# A Simple Sufficient Condition for Convergence of Projected Consensus Algorithm

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***Abstract*—This letter studies the convergence property of the projected consensus algorithm. Under the assumption that: 1) the intersection of all constraint sets is nonempty; 2) the directed graph representing the communication among agents is time-invariant, strongly connected and aperiodic; and 3) the sum of the weights on incoming edges to each vertex of the graph is one, we prove that the states of all agents converge to a common point in the intersection of all constraint sets. Our proof does not need the assumption that every vertex has a self-loop. The validity of the theoretical analysis is confirmed through numerical experiments performed for a system of linear equations with nonnegativity constraints.**

***Index Terms*—Distributed control, decentralized control, optimization algorithms.**

## I. Introduction

Distributed computations by multi-agent networks such as average consensus [1]–[4], dynamic consensus [5], [6], constrained consensus [7]–[11], distributed formation [12]–[14], distributed optimization [7], [15]–[19], algebraic connectivity estimation [20]–[23] have attracted a great deal of attention. In these problems, agents are interconnected over an underlying communication network, and each agent computes or estimates the value of some global feature only from the information of itself and the agents in its neighborhood.

In this letter, we focus our attention on the constrained consensus problem [7], [9] where the states of $m$ agents have to converge to a common vector in the intersection of $m$ constraint sets which are closed and convex, under the condition that agent $i$ only knows the $i$-th constraint set. For this problem, Nedić *et al.* [7] proposed a distributed algorithm called the projected consensus algorithm, in which each agent performs the projection onto its constraint set and the weighted averaging of the states of the agents in its neighborhood, alternately. They considered the case where the directed graph, which we call the communication graph in this letter, representing the information exchange among agents is time-varying, and proved under some assumptions that a constrained consensus is reached by their algorithm for any initial condition. The assumptions include the double stochasticity of the weights and the existence of a positive lower bound for the weights on self-loops. By the double stochasticity, we mean that not only the sum of the weights on incoming edges but also the sum of the weights on outgoing edges from each vertex are one. Recently, Nedić and Liu [9] proved the convergence of the algorithm under weaker assumptions. In their analysis, the double stochasticity of the weights is relaxed to the single stochasticity, which means that only the sum of the weights on incoming edges to each vertex is one. However, the existence of a positive lower bound for the weights on self-loops is still needed.

The objective of this letter is to show that the above-mentioned assumption on the weights on self-loops is not needed to guarantee the convergence of the projected consensus algorithm when the communication graph is time-invariant. To be more specific, we prove that the states of all agents converge to a common vector in the intersection of all constraint sets under the following assumptions: i) the intersection of all constraint sets is nonempty, ii) the communication graph is time-invariant, strongly connected and aperiodic, and iii) the sum of the weights on incoming edges to each vertex of the graph is one. A directed graph is said to be aperiodic if the greatest common divisor of the lengths of cycles is one. For example, if there exists a vertex having a self-loop then the communication graph is aperiodic. A key difference between the proof given in this letter and the one in [9] is that $\max_{1\leq i\leq m}\{\|\boldsymbol{x}^i(k)-\boldsymbol{x}\|\}$ is employed as a candidate Lyapunov function in the former while $\sum_{i=1}^{m}\pi_i(k)\|\boldsymbol{x}^i(k)-\boldsymbol{x}\|^2$ is used in the latter, where $\boldsymbol{x}^i(k)$ is the state of agent $i$ at time $k$ which is expected to converge to a common point in the intersection of $m$ constraints, $\boldsymbol{x}$ is any vector in the intersection of all constraint sets, and $\pi_i(k)$ are the suitably defined time-varying positive weights.[1] Our Lyapunov function is similar to the one used in [24] where multi-agent systems with continuous-time dynamics is studied.

In order to confirm the validity of the theoretical analysis, we perform some numerical experiments for a system of

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[1] Note that the communication graph is assumed to be time-varying in [9]. If it is time-invariant, $\pi_i(k)$ become positive constants.

linear equations with nonnegativity constraints [25], [26]. In this problem, the projection onto each constraint set is easily done by using the algorithm presented in [25]. It is shown experimentally that the states of all agents converge to a common solution vector exponentially if our sufficient condition is satisfied.

## II. Projected Consensus Algorithm

In what follows, we denote the set of real numbers and the set of nonnegative integers by $\mathbb{R}$ and $\mathbb{Z}_+$, respectively. Also, we use $\|\boldsymbol{x}\|$ to denote the Euclidean norm of a vector $\boldsymbol{x}$. Let us consider a network of $m$ agents labeled by $1, 2, \dots, m$. Given $m$ closed convex sets $X_1, X_2, \dots, X_m \subseteq \mathbb{R}^n$, we want to make agents in the network find a point $\boldsymbol{x} \in X = \cap_{i=1}^m X_i$ in a distributed manner. That is, we assume that i) agent $i$ only knows $X_i$, ii) agent $i$ has its own state $\boldsymbol{x}^i \in X_i$, iii) agents exchange their states through communication channels, iv) agent $i$ updates its own state $\boldsymbol{x}^i$ based on the states of the agents in its neighborhood.

Nedić *et al.* [7] proposed a distributed algorithm called the projected consensus algorithm for agents to solve the constrained consensus problem. When the communication among agents is time-invariant, their algorithm is described by

$$\boldsymbol{x}^i(k) = \begin{cases} P_{X_i}[\boldsymbol{s}^i], & \text{if } k = 0, \\ P_{X_i}\big[\boldsymbol{w}^i(k-1)\big], & \text{if } k \geq 1, \end{cases} \tag{1}$$

with

$$\boldsymbol{w}^i(k) = \sum_{j=1}^m a_j^i \boldsymbol{x}^j(k)$$

where $\boldsymbol{x}^i(k) \in X_i$ denotes the state of agent $i$ at time $k \in \mathbb{Z}_+$, $\boldsymbol{s}^i \in \mathbb{R}^n$ is the seed to determine the initial state $\boldsymbol{x}^i(0)$ of agent $i$, $\boldsymbol{w}^i(k) \in \mathbb{R}^n$ is the weighted average of the states of all agents at time $k$ computed by agent $i$, $a_j^i$ is a nonnegative number representing the weight of the state $\boldsymbol{x}^j(k-1)$ in $\boldsymbol{w}^i(k)$, $P_{X_i}[\bar{\boldsymbol{x}}]$ is the projection of $\bar{\boldsymbol{x}}$ on $X_i$, that is, $P_{X_i}[\bar{\boldsymbol{x}}] = \arg\min_{\boldsymbol{x} \in X_i} \|\bar{\boldsymbol{x}} - \boldsymbol{x}\|$.

As is easily seen from (1), agent $i$ receives information from agent $j$ if and only if $a_j^i > 0$. In this sense, the information exchange or communication among agents is determined by the values of $a_j^i$, and can be represented by the directed graph $G = (V, E)$ where $V = \{1, 2, \dots, m\}$ is the vertex set, $E = \{(j, i) \mid a_j^i > 0\}$ is the edge set. In what follows, this directed graph is called the communication graph of the multi-agent network.

As far as (1) is considered, the result of the convergence analysis done by Nedić and Liu [9] can be stated as follows.

*Theorem 1:* Let the set $X = \cap_{i=1}^m X_i$ be nonempty. If the three assumptions below are valid then for any seeds $\boldsymbol{s}^1, \boldsymbol{s}^2, \dots, \boldsymbol{s}^m$ there exists a point $\boldsymbol{x} \in X$ such that $\lim_{k\to\infty} \boldsymbol{x}^i(k) = \boldsymbol{x}$ for all $i$.

1) $\sum_{j=1}^m a_j^i = 1$ for all $i$.
2) The communication graph $G$ is strongly connected.
3) $a_i^i > 0$ for all $i$.

The first assumption is natural because $\boldsymbol{w}^i(k-1) = \sum_{j=1}^m a_j^i \boldsymbol{x}^j(k-1)$ can be seen as a weighted average of the states $\boldsymbol{x}^j(k-1)$ $(j = 1, 2, \dots, m)$. The second assumption is also natural because agents cannot reach a consensus without the strong connectivity of the network in general. On the other hand, it is not clear whether the third assumption is really needed, though it plays an important role in the convergence analysis of Nedić and Liu [9]. In the next section, we show that the convergence of (1) is guaranteed even if the third assumption is not satisfied.

## III. Convergence Analysis

In this section, we prove the convergence of (1) under the following assumption.

*Assumption 1:* The weights $\{a_j^i\}$ and the associated communication graph $G$ satisfy the following three conditions.

1) $\sum_{j=1}^m a_j^i = 1$ for all $i$.
2) $G$ is strongly connected.
3) $G$ is aperiodic.

A directed graph is said to be aperiodic if the greatest common divisor of the lengths[2] of its cycles is one. For example, if there exists a vertex having a self-loop then the communication graph is aperiodic, because it has a cycle with length one. The following lemma describes an important property of strongly connected aperiodic directed graphs, which plays a crucial role in later analysis.

*Lemma 1 (See [27]):* If a directed graph $G = (V, E)$ is strongly connected and aperiodic then there exists a positive integer $r$ such that there exists a directed walk from vertex $j$ to vertex $i$ with length $r$ for all $(i, j) \in V \times V$.

For a point $\boldsymbol{x} \in X = \cap_{i=1}^m X_i$, we define the function $f_{\boldsymbol{x}} : \prod_{i=1}^m X_i \to \mathbb{R}$ as

$$f_{\boldsymbol{x}}(\boldsymbol{x}^1(k), \boldsymbol{x}^2(k), \dots, \boldsymbol{x}^m(k)) = \max_{1\leq i\leq m} \left\{\|\boldsymbol{x}^i(k) - \boldsymbol{x}\|\right\}.$$

The left-hand side of this equation is denoted by $f_{\boldsymbol{x}}(k)$ for simplicity. We also define the set $\Omega \subset \prod_{i=1}^m X_i$ as

$$\Omega = \left\{ (\boldsymbol{x}^1, \dots, \boldsymbol{x}^m) \in \prod_{i=1}^m X_i \;\middle|\; \boldsymbol{x}^1 = \cdots = \boldsymbol{x}^m \in X \right\}.$$

We first give two lemmas regarding the monotonicity of the sequence $\{f_{\boldsymbol{x}}(k)\}_{k=0}^\infty$.

*Lemma 2:* Suppose that the first condition in Assumption 1 holds. Let $\boldsymbol{x}$ be any point in $X$. Then $\|\boldsymbol{w}^i(k) - \boldsymbol{x}\| \leq f_{\boldsymbol{x}}(k)$ holds for all $i \in \{1, 2, \dots, m\}$ and $k \in \mathbb{Z}_+$. In particular, if there exist $j_1$ and $j_2$ such that $a_{j_1}^i > 0$, $a_{j_2}^i > 0$ and $\boldsymbol{x}^{j_1}(k) \neq \boldsymbol{x}^{j_2}(k)$ then $\|\boldsymbol{w}^i(k) - \boldsymbol{x}\| < f_{\boldsymbol{x}}(k)$ holds.

*Proof:* Using the first condition in Assumption 1 and the strong convexity of the function $g(\boldsymbol{y}) = \|\boldsymbol{y}\|^2$ defined on $\mathbb{R}^n$ (see also [9, Lemma 5]), we have

$$\begin{aligned} &\|\boldsymbol{w}^i(k) - \boldsymbol{x}\|^2 \\ &\quad= \left\| \sum_{j=1}^m a_j^i (\boldsymbol{x}^j(k) - \boldsymbol{x}) \right\|^2 \\ &\quad= \sum_{j=1}^m a_j^i \|\boldsymbol{x}^j(k) - \boldsymbol{x}\|^2 - \frac{1}{2} \sum_{j_1=1}^m \sum_{j_2=1}^m a_{j_1}^i a_{j_2}^i \|\boldsymbol{x}^{j_1}(k) - \boldsymbol{x}^{j_2}(k)\|^2 \\ &\quad\leq \sum_{j=1}^m a_j^i f_{\boldsymbol{x}}(k)^2 - \frac{1}{2} \sum_{j_1=1}^m \sum_{j_2=1}^m a_{j_1}^i a_{j_2}^i \|\boldsymbol{x}^{j_1}(k) - \boldsymbol{x}^{j_2}(k)\|^2 \end{aligned}$$

[2] We are assuming that all edges have length one.

$$
\begin{aligned}
&= f_{\boldsymbol{x}}(k)^2 - \frac{1}{2}\sum_{j_1=1}^{m}\sum_{j_2=1}^{m} a^i_{j_1} a^i_{j_2} \left\| \boldsymbol{x}^{j_1}(k) - \boldsymbol{x}^{j_2}(k) \right\|^2 \\
&\le f_{\boldsymbol{x}}(k)^2.
\end{aligned}
$$

The last inequality holds strictly if there exist $j_1$ and $j_2$ such that $a^i_{j_1} > 0$, $a^i_{j_2} > 0$ and $\boldsymbol{x}^{j_1}(k) \neq \boldsymbol{x}^{j_2}(k)$. ■

*Lemma 3:* Suppose that the first condition in Assumption 1 holds. Let $\boldsymbol{x}$ be any point in $X$. Then the sequence $\{f_{\boldsymbol{x}}(k)\}_{k=0}^{\infty}$ is monotone nonincreasing.

*Proof:* Because $X_i$ is a closed convex set, $\|P_{X_i}[\boldsymbol{y}] - \boldsymbol{x}\| \le \|\boldsymbol{y} - \boldsymbol{x}\|$ holds for any $\boldsymbol{y} \in \mathbb{R}^n$ [7]. Taking this inequality and Lemma 2 into account, we have

$$
\begin{aligned}
f_{\boldsymbol{x}}(k+1) &= \max_{1\le i\le m}\left\{\left\|\boldsymbol{x}^i(k+1) - \boldsymbol{x}\right\|\right\} \\
&= \max_{1\le i\le m}\left\{\left\|P_{X_i}[\boldsymbol{w}^i(k)] - \boldsymbol{x}\right\|\right\} \\
&\le \max_{1\le i\le m}\left\{\left\|\boldsymbol{w}^i(k) - \boldsymbol{x}\right\|\right\} \\
&\le f_{\boldsymbol{x}}(k)
\end{aligned}
$$

which means that $\{f_{\boldsymbol{x}}(k)\}_{k=0}^{\infty}$ is monotone nonincreasing. ■

We next give two lemmas to show that if a constrained consensus is not reached at some time instance then $f_{\boldsymbol{x}}(k)$ strictly decreases within a fixed number of iterations.

*Lemma 4:* Suppose that the first condition in Assumption 1 holds. Let $\boldsymbol{x}$ be any point in $X$. Then $\|\boldsymbol{x}^i(k+1) - \boldsymbol{x}\| = f_{\boldsymbol{x}}(k)$ holds if and only if there exists a vector $\boldsymbol{y}$ such that i) $\boldsymbol{y} \in X_i$, ii) $\boldsymbol{x}^j(k) = \boldsymbol{y}$ for all $j$ with $a^i_j > 0$ and iii) $\|\boldsymbol{y} - \boldsymbol{x}\| = f_{\boldsymbol{x}}(k)$.

*Proof:* Suppose that there exists a vector $\boldsymbol{y}$ satisfying the three conditions. Then we have

$$
\begin{aligned}
\|\boldsymbol{x}^i(k+1) - \boldsymbol{x}\| &= \|P_{X_i}[\boldsymbol{w}^i(k)] - \boldsymbol{x}\| \\
&= \|P_{X_i}[\boldsymbol{y}] - \boldsymbol{x}\| \\
&= \|\boldsymbol{y} - \boldsymbol{x}\| \\
&= f_{\boldsymbol{x}}(k).
\end{aligned}
$$

Suppose next that there does not exist such a vector $\boldsymbol{y}$. In this case, the following three cases have to be considered.

1) There exist $j_1$ and $j_2$ such that $a^i_{j_1} > 0$, $a^i_{j_2} > 0$ and $\boldsymbol{x}^{j_1}(k) \neq \boldsymbol{x}^{j_2}(k)$.
2) There exists a vector $\boldsymbol{y}$ such that $\boldsymbol{x}^j(k) = \boldsymbol{y}$ for all $j$ with $a^i_j > 0$, but $\boldsymbol{y} \notin X_i$.
3) There exists a vector $\boldsymbol{y} \in X_i$ such that $\boldsymbol{x}^j(k) = \boldsymbol{y}$ for all $j$ with $a^i_j > 0$, but $\|\boldsymbol{y} - \boldsymbol{x}\| < f_{\boldsymbol{x}}(k)$.

In the first case, by Lemma 2, we have

$$
\begin{aligned}
\|\boldsymbol{x}^i(k+1) - \boldsymbol{x}\| &= \left\|P_{X_i}[\boldsymbol{w}^i(k)] - \boldsymbol{x}\right\| \\
&\le \|\boldsymbol{w}^i(k) - \boldsymbol{x}\| \\
&< f_{\boldsymbol{x}}(k).
\end{aligned}
$$

In the second case, it follows from [7, Lemma 1 (b)] that $\|\boldsymbol{x}^i(k+1) - \boldsymbol{x}\| = \|P_{X_i}[\boldsymbol{y}] - \boldsymbol{x}\| < \|\boldsymbol{y} - \boldsymbol{x}\| \le f_{\boldsymbol{x}}(k)$. In the last case, we have $\|\boldsymbol{x}^i(k+1) - \boldsymbol{x}\| = \|P_{X_i}[\boldsymbol{y}] - \boldsymbol{x}\| \le \|\boldsymbol{y} - \boldsymbol{x}\| < f_{\boldsymbol{x}}(k)$. ■

*Lemma 5:* Suppose that all of the three conditions in Assumption 1 hold. Let $\boldsymbol{x}$ be any point in $X$. If $(\boldsymbol{x}^1(k^*), \boldsymbol{x}^2(k^*), \dots, \boldsymbol{x}^m(k^*)) \notin \Omega$ for some $k^* \in \mathbb{Z}_+$ then $f_{\boldsymbol{x}}(k^* + r) < f_{\boldsymbol{x}}(k^*)$ holds, where $r$ is the integer having the property given in Lemma 1 for the communication graph.

*Proof:* Suppose that $(\boldsymbol{x}^1(k^*), \boldsymbol{x}^2(k^*), \dots, \boldsymbol{x}^m(k^*)) \notin \Omega$ for some $k^* \in \mathbb{Z}_+$. We prove the inequality $f_{\boldsymbol{x}}(k^*+r) < f_{\boldsymbol{x}}(k^*)$ by contradiction. Assume that this is not the case. Then $\|\boldsymbol{x}^i(k^* + r) - \boldsymbol{x}\| = f_{\boldsymbol{x}}(k^*+r) = f_{\boldsymbol{x}}(k^*+r-1) = \dots = f_{\boldsymbol{x}}(k^*)$ for some $i$. Using Lemma 4, we can say that $\boldsymbol{x}^j(k^*+r-1) = \boldsymbol{x}^i(k^*+r) \in X_j$ for all $j$ with $a^i_j > 0$. This means that $\|\boldsymbol{x}^j(k^* + r - 1) - \boldsymbol{x}\| = f_{\boldsymbol{x}}(k^* + r - 2)$ for all $j$ with $a^i_j > 0$. Using Lemma 4 again, we can say that $\boldsymbol{x}^j(k^* + r - 2) = \boldsymbol{x}^i(k^* + r) \in X_j$ for all $j$ such that there exists a directed walk from vertex $j$ to vertex $i$ with length two. Repeating this argument, we can finally say that $\boldsymbol{x}^j(k^*) = \boldsymbol{x}^i(k^* + r) \in X_j$ for all $j$ such that there exists a directed walk from vertex $j$ to vertex $i$ with length $r$. This implies by Lemma 1 that $\boldsymbol{x}^j(k^*) = \boldsymbol{x}^i(k^* + r) \in X$ for all $j \in V$, which contradicts the assumption that $(\boldsymbol{x}^1(k^*), \boldsymbol{x}^2(k^*), \dots, \boldsymbol{x}^m(k^*)) \notin \Omega$. ■

Now we are ready to prove the main result of this letter.

*Theorem 2:* Let $X = \cap_{i=1}^m X_i$ be nonempty. Suppose that all of the three conditions in Assumption 1 hold. Then, for any seeds $\boldsymbol{s}^1, \boldsymbol{s}^2, \dots, \boldsymbol{s}^m$, the sequence $\{(\boldsymbol{x}^1(k), \boldsymbol{x}^2(k), \dots, \boldsymbol{x}^m(k))\}_{k=0}^{\infty}$ generated by the projected consensus algorithm (1) converges to a point in $\Omega$.

*Proof:* Let $r$ be the positive integer having the property given in Lemma 1 for the communication graph. Let the mapping that maps $(\boldsymbol{x}^1(k), \boldsymbol{x}^2(k), \dots, \boldsymbol{x}^m(k))$ to $(\boldsymbol{x}^1(k + r), \boldsymbol{x}^2(k + r), \dots, \boldsymbol{x}^m(k + r))$ by using (1) be denoted by $A : \prod_{i=1}^m X_i \to \prod_{i=1}^m X_i$. Then, the following statements hold true.

1) The sequence $\{A^l(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0))\}_{l=0}^{\infty}$ is contained in a bounded subset of $\prod_{i=1}^m X_i$, because it follows from Lemma 3 that
$$
\begin{aligned}
&f_{\boldsymbol{x}}(A^l(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0))) \\
&\quad \le f_{\boldsymbol{x}}(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0)) = \max_{1\le i\le m}\left\{\|\boldsymbol{x}^i(0) - \boldsymbol{x}\|\right\}
\end{aligned}
$$
holds for all $l \in \mathbb{Z}_+$.
2) If $A^l(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0)) \notin \Omega$ then
$$
\begin{aligned}
&f_{\boldsymbol{x}}(A^{l+1}(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0))) \\
&\quad < f_{\boldsymbol{x}}(A^l(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0))),
\end{aligned}
$$
and if $A^l(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0)) \in \Omega$ then
$$
\begin{aligned}
&A^{l+1}(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0)) \\
&\quad = A^l(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0)).
\end{aligned}
$$
3) $A$ is continuous because it is a composite mapping of two kinds of continuous mappings: the weighted averaging and the projection onto a closed convex set.

From these observations and Zangwill's global convergence theorem [28], we can say that the sequence $\{A^l(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0))\}_{l=0}^{\infty}$ generated by (1) has at least one convergent subsequence and the limit of any convergent subsequence is a point in $\Omega$. Let $\{A^l(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0))\}_{l\in\mathcal{L}\subset\mathbb{Z}_+}$ be any convergent subsequence of $\{A^l(\boldsymbol{x}^1(0), \boldsymbol{x}^2(0), \dots, \boldsymbol{x}^m(0))\}_{l=0}^{\infty}$ and let $(\boldsymbol{u}, \boldsymbol{u}, \dots, \boldsymbol{u}) \in \Omega$ be its limit point. Then the corresponding subsequence $\{f_{\boldsymbol{u}}(rl)\}_{l\in\mathcal{L}}$ of $\{f_{\boldsymbol{u}}(k)\}_{k=0}^{\infty}$ converges to zero. Taking this result and Lemma 3 into account, we can conclude that the entire sequence $\{f_{\boldsymbol{u}}(k)\}_{k=0}^{\infty}$ converges to zero, which means that $\{(\boldsymbol{x}^1(k), \boldsymbol{x}^2(k), \dots, \boldsymbol{x}^m(k))\}_{k=0}^{\infty}$ converges to $(\boldsymbol{u}, \boldsymbol{u}, \dots, \boldsymbol{u})$. ■

**Algorithm 1** Projection Onto $X_i = \{\boldsymbol{x} \mid \boldsymbol{b}_i\boldsymbol{x} = c_i, \boldsymbol{x} \geq \boldsymbol{0}\}$

**Input:** $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $\boldsymbol{b}_i = (b_{i1}, b_{i2}, \ldots, b_{in}) \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$.

**Output:** $\boldsymbol{y} \in \{P_{X_i}[\boldsymbol{x}], -\boldsymbol{1}\}$ ($\boldsymbol{y} = -\boldsymbol{1}$ means $X_i$ is empty)

1: Set $\boldsymbol{y} \leftarrow \boldsymbol{x}$.
2: If $\boldsymbol{b}_i = \boldsymbol{0}$ then go to Step 5.
3: Set $\boldsymbol{y} \leftarrow \left(\boldsymbol{I} - \boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{b}_i/\|\boldsymbol{b}_i\|^2\right)\hat{\boldsymbol{x}} + c_i\boldsymbol{b}_i^{\mathrm{T}}/\|\boldsymbol{b}_i\|^2$.
4: If $\boldsymbol{y} \geq \boldsymbol{0}$ then return $\boldsymbol{y}$ and stop.
5: Set $J \leftarrow \{j \mid y_j < 0\}$.
6: Set $y_j \leftarrow 0$ and $b_{ij} \leftarrow 0$ for all $j \in J$.
7: If $\boldsymbol{b}_i \neq \boldsymbol{0}$ then go to Step 3.
8: If $c_i \neq 0$ set $\boldsymbol{y} \leftarrow -\boldsymbol{1}$. Return $\boldsymbol{y}$ and stop.

The convergence of the algorithm (1) is now guaranteed by Theorem 2. What we need to do next is to determine the convergence rate. The results given in [9] can be used when $a_i^i > 0$ for all $i$. However, when $a_i^i = 0$ for some $i$, no meaningful conclusion can be drawn by the direct use of the results in [9]. The analysis of the convergence rate for this case is left for a future work.

The extension of our results to multi-agent networks having switching topology is also an important issue. It is easy to see from the proof of Theorem 2 that if the time-varying weights always satisfy Assumption 1 and each topology lasts at least $r$ time steps, where $r$ is the integer given in Lemma 1 and determined from the topology, then the same conclusion as Theorem 2 is drawn. However, this condition is too severe. Further studies are needed to obtain milder condition.

## IV. Application to Linear Equations With Nonnegativity Constraints

As an example of the constrained consensus problem, we consider the problem of finding a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ satisfying

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{c}, \quad \boldsymbol{x} \geq \boldsymbol{0} \tag{2}$$

where $\boldsymbol{B} = (b_{ij}) \in \mathbb{R}^{m\times n}$ is a constant matrix and $\boldsymbol{c} \in \mathbb{R}^m$ is a constant vector. The inequality $\boldsymbol{x} \geq \boldsymbol{0}$ means that all entries of $\boldsymbol{x}$ are nonnegative. Let $\boldsymbol{b}_i$ be the $i$-th row of $\boldsymbol{B}$ and let $X_i \triangleq \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{b}_i\boldsymbol{x} = c_i,\ \boldsymbol{x} \geq \boldsymbol{0}\}$ for $i = 1, 2, \ldots, m$. Then the problem is equivalent to finding a point in $X = \cap_{i=1}^m X_i$, and thus can be solved by (1) if $X$ is nonempty.

In order to confirm the validity of the theoretical analysis in the previous section, we conduct some numerical experiments. Various kinds of methods can be used for computing $P_{X_i}[\boldsymbol{x}]$. In our experiments, we use the algorithm proposed by Michelot [25], which is described in Algorithm 1, because it is simple and easy to implement.

In the first experiment, we consider the situation in which the problem (2) with

$$\boldsymbol{B} = \begin{pmatrix} -65 & 33 & 28 & 71 & 59 & -66 \\ 56 & 29 & 38 & -75 & -33 & 86 \\ 72 & -91 & 73 & 14 & -70 & 43 \\ 16 & 9 & -23 & 63 & -62 & 85 \\ -21 & 86 & -6 & 23 & 56 & 24 \end{pmatrix},$$
$$\boldsymbol{c} = (-115, 978, 612, 148, 399)^{\mathrm{T}}$$

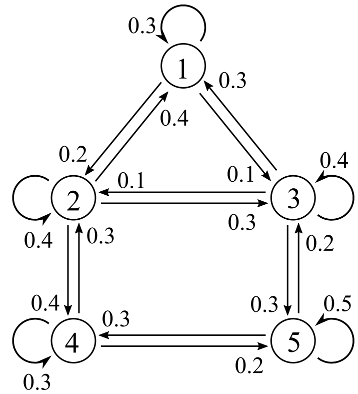



Fig. 1. Communication graph of the multi-agent network used for the first experiment. Numbers beside directed edges represent weights.

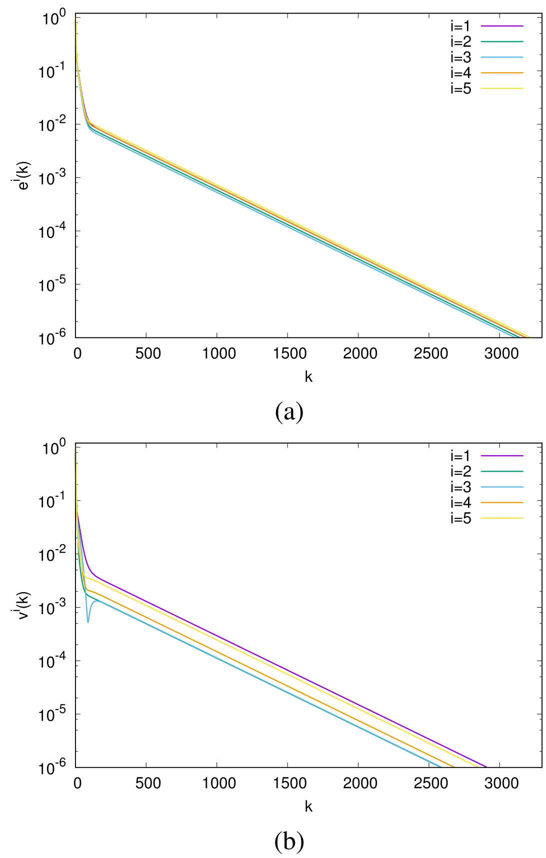
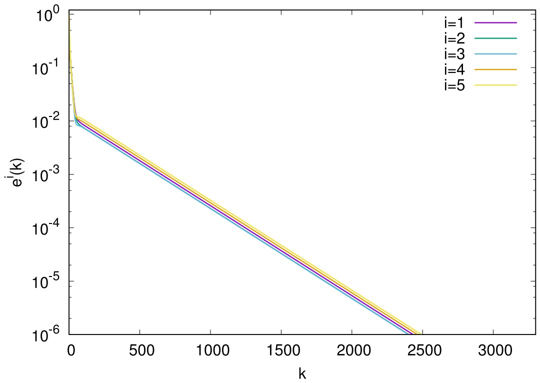


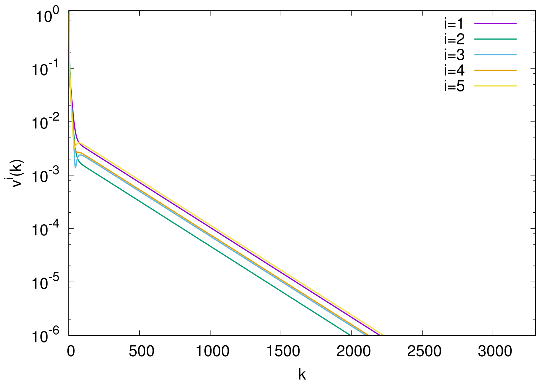

Fig. 2. Results of the first experiment. (a) Time evolution of $e^i(k) = \|\boldsymbol{B}\boldsymbol{x}^i(k) - \boldsymbol{c}\|/\|\boldsymbol{c}\|$. (b) Time evolution of $v^i(k) = \|\boldsymbol{x}^i(k) - (1/5)\sum_{j=1}^5 \boldsymbol{x}^j(k)\|/\|(1/5)\sum_{j=1}^5 \boldsymbol{x}^j(k)\|$.

is solved by a five-agent network. Note that $X$ is nonempty in this case because $\boldsymbol{x} = (5, 4, 8, 0, 2, 4)^{\mathrm{T}}$ is a solution of $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{c}$. The weights on the states are set to

$$\boldsymbol{A} = \begin{pmatrix} a_1^1 & \cdots & a_5^1 \\ \vdots & \ddots & \vdots \\ a_1^5 & \cdots & a_5^5 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.4 & 0.3 & 0.0 & 0.0 \\ 0.2 & 0.4 & 0.1 & 0.3 & 0.0 \\ 0.1 & 0.3 & 0.4 & 0.0 & 0.2 \\ 0.0 & 0.4 & 0.0 & 0.3 & 0.3 \\ 0.0 & 0.0 & 0.3 & 0.2 & 0.5 \end{pmatrix}$$

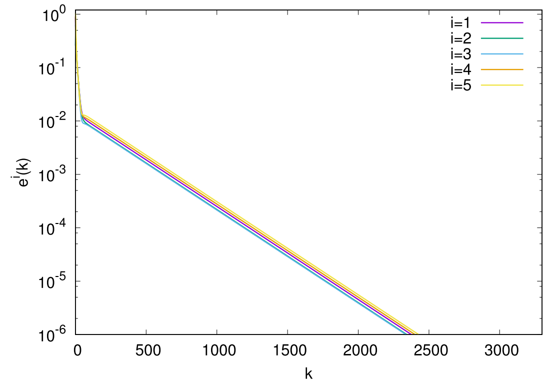



(a)

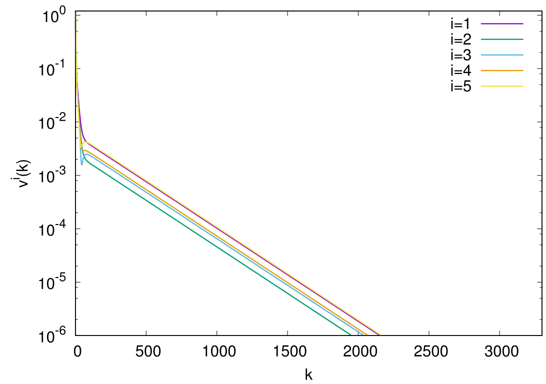



(b)

Fig. 3. Results of the second experiment. (a) Time evolution of $e^i(k) = \|\boldsymbol{B}\boldsymbol{x}^i(k) - \boldsymbol{c}\|/\|\boldsymbol{c}\|$. (b) Time evolution of $v^i(k) = \|\boldsymbol{x}^i(k) - (1/5)\sum_{j=1}^{5} \boldsymbol{x}^j(k)\|/\|(1/5)\sum_{j=1}^{5} \boldsymbol{x}^j(k)\|$.




(a)




(b)

Fig. 4. Results of the third experiment. (a) Time evolution of $e^i(k) = \|\boldsymbol{B}\boldsymbol{x}^i(k) - \boldsymbol{c}\|/\|\boldsymbol{c}\|$. (b) Time evolution of $v^i(k) = \|\boldsymbol{x}^i(k) - (1/5)\sum_{j=1}^{5} \boldsymbol{x}^j(k)\|/\|(1/5)\sum_{j=1}^{5} \boldsymbol{x}^j(k)\|$.

and the entries of the seeds $\boldsymbol{s}^1, \boldsymbol{s}^2, \ldots, \boldsymbol{s}^m$ are chosen randomly from the interval $[-1, 1]$. Note that $\boldsymbol{A}$ satisfies all conditions in Assumption 1. The communication graph of the multi-agent network is shown in Fig. 1. It is easy to see that the graph is strongly connected.

The results of the first experiment are shown in Fig. 2. Figure 2 (a) shows the time evolution of $e^i(k) = \|\boldsymbol{B}\boldsymbol{x}^i(k) - \boldsymbol{c}\|/\|\boldsymbol{c}\|$ for $i = 1, 2, \ldots, 5$. It is clear from the figure that $e^i(k)$ converges exponentially to zero for all $i$. This means that each agent eventually finds a vector $\boldsymbol{x}$ satisfying (2). Figure 2 (b) shows the time evolution of $v^i(k) = \|\boldsymbol{x}^i(k) - (1/5)\sum_{j=1}^{5} \boldsymbol{x}^j(k)\|/\|(1/5)\sum_{j=1}^{5} \boldsymbol{x}^j(k)\|$ for $i = 1, 2, \ldots, 5$. It is clear from the figure that $v^i(k)$ also converges exponentially to zero for all $i$. This means that the states $\boldsymbol{x}^i(k)$ $(i = 1, 2, \ldots, 5)$ converge to the same value, that is, five agents eventually reach a consensus.

In the second experiment, we consider the situation in which the same problem as in the first experiment is solved by a different five-agent network. The weights on the states are set to

$$\boldsymbol{A} = \begin{pmatrix} 0.3 & 0.4 & 0.3 & 0.0 & 0.0 \\ 0.2 & 0.0 & 0.5 & 0.3 & 0.0 \\ 0.5 & 0.3 & 0.0 & 0.0 & 0.2 \\ 0.0 & 0.4 & 0.0 & 0.0 & 0.6 \\ 0.0 & 0.0 & 0.3 & 0.7 & 0.0 \end{pmatrix}$$

and the entries of the seeds $\boldsymbol{s}^1, \boldsymbol{s}^2, \ldots, \boldsymbol{s}^m$ are set to the same values as in the first experiment. The matrix $\boldsymbol{A}$ above is obtained from the one in the first experiment by setting $a_i^i$ $(i = 2, 3, 4, 5)$ to zero and then by increasing the values of the smallest offdiagonal entries in each row so that the first condition in Assumption 1 is satisfied. The communication graph of this multi-agent network is strongly connected. In addition, it is aperiodic because vertex 1 has a self-loop.

The results of the second experiment are shown in Fig. 3. Figures 3 (a) and 3 (b) show the time evolution of $e^i(k) = \|\boldsymbol{B}\boldsymbol{x}^i(k) - \boldsymbol{c}\|/\|\boldsymbol{c}\|$ and $v^i(k) = \|\boldsymbol{x}^i(k) - (1/5)\sum_{j=1}^{5} \boldsymbol{x}^j(k)\|/\|(1/5)\sum_{j=1}^{5} \boldsymbol{x}^j(k)\|$, respectively, for $i = 1, 2, \ldots, 5$. As in the first experiment, both $e^i(k)$ and $v^i(k)$ converge exponentially to zero for all $i$. But the more important point is that their convergence rates are faster than those in the first experiment.

In the third experiment, we consider the situation in which the same problem as in the first experiment is solved by a five-agent network with $a_i^i = 0$ for all $i$. The weights on the states are set to

$$\boldsymbol{A} = \begin{pmatrix} 0.0 & 0.4 & 0.6 & 0.0 & 0.0 \\ 0.2 & 0.0 & 0.5 & 0.3 & 0.0 \\ 0.5 & 0.3 & 0.0 & 0.0 & 0.2 \\ 0.0 & 0.4 & 0.0 & 0.0 & 0.6 \\ 0.0 & 0.0 & 0.3 & 0.7 & 0.0 \end{pmatrix}$$

and the entries of the seeds $\boldsymbol{s}^1, \boldsymbol{s}^2, \ldots, \boldsymbol{s}^m$ are set to the same values as in the first experiment. The matrix $\boldsymbol{A}$ above is obtained from the one in the first experiment by setting $a_i^i$ to zero for all $i$ and then by increasing the values of the smallest offdiagonal entries in each row so that the first condition in Assumption 1 is satisfied. The communication graph of this multi-agent network is strongly connected. In addition, it is aperiodic because the greatest common divisor of the lengths of cycles, which are three, four and five, is one.

The results of the third experiment are shown in Fig. 4. Figures 4 (a) and 4 (b) show the time evolution of $e^i(k) = \|\boldsymbol{B}\boldsymbol{x}^i(k) - \boldsymbol{c}\|/\|\boldsymbol{c}\|$ and $v^i(k) = \|\boldsymbol{x}^i(k) - (1/5)\sum_{j=1}^{5} \boldsymbol{x}^j(k)\|/\|(1/5)\sum_{j=1}^{5} \boldsymbol{x}^j(k)\|$, respectively, for $i = 1, 2, \ldots, 5$. As in the first and second experiments, both $e^i(k)$ and $v^i(k)$ converge exponentially to zero for all $i$. In addition, their convergence rates are faster than those in the second experiment.

The results of these experiments suggest that the smaller the weights on self-loops are, the faster the convergence rate of the algorithm is. We do not address this issue in this letter. Further studies are needed to determine whether this statement holds true.

## V. Conclusion

The convergence property of the projected consensus algorithm proposed by Nedić *et al.* [7] was studied in this letter. Restricting ourselves to the case where the communication graph is time-invariant, we have proved, under weaker conditions than those in the literature, that a constrained consensus is achieved by the algorithm. In our proof, self-loops in the communication graph are not necessarily needed, while it is assumed that every vertex has a self-loop in, for example, the convergence analysis done by Nedić and Liu [9]. We have also confirmed the validity of the theoretical analysis through numerical experiments in which a system of linear equations with nonnegativity constraints are solved by the algorithm. Experimental results indicate that the convergence rate may become faster as the weights on self-loops become smaller. Clarifying the relationship between the weights and the convergence rate is left for a future work.

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