Problem Collection for Introduction to Mathematical Reasoning

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Problem 1. Three strangers meet at a taxi stand and decide to share a cab to cut down the cost. Each has a different destination but all are heading in more-or-less the same direction. Bob is traveling 10 miles, Sally is traveling 20 miles, and Mike is traveling 30 miles. If the taxi costs \$2 per mile, how much should each contribute to the total fare? What do you think is the most common answer to this question?

Problem 2. Christine wants to take yoga classes to increase her strength and flexibility. In her neighborhood, there are two yoga studios: Namaste Yoga and Yoga Spirit. At Namaste Yoga, a student's first class costs \$12, and additional classes cost \$10 each. At Yoga Spirit, a student's first class costs \$24, and additional classes cost \$8 each. Because Christine wants to save money, she is interested in comparing the costs of the two studios. For what number of yoga classes do the two studios cost the same amount?

Problem 3. Imagine a hallway with 1000 doors numbered consecutively 1 through 1000. Suppose all of the doors are closed to start with. Then some dude with nothing better to do walks down the hallway and opens all of the doors. Because the dude is still bored, he decides to close every other door starting with door number 2. Then he walks down the hall and changes (i.e., if open, he closes it; if closed, he opens it) every third door starting with door 3. Then he walks down the hall and changes every fourth door starting with door 4. He continues this way, making a total of 1000 passes down the hallway, so that on the 1000th pass, he changes door 1000. At the end of this process, which doors are open and which doors are closed?

Problem 4. The Sunny Day Juice Stand sells freshly squeezed lemonade and orange juice at the farmers' market. The juices are ladled out of large glass jars, each holding exactly the same amount of juice. Linda and Julie set up their stand early one Saturday morning. The first customer of the day ordered orange juice and Linda carefully ladled out 8 ounces into a paper cup. As she was about to hand the cup to the customer, he changed his mind and asked for lemonade instead. Accidentally, Linda dumped the cup of orange juice into the jar of lemonade. She quickly mixed up the juices, ladled out a cup of the mixture (mostly lemonade) and turned to hand it to the customer. "I've decided I don't want anything to drink right now," he said, and frazzled, Linda dumped the cupful of juice mixture into the orange juice jar. Linda's assistant, Julie, watched all of this with amusement. As the man walked away, she wondered aloud, "Now is there more orange juice in the lemonade or more lemonade in the orange juice?"

Problem 5. Imagine you have 25 pebbles, each occupying one square on a 5 by 5 chess board. Tackle each of the following variations of a puzzle.

(a) Variation 1: Suppose that each pebble must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(b) Variation 2: Suppose that all but one pebble (your choice which one) must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(c) Variation 3: Consider Variation 1 again, but this time also allow diagonal moves to adjacent squares. If this is possible, describe a solution. If this is impossible, explain why.

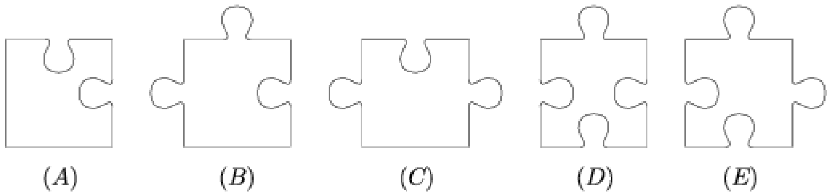
Problem 6. Consider an $n \times n$ chess board and variation 1 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 7. Consider an $n \times n$ chess board and variation 2 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 8. Imagine a 5×5 grid of squares, where each square is occupied by a kangaroo. If each kangaroo can hop only one square to the left, right, up, or down and a square can hold more than one kangaroo, what is the maximum number of unoccupied squares after the kangaroos are done hopping? What about a larger grid?

Problem 9. What four-digit number reverses its digits when multiplied by 4?

Problem 10. A rectangular puzzle that says "850 pieces" actually consists of 851 pieces. Each piece is identical to one of the 5 samples shown in the diagram. How many pieces of type (E) are there in the puzzle?



Problem 11. Describe where on Earth from which you can travel one mile south, then one mile east, and then one mile north and arrive at your original location. There is more than one such location. Find them all.

Problem 12. You are in a big city where all the streets go in one of two perpendicular directions. You take your car from its parking place and drive on a tour of the city such that you do not pass through the same intersection twice and return back to where you started. If you made 100 left turns, how many right turns did you make?

Problem 13. You are in a big city grid where all the streets go in one of two perpendicular directions and every city block is the same size. Imagine you take your bicycle on a tour of the city, where your tour starts at an intersection, you may retrace part of your path, you may visit the same intersection more than once, and you finish your tour where you started. What can you say about the number of city blocks that you traveled? Note that if you travel along the same city block twice (possibly in opposite directions), this would contribute two to your count. Can you conclude anything about the number of distinct city blocks that you traveled? Justify your answers.

Problem 14. Find the rational number with smallest denominator between $1/3$ and $3/8$.

Problem 15. Suppose there are two bags of candy containing 8 pieces and 6 pieces, respectively. You and your friend are going to play a game and the winner gets to eat all of the candy. Here are the rules for the game:

1. You and your friend will alternate removing pieces of candy from the bags. Let's assume that you go first.

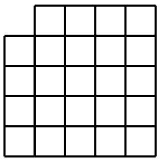
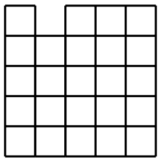
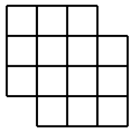
2. On each turn, the designated player selects a bag that still has candy in it and then removes at least one piece of candy. The designated player can only remove candy from a single bag and he/she must remove at least one piece.

3. The winner is the one that removes all the candy from the last remaining bag.

Does one of you have a guaranteed winning strategy? If so, describe that strategy. Can you generalize to handle any number of pieces of candy in either of the two bags?

Problem 16. I have 10 sticks in my bag. The length of each stick is an integer. No matter which 3 sticks I try to use, I cannot make a triangle out of those sticks. What is the minimum length of the longest stick?

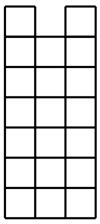
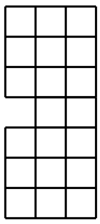
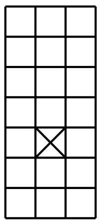
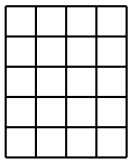
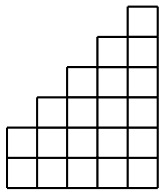
Problem 17. Tile the following grids with dominoes. If a tiling is not possible, explain why.

(a)  (b)  (c) 

Problem 18. Rufus and Dufus are identical twins. They are each independently given the same 4-digit number. Rufus takes the number and converts it from decimal (base 10) to base 4, and writes down the 6-digit result. Dufus simply writes the first and last digits of the number followed by the number in its entirety. Rufus is shocked to discover that Dufus has written down exactly the same number has him. What was the original number? In other words, if the original number was $xyzw$, which number $xyzw$, when converted from decimal to base 4 becomes $xwxyzw$?

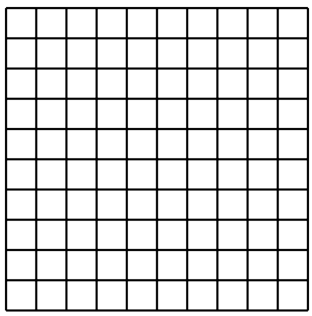
Problem 19. Find all tetrominoes (polyomino with 4 cells). Note that two tetrominoes are considered the same if we can obtain one from the other by rotation or flipping it over. The next problem gives you a hint as to how many there are.

Problem 20. Tile the following grids using every tetromino exactly once. The X in (c) denotes an absence of an available square in the grid. If a tiling is not possible, explain why.

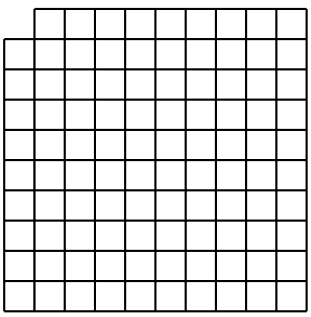
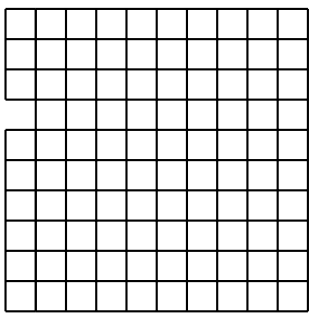
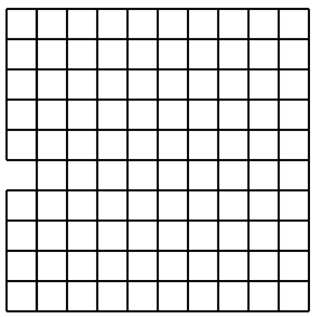
(a)  (b)  (c)  (d)  (e) 

Problem 21. How many factors of 10 are there in 50! (i.e., 50 factorial)?

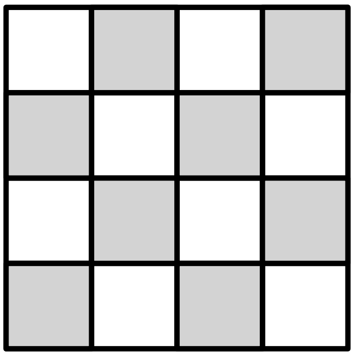
Problem 22. Consider the 10×10 grid of squares below. Show that you can color the squares of the grid with 3 colors so that every consecutive row of 3 squares and every consecutive column of 3 squares uses all 3 colors.



Problem 23. Tile each of the grids below with trominoes that consist of 3 squares in a line. If a tiling is not possible, explain why.

(a)  (b)  (c) 

Problem 24. Pennies and Paperclips is a two-player game played on a 4×4 checkerboard as shown below.



One player, "Penny", gets two pennies as her pieces. The other player, "Clip", gets a pile of paperclips as his pieces. Penny places her two pennies on any two different squares on the board. Once the pennies are placed, Clip attempts to cover the remainder of the board with paperclips - with each paperclip being required to cover two vertically or horizontally adjacent squares. Paperclips are not allowed to overlap. If the remainder of the board can be covered with paperclips then Clip is declared the winner. If the remainder of the board cannot be covered with paperclips then Penny is the winner.

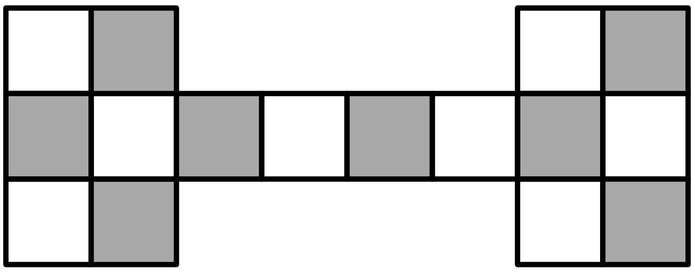
(a) Does either player have a winning strategy? If so, describe the winning strategies.

(b) State and prove a conjecture that determines precisely every situation in which Penny wins based on the placement of the pennies.

(c) State and prove a conjecture that determines precisely every situation in which Clip wins based on the placement of the pennies.

(d) Are there any situations in which neither player wins, or have you characterized all possible outcomes? Explain.

Problem 25. Consider the game Pennies and Paperclips described in the previous problem, but instead of playing on a 4×4 checkerboard, let's play on the following board.



State and prove a conjecture that determines precisely every situation in which Clip wins based on the placement of the pennies.

Problem 26. We call a game board for the Pennies and Paperclips game **fair**, if for each player there is at least one scenario in which they can win.

(a) Is the board from Problem 24 fair?

(b) Is the board from Problem 25 fair?

(c) Are there game boards that are not fair? That is, are there game boards on which one player can never win? If so, provide such a board and explain why it must be unfair. If not, explain why no such board exists.

(d) Can you create a fair board in which your conjecture from Problem 24(c) does not always hold?

Problem 27. There is a plate of 40 cookies. You and your friend are going to take turns taking either 1 or 2 cookies from the plate. However, it is a faux pas to take the last cookie, so you want to make sure that you do not take the last cookie. How can you guarantee that you will never be the one taking the last cookie? What about n cookies?

Problem 28. The Sylver Coinage Game is a game in which 2 players alternately name positive integers that are not the sum of nonnegative multiples of previously named integers. The person who names 1 is the loser! Here is a sample game between A and B:

1. A opens with 5. Now neither player can name $5, 10, 15, \ldots$
2. B names 4. Now neither player can name 4, 5, 8, 9, 10, or any number greater than 11.
3. A names 11. Now the only remaining numbers are 1, 2, 3, 6, and 7.
4. B names 6. Now the only remaining numbers are 1, 2, 3, and 7.
5. A names 7. Now the only remaining numbers are 1, 2, and 3.
6. B names 2. Now the only remaining numbers are 1 and 3.
7. A names 3, leaving only 1.
8. B is forced to name 1 and loses.

If player A names 3, can you find a strategy that guarantees that the second player wins? If so, describe the strategy? If such a strategy is not possible, then explain why?

Problem 29. Find all distinct pairs of numbers with largest gcd between and including 51 and 100. By distinct pair, we mean that you cannot choose the same number twice. Note that gcd is short for greatest common divisor. For example, $\gcd(14, 20) = 2$.

Problem 30. Four red ants and two black ants are walking along the edge of a one meter stick. The four red ants, called Albert, Bart, Debbie, and Edith, are all walking from left to right, and the two black ants, Cindy and Fred, are walking from right to left. The ants always walk at exactly one centimeter per second. Whenever they bump into another ant, they immediately turn around and walk in the other direction. And whenever they get to the end of a stick, they fall off. Albert starts at the left hand end of the stick, while Bart starts 20.2 cm from the left, Debbie is at 38.7cm, Edith is at 64.9cm and Fred is at 81.8cm. Cindy's position is not known—all we know is that he starts somewhere between Bart and Debbie. Which ant is the last to fall off the stick? And how long will it be before he or she does fall off?

Problem 31. A certain fast-food chain sells a product called "nuggets" in boxes of $6, 9$, and 20. A number n is called *nuggetable* if one can buy exactly n nuggets by buying some number of boxes. For example, 21 is nuggetable since you can buy two boxes of six and one box of nine to get 21. Here are the first few nuggetable numbers:

$$6, 9, 12, 15, 18, 20, 21, 24, 26, 27, \ldots$$

and here are the first few non-nuggetable numbers:

$$1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots$$

What is the largest non-nuggetable number?

Problem 32. Take 15 poker chips or coins, divide into any number of piles with any number of chips in each pile. Arrange piles in adjacent columns. Take the top chip off every column and make a new column to the left. Repeat forever. What happens? Make conjectures about what happens when we change the number of chips.

Problem 33. The nth *triangular number* is defined via $t_n := 1+2+\cdots+n$. For example, $t_4 = 1+2+3+4 = 10$. Find a visual proof of the following fact. By "visual proof" we mean a sufficiently general picture that is convincing enough to justify the claim.

For all $n \in \mathbb{N}$, $t_n = \dfrac{n(n+1)}{2}$.

Problem 34. Let t_n denote the nth triangular number. Find both an algebraic proof and a visual proof of the following fact.

For all $n \in \mathbb{N}$, $t_n + t_{n+1} = (n+1)^2$.

Problem 35. Find a visual proof of the following fact. *Warning:* This problem is not about triangular numbers.

For $n \in \mathbb{N}$, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Problem 36. Suppose someone draws 20 distinct random lines in the plane. What is the maximum number of intersections of these lines?

Problem 37. Let t_n denote the nth triangular number. Find an algebraic and a visual proof of the following fact.

For all $a, b \in \mathbb{N}$, $t_{ab} = t_a t_b + t_{a-1} t_{b-1}$.

Problem 38. Consider a room with 25 people. If every person shakes hands with every other person (only once), then how many handshakes occurred? How about n people?

Problem 39. How many ways can 110 be written as the sum of 14 different positive integers? *Hint:* First, figure out what the largest possible integer could be in the sum. Note that the largest integer in the sum will be maximized when the other 13 numbers are as small as possible. Finish off the problem by doing an analysis of cases.

Problem 40. In the game Turnaround, you are given a permutation of the numbers from 1 to n. Your goal is to get them in the natural order $12 \cdots n$. At each stage, your only option is to reverse the order of the first k places (you get to pick k at each stage). For instance, given 6375142, you could reverse the first four to get 5736142 and then reverse the first six to get 4163752. Solve the following sequence in as few moves as possible: 352614.

Problem 41. A signed permutation of the numbers 1 through n is a fixed arrangement of the numbers 1 through n, where each number can be either be positive or negative. For example, $(-2, 1, -4, 5, 3)$ is a signed permutation of the numbers 1 through 5. In this case, think of positive numbers as being right-side-up and negative numbers as being upside-down. A *reversal* of a signed permutation is the act of performing a 180-degree rotation to some consecutive subsequence of the permutation. That is, a reversal swaps the order of a subsequence of numbers while changing the sign of each number in the subsequence. Performing a reversal to a signed permutation results in a new signed permutation. For example, if we perform a reversal on the second, third, and fourth entries in $(-2, 1, -4, 5, 3)$, we obtain $(-2, -5, 4, -1, 3)$. The *reversal distance* of a signed permutation of 1 through n is the minimum number of reversals required to transform the given signed permutation into $(1, 2, \ldots, n)$. It turns out that the reversal distance of $(3, 1, 6, 5, -2, 4)$ is 5. Find a sequence of 5 reversals that transforms $(3, 1, 6, 5, -2, 4)$ into $(1, 2, 3, 4, 5, 6)$.

Problem 42. A soul swapping machine swaps the souls inside two bodies placed in the machine. Soon after the invention of the machine an unforeseen limitation is discovered: swapping only works on a pair of bodies once. Souls get more and more homesick as they spend time in another body and if a soul is not returned to its original body after a few days, it will kill its current host.

(a) Suppose Tom and Jerry swap souls and Garfield and Odie swap souls. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(b) Suppose Batman and Robin swap souls and then Robin's body and Flash utilize the machine. Argue that it is not possible to return the swapped souls to their original bodies using only Batman, Robin, and Flash.

(c) Consider the scenario of the previous problem. Suppose Wonder Woman and Superman are now available to sit in the machine after Batman, Robin, and Flash have already swapped souls. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(d) Now, suppose the soul swapping machine is used by the following pair of bodies (in the order listed): Adam and Alicia, Alicia and Gwen, Gwen and Blake. In addition, Pharrell and Miley are standing nearby. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

Problem 43. Four prisoners are making plans to escape from jail. Their current plan requires them to cross a narrow bridge in the dark that has no handrail. In order to successfully cross the bridge, they must use a flashlight. However, they only have a single flashlight. To complicate matters, at most two people can be on the bridge at the same time. So, they will need to make multiple trips across the bridge, returning the flashlight back to the first side of the bridge by having someone walk it back. Unfortunately, they can't throw the flashlight. It takes 1, 2, 5, and 10 minutes for prisoner A, prisoner B, prisoner C, and prisoner D to cross the bridge and when two prisoners are walking together with the flashlight, it takes the time of the slower prisoner. What is the minimum total amount of time it takes all four prisoners to get across the bridge?

Problem 44. You need to pack several items into your shopping bag without squashing anything. The items are to be placed one on top of the other. Each item has a weight and a strength, defined as the maximum weight that can be placed above that item without it being squashed. A packing order is safe if no item in the bag is squashed, that is, if, for each item, that item's strength is at least the combined weight of what's placed above that item. For example, here are three items and a packing order:

Ordering	Item	Weight	Strength
Top	Apples	5	6
Middle	Bread	4	4
Bottom	Carrots	12	9

This packing is not safe. The bread is squashed because the weight above it, 5, is greater than its strength, 4.

(a) Find all safe orderings of the three items above.

(b) Consider packing items in weight order, with the heaviest at the bottom. Show by giving an example (i.e., invent some items and give them weights and strengths of your choosing) that this strategy might not produce a safe packing order, even if one exists.

(c) Consider packing items in strength order, with the strongest at the bottom. Show by giving an example (i.e., invent some items and give them weights and strengths of your choosing) that this strategy might not produce a safe packing order, even if one exists.

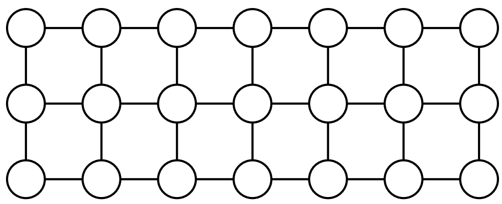
(d) Consider we have a safe packing order in our bag. Assume that item j sits directly on item i. Suppose also that

$$(\text{weight of } j)-(\text{strength of } i) \geq (\text{weight of } i)-(\text{strength of } j).$$

Show that if we swap items i and j we still have a safe packing order.

(e) Suggest a practical method of producing a safe packing order if one exists. Explain your answer.

Problem 45. In the lattice below, we color 11 vertices points black. Prove that no matter which 11 are colored black, we always have a rectangle with black corners.



Problem 46. Each point of the plane is colored red or blue. Show that there is a rectangle whose corners are all the same color.

Problem 47. Our space ship is at a Star Base with coordinates $(1,2)$. Our hyper drive allows us to jump from coordinates (a,b) to either coordinates $(a,a+b)$ or to coordinates $(a+b,b)$. How can we reach the impending enemy attack at coordinates $(8,13)$?

Problem 48. Consider our Star Base from the previous problem. Recall that our hyper drive allows us to jump from coordinates (a,b) to either coordinates $(a,a+b)$ or to coordinates $(a+b,b)$. If we start at $(1,0)$, which points in the plane can we get to by using our hyper drive? Justify your answer.

Problem 49. Suppose you randomly cut a stick into 3 pieces. What is the probability that you can form a triangle out of these 3 pieces?

Problem 50. Suppose you randomly pick 3 distinct points on a circle. What is the probability that the center of the circle lies in the interior of the triangle formed by these 3 points?

Problem 51. You have 14 coins, dated 1901 through 1914. Seven of these coins are real and weigh 1.000 ounce each. The other seven are counterfeit and weigh 0.999 ounces each. You do not know which coins are real or counterfeit. You also cannot tell which coins are real by look or feel. Fortunately for you, Zoltar the Fortune-Weighing Robot is capable of making very precise measurements. You may place any number of coins in each of Zoltar's two hands and Zoltar will do the following:

- If the weights in each hand are equal, Zoltar tells you so and returns all of the coins.
- If the weight in one hand is heavier than the weight in the other, then Zoltar takes one coin, at random, from the heavier hand as tribute. Then Zoltar tells you which hand was heavier, and returns the remaining coins to you.

Your objective is to identify a single real coin that Zoltar has not taken as tribute.

Problem 52. Welcome to Circle-Dot[1]. We'll approach Circle-Dot as a game, where the object of the game is to construct a word made entirely of ◦'s and •'s. Circle-Dot begins with two words; called axioms. Using the two axioms and three rules of inference, we can create new Circle-Dot words, which are theorems in the Circle-Dot System. The process of creating Circle-Dot words using the axioms and rules of inference are proofs in the system.

On each of your "turns" in the game you can apply one of the 5 available axioms or rules to your current list of constructed Circle-Dot words to produce a new word. Also, once you have produced a new word, you can use this theorem in future "games."

Below are the axioms for Circle-Dot. Note that ◦ and • are valid symbols in the system while w and v are variables that stand for any sequence of ◦'s and •'s.

Axiom A. ◦•

Axiom B. •◦

At any time in your proof, you may quote an axiom. Below are the rules for generating new statements from known statements.

Rule 1. Given wv and vw, conclude w

Rule 2. Given w and v, conclude $w \bullet v$

Rule 3. Given $wv\bullet$, conclude $w\circ$

As an example, let's try to prove the following theorem.

Theorem C. • (just a single dot)

At the moment, the only tools we have for getting started are the axioms. As we prove theorems, we'll be able to incorporate them into our proofs, as well. To get started, let's apply Axiom A and see what that gets us. Applying Axiom A, we get ◦•. Looking at Rules 2 and 3, it should be moderately clear that they won't help us get a single dot. So, perhaps Rule 1 will be useful, but to use it, we see that we need to have wv and vw. Applying Axiom B, we get •◦. Now, if we let $w = \bullet$ and $v = \circ$, then $wv = \bullet\circ$ and $vw = \circ\bullet$. Applying Rule 1, we can conclude that • holds. Putting this all together, we can write something like the following.

Proof of Theorem C.

1. ◦• by Axiom A
2. •◦ by Axiom B

[1] The Circle-Dot System was developed by Ken Monks from the University of Scranton.

3. ● by Rule 1 (using lines 2 and 1)

Now, try proving the following theorems.

Theorem D. ○

Theorem E. ●●●

Theorem F. ●●○

Theorem G. ●○○

Theorem H. ○●●○

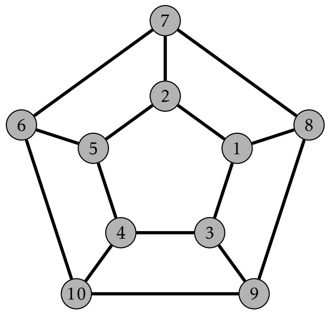
Theorem I. ○○○○

Theorem J. ●○●

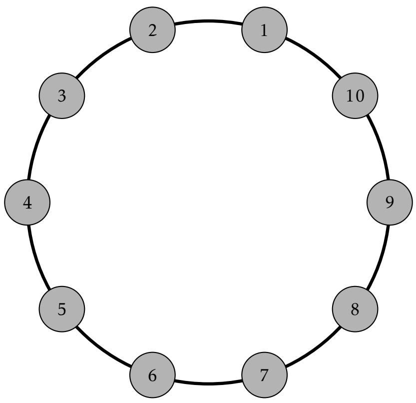
Theorem K. ●○○○

Make a conjecture about which sequences of ○'s and ●'s are theorems in the Circle-Dot system.

Problem 53. The graph depicted below is an example of a Hastings Helm. Notice that we have labeled the 10 vertices of the graph with the natural numbers 1 through 10. Two vertices are said to be *adjacent* if they are joined by an edge. For example, the vertex currently labeled by 4 is adjacent to the vertices labeled by 3, 5, and 10. Is it possible to relabel the vertices so that the labels of adjacent vertices have no factors other than 1 in common? Notice that since the vertices currently labeled by 3 and 9 are adjacent and have a factor of 3 in common, the current labeling will not do the job. If you can find an appropriate labeling, then show it. If no such labeling exists, then explain why.



Problem 54. Ten people form a circle. Each picks a number and tells it to the two neighbors adjacent to him/her in the circle. Then each person computes and announces the average of the numbers of his/her two neighbors. The figure shows the average announced by each person. What is the number picked by the person who announced 6?



Problem 55. Consider a tournament with 30 teams. If every team plays every other team, how many games were played?

Problem 56. Find a solution to the equation $28x + 30y + 31z = 365$, where x, y, and z are positive whole numbers.

Problem 57. Find all integers a, b, c, d, and e, such that

$$
\begin{aligned}
a^2 &= a + b - 2c + 2d + e - 8 \\
b^2 &= -a - 2b - c + 2d + 2e - 6 \\
c^2 &= 3a + 2b + c + 2d + 2e - 31 \\
d^2 &= 2a + b + c + 2d + 2e - 2 \\
e^2 &= a + 2b + 3c + 2d + e - 8.
\end{aligned}
$$

Problem 58. Consider the equation below. If a is a number, what number is it?

$$
a = \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cdots}}}}}
$$

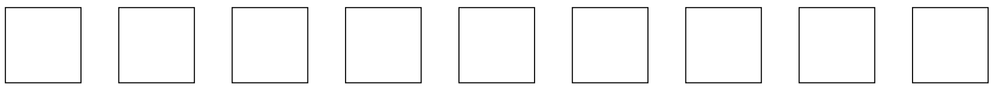
Problem 59. A colony of chameleons on an island currently comprises 13 green, 15 blue, and 17 red individuals. When two chameleons of different colors meet, they both change their colors to the third color. Is it possible that all chameleons in the colony eventually have the same color?

Problem 60. Let X be the intersection of the diagonals of the trapezoid $ABCD$ with parallel sides AB and CD. Show that the areas of triangles AXD and BXC are the same.

Problem 61. Consider the regular hexagon $ABCDEF$. Let X be the midpoint of CD and let Y be the midpoint of DE. Let Z be the common point of AX and BY. Which polygon has larger area, ABZ or $DXZY$?

Problem 62. There are 8 frogs and 9 rocks on a field. The 9 rocks are laid out in a horizontal line. The 8 frogs are evenly divided into 4 green frogs and 4 brown frogs. The green frogs sit on the first 4 rocks facing right and the brown frogs sit on the last 4 rocks facing left. The fifth rock is vacant for now. Switch the places of the green and brown frogs by using the following rules:

- A frog can only jump forward
- A frog can hop to an vacant rock one place ahead
- A frog can leap over its neighbor frog to a vacant rock two places ahead



Can we generalize this problem and find how many jumps are necessary to switch n green and n brown frogs?

Problem 63. I've got three identical-looking eggs and one long thin eggbox which has six spaces for eggs in a row. It turns out that that are 20 configurations of placing 3 eggs in the 6 spaces (where order of the eggs does not matter). *Note:* One way to count the number of configurations is by computing $\binom{6}{3}$ ("6 choose 3"). Suppose you start with 3 eggs in the 3 leftmost spaces. Is it possible to obtain all 20 configurations by only moving one egg at a time to an adjacent open space? If so, find a sequence of moves that accomplishes this task or argue that no such sequence exists.

Problem 64. Suppose you have 6 toothpicks that are exactly the same length. Can you arrange the toothpicks so that exactly 4 identical triangles are formed? You cannot cut, break, or bend the toothpicks. Moreover, each vertex of a triangle must be formed when the tips of two toothpicks meet.

Problem 65. Consider a gambler who tosses a coin at most 6 times, and if it comes out heads (H), wins a dollar, and if it comes out tails (T), loses a dollar. He is kicked out as soon as he is in the red, i.e., has negative capital. In how many ways can he survive to 6 rounds, but at the end break even?

Problem 66. There are 30 red, 40 yellow, 50 blue, and 60 green balls in a box. We take out balls from the box with closed eyes. On the first turn we take out 1 ball, on the second turn we take out 2, and so on. On the nth turn we take out n balls. What is the minimum number of balls we need to take out to guarantee the following:

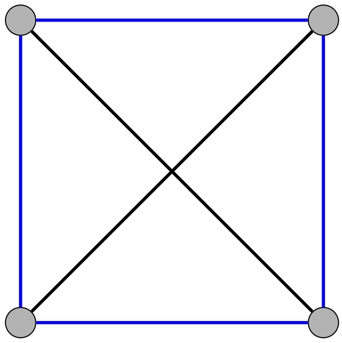
(a) We have a blue ball;

(b) We have a red and a green ball;

(c) We have all four colors.

Problem 67. Suppose you have 12 coins, all identical in appearance and weight except for one that is either heavier or lighter than the other 11 coins. What is the minimum number of weighings one must do with a two-pan scale in order to identify the counterfeit?

Problem 68. Find all the ways to arrange four points in the plane so that only two distances occur between any two points. Below is one possibility. Find the remaining configurations.



Problem 69. Three boxes, one with black, one with white, and one with black and white balls. Each of the boxes is labeled B, W, and BW, but unfortunately, *all* the boxes are labeled incorrectly. Moreover, you cannot see inside each of the boxes, but you can reach in and pull a ball out. What is the minimum number of balls that need to be pulled before you can relabel all the boxes correctly?

Problem 70. Which of the following statements is/are true?

1. Exactly one of the statements in this list is false.
2. Exactly two of the statements in this list are false.
3. Exactly three of the statements in this list are false.
4. Exactly four of the statements in this list are false.
5. Exactly five of the statements in this list are false.
6. Exactly six of the statements in this list are false.
7. Exactly seven of the statements in this list are false.
8. Exactly eight of the statements in this list are false.
9. Exactly nine of the statements in this list are false.
10. Exactly ten of the statements in this list are false.

Problem 71 (The Good Teacher)**.** You are teaching Calculus I, and you wish to give the students a cubic polynomial and have them find its three x-intercepts, its two critical points and its one inflection point. Because you are a kind person, you want these 6 points to all have integer coordinates and you want the cubic to have integer coefficients that are not too horribly large. Find one.

Problem 72. Consider an equilateral triangle with side lengths of 2 units. Find an arrangement of 5 distinct points or argue that no such arrangement exists such that all 5 points are in the interior of the triangle and every pair of points is at least 1 unit apart.

Problem 73 (The Martian Artifacts)**.** Recent archaeological work on Mars discovered a site containing a pile of white spheres, each about the size of a tennis ball. A plaque near the mound states that each sphere contains a jewel that come in many different colors while strictly more than half of the spheres contain jewels of the same color. When two spheres are brought together, they both glow white if their internal jewels are the same color; otherwise, no glow. In how few tests can you find a sphere that you are certain holds a jewel of the majority color if the number of spheres in the pile is $2,3,4,5,6,7,8$, or 9? You should provide an answer with justification for each of the different values. Can you make a conjecture about how many tests are required if the number of spheres in the pile is $n \geq 2$?

Problem 74. We have the following information about three integers:

(a) Their product is a prime;

(b) One of them is the average of the other two.

What are these numbers? *Hint*: You need to find all such triples and show that there are no others. Also, 1 is not a prime number.

Problem 75. Four people are lined up on some steps. They are all looking down the steps and a wall separates the fourth person from the other three. In particular:

- Person 1 can see persons 2 and 3.
- Person 2 can see person 3.
- Person 3 cannot see anyone.
- Person 4 cannot see anyone.

All four people are wearing hats. They are told that there are two white hats and two black hats. Initially, no one knows what color hat they are wearing. They are told to shout out the color of the hat that they are wearing as soon as they know for certain what color it is. Additional constraints:

- They are not allowed to turn around or move.
- They are not allowed to talk to each other.
- They are not allowed to take their hats off.

Who is the first person to shout out the color of his/her hat and why?

Problem 76. The first vote counts of the papal conclave resulted in 33 votes each for candidates A and B and 34 votes for candidate C. The cardinals then discussed the candidates in pairs. In the second round each pair of cardinals with differing first votes changed their votes to the third candidate they did not vote for in the first round. The new vote counts were 16, 17 and 67. They were about to start the smoke signal when Cardinal Ordinal shouted "wait". What was his reason?

Problem 77. Suppose I have n coins that I repeatedly toss all at once. After each toss, I count how many coins turned up heads versus how many turned up tails, and then multiply these two numbers together. For example, if I had 3 coins and one of them landed on tails and the other 2 on heads, then my product is 2. After some tinkering, I discover that the average value of my possible products is exactly 3 times the number of coins I have. How many coins do I have?

Problem 78. Which answer in the list is the correct answer to this question?

1. All of the below.
2. None of the below.
3. All of the above.
4. One of the above.
5. None of the above.
6. None of the above.

Problem 79. We have two strings of pyrotechnic fuse. The strings do not look homogeneous in thickness but both of them have a label saying 4 minutes. So we can assume that it takes 4 minutes to burn through either of these fuses. How can we measure a one minute interval?

Problem 80. My Uncle Robert owns a stable with 25 race horses. He wants to know which three are the fastest. He owns a race track that can accommodate five horses at a time. What is the minimum number of races required to determine the fastest three horses?

Problem 81. Annie, Bob, and Cristy are sitting by a campfire when Cristy announces that she is thinking of a 3-digit number. She then tells Annie and Bob that the number she is thinking of is one of the following:

$$515, 516, 519, 617, 618, 714, 716, 814, 815, 817.$$

Next, Cristy whispers the leftmost digit in Annie's ear and then whispers the remaining two digits in Bob's ear. The following conversation then takes place:

> Annie: I don't know what the number is, but I know Bob doesn't know too.
>
> Bob: At first I didn't know what the number was, but now I know.
>
> Annie: Ah, then I know the number, too.

From that information, determine Cristy's number.

Problem 82. A father has 20 one dollar bills to distribute among his five sons. He declares that the oldest son will propose a scheme for dividing up the money and all five sons will vote on the plan. If a majority agree to the plan, then it will be implemented, otherwise dad will simply split the money evenly among his sons. Assume that all the sons act in a manner to maximize their monetary gain but will opt for evenly splitting the money, all else being equal. What proposal will the oldest son put forth, and why?

Problem 83. Imagine that in the scenario of the previous problem the father decides that after the oldest son's plan is unveiled, the second son will have the opportunity to propose a different division of funds. The sons will then vote on which plan they prefer. Assume that the sons still act to maximize their monetary gain, but will vote for the older son's plan if they stand to receive the same amount of money either way. What will transpire in this case, and why?

Problem 84. Show that in any group of 6 students there are 3 students who know each other or 3 students who do not know each other.

Problem 85. Show that in any set of seven different positive integers there are three numbers such that the greatest common divisor of any two of them leaves the same remainder when divided by three.

Problem 86. Given enough space, the population of a certain type of bacteria doubles every minute. Suppose one bacterium is placed in a bottle at 11:00AM and an hour later, the bottle is full.

(a) At what time is the bottle half full?

(b) Suppose that at 11:15AM an intelligent bacterium recognizes the space limitations her fellow bacteria are going to have in 45 minutes. The bacteria look around the room and notice 3 empty bottles nearby. Shortly thereafter, the bacteria start emigrating to the empty bottles in an attempt to prolong their existence. At what time will all 4 bottles be full?

Problem 87. In a PE class, everyone has 5 friends. Friendships are mutual. Two students in the class are appointed captains. The captains take turns selecting members for their teams, until everyone is selected. Prove that at the end of the selection process there are the same number of friendships within each team.

Problem 88. In the senate of the Klingon home world no senator has more than three enemies. Show that the senate can be separated into two houses so that nobody has more than one enemy in the same house.

Problem 89. After Thor is captured by Loki, Loki sets Thor the following challenge in order to gain his freedom. Thor is presented three closed doors, numbered 1–3. Thor's hammer (which he is unable to summon due to a spell Loki cast on the hammer) is behind one of the doors and there are wolves behind the other two doors. If Thor can guess which door his hammer is behind, Loki will return the hammer and let Thor go. Otherwise, Loki will cast a spell that turns Thor into a goat. Thor picks door number 1. Because Loki is mischievous and knows what is behind each door, he decides to show Thor what is behind door number 3, which happens to be a wolf. Loki says, "Do you want to pick door number 2 or stick with your original choice of door 1?" Is it to Thor's advantage to switch his choice?

Problem 90. You and your two friends Thor and Valkyrie are captured by Loki. In order to gain your freedom, Loki sets you the following challenge. The three of you are put in adjacent cells. In each cell is a quantity of stones. Each of you can count the number of stones in your own cell, but not in anyone else's. You are told that each cell has at least one stone but at most nine stones, and no two cells have the same number of stones. The rules of the challenge are as follows: The three of you will ask Loki a single question each, which he will answer truthfully "Yes" or "No". Every one hears the questions and the answers. Loki will set all of you free only if one of you can correctly determine the total number of stones in all the cells. Here is the initial conversation.

> Thor: Is the total an even number?
>
> Loki: No.
>
> Valkyrie: Is the total a prime number?
>
> Loki: No

You have five stones in your cell. What question will you ask? You should assume that Thor and Valkyrie are just as good at logic as you are.

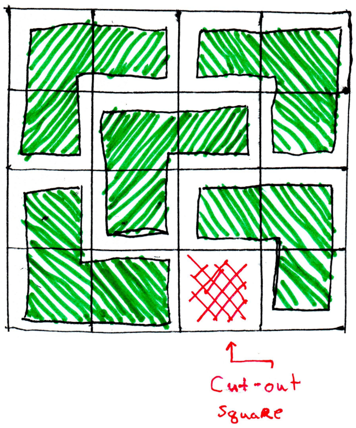
Problem 91. 100 prisoners are isolated in individual jail cells with no way to communicate. They are currently serving life sentences. Due to an overcrowded prison, the jailer decides to offer the prisoners the following deal. There is a room with nothing in it except a light switch (that starts in the off position). At random, the jailer will escort a single prisoner into the room with the light switch. After 5 seconds, the jailer will escort the prisoner back to his/her jail cell. The jailer will repeat this over and over again. He tells each of the prisoners that if one of the prisoners can indicate when every prisoner has been in the room with the light switch at least once, he will let all the prisoners go. However, if a prisoner erroneously states that each prisoner has been in the room with the light switch, then all the prisoners will be executed. Before beginning, the jailer gets all 100 prisoners together and gives them 5 minutes to come up with a plan. What should their plan be? It's important to note that the jailer is choosing prisoners at random to take in the room. That is, by chance, the same prisoner may be escorted to the room several times in a row. Also, your task is to devise a scheme for the prisoners to communicate with the light switch. You shouldn't bother searching for other ways for the prisoners to communicate.

Problem 92. Recall that the nth triangular number is defined via $t_n := 1+2+3+\cdots+n$. In an earlier problem, we utilized a visual proof to show that $t_n = \frac{n(n+1)}{2}$. Prove that this is true using induction.

Problem 93. Suppose we draw n lines in the plane that have the maximum number of unique intersections. This partitions the plane into disjoint regions (some of which are polygons with finite area and some are not). Suppose we color each of the regions so that no two adjacent regions (i.e., share a common edge) have the same color. What is the fewest colors we could use to accomplish this? Justify your answer.

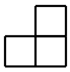
Problem 94. Prove that every natural number can be written as the sum of distinct powers of two.

Problem 95. Consider a grid of squares that is 2^n squares wide by 2^n squares tall such that one of the squares has been cut out, but you don't know which one! You have a bunch of L-shaped trominoes made up of 3 squares. Prove that you can perfectly cover this grid with trominoes (with no overlap) for any $n \in \mathbb{N}$. The figure below depicts one possible covering for the case involving $n = 2$. *Hint:* Use induction.



Problem 96. In a certain kind of tournament, every player plays every other player exactly once and either wins or loses (there are no ties). Define a *top player* to be a player who, for every other player x, either beats x or beats a player y who beats x. (There may be more than one top player.) Prove that every n-player tournament has a top player. *Hint:* Use induction. For the inductive step, start with a tournament with $k+1$ players and remove a single player that has the lowest number of wins. There might be lots of players tied for lowest number of wins, in which case just pick one of them at random to remove.

Problem 97. What size rectangles can be tiled with the following tromino?



Problem 98. Use induction to prove if a string in the Circle-Dot system ends in $\circ\circ\bullet$, then we cannot prove it in the system. *Hint:* Induct on the number of steps in a proof in the Circle-Dot system.