjacent regions (i.e., share a common edge) have the same color. What is the fewest colors we could use to accomplish this? Justify your answer.

C2. Consider a grid of squares that is 2^n squares wide by 2^n squares tall such that one of the squares has been cut out, but you don't know which one! You have a bunch of L-shaped trominoes made up of 3 squares. Prove that you can perfectly cover this grid with trominoes (with no overlap) for any $n \in \mathbb{N}$.

C3. Use induction to prove if a string in the Circle-Dot system ends in $\circ \circ \bullet$, then we cannot prove it in the system. *Hint:* Induct on the number of steps in a proof in the Circle-Dot system. I've included the axioms and rules for Circle-Dot on the last page in case you need them.

Part D

Complete $\boxed{\textbf{two}}$ of the following problems. (4 points)

D1. Nine balls—marked 1, 2, 3, 4, 5, 6, 7, 8 and 9—are in a lottery machine. The machine dispenses one ball each to three people, Amy, Bart, and Cathy. Each person knows only the number of their own ball; they do not know the balls that the others were given, nor the ones left in the machine. Before the game begins, each of them show their balls to a fourth person, Zog, who says: "On one of the balls is a number that is the sum of the numbers on the other two balls." At which point the following discussion occurs:

Amy: "There are 8 possibilities for Bart's ball."

Bart: "There are 8 possibilities for Cathy's ball."

Cathy: "There a are 4 possibilities for Amy's ball."

Amy: "I know Bart's ball!"

Bart: "I know Cathy's ball!"

What is Cathy's ball?

D2. A signed permutation of the numbers 1 through n is a fixed arrangement of the numbers 1 through n, where each number can be either be positive or negative. For example, $(-2, 1, -4, 5, 3)$ is a signed permutation of the numbers 1 through 5. A *reversal* of a signed permutation swaps the order of a consecutive subsequence of numbers while changing the sign of each number in the subsequence. Performing a reversal to a signed permutation results in a new signed permutation. For example, if we perform a reversal on the second, third, and fourth entries in $(-2, 1, -4, 5, 3)$, we obtain $(-2, -5, 4, -1, 3)$. The *reversal distance* of a signed permutation of 1 through n is the minimum number of reversals required to transform the given signed permutation into $(1, 2, \ldots, n)$. It turns out that $(4, 3, 2, 1)$ has reversal distance 5. Find another signed permutation of 1 through 4 that has reversal distance 5. Demonstrate the reversals needed to transform your permutation into $(1, 2, 3, 4)$.

D3. A soul swapping machine swaps the souls inside two bodies placed in the machine. Soon after the invention of the machine an unforeseen limitation is discovered: swapping only works on a pair of bodies once. Souls get more and more homesick as they spend time in another body and if a soul is not returned to its original body after a few days, it will kill its current host. Suppose the following sequence of soul swaps take place (where the names indicate the bodies that sit in the machine):

(a) Iron Man $\leftrightarrow$ Hulk
(b) Hulk $\leftrightarrow$ Captain America
(c) Thor $\leftrightarrow$ Hulk
(d) Nick Fury $\leftrightarrow$ Agent Coulson

Find a way to put each soul back in the appropriate body. Your solution should use the minimum number of soul swaps possible.

D4. You are playing a game of hide and seek with a clever rabbit. There are 6 boxes in a row, numbered 1 through 6. The rabbit is always hiding in one of these boxes. Each time you look for the rabbit, you can look inside a single box. If you see the rabbit, you win! Otherwise, you leave the room and the rabbit must move to an adjacent box while you are outside the room. After the rabbit has relocated, you can enter the room and try again. Can you devise a strategy for catching the rabbit or is the rabbit clever enough to avoid being captured?

D5. Suppose there are three bags of marbles containing 17, 17, and 21 pieces, respectively. You and your friend are going to play. Here are the rules for the game:

(a) You and your friend will alternate removing marbles from the bags. Let's assume that you go first.

(b) On each turn, the designated player selects a bag that still has marbles in it and then removes at least one marble. The designated player can only remove marbles from a single bag and he/she must remove at least one marble.

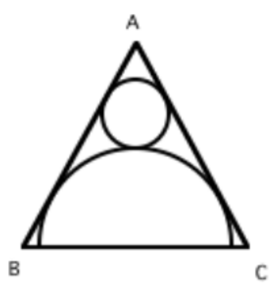
(c) The *loser* is the one that is forced to remove all the marbles from the last remaining bag.

Does one of you have a guaranteed winning strategy? If so, describe that strategy.

Part E

Complete **two** of the following problems. (4 points each)

E1. The figure below shows an equilateral triangle ABC with an inscribed semicircle of radius R that is tangent to sides AB and AC, and inscribed circle of radius r that is tangent to the triangle and the semicircle. Find the value of r/R.



E2. You and your spouse are at a party of five couples (including you). As the party goes on, various introductions and hand-shakings occur. You observe that nobody shakes hands with their spouse, as presumably they already know them. At some point, you stop the party and ask each of the nine other people how many hands they have shaken. You are surprised to obtain nine distinct answers (i.e., one person shook no hands, one person shook hands with exactly one other person, one person shook hands with exactly two other people, and so on). How many people did your spouse shake hands with?

E3. Consider the following two-player game played on an $n \times n$ grid with light up squares. Initially, all of the lights are off. On the opening move, player A chooses one of the squares to light up. Next, player B chooses any square that is not lit up and then additional squares light up if at least two of its immediate neighbors (horizontal or vertical) are lit up. Lit up squares stay on during the rest of the game. Players A and B continue to alternate moves. On each turn, a player may light up a square that is not lit up and then additional squares light up if at least two of its immediate neighbors are lit up. The loser of the game is the one that cannot make a move on his or her turn. Does one of the players have a guaranteed strategy for winning on the 3×3 grid? If so, describe the strategy.

E4. Sally has a long straight tree branch. She cut off $2/5$ of the branch. Later, she cut off another 14 cm. The ratio of the length of branch remaining to the total length cut off is $1 : 3$. What is the length of the remaining branch?

E5. Each integer on the number line is colored with exactly one of three possible colors—red, green, or blue—according to the following two rules:

- The negative of a red number must be colored blue;
- The sum of two blue numbers (not necessarily distinct) must be colored red.

Using this information, determine all possible colorings of the integers that satisfy these rules. *Hint:* First show that the negative of a blue number must be colored red and the sum of two red numbers must be colored blue. Also, show that 0 must be green and that a red plus a blue is green.

Part F

Complete **both** questions below. (1 point each)

F1. What was your favorite problem from the course and why?

F2. In this course, what did you learn about the process of doing or creating mathematics?

Circle-Dot

In case you decide to Problem C3, here is a reminder about Circle-Dot.

Circle-Dot begins with two words; called axioms. Using the two axioms and three rules of inference, we can create new Circle-Dot words, which are theorems in the Circle-Dot System. The process of creating Circle-Dot words using the axioms and rules of inference are proofs in the system. Below are the axioms for Circle-Dot. Note that $\circ$ and $\bullet$ are valid symbols in the system while w and v are variables that stand for any nonempty sequence of $\circ$'s and $\bullet$'s.

Axiom A. $\circ\bullet$

Axiom B. $\bullet\circ$

At any time in your proof, you may quote an axiom. Below are the rules for generating new statements from known statements.

Rule 1. Given wv and vw, conclude w

Rule 2. Given w and v, conclude $w \bullet v$

Rule 3. Given $wv\bullet$, conclude $w\circ$

