Chapter **8**

Lines and Planes

In this chapter, you will revisit your knowledge of intersecting lines in two dimensions and extend those ideas into three dimensions. You will investigate the nature of planes and intersections of planes and lines, and you will develop tools and methods to describe the intersections of planes and lines.

By the end of this chapter, you will

- recognize that the solution points (x, y) in two-space of a single linear equation in two variables form a line and that the solution points (x, y) in two-space of a system of two linear equations in two variables determine the point of intersection of two lines, if the lines are not coincident or parallel
- determine, through investigation with and without technology, that the solution points (x, y, z) in three-space of a single linear equation in three variables form a plane and that the solution points (x, y, z) in three-space of a system of two linear equations in three variables form the line of intersection of two planes, if the planes are not coincident or parallel
- determine, through investigation using a variety of tools and strategies, different geometric configurations of combinations of up to three lines and/or planes in three-space; organize the configurations based on whether they intersect and, if so, how they intersect
- recognize a scalar equation for a line in two-space to be an equation of the form $Ax + By + C = 0$, represent a line in two-space using a vector equation and parametric equations, and make connections between a scalar equation, a vector equation, and parametric equations of a line in two-space
- recognize that a line in three-space cannot be represented by a scalar equation, and represent a line in three-space using the scalar equations of two intersecting planes and using vector and parametric equations
- recognize a normal to a plane geometrically and algebraically, and determine, through investigation, some geometric properties of the plane
- recognize a scalar equation for a plane in three-space to be an equation of the form $Ax + By + Cz + D = 0$ whose solution points make up the plane, determine the intersection of three planes represented using scalar equations by solving a system of three linear equations in three unknowns algebraically, and make connections between the algebraic solution and the geometric configuration of three planes
- determine, using properties of a plane, the scalar, vector, or parametric form, given another form
- determine the equation of a plane in its scalar, vector, or parametric form, given another of these forms
- solve problems relating to lines and planes in three-space that are represented in a variety of ways and involving distances or intersections, and interpret the result geometrically

Prerequisite Skills

Linear Relations

1. Make a table of values and graph each linear function.

a) $y = 4x - 8$

b) $y = -2x + 3$

c) $3x - 5y = 15$

d) $5x + 6y = 20$

2. Find the x- and y-intercepts of each linear function.

a) $y = 3x + 7$

b) $y = 5x - 10$

c) $2x - 9y = 18$

d) $4x + 8y = 9$

3. Graph each line.

a) slope $= -2$; y-intercept $= 5$

b) slope $= 0.5$; x-intercept $= 15$

c) slope $= \frac{3}{5}$, through $(1, -3)$

d) slope $= \frac{-8}{3}$, through $(-5, 6)$

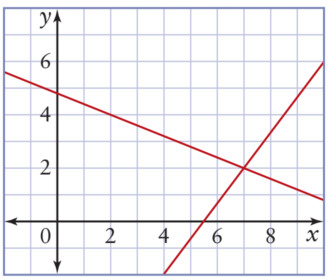
e) $2x - 6 = 0$

f) $y + 4 = 0$

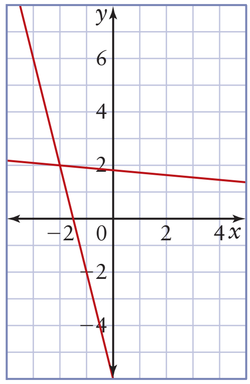
Solving Linear Systems

4. Determine the coordinates of the point of intersection of each linear system.

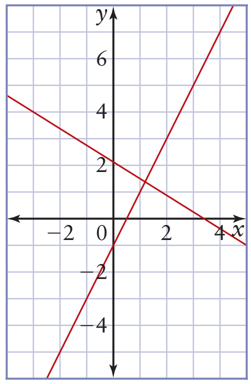
a)



b)



c)



5. Solve each linear system.

a) $y = 3x + 2$
$y = -x - 2$

b) $x + 2y = 11$
$x + 3y = 16$

c) $4x + 3y = -20$
$5x - 2y = 21$

d) $2x + 4y = 15$
$4x - 6y = -15$

Writing Equations of Lines

6. Write the equation of each line.

a) parallel to $y = 3x + 5$ with x-intercept 10

b) parallel to $4x + 5y = 7$ and through P$(-2, 6)$

c) perpendicular to $y = -\frac{3}{2}x + 6$ with the same x-intercept as $5x - 2y = 20$

d) perpendicular to $7x + 5y = 20$ with the same y-intercept as $6x - 5y = 15$

e) parallel to the y-axis and through B$(-3, 0)$

Dot and Cross Products

7. Use $\vec{a}\cdot\vec{b}$ to determine if $\vec{a}$ and $\vec{b}$ are perpendicular.

a) $\vec{a}=[3, 1]$, $\vec{b}=[5, 7]$

b) $\vec{a}=[-4, 5]$, $\vec{b}=[-9, 1]$

c) $\vec{a}=[6, 1]$, $\vec{b}=[-2, 12]$

d) $\vec{a}=[1, 9, -4]$, $\vec{b}=[3, -6, -2]$

e) $\vec{a}=[3, 4, 1]$, $\vec{b}=[1, -1, 1]$

f) $\vec{a}=[7, -3, 2]$, $\vec{b}=[1, 8, 10]$

8. Find $\vec{a}\times\vec{b}$.

a) $\vec{a}=[2, -7, 3]$, $\vec{b}=[1, 9, 6]$

b) $\vec{a}=[8, 2, -4]$, $\vec{b}=[3, 7, -1]$

c) $\vec{a}=[3, 3, 5]$, $\vec{b}=[5, 1, -1]$

d) $\vec{a}=[2, 0, 0]$, $\vec{b}=[0, 7, 0]$

9. Find a vector parallel to each given vector. Is your answer unique? Explain.

a) $\vec{a}=[1, 5]$

b) $\vec{a}=[-3, 4]$

c) $\vec{a}=[2, 1, 7]$

d) $\vec{a}=[-1, -4, 5]$

10. Find a vector perpendicular to each vector in question 9. Is your answer unique? Explain.

11. Find the measure of the angle between the vectors in each pair.

a) $\vec{a}=[1, 3]$, $\vec{b}=[2, 5]$

b) $\vec{a}=[-4, 1]$, $\vec{b}=[7, 2]$

c) $\vec{a}=[1, 0, 2]$, $\vec{b}=[5, 3, 4]$

d) $\vec{a}=[-3, 2, -8]$, $\vec{b}=[1, -2, 6]$

CHAPTER PROBLEM

Computer graphics have come a long way since the first computer game. One of the first computer games, called Spacewar!, was available in the 1960s. It was based on two simple-looking spaceships. Players could move and rotate the ships to try to eliminate their opponent's ship. The graphics were simple, but the movement of the images was controlled by mathematical equations.

Over 40 years later, computer games span the gap between fantasy role-playing games and simulations. They use rich textures, 3-D graphics, and physics engines that make them look very realistic. The movements of the images in these new games are still controlled by mathematical equations. Suppose you work for a gaming company, Aftermath Animations, that is producing a series of games that involve motion. How will you use your mathematical expertise to design a new action game?

8.1 Equations of Lines in Two-Space and Three-Space

Any non-vertical line in two-space can be defined using its slope and y-intercept. The slope defines the direction of a line and the y-intercept defines its exact position, distinguishing it from other lines with the same slope. In three-space, the slope of a line is not defined, so a line in three-space must be defined in other ways.

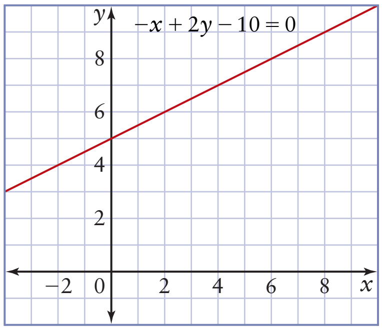
Investigate How can you use vectors to define a line in two-space?

Tools

- grid paper

Method 1: ***Use Paper and Pencil***

The equation of this line is $-x + 2y - 10 = 0$.



1. Copy the graph.
   a) Draw a vector that is parallel to the line. Label the vector $\vec{m}$. How many possible vectors $\vec{m}$ exist that are parallel to this line? Explain.
   b) Draw a position vector starting at the origin with its tip on the line. Label this vector $\vec{r}_0$. How many possible position vectors $\vec{r}_0$ are there for this line? Explain.
2. **Reflect** In the equation $y = mx + b$, the slope, m, changes the orientation of the line, and the y-intercept, b, changes the position of the line. How do the roles of $\vec{m}$ and $\vec{r}_0$ compare to those of m and b?
3. Draw the vector $\vec{a} = [-1, 2]$. Compare the vector $\vec{a}$ with the equation of the line and with the graph of the line.
4. On a new grid, graph the line $3x + 4y - 20 = 0$. On the same grid, draw the vector $\vec{b} = [3, 4]$. Compare the vector $\vec{b}$ with the equation of the line and the graph of the line.
5. **Reflect** Describe how your answers for steps 3 and 4 are similar. Give an example of a different line and vector pair that have the same property as those from steps 3 and 4.

Method 2: ***Use*** **The Geometer's Sketchpad®**

Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.1. Download the file **8.1Vector2D.gsp**. Open the sketch.

Tools

- computer with *The Geometer's Sketchpad®*
- 8.1 Vector2D.gsp

1. The vector $\vec{m}$ is parallel to the blue line. Move the vectors $\vec{m}$ and $\vec{r}_0$ by dragging their tips. What happens to the blue line as you move vector $\vec{m}$? vector $\vec{r}_0$? Record your observations.

2. Reflect How does changing vector $\vec{m}$ affect the line? How does changing vector $\vec{r}_0$ affect the line?

3. Link to page 2. Try to change vectors $\vec{m}$ and $\vec{r}_0$ so the blue line matches the green line.

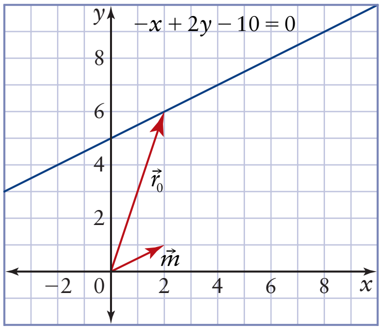
4. Reflect How many different $\vec{m}$ vectors exist that will still produce the same line? How many different $\vec{r}_0$ vectors exist that will still produce the same line?

5. Link to page 3. How does vector $\vec{s}$ compare with vector $\vec{m}$? Which vectors will have the resultant $\vec{r}$?

6. Link to page 4. Compare the coefficients of the standard form equation with the values in the perpendicular vector.

7. Reflect Given the equation of a line in standard form, how could you determine a vector perpendicular to the line without graphing?

In two-space, a line can be defined by an equation in standard form, $Ax + By + C = 0$ (also called the **scalar equation**), or in slope−intercept form, $y = mx + b$. Vectors can also be used to define a line in two-space. Consider the line defined by $-x + 2y - 10 = 0$ from the Investigate. A **direction vector** parallel to the line is $\vec{m} = [2, 1]$. A position vector that has its tip on the line is $\vec{r}_0 = [2, 6]$.

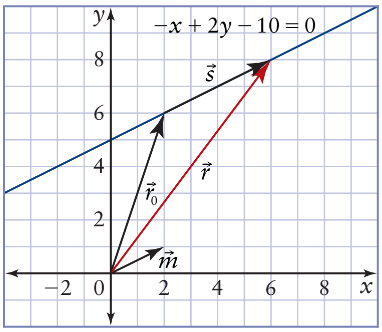


CONNECTIONS

When mathematicians use the subscript "0," it usually refers to an initial known value. The "0" is read "naught," a historical way to say "zero." So, a variable such as $\vec{r}_0$ is read "r naught."

Another position vector, such as $\vec{r} = [6, 8]$, can be drawn to form a closed triangle. The third side of the triangle is the vector $\vec{s}$. The closed triangle can be represented by the vector sum

$\vec{r} = \vec{r}_0 + \vec{s}$



Since $\vec{s}$ is parallel to $\vec{m}$, it is a scalar multiple of $\vec{m}$. So, $\vec{s} = t\vec{m}$. Substitute $\vec{s} = t\vec{m}$ into $\vec{r} = \vec{r}_0 + \vec{s}$, to get

$\vec{r} = \vec{r}_0 + t\vec{m}$, $t \in \mathbb{R}$

This is the **vector equation** of a line in two-space. In a vector equation, the variable t is a parameter. By changing the value of t, the vector $\vec{r}$ can have its tip at any specific point on the line.

Substitute $\vec{r} = [x, y]$, $\vec{r}_0 = [x_0, y_0]$, and $\vec{m} = [m_1, m_2]$ to write an alternate form of the vector equation.

Vector Equation of a Line in Two-Space

$\vec{r} = \vec{r}_0 + t\vec{m}$ or $[x, y] = [x_0, y_0] + t[m_1, m_2]$,

where

- $t \in \mathbb{R}$,
- $\vec{r} = [x, y]$ is a position vector to any unknown point on the line,
- $\vec{r}_0 = [x_0, y_0]$ is a position vector to any known point on the line, and
- $\vec{m} = [m_1, m_2]$ is a direction vector parallel to the line.

Example 1 Vector Equation of a Line in Two-Space

a) Write a vector equation for the line through the points A(1, 4) and B(3, 1).

b) Determine three more position vectors to points on the line. Graph the line.

c) Draw a closed triangle that represents the vector equation using points A and B.

d) Determine if the point (2, 3) is on the line.

Solution

a) Determine the vector from point A to point B to find the direction vector.

$$\begin{aligned}\vec{m} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= [3, 1] - [1, 4] \\ &= [2, -3]\end{aligned}$$

Choose one of points A or B to be the position vector, $\vec{r}_0 = [3, 1]$.

A vector equation is

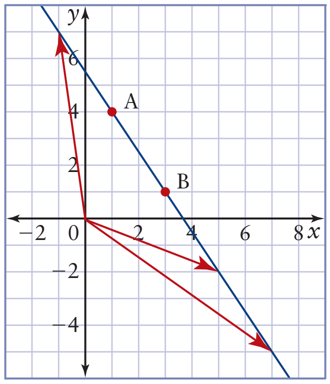
$[x, y] = [3, 1] + t[2, -3]$.

b) Let $t = 1$, 2, and -2

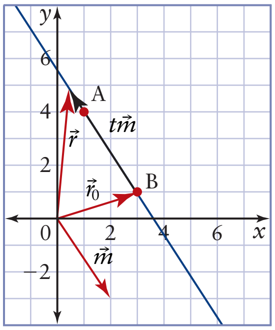
$$[x, y] = [3, 1] + (1)[2, -3] \\ = [5, -2]$$

$$[x, y] = [3, 1] + (2)[2, -3] \\ = [7, -5]$$

$$[x, y] = [3, 1] + (-2)[2, -3] \\ = [-1, 7]$$



c)



d) If the point (2, 3) is on the line, then the position vector [2, 3] has its tip on the line. So, there exists a single value of t that makes the equation true.

$[x, y] = [3, 1] + t[2, -3]$ becomes
$[2, 3] = [3, 1] + t[2, -3]$

Equate the x-coordinates.
$2 = 3 + 2t$
$t = -\dfrac{1}{2}$

Equate the y-coordinates.
$3 = 1 - 3t$
$t = \dfrac{2}{3}$

Since the t-values are not equal, the point (2, 3) does not lie on the line.

The vector equation of a line can be separated into two parts, one for each variable. These are called the **parametric equations** of the line, because the result is governed by the parameter t, $t \in \mathbb{R}$.

The vector equation $\vec{r} = \vec{r}_0 + t\vec{m}$ can be written $[x, y] = [x_0, y_0] + t[m_1, m_2]$.

The parametric equations of a line in two-space are

$$x = x_0 + tm_1$$
$$y = y_0 + tm_2, \qquad t \in \mathbb{R}$$

Example 2 Parametric Equations of a Line in Two-Space

Consider line ℓ_1.

$$\ell_1: \begin{cases} x = 3 + 2t \\ y = -5 + 4t \end{cases}$$

a) Find the coordinates of two points on the line.

b) Write a vector equation of the line.

c) Write the scalar equation of the line.

d) Determine if line ℓ_1 is parallel to line ℓ_2.

$$\ell_2: \begin{cases} x = 1 + 3t \\ y = 8 + 12t \end{cases}$$

Solution

a) Let $t = 0$. This gives the point $P_1 = (3, -5)$.

To find another point, choose any value of t. Let $t = 1$.

$x = 3 + 2(1) = 5$
$y = -5 + 4(1) = -1$

The coordinates of two points on the line are $(5, -1)$ and $(3, -5)$.

b) From the parametric equations, we can choose $\vec{r}_0 = [3, -5]$ and $\vec{m} = [2, 4]$.

A possible vector equation is

$[x, y] = [3, -5] + t[2, 4]$

The parameter t is usually placed before the direction vector to avoid ambiguity.

c) For the scalar equation of the line, isolate t in both of the parametric equations.

$x = 3 + 2t$ $\quad$ $y = -5 + 4t$

$t = \dfrac{x-3}{2}$ $\quad$ $t = \dfrac{y+5}{4}$

$$\frac{x-3}{2} = \frac{y+5}{4}$$

$4x - 12 = 2y + 10$

The scalar equation is $2x - y - 11 = 0$.

CONNECTIONS

The scalar equation of a line is $Ax + By + C = 0$. This is also called the Cartesian equation of a line.

d) The direction vectors of parallel lines are scalar multiples of one another.

For line ℓ_1, $\vec{m}_1 = [2, 4]$ $\qquad$ For line ℓ_2, $\vec{m}_2 = [3, 12]$

If the lines are parallel, we need $[2, 4] = k[3, 12]$.

$2 = 3k$ $\qquad$ $4 = 12k$

$k = \dfrac{2}{3}$ $\qquad$ $k = \dfrac{1}{3}$

The required k-value does not exist, so lines ℓ_1 and ℓ_2 are not parallel.

Example 3 Scalar Equation of a Line in Two-Space

Consider the line with scalar equation $4x + 5y + 20 = 0$.

a) Graph the line.

b) Determine a position vector that is perpendicular to the line.

c) How does the position vector from part b) compare to the scalar equation?

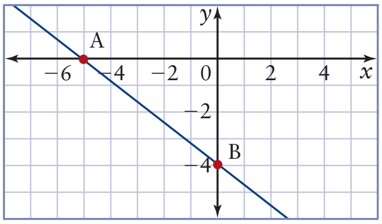
d) Write a vector equation of the line.

Solution

a) Find the intercepts A and B.

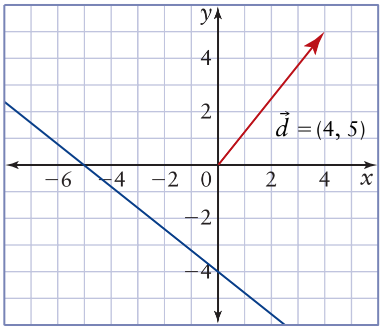
Let $y = 0$. The x-intercept is -5.

Let $x = 0$. The y-intercept is -4.



b) Find a direction vector for the line.

$$\begin{aligned}\vec{m} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= [0, -4] - [-5, 0]\\ &= [5, -4]\end{aligned}$$



To find a vector $\vec{d} = [x, y]$ that is perpendicular to $\vec{m}$,

$$\begin{aligned}[5, -4]\cdot[x, y] &= 0\\ 5x - 4y &= 0\\ y &= \frac{5}{4}x\end{aligned}$$

Choose any value for x; $x = 4$ is a convenient choice

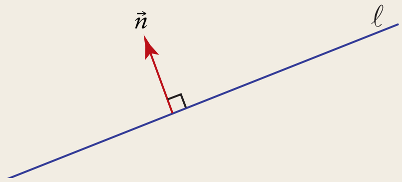
$\vec{d} = [4, 5]$ is perpendicular to the line.

c) The components of $\vec{d}$ correspond to the coefficients of x and y in the scalar equation.

d) To write a vector equation, we need a position vector and a direction vector. A vector equation of the line is $[x, y] = [-5, 0] + t[5, -4]$, $t \in \mathbb{R}$.

A **normal vector** to a line ℓ is a vector $\vec{n}$ that is perpendicular to the line.

If ℓ: $Ax + By + C = 0$, then $\vec{n} = [A, B]$.



A line in two-space can be represented in many ways: a vector equation, parametric equations, a scalar equation, or an equation in slope–intercept form. The points (x, y) that are the solutions of these equations form a line in two-space.

A line in three-space can also be defined by a vector equation or by parametric equations. It cannot, however, be defined by a scalar equation. In three-space, we will see that a scalar equation defines a **plane**. A plane is a two-dimensional flat surface that extends infinitely far in all directions.

As in two-space, a direction vector and a position vector to a known point on the line are needed to define a line in three-space.

The line passing through the point P_0 with position vector $\vec{r}_0 = [x_0, y_0, z_0]$ and having direction vector $\vec{m} = [m_1, m_2, m_3]$ has vector equation

$\vec{r} = \vec{r}_0 + t\vec{m}, t \in \mathbb{R}$ or
$[x, y, z] = [x_0, y_0, z_0] + t[m_1, m_2, m_3], t \in \mathbb{R}$

The parametric equations of a line in three-space are

$$x = x_0 + tm_1$$
$$y = y_0 + tm_2$$
$$z = z_0 + tm_3, t \in \mathbb{R}$$

Example 4 Equations of Lines in Three-Space

A line passes through points A(2, −1, 5) and B(3, 6, −4).

a) Write a vector equation of the line.

b) Write parametric equations for the line.

c) Determine if the point C(0, −15, 9) lies on the line.

Solution

a) Find a direction vector.

$$\vec{m} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= [3, 6, -4] - [2, -1, 5]$$
$$= [1, 7, -9]$$

A vector equation is

$[x, y, z] = [2, -1, 5] + t[1, 7, -9]$

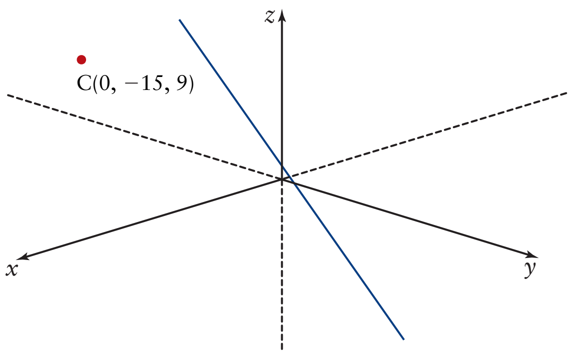
b) The corresponding parametric equations are

$x = 2 + t$
$y = -1 + 7t$
$z = 5 - 9t$

c) Substitute the coordinates of C(0, −15, 9) into the parametric equations and solve for t.

$0 = 2 + t$	$-15 = -1 + 7t$	$9 = 5 - 9t$
$t = -2$	$t = -2$	$t = -\frac{4}{9}$

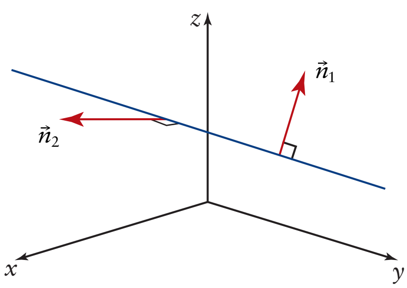
The t-values are not equal, so the point does not lie on the line. This can be seen on the graph.



CONNECTIONS

Diagrams involving objects in three-space are best viewed with 3D graphing software. Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.1 for instructions on how to access 3-D Grapher software.

A line in two-space has an infinite number of normal vectors that are all parallel to one another. A line in three-space also has an infinite number of normal vectors, but they are not necessarily parallel to one another.



CONNECTIONS

Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.1 to see an applet showing normals to a 3D line.

KEY CONCEPTS

- In two-space, a line can be defined by an equation in slope–intercept form, a vector equation, parametric equations, or a scalar equation.

<table>
<tr><th colspan="3">Equations of Lines in Two-Space</th></tr>
<tr><td>Slope–Intercept</td><td>$y = mx + b$</td><td>m is the slope of the line
b is the y-intercept</td></tr>
<tr><td>Vector</td><td>$\vec{r} = \vec{r}_0 + t\vec{m}, t \in \mathbb{R}$
or
$[x, y] = [x_0, y_0] + t[m_1, m_2], t \in \mathbb{R}$</td><td rowspan="2">$\vec{r} = [x, y]$ is a position vector to any point on the line
$\vec{r}_0 = [x_0, y_0]$ is a position vector to a known point on the line
$\vec{m} = [m_1, m_2]$ is a direction vector for the line</td></tr>
<tr><td>Parametric</td><td>$x = x_0 + tm_1$
$y = y_0 + tm_2, t \in \mathbb{R}$</td></tr>
<tr><td>Scalar</td><td>$Ax + By + C = 0$</td><td>$\vec{n} = [A, B]$ is a normal vector to the line</td></tr>
</table>

- In three-space, a line can be defined by a vector equation or by parametric equations.

<table>
<tr><th colspan="3">Equations of Lines in Three-Space</th></tr>
<tr><td>Vector</td><td>$\vec{r} = \vec{r}_0 + t\vec{m}, t \in \mathbb{R}$
or
$[x, y, z] = [x_0, y_0, z_0] + t[m_1, m_2, m_3], t \in \mathbb{R}$</td><td rowspan="2">$\vec{r} = [x, y, z]$ is a position vector to any point on the line
$\vec{r}_0 = [x_0, y_0, z_0]$ is a position vector to a known point on the line
$\vec{m} = [m_1, m_2, m_3]$ is a direction vector for the line</td></tr>
<tr><td>Parametric</td><td>$x = x_0 + tm_1$
$y = y_0 + tm_2$
$z = z_0 + tm_3, t \in \mathbb{R}$</td></tr>
</table>

- A normal vector to a line is perpendicular to that line.
- To define a line in two-space or three-space, two pieces of information are needed: two points on the line, or a point and a direction vector. For lines in two-space, a perpendicular vector and a point can also be used to define a line.

Communicate Your Understanding

C1 Why are you asked to find *a* vector equation of a line and not *the* vector equation?

C2 Given the equation of a line, how can you tell if the line is in two-space or three-space?

C3 Given a normal to a line in two-space and a point on that line, explain how to find the equation of the line.

C4 Given the equation $[x, y] = [3, 5, -2] + t[4, 7, 1]$, a student says, "That equation says that one point on the line is $(3, 5, -2)$." Do you agree?

C5 All normals to a line are parallel to each other. Do you agree? Explain.

A Practise

1. Write a vector equation for a line given each direction vector $\vec{m}$ and point P_0.

   a) $\vec{m} = [3, 1]$, $P_0(2, 7)$

   b) $\vec{m} = [-2, 5]$, $P_0(10, -4)$

   c) $\vec{m} = [10, -3, 2]$, $P_0(9, -8, 1)$

   d) $\vec{m} = [0, 6, -1]$, $P_0(-7, 1, 5)$

2. Write a vector equation of the line through each pair of points.

   a) A(1, 7), B(4, 10)

   b) A(−3, 5), B(−2, −8)

   c) A(6, 2, 5), B(9, 2, 8)

   d) A(1, 1, −3), B(1, −1, −5)

3. Draw a closed triangle that represents each vector equation.

   a) $[x, y] = [1, -3] + t[2, 5]$

   b) $[x, y] = [-5, 2] + t[4, -1]$

   c) $[x, y] = [2, 5] + t[4, -3]$

   d) $[x, y] = [-2, -1] + t[-5, 2]$

4. **Use Technology** Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.1. Download the file **8.1 VectorEquation.gsp** to check your answers to question 3.

5. Determine if each point P is on the line $[x, y] = [3, 1] + t[-2, 5]$.

   a) P(−1, 11)

   b) P(9, −15)

   c) P(−9, 21)

   d) P(−2, 13.5)

6. Write the parametric equations for each vector equation.

   a) $[x, y] = [10, 6] + t[13, 1]$

   b) $[x, y] = [0, 5] + t[12, -7]$

   c) $[x, y, z] = [3, 0, -1] + t[6, -9, 1]$

   d) $[x, y, z] = [11, 2, 0] + t[3, 0, 0]$

7. Write a vector equation for each line, given the parametric equations.

   a) $x = 3 + 5t$
      $y = 9 + 7t$

   b) $x = -5 - 6t$
      $y = 11t$

   c) $x = 1 + 4t$
      $y = -6 + t$
      $z = 2 - 2t$

   d) $x = 7$
      $y = -t$
      $z = 0$

8. Given each set of parametric equations, write the scalar equation.

   a) $x = 1 + 2t$
      $y = 1 - 3t$

   b) $x = -2 + t$
      $y = 4 + 5t$

   c) $x = 5 + 7t$
      $y = -2 - 4t$

   d) $x = 0.5 + 0.3t$
      $y = 1.5 - 0.2t$

9. Graph each line.

   a) $3x + 6y + 36 = 0$

   b) $[x, y] = [-1, 7] + t[2, -5]$

   c) $x = 4 + t$
      $y = 3t - 2$

   d) $4x - 15y = 10$

10. Write the scalar equation of each line given the normal vector $\vec{n}$ and point P_0.

    a) $\vec{n} = [3, 1]$, $P_0(2, 4)$

    b) $\vec{n} = [1, -1]$, $P_0(-5, 1)$

    c) $\vec{n} = [0, 1]$, $P_0(-3, -7)$

    d) $\vec{n} = [1.5, -3.5]$, $P_0(0.5, -2.5)$

11. Given each scalar equation, write a vector equation and the parametric equations.

    a) $x + 2y = 6$

    b) $4x - y = 12$

    c) $5x - 2y = 13$

    d) $8x + 9y = -45$

12. Given the points A(3, 4, −5) and B(9, −2, 7), write a vector equation and the parametric equations of the line through A and B.

13. Which points are on the line $[x, y, z] = [1, 3, -7] + t[2, -1, 3]$?

    a) $P_0(7, 0, 2)$

    b) $P_0(2, 1, -3)$

    c) $P_0(13, -3, 11)$

    d) $P_0(-4, 0.5, -14.5)$

B Connect and Apply

14. In each case, determine if ℓ_1 and ℓ_2 are parallel, perpendicular, or neither. Explain.

a) ℓ_1: $4x - 6y = 9$
ℓ_2: $[x, y] = [6, 3] + t[3, 2]$

b) ℓ_1: $x + 9y = 2$
$\ell_2: \begin{cases} x = t \\ y = 15 + 9t \end{cases}$

15. Describe the line defined by each equation.

a) $[x, y] = [-2, 3] + t[1, 0]$

b) $[x, y, z] = t[0, 0, 1]$

c) $[x, y] = [1, 1] + s[0, 5]$

d) $[x, y, z] = [-1, 3, 2] + s[1, 0, 1]$

16. Determine the vector equation of each line.

a) parallel to the x-axis and through $P_0(3, -8)$

b) perpendicular to $4x - 3y = 17$ and through $P_0(-2, 4)$

c) parallel to the z-axis and through $P_0(1, 5, 10)$

d) parallel to $[x, y, z] = [3, 3, 0] + t[3, -5, -9]$ with x-intercept -10

e) with the same x-intercept as $[x, y, z] = [3, 0, 0] + t[4, -4, 1]$ and the same z-intercept as $[x, y, z] = [6, 3, -1] + t[3, 6, -2]$

17. Explain why it is not possible for a line in three-space to be represented by a scalar equation.

18. **Use Technology** Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.1 for instructions on how to access 3-D Grapher software.
Use the 3-D Grapher to determine if

a) the point $(7, -21, 7)$ is on the line ℓ_1: $[x, y, z] = [4, -3, 2] + t[1, 8, -3]$

b) the line $[x, y, z] = [2, -19, 8] + s[4, -5, -9]$ and the line ℓ_1 intersect

c) the line $[x, y, z] = [1, 0, 3] + s[4, -5, -9]$ and the line ℓ_1 intersect

19. **Chapter Problem** In computer animation, the motion of objects is controlled by mathematical equations. But more than that, objects have to move in time. One way to facilitate this movement is to use a parameter, t, to represent time (in seconds, hours, days, etc.). Games like Pong or Breakout require only simple motion. In more complex simulation games, positions in three dimensions—as well as colour, sound, reflections, and point of view—all have to be controlled in real time.

a) Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.1. Download the file **8.1 VectorEquation.gsp**. Follow the instructions in the file.

b) How did this file help describe how simple animation works?

20. Line ℓ_1 in three-space is defined by the vector equation $[x, y, z] = [4, -3, 2] + t[1, 8, -3]$. Answer the following without using technology.

a) Determine if the point $(7, -21, 7)$ lies on line ℓ_1.

b) Determine if line ℓ_1 intersects line ℓ_2, defined by $[x, y, z] = [2, -19, 8] + s[4, -5, -9]$.

c) Determine if line ℓ_1 intersects line ℓ_3, defined by $[x, y, z] = [1, 0, 3] + v[4, -5, -9]$.

21. Consider the lines
ℓ_1: $[x, y] = [3, -2] + t[4, -5]$ and
ℓ_2: $[x, y] = [1, 1] + s[7, k]$.

a) For what value of k are the lines parallel?

b) For what value of k are the lines perpendicular?

22. Are these vector equations different representations of the same line? Explain.

a) ℓ_1: $[x, y, z] = [11, -2, 17] + t[3, -1, 4]$

b) ℓ_2: $[x, y, z] = [-13, 6, -10] + s[-3, 1, -4]$

c) ℓ_3: $[x, y, z] = [-4, 3, -3] + t[-6, 2, -8]$

23. A line passes through the points A(3, −2) and B(−5, 4).

a) Find the vector $\overrightarrow{AB}$. How will this vector relate to the line through A and B?

b) Write a vector equation and the parametric equations of the line.

c) Determine a vector perpendicular to $\overrightarrow{AB}$. Use this vector to write the scalar equation of the line AB.

d) Find three more points on the same line.

e) Determine if the points C(35, −26) and D(−9, 8) are on the line.

24. a) Determine if each vector is perpendicular to the line $[x, y, z] = [3, -1, 5] + t[2, -3, -1]$.

i) $\vec{a} = [1, -1, 5]$

ii) $\vec{b} = [2, 2, 2]$

iii) $\vec{c} = [-4, -7, 13]$

b) Find three vectors that are perpendicular to the line but not parallel to any of the vectors from part a).

25. Determine the equations of the lines that form the sides of the triangle with each set of vertices.

a) A(7, 4), B(4, 3), C(6, −3)

b) D(1, −3, 2), E(1, −1, 8), F(5, −17, 0)

26. a) Determine if the line $[x, y, z] = [4, 1, -2] + t[3, 1, -5]$ has x-, y-, and/or z-intercepts.

b) Under what conditions will a line parallel to $[x, y, z] = [4, 1, -2] + t[3, 1, -5]$ have *only*

i) an x-intercept?

ii) a y-intercept?

iii) a z-intercept?

c) Under what conditions (if any) will a line parallel to the one given have an x-, a y-, and a z-intercept?

C Extend and Challenge

Parametric curves have coordinates defined by parameters. Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.1. Download the file **8.1 ParametricCurves.gsp** to trace graphs for questions 27 to 30.

27. Graph $x = 2\sin(t)$, $y = 2\cos(t)$, $t \geq 0$.

a) Describe the graph.

b) What effect does the value 2 have on the graph?

c) How does the shape of the graph change if the coefficients of the sine and cosine functions change?

d) How does the graph change if cosine changes to sine in the original y-equation?

28. Graph $x = 0.5t\sin(t)$, $y = 0.5t\cos(t)$, $t \geq 0$. Describe the graph.

29. Graph $x = t - \sin(t)$, $y = 1 - \cos(t)$, $t \geq 0$.

a) This graph is called a cycloid. Describe the graph.

b) What is the significance of the distance between successive x-intercepts?

30. Research and find the equation of each parametric curve. Describe the shape of each graph.

a) tricuspoid **b)** lissajous **c)** epicycloid

31. Compare the lines in each pair. Are the lines parallel, perpendicular, or neither? Explain.

a) ℓ_1: $[x, y, z] = [3, -1, 8] + t[4, -6, -15]$
ℓ_2: $[x, y, z] = [1, 1, 0] + s[-8, 12, 20]$

b) ℓ_1: $[x, y, z] = [10, 2, -3] + t[5, 1, -5]$
ℓ_2: $[x, y, z] = [1, 1, 0] + s[1, 5, 2]$

32. Consider the line defined by the equation $[x, y] = [3, 4] + t[2, 5]$.

a) Write the parametric equations for the line.

b) Isolate t in each of the parametric equations.

c) Equate the equations from part b). This form of the equation is called the **symmetric equation**.

d) Compare the symmetric equation from part c) with the original vector equation.

e) Use your answer to part d). Write the symmetric equation for each vector equation.

i) $[x, y] = [1, 7] + t[3, 8]$

ii) $[x, y] = [4, -2] + t[1, 9]$

iii) $[x, y] = [-5, 2] + t[-3, -4]$

33. For each symmetric equation, write the vector and scalar equations.

a) $\dfrac{x-6}{2} = \dfrac{y-9}{7}$

b) $\dfrac{x+3}{4} = \dfrac{y+9}{-5}$

c) $\dfrac{4-x}{7} = y+10$

34. In three-space, the **symmetric equations** of a line are given by $\dfrac{x-x_0}{m_1} = \dfrac{y-y_0}{m_2} = \dfrac{z-z_0}{m_3}$, where $[x_0, y_0, z_0]$ is a point on the line and $[m_1, m_2, m_3]$ is a direction vector for the line, provided none of m_1, m_2, or m_3 is 0.

a) Given each vector equation, write the symmetric equations.

i) $[x, y, z] = [1, 3, 9] + t[5, 4, 2]$

ii) $[x, y, z] = [-4, -1, 7] + t[-2, 8, 1]$

iii) $[x, y, z] = [5, 1, -9] + t[-1, -3, 11]$

b) Given the symmetric equations, write the corresponding vector equation.

i) $\dfrac{x-4}{8} = \dfrac{y-12}{5} = \dfrac{z-15}{2}$

ii) $x-6 = \dfrac{y+1}{7} = \dfrac{z+5}{-3}$

iii) $\dfrac{5-x}{6} = \dfrac{-y-3}{10} = \dfrac{z}{11}$

35. Determine the angle between the lines in each pair.

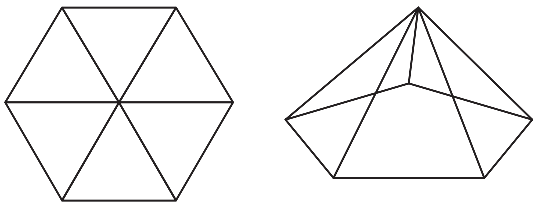
a) ℓ_1: $[x, y] = [4, 1] + t[1, 5]$
ℓ_2: $[x, y] = [-1, 3] + s[-2, 7]$

b) ℓ_1: $[x, y, z] = [3, 1, -1] + t[2, -2, 3]$
ℓ_2: $[x, y, z] = [5, -1, 2] + s[1, -3, 5]$

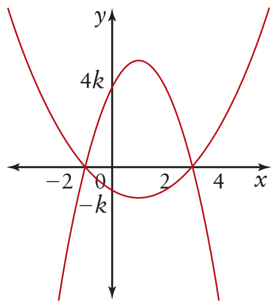
36. Write the parametric equations of the line that goes through the point $(6, -2, 1)$ and is perpendicular to both $[x, y, z] = [1, 4, -2] + t[3, -1, 1]$ and $[x, y, z] = [9, 5, -3] + s[1, -3, 7]$.

37. Write the equations of two lines that intersect at the point $A(3, 1, -1)$ and are perpendicular to each other.

38. **Math Contest** A hexagon with an area of 36 cm^2 is made up of six tiles that are equilateral triangles, as shown. One of the tiles is removed and the remaining five are bent so that the two free edges come together, forming the object into a pentagonal pyramid (with no bottom), as shown. Determine the height of the pyramid.



39. **Math Contest** Two quadratic functions have x-intercepts at -1 and 3, and y-intercepts at $-k$ and $4k$, as shown. The quadrilateral formed by joining the four intercepts has an area of 30 units^2. Determine the value of k.



8.2 Equations of Planes

In two-space, you have studied points, vectors, and lines. Lines in two-space can be represented by vector or parametric equations and also scalar equations of the form $Ax + By + C = 0$, where the vector $\vec{n} = [A, B]$ is a normal to the line.

In three-space, you can study lines and planes. You can use concepts from two-space to represent planes in a variety of ways, including vector, parametric, and scalar equations.

Investigate The equation of a plane

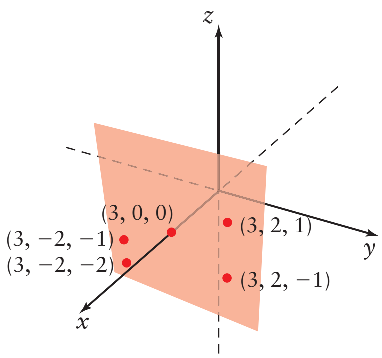
For the line $x = 5$ in two-space, the x-coordinate of every point on the line is 5. It is a vertical line with x-intercept 5. Each point on the line $x = 5$ is defined by an ordered pair of the form $(5, y)$, where y can have any value.

CONNECTIONS

Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.2 to download dynamic versions of these diagrams.

1. Consider these planes.

i)



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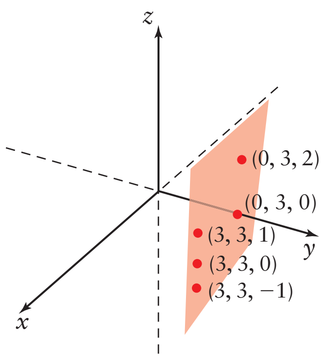

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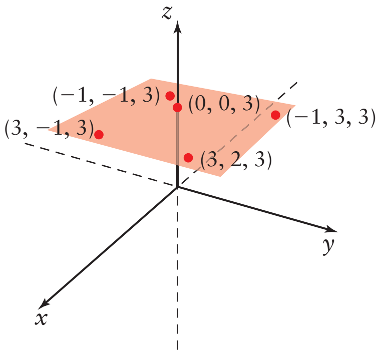


a) On each plane, what do all of the points have in common?

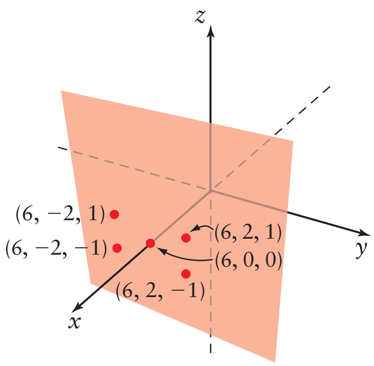
b) **Reflect** Which equation do you think defines each plane? Explain your choice.

I $z = 3$ **II** $x = 3$ **III** $y = 3$

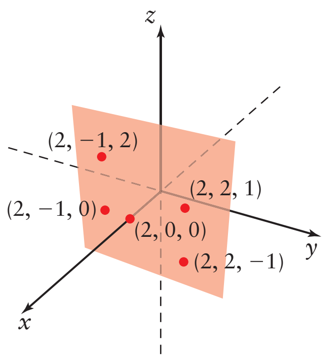
c) Describe each plane in words.

2. Repeat step 1 for these planes. Consider the equations $z = -2$, $x = 2$, and $x = 6$.

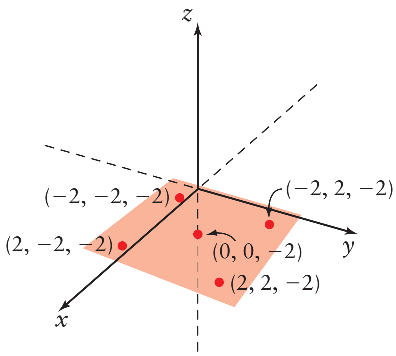
i)



ii)



iii)



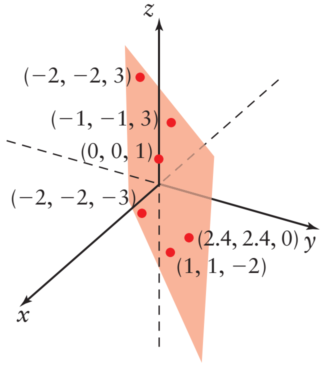
3. How would you describe a plane with equation $y = 6$? List the coordinates of six points on this plane.

4. How does each plane change for different values of k?

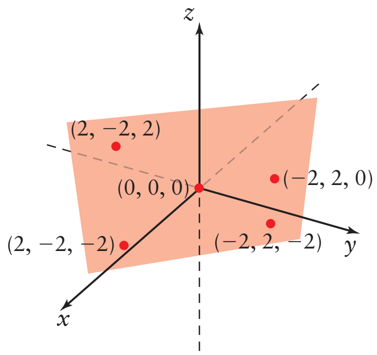
a) $x = k$ **b)** $y = k$ **c)** $z = k$

5. Consider these planes. What do all the points on each plane have in common?

i)



ii)



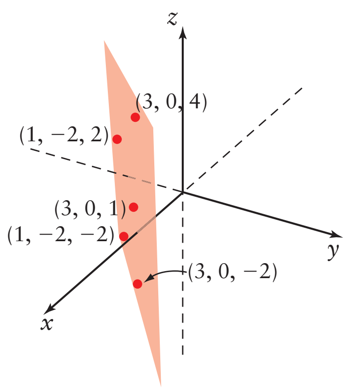
a) **Reflect** Which equation do you think defines each plane? Explain your choice.

I $x + y = 0$ **II** $x - y = 0$

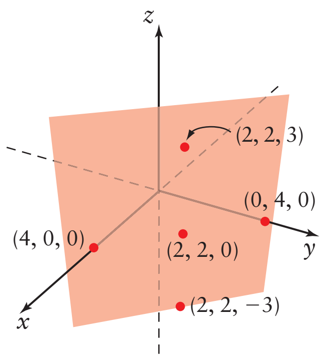
b) Describe each plane in words.

6. Consider these planes and their equations. How do the points shown on each plane relate to the equation of the plane?

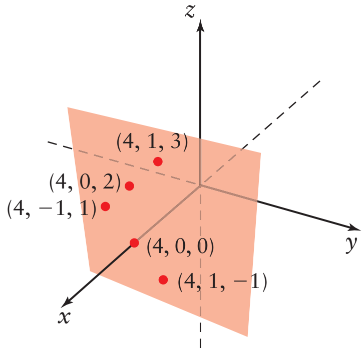
a) $x - y = 3$



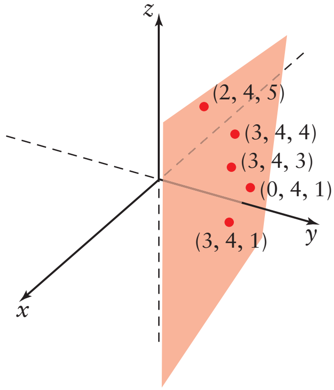
b) $x + y = 4$



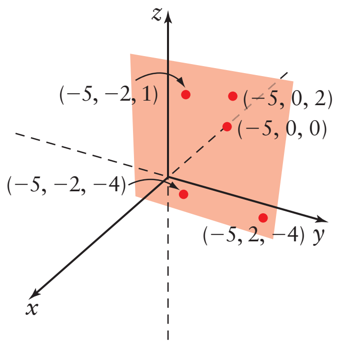
c) $x = 4$



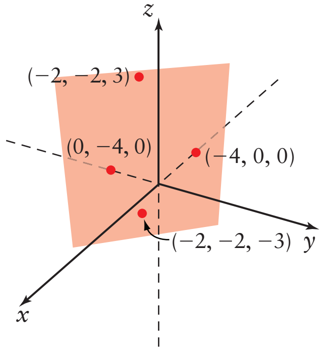
d) $y = 4$



e) $x = -5$

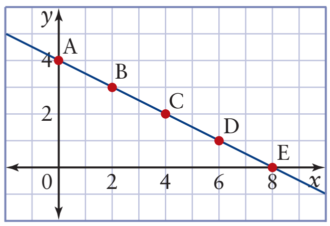


f) $x + y = -4$



7. **Reflect** Refer to step 6. Write the equation of a plane parallel to the y-axis. Explain your thinking.

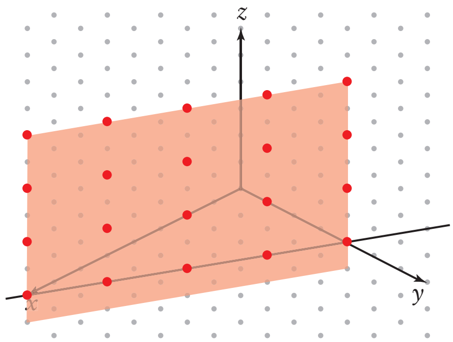
In two-space, the equation $x + 2y - 8 = 0$ defines a line passing through the points A(0, 4), B(2, 3), C(4, 2), D(6, 1), and E(8, 0).



This same equation can be graphed in three-space. Since there is a third coordinate, z, in three-space, each point will have a z-coordinate associated with it. The three-space version of this equation would be $x + 2y + 0z = 8$. Since the z-coefficient is zero, points such as A(0, 4, z), B(2, 3, z), C(4, 2, z), D(6, 1, z), and E(8, 0, z), where z can be any value, will satisfy the equation.

Plotting sets of these points on isometric graph paper suggests a surface that is parallel to the z-axis and contains the line $x + 2y - 8 = 0$. This surface is a plane.

In general, the scalar equation of a plane in three-space is $Ax + By + Cz + D = 0$.



CONNECTIONS

Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.2 to see a dynamic sketch of this plane

Example 1 Points on a Plane

Consider the plane defined by the scalar equation $x + 2y - z - 8 = 0$.

a) Determine if the points A(1, 3, −1), B(3, 5, 1), and C(1, 6, 5) are on the plane.

b) Determine the x-, y-, and z-intercepts of the plane.

c) Determine the coordinates of another point on the plane.

d) Write two vectors that are parallel to the plane.

Solution

a) If a point lies on the plane, its coordinates must satisfy the equation.

Check A(1, 3, −1)

$$\begin{aligned}\text{LS} &= (1) + 2(3) - (-1) - 8\\ &= 0\\ &= \text{RS}\end{aligned}$$

Check B(3, 5, 1)

$$\begin{aligned}\text{LS} &= (3) + 2(5) - (1) - 8\\ &= 4\\ &\neq \text{RS}\end{aligned}$$

Check C(1, 6, 5)

$$\begin{aligned}\text{LS} &= (1) + 2(6) - (5) - 8\\ &= 0\\ &= \text{RS}\end{aligned}$$

Points A and C are on the plane, but point B is not.

b) To find an intercept in three-space, set the other coordinates equal to zero.

At the x-intercept, both the y- and z-coordinates equal zero.

$$\begin{aligned}x + 2(0) - (0) - 8 &= 0\\ x &= 8\end{aligned}$$

The x-intercept is 8.

At the y-intercept, both the x- and z-coordinates equal zero.

$$\begin{aligned}(0) + 2y - (0) - 8 &= 0\\ y &= 4\end{aligned}$$

The y-intercept is 4.

At the z-intercept, both the x- and y-coordinates equal zero.

$$\begin{aligned}(0) + 2(0) - z - 8 &= 0\\ z &= -8\end{aligned}$$

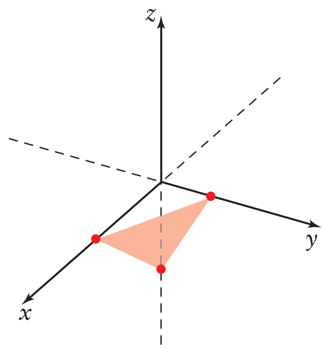
The z-intercept is −8.

c) To find the coordinates of another point on the plane, choose arbitrary values for two of the variables and solve for the third.

Let $x = 1$ and $y = 1$.

$$\begin{aligned}(1) + 2(1) - z - 8 &= 0\\ z &= -5\end{aligned}$$

Therefore (1, 1, −5) is a point on the plane.

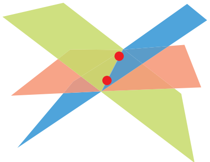


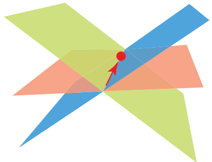
d) Vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are parallel to the plane.

$$\begin{aligned}\overrightarrow{AC} &= [1, 6, 5] - [1, 3, -1]\\ &= [0, 3, 6]\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC} &= [1, 6, 5] - [3, 5, 1]\\ &= [-2, 1, 4]\end{aligned}$$

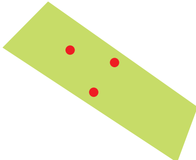
In two-space, a line can be uniquely defined either by two points or by a direction vector and a point. In three-space, more than one plane is possible given two points or a direction vector and a point.

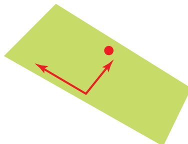




To uniquely define a plane in three-space, we need

three non-collinear points OR two non-parallel direction vectors and a point





A plane in three-space can be defined with a vector equation.

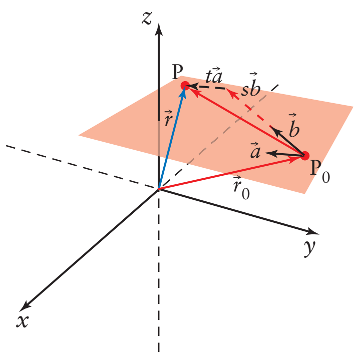
Vector Equation of a Plane in Three-Space

$\vec{r} = \vec{r}_0 + t\vec{a} + s\vec{b}$ or $[x, y, z] = [x_0, y_0, z_0] + t[a_1, a_2, a_3] + s[b_1, b_2, b_3]$,

where

- $\vec{r} = [x, y, z]$ is a position vector for any point on the plane,
- $\vec{r}_0 = [x_0, y_0, z_0]$ is a position vector for a known point on the plane,
- $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ are non-parallel direction vectors parallel to the plane, and
- t and s are scalars, $t, s \in \mathbb{R}$

On a graph, the vectors $\vec{r}$, $\vec{r}_0$, $\vec{a}$, and $\vec{b}$ form a closed loop, so algebraically they can form an equation, $\vec{r} = \vec{r}_0 + t\vec{a} + s\vec{b}$. Starting from the origin, $\vec{r}_0$ goes to a known point on the plane, then to two vectors in the plane that are scalar multiples of the direction vectors, $\vec{a}$ and $\vec{b}$. The resultant of these three vectors is the vector $\vec{r}$.



Parametric equations can also be used to define a plane.

Parametric Equations of a Plane in Three-Space

$x = x_0 + ta_1 + sb_1$
$y = y_0 + ta_2 + sb_2$
$z = z_0 + ta_3 + sb_3,\ t, s \in \mathbb{R}$

Example 2 Vector and Parametric Equations

Consider the plane with direction vectors $\vec{a} = [8, -5, 4]$ and $\vec{b} = [1, -3, -2]$ through $P_0(3, 7, 0)$.

a) Write the vector and parametric equations of the plane.

b) Determine if the point $Q(-10, 8, -6)$ is on the plane.

c) Find the coordinates of two other points on the plane.

d) Find the x-intercept of the plane.

CONNECTIONS

Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.2 to manipulate a dynamic version of the vector equation of a plane.

Solution

a) A vector equation of the plane is

$$[x, y, z] = [3, 7, 0] + t[8, -5, 4] + s[1, -3, -2]$$

Use any letters other than x, y, and z for the parameters.

Parametric equations for the plane are

$$x = 3 + 8t + s$$
$$y = 7 - 5t - 3s$$
$$z = 4t - 2s$$

b) If the point $(-10, 8, -6)$ is on the plane, then there exists a single set of t- and s-values that satisfy the equations.

$$\begin{aligned} -10 &= 3 + 8t + s && ① \\ 8 &= 7 - 5t - 3s && ② \\ -6 &= 4t - 2s && ③ \end{aligned}$$

Solve ① and ② for t and s using elimination.

$$\begin{aligned} -39 &= 24t + 3s && 3① \\ 1 &= -5t - 3s && ② \\ \hline -38 &= 19t && 3① + ② \\ t &= -2 \end{aligned}$$

$$\begin{aligned} -13 &= 8(-2) + s && \text{Substitute } t = -2 \text{ into either equation ① or ②.} \\ s &= 3 \end{aligned}$$

Now check if $t = -2$ and $s = 3$ satisfy ③.

$$\begin{aligned} \text{RS} &= 4(-2) - 2(3) \\ &= -14 \\ &\neq \text{LS} \end{aligned}$$

Since the values for t and s do not satisfy equation ③, the point $(-10, 8, -6)$ does not lie on the plane.

c) To find other points on the plane, use the vector equation and choose arbitrary values for the parameters t and s.

Let $t = 1$ and $s = -1$.

$$\begin{aligned}[x, y, z] &= [3, 7, 0] + (1)[8, -5, 4] + (-1)[1, -3, -2] \\ &= [3, 7, 0] + [8, -5, 4] + [-1, 3, 2] \\ &= [10, 5, 6]\end{aligned}$$

Let $t = 2$ and $s = 1$.

$$\begin{aligned}[x, y, z] &= [3, 7, 0] + (2)[8, -5, 4] + (1)[1, -3, -2] \\ &= [3, 7, 0] + [16, -10, 8] + [1, -3, -2] \\ &= [20, -6, 6]\end{aligned}$$

The coordinates of two other points on the plane are (10, 5, 6) and (20, −6, 6).

d) To find the x-intercept, set $y = z = 0$ and solve for s and t.

$$\begin{aligned} x &= 3 + 8t + s && ① \\ 0 &= 7 - 5t - 3s && ② \\ 0 &= 4t - 2s && ③ \end{aligned}$$

Solve ② and ③ for t and s using elimination.

$$\begin{aligned} -14 &= -10t - 6s && 2② \\ 0 &= 12t - 6s && 3③ \\ \hline -14 &= -22t && 2② - 3③ \end{aligned}$$

$$t = \frac{7}{11}$$

$$0 = 4\left(\frac{7}{11}\right) - 2s \qquad ③$$

$$s = \frac{14}{11}$$

Now, substitute $t = \frac{7}{11}$ and $s = \frac{14}{11}$ into ①.

$$\begin{aligned} x &= 3 + 8\left(\frac{7}{11}\right) + \left(\frac{14}{11}\right) \\ &= \frac{103}{11} \end{aligned}$$

The x-intercept is $\frac{103}{11}$.

You can check your answers using 3-D graphing software or isometric graph paper.

Technology Tip

Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.1 for instructions on how to access 3-D Grapher software.

Example 3 Write Equations of Planes

Find the vector and parametric equations of each plane.

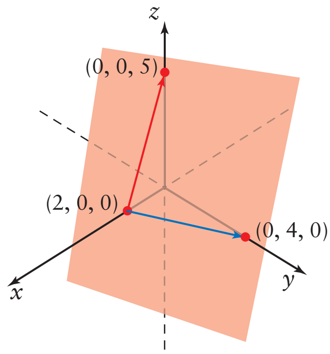
a) the plane with x-intercept $= 2$, y-intercept $= 4$, and z-intercept $= 5$

b) the plane containing the line $[x, y, z] = [0, 3, -5] + t[6, -2, -1]$ and parallel to the line $[x, y, z] = [1, 7, -4] + s[1, -3, 3]$

Solution

a) Use the intercepts to find two direction vectors.

$$\vec{a} = [2, 0, 0] - [0, 4, 0]$$
$$= [2, -4, 0]$$
$$\vec{b} = [0, 0, 5] - [0, 2, 0]$$
$$= [0, -2, 5]$$



For a point on the plane, choose (4, 0, 0).

A possible vector equation of the plane is

$[x, y, z] = [2, 0, 0] +$
$t[2, -4, 0] + k[0, -2, 5]$, $t, k \in \mathbb{R}$.

This equation is not unique. Different vector equations could be written using different points to find the position and direction vectors.

The corresponding parametric equations are

$$x = 4 + 4t$$
$$y = 2t + 2k$$
$$z = 5k \qquad t, k \in \mathbb{R}.$$

b) The plane contains the line $[x, y, z] = [0, 3, -5] + t[6, -2, -1]$, so $(0, 3, -5)$ is on the plane. Use $[0, 3, -5]$ as the position vector. Since $[6, -2, -1]$ is parallel to the plane, it can be used as direction vectors.

The plane is also parallel to the line $[x, y, z] = [1, 7, -4] + s[1, -3, 3]$.

Use direction vector $[1, -3, 3]$ as a second direction vector for the plane.

Since $[6, -2, -1]$ is not a scalar multiple of $[1, -3, 3]$, the direction vectors are not parallel.

A vector equation of the plane is

$[x, y, z] = [0, 3, -5] + k[6, -2, -1] + l[1, -3, 3]$, $k, l \in \mathbb{R}$

The corresponding parametric equations for the plane are

$$x = 6k + l$$
$$y = 3 - 2k - 3l$$
$$z = -5 - k + 3l \qquad k, l \in \mathbb{R}$$

KEY CONCEPTS

- In two-space, a scalar equation defines a line. In three-space, a scalar equation defines a plane.
- In three-space, a plane can be defined by a vector equation, parametric equations, or a scalar equation.

Equations of Planes in Three-Space	
Vector	$\vec{r} = \vec{r}_0 + t\vec{a} + s\vec{b}$ or $[x, y, z] = [x_0, y_0, z_0] + t[a_1, a_2, a_3] + s[b_1, b_2, b_3]$, where • $\vec{r} = [x, y, z]$ is a position vector for any point on the plane, • $\vec{r}_0 = [x_0, y_0, z_0]$ is a position vector for a known point on the plane, • $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ are non-parallel direction vectors parallel to the plane, and • t and s are scalars, $t, s \in \mathbb{R}$
Parametric	$x = x_0 + ta_1 + sb_1$ $y = y_0 + ta_2 + sb_2$ $z = z_0 + ta_3 + sb_3$, $t, s \in \mathbb{R}$
Scalar	$Ax + By + Cz + D = 0$

- A plane can be uniquely defined by three non-collinear points or by a point and two non-parallel direction vectors.
- For a scalar equation, any point (x, y, z) that satisfies the equation lies on the plane.
- For vector and parametric equations, any combination of values of the parameters t and s will produce a point on the plane.
- The x-intercept of a plane is found by setting $y = z = 0$ and solving for x. Similarly, the y- and z-intercepts are found by setting $x = z = 0$ and $x = y = 0$, respectively.

Communicate Your Understanding

C1 For scalar equations, such as $x = 5$ and $6x + 2y + 5 = 0$, why is it necessary to specify if you are working in two-space or three-space?

C2 Explain why the two direction vectors in the vector equation of a plane cannot be parallel.

C3 Which form of the equation of the plane—scalar, vector, or parametric—would you use to find the x-, y-, and z-intercepts? Explain your reasons.

C4 Describe a situation in which three points do not describe a unique plane.

A Practise

1. Write the coordinates of three points on each plane.
   a) $x = 8$
   b) $y + 3z - 9 = 0$
   c) $3x - 7y + z + 8 = 0$
   d) $2x + 3y - 4z = 8$

2. For each plane in question 1, determine two vectors parallel to the plane.

3. Does each point lie on the plane $4x + 3y - 5z = 10$?
   a) A(1, 2, 0)
   b) B(−7, 6, 4)
   c) C(−2, 1, −3)
   d) D(1.2, −2.4, 6.2)

4. Find the x-, y-, and z-intercepts of each plane.
   a) $3x - 2y + 4z = 12$
   b) $x + 5y - 6z = 30$
   c) $4x + 2y - 7z + 14 = 0$
   d) $3x + 6z + 18 = 0$

5. Write the parametric equations of each plane given its vector equation.
   a) $[x, y, z] = [1, 3, -2] + s[-3, 4, -5] + t[9, 2, -1]$
   b) $[x, y, z] = [0, -4, 1] + s[1, 10, -1] + t[0, 3, 4]$
   c) $[x, y, z] = [0, 0, 5] + s[0, 3, 0] + t[1, 0, 5]$

6. Write the vector equation of a plane given its parametric equations.
   a) $x = 9 + 3s - 2t$
      $y = 4 - 7s + t$
      $z = -1 - 5s - 4t$
   b) $x = 2 + s + 7t$
      $y = 12s - 8t$
      $z = 11 + 6s$
   c) $x = -6$
      $y = 8s$
      $z = 5 - 13t$

7. Determine if each point is on the plane $[x, y, z] = [6, -7, 10] + s[1, 3, -1] + t[2, -2, 1]$.
   a) P(10, −19, 15)
   b) P(−4, −13, 10)
   c) P(8.5, −3.5, 9)

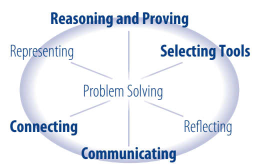
8. Determine the coordinates of two other points on the plane in question 7.

9. Determine the x-, y-, and z-intercepts of each plane.
   a) $[x, y, z] = [1, 8, 6] + s[1, -12, -12] + t[2, 4, -3]$
   b) $[x, y, z] = [6, -9, -8] + s[1, -4, -4] + t[3, 3, 8]$

B Connect and Apply

10. Write a vector equation and parametric equations for each plane.
    a) contains the point $P_0(6, -1, 0)$; has direction vectors $\vec{a} = [2, 0, -5]$ and $\vec{b} = [1, -3, 1]$
    b) contains the point $P_0(9, 1, -2)$; is parallel to $[x, y, z] = [4, 1, 8] + s[1, -1, 1]$ and $[x, y, z] = [-5, 0, 10] + t[-6, 2, 5]$
    c) contains the points A(1, 3, −2), B(3, −9, 7), C(4, −4, 5)
    d) has x-intercept 8, y-intercept −3, and z-intercept 2

11. **Use Technology** Describe each plane. Verify your answers using 3-D graphing technology.



    a) $x = 5$
    b) $y = -7$
    c) $z = 10$
    d) $x + y = 8$
    e) $x + 2z = 4$
    f) $3y - 2z - 12 = 0$
    g) $x + y + z = 0$
    h) $3x + 2y - z = 6$
    i) $4x - 5y + 2z + 20 = 0$

CONNECTIONS

Go to *www.mcgrawhill.ca/links/calculus12* and follow the links for instructions on how to download 3-D Grapher software.

12. Use Technology Determine an equation for each plane. Verify your answers using 3-D graphing technology.

a) parallel to both the x-axis and z-axis and through the point A(3, 1, 5)

b) parallel to the xy-plane; does not pass through the origin

c) containing the points A(2, 1, 1), B(5, −3, 2), and C(0, −1, 4)

d) perpendicular to vector $\vec{a} = [4, 5, -2]$; does not pass through the origin

e) containing the x-axis and parallel to the vector $\vec{a} = [4, 1, -3]$

f) parallel to, but not touching, the y-axis

g) two planes that are perpendicular to each other but neither parallel nor perpendicular to the xy-, xz-, or yz-planes.

13. In each case, explain why the given information does not define a unique plane in three-space.

a) y-intercept −3 and points R(2, −3, 4) and S(−2, −3, −4)

b) points A(2, 0, 1), B(5, −15, 7), and C(0, 10, −3)

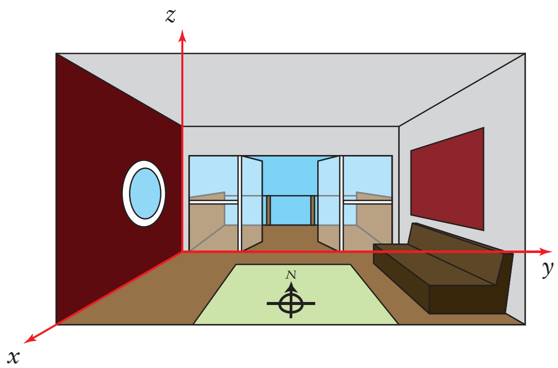
c) vector equation $[x, y, z] = [1, -10, 3] + s[8, -12, 4] + t[-6, 9, -3]$

d) point P(3, −1, 4) and line $[x, y, z] = [-5, 3, 10] + t[4, -2, -3]$

e) parametric equations:
$x = 3 + 2s - 4t$
$y = 2 - 3s + 6t$
$z = 1 + s - 2t$

f) line $[x, y, z] = [0, 1, 1] + s[-1, 5, -2]$ and line $[x, y, z] = [-1, 5, -2] + t[1, -5, 2]$

14. Chapter Problem You are designing a room for a role-playing game. The dimensions of the room are 8 m long, 6 m wide, and 4 m high. Suppose the floor at the southwest corner of the room is the origin, the positive x-axis represents east, the positive y-axis represents north, and the positive z-axis represents up.



Write equations for each of the walls, the floor, and the ceiling.

C Extend and Challenge

15. A plane is perpendicular to both $[x, y, z] = [1, -10, 8] + s[1, 2, -1]$ and $[x, y, z] = [2, 5, -5] + t[2, 1, -3]$, and contains the point P(−1, 4, −2). Determine if the point A(7, 10, 16) is also on this plane.

16. Write the equation of the plane that contains points A(3, 0, 4), B(2, −3, 1), C(−5, 8, −4), and D(1, 4, 3), if it exists.

17. Determine the value of k such that the points A(4, −2, 6), B(0, 1, 0), C(1, 0, −5), and D(1, k, −2) lie on the same plane.

18. Consider a plane defined by $x + 2y - z = D$.

a) Determine four points on the plane.

b) Use the points from part a) to write three vectors.

c) Use the triple scalar product with the vectors from part b) to show that the vectors are coplanar.

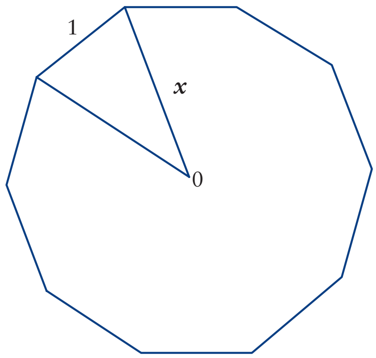
d) Use two of the vectors to find a fourth vector perpendicular to the plane.

CONNECTIONS

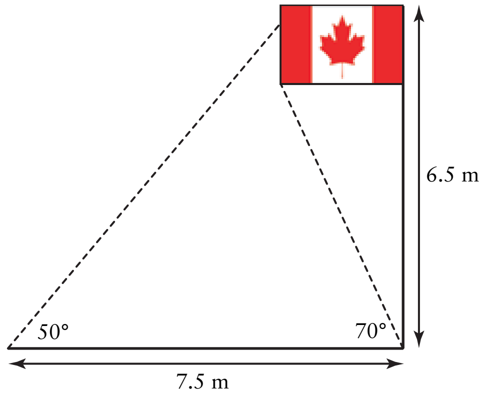
The triple scalar product is $\vec{a} \cdot \vec{b} \times \vec{c}$.

19. In two-space, the equation of the line $\frac{x}{a} + \frac{y}{b} = 1$ has x-intercept at a and y-intercept at b. Is there an analogous equation for a plane in three-space? Use a diagram to explain.

20. **Math Contest** The figure shown is a regular decagon with side length 1. Determine the exact value of x.



21. **Math Contest** The angle of elevation of the top corner of the flag shown is 60° from one position, and of the bottom corner is 80° from a different position, as shown. Determine the area of the flag. (The diagram is not to scale.)



CAREER CONNECTION

The need for communication satellites has grown dramatically in recent years. There is continual demand for more TV channels, cell phones and Internet access. More aerospace engineers, like Roland, will be needed to design and build such satellites. To become qualified as an aerospace engineer, Roland completed a 4-year bachelor's program in at Carleton University. He also completed a master's degree and PhD, so that he could work in a senior research position within his company. His education gave him a thorough understanding of subjects such as aerodynamics, propulsion, thermodynamics, and guidance and control systems. In the process of creating a new satellite or part needed, Roland uses advanced equipment, such as CAD software, robotics, lasers, and advanced electronic optics.

8.3

Properties of Planes

A flight simulator allows pilots in training to learn how to fly an airplane while still on the ground. It starts out with its floor parallel to the ground, and then it is articulated on hydraulics. Careful coordination of the position and orientation of the simulator with the images on the computer screens makes the experience very realistic. The simulator can move in directions along the x-, y-, and z-axes. An understanding of how planes are oriented in three-space allows the simulator to be programmed so its motion matches the images on the simulator software.

Investigate — How is the normal vector to a plane related to the scalar equation of a plane?

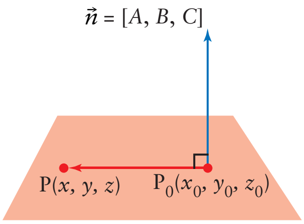
CONNECTIONS

The cross product of two vectors yields a vector perpendicular to both original vectors.

1. Determine the x-, y-, and z-intercepts of the plane defined by the scalar equation $x - 2y + 3z - 6 = 0$.
2. Determine two direction vectors parallel to the plane but not to each other. Determine a single vector perpendicular to both of the direction vectors.
3. Find three other points on the plane. Repeat step 2 to determine two other direction vectors parallel to the plane but not to each other; then determine a vector perpendicular to both of these direction vectors.
4. Repeat steps 1 to 3 for each scalar equation.
   a) $2x + y - 4z - 8 = 0$
   b) $x - y + 3z - 9 = 0$
5. **Reflect** How do the perpendicular vectors found in steps 2 and 3 compare? How do these vectors compare to the coefficients in the scalar equation?
6. **Reflect** A vector perpendicular to the plane is called a normal vector. How could you use the normal vector and a point on the plane to determine the scalar equation of the plane?

Determining the Scalar Equation of a Plane

If we are given a vector $\vec{n} = [A, B, C]$ that is normal to a plane and a point $P_0(x_0, y_0, z_0)$ on the plane, we can determine the scalar equation of the plane.



If $P(x, y, z)$ is any point on the plane, the vector $\overrightarrow{P_0P}$ is parallel to the plane but perpendicular to the normal, $\vec{n}$.

$$\begin{aligned}\overrightarrow{P_0P} &= [x, y, z] - [x_0, y_0, z_0] \\ &= [x - x_0, y - y_0, z - z_0]\end{aligned}$$

Since $\overrightarrow{P_0P} \perp \vec{n}$, $\overrightarrow{P_0P} \cdot \vec{n} = 0$.

CONNECTIONS

The dot product of perpendicular vectors is zero.

$[x - x_0, y - y_0, z - z_0] \cdot [A, B, C] = 0$

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ Expand and rearrange.

$Ax + By + Cz + (-Ax_0 - By_0 - Cz_0) = 0$

The expression in the brackets is made up of known values, so it is a constant. Use D to represent this constant.

Therefore, $Ax + By + Cz + D = 0$ is the scalar equation of the plane having normal vector $\vec{n} = [A, B, C]$.

Example 1 Write the Scalar Equation of a Plane Given the Normal and a Point

Consider the plane that has normal vector $\vec{n} = [3, -2, 5]$ and contains the point $P_0(1, 2, -3)$.

a) Write the scalar equation of the plane.

b) Is vector $\vec{a} = [4, 1, -2]$ parallel to the plane?

c) Is vector $\vec{b} = [15, -10, 25]$ normal to the plane?

d) Find another vector that is normal to the plane.

Solution

a) Method 1: ***Use the Dot Product***

Let $P(x, y, z)$ be any point on the plane.

$\overrightarrow{P_0P} = [x - 1, y - 2, z + 3]$ is a vector in the plane.

$$\begin{aligned} \overrightarrow{P_0P} \cdot \vec{n} &= 0 \\ [x - 1, y - 2, z + 3] \cdot [3, -2, 5] &= 0 \\ (x - 1)(3) + (y - 2)(-2) + (z + 3)(5) &= 0 \\ 3x - 2y + 5z + 16 &= 0 \end{aligned}$$

The scalar equation is $3x - 2y + 5z + 16 = 0$.

Method 2: ***Use*** $Ax + By + Cz + D = 0$

The components of the normal vector are the first three coefficients of the scalar equation.

$3x - 2y + 5z + D = 0$

Since the point $P_0(1, 2, -3)$ lies on the plane, substitute its coordinates to find D.

$$\begin{aligned} 3(1) - 2(2) + 5(-3) + D &= 0 \\ D &= 16 \end{aligned}$$

The scalar equation is $3x - 2y + 5z + 16 = 0$.

b) $\vec{a}$ is parallel to the plane only if $\vec{a} \cdot \vec{n} = 0$.

$$\begin{aligned} \vec{a} \cdot \vec{n} &= [4, 1, -2] \cdot [3, -2, 5] \\ &= 4(3) + 1(-2) + (-2)(5) \\ &= 0 \end{aligned}$$

Vector $\vec{a}$ is parallel to the plane.

c) Vector $\vec{b} = [15, -10, 25]$ is perpendicular to the plane only if it is parallel to the normal vector, $\vec{n}$, that is, only if $\vec{b} = k\vec{n}$, where k is a constant.

$[15, -10, 25] = k[3, -2, 5]$

$$\begin{aligned} 15 &= 3k & \qquad -10 &= -2k & \qquad 25 &= 5k \\ k &= 5 & k &= 5 & k &= 5 \end{aligned}$$

Clearly, $\vec{b} = 5\vec{n}$, $\vec{b}$ is parallel to $\vec{n}$, and $\vec{b}$ is normal to the plane.

d) Any scalar multiple of $\vec{n} = [3, -2, 5]$ is normal to the plane. The vectors $[15, -10, 25]$ and $[6, -4, 10]$ are two such vectors.

CONNECTION

Solutions to Examples 1, 2, and 3 can be verified using 3D graphing software

Example 2 Write Equations of a Plane Given Points on the Plane

Find the scalar equation of the plane containing the points A(−3, −1, −2), B(4, 6, 2), and C(5, −4, 1).

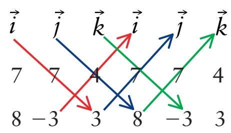
Solution

First, find a normal vector to the plane.

$\overrightarrow{AB} = [7, 7, 4]$ and $\overrightarrow{AC} = [8, -3, 3]$ are vectors in the plane.

$\overrightarrow{AB} \times \overrightarrow{AC}$ will give a vector normal to the plane.

$$\begin{aligned}[7, 7, 4] \times [8, -3, 3] &= [7(3) - 4(-3), 4(8) - 7(3), 7(-3) - 7(8)] \\ &= [33, 11, -77]\end{aligned}$$



Thus $\vec{n}$ can be [33, 11, −77] or, more conveniently, [3, 1, −7].

The scalar equation of the plane is of the form

$3x + y - 7z + D = 0$

Use one of the given points, say A, to determine D.

$$\begin{aligned}3(-3) + (-1) - 7(-2) + D &= 0 \\ D &= -4\end{aligned}$$

The scalar equation of the plane is $3x + y - 7z - 4 = 0$.

Example 3 Use Properties of Planes to Write Equations

Determine the scalar equation of each plane.

a) parallel to the xz-plane; through the point (−7, 8, 9)

b) containing the line $[x, y, z] = [1, 2, 4] + t[4, 1, 11]$ and perpendicular to $[x, y, z] = [4, 15, 8] + s[2, 3, -1]$

Solution

a) The xz-plane has a normal vector perpendicular to both the x- and z-axes.

One possible normal is $\vec{j} = [0, 1, 0]$.

Thus the scalar equation will have the form $(0)x + (1)y + (0)z + D = 0$ or $y + D = 0$

Point (−7, 8, 9) is on the plane.

$$\begin{aligned}8 + D &= 0 \\ D &= -8\end{aligned}$$

The scalar equation is $y - 8 = 0$.

b) If the plane is perpendicular to the line $[x, y, z] = [4, 15, 8] + s[2, 3, -1]$, then the direction vector of the line is a normal vector to the plane and $\vec{n} = [2, 3, -1]$.

The scalar equation has the form $2x + 3y - z + D = 0$.

Since the line $[x, y, z] = [1, 2, 4] + t[4, 1, 11]$ is contained in the plane, the point $(1, 2, 4)$ is on the plane.

$$2(1) + 3(2) - (4) + D = 0$$
$$D = -4$$

The scalar equation is $2x + 3y - z - 4 = 0$.

KEY CONCEPTS

- The scalar equation of a plane in three-space is $Ax + By + Cz + D = 0$, where $\vec{n} = [A, B, C]$ is a normal vector to the plane.
- Any vector parallel to the normal of a plane is also normal to that plane.
- The coordinates of any point on the plane satisfy the scalar equation.
- A normal vector (for orientation) and a point (for position) can be used to define a plane.

Communicate Your Understanding

C1 Explain why three non-collinear points always define a plane, but four non-collinear points may not always define a plane. How is this similar to the idea that, in two-space, two distinct points will always define a line while three distinct points may not?

C2 Given a scalar equation of a plane, explain how you would find a vector equation for that plane.

C3 Describe a situation in which it is easiest to find the scalar equation of a plane. Describe another situation in which it is easiest to find the vector equation of a plane.

C4 A direction vector indicates the orientation of a line. Why must a normal be used, instead of a single direction vector, to indicate the orientation of a plane?

A Practise

1. Determine if each point lies on the plane $x + 2y - 3z - 5 = 0$.
   a) M(5, −3, −2)
   b) N(3, 2, −1)
   c) P(−7, 0, −4)
   d) Q(6, 1, 1)
   e) R(0, 0, 5)
   f) S(1, 2, −3)

2. Find two vectors normal to each plane.
   a) $x + 2y + 2z - 5 = 0$
   b) $6x - y + 4z + 8 = 0$
   c) $5x + 2z = 7$
   d) $5y = 8$
   e) $3x + 4y - 7 = 0$
   f) $-x - 3y + z = 0$

3. Determine a vector parallel to each plane in question 2.

4. Write the scalar equation of each plane given the normal $\vec{n}$ and a point P on the plane.
   a) $\vec{n} = [1, -1, 1]$, P(2, −1, 8)
   b) $\vec{n} = [3, 7, 1]$, P(3, −6, 4)
   c) $\vec{n} = [2, 0, -5]$, P(1, 10, −3)
   d) $\vec{n} = [-9, 0, 0]$, P(−2, 3, −15)
   e) $\vec{n} = [4, -3, 2]$, P(6, 3, −4)
   f) $\vec{n} = [4, -3, 4]$, P(−2, 5, 3)

5. Consider the plane $-x + 4y + 2z + 6 = 0$.
   a) Determine a normal vector, $\vec{n}$, to the plane.
   b) Determine the coordinates of two points, S and T, on the plane.
   c) Determine $\overrightarrow{ST}$.
   d) Show that $\overrightarrow{ST}$ is perpendicular to $\vec{n}$.

6. Write a scalar equation of each plane, given its vector equation.
   a) $[x, y, z] = [3, 7, -5] + s[1, 2, -1] + t[1, -2, 3]$
   b) $[x, y, z] = [5, -2, 3] + s[3, -2, 4] + t[5, -2, 6]$
   c) $[x, y, z] = [6, 8, 2] + s[2, -1, -1] + t[1, 3, 3]$
   d) $[x, y, z] = [9, 1, -8] + s[6, 5, 2] + t[3, -3, 1]$
   e) $[x, y, z] = [0, 0, 1] + s[0, 1, 0] + t[0, 0, -1]$
   f) $[x, y, z] = [3, 2, 1] + s[2, 0, 3] + t[3, 0, 2]$

7. Write a scalar equation of each plane, given its parametric equations.
   a) $\pi_1: \begin{cases} x = 3 - 2s + 2t \\ y = 1 + 3s + t \\ z = 5 - s - 2t \end{cases}$
   b) $\pi_2: \begin{cases} x = -1 + 5t \\ y = 3 - s - 2t \\ z = -2 + s \end{cases}$
   c) $\pi_3: \begin{cases} x = -1 - s + 2t \\ y = 1 - s + 4t \\ z = 2 + 3s + t \end{cases}$
   d) $\pi_4: \begin{cases} x = 2 + s + t \\ y = 3 + 2s \\ z = 5 + 4s + 2t \end{cases}$
   e) $\pi_5: \begin{cases} x = 5s + 2t - 3 \\ y = 2 + 3s \\ z = 2t - s - 1 \end{cases}$
   f) $\pi_6: \begin{cases} x = 3s + t \\ y = 2 - s - 5t \\ z = 1 + 2s + 3t \end{cases}$

B Connect and Apply

8. For each situation, write a vector equation and a scalar equation of the plane.

a) perpendicular to the line $[x, y, z] = [2, 4, -9] + t[3, 5, -3]$ and including the point $(4, -2, 7)$

b) parallel to the yz-plane and including the point $(-1, -2, 5)$

c) parallel to the plane $3x - 9y + z - 12 = 0$ and including the point $(-3, 7, 1)$

d) containing the lines $[x, y, z] = [-2, 3, 12] + s[-2, 1, 5]$ and $[x, y, z] = [1, -4, 4] + t[-6, 3, 15]$

9. Refer to your answers to question 8. Based on the information given each time, was it easier to write the vector equation or the scalar equation? Explain.

10. Determine if the planes in each pair are parallel, perpendicular, or neither.

a) π_1: $4x - 5y + z - 9 = 0$
π_2: $2x - 9y + z - 2 = 0$

b) π_1: $5x - 6y + 2z - 2 = 0$
π_2: $2x - 5y - 20z + 13 = 0$

c) π_1: $12x - 6y + 3z = 1$
π_2: $-6x + 3y - 9z = 8$

11. Write the equation of the line perpendicular to the plane $3x - 7y + 3z - 5 = 0$ and through $(3, 9, -2)$.

12. Write a vector equation of each plane.

a) $y - 3 = 0$

b) $x + y + 8 = 0$

c) $x + y + z = 10$

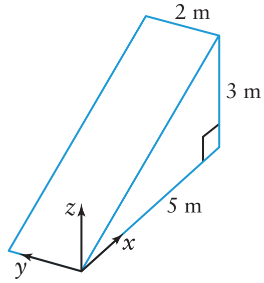
d) $4x - y + 8z = 4$

e) $3x + 2y - z = 12$

f) $2x - 5y - 3z = 30$

13. **Use Technology** Use 3-D graphing technology to verify any of your answers for questions 1 to 7 by graphing the given information and the plane found.

14. **Chapter Problem**
In the computer game, there must be a ramp that is 2 m wide, with height 3 m and length 5 m. Assume that one corner of the bottom of the ramp is at the origin.

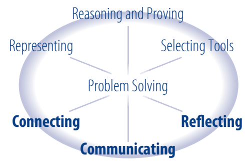


a) Determine the scalar equations of the planes that include the sides and top of the ramp.

b) Determine the equation of the line across the top edge of the ramp.

c) Are there any restrictions for the equations you have created in parts a) and b)?

15. **Use Technology**
Use 3-D graphing technology.



a) Determine the effect of changing k in the plane given by $3x + 6y - 4z = k$.

b) Change the coefficients of the scalar equation and repeat part a). What generalization can you make about the effect of the constant term?

c) What is true about all planes with $k = 0$?

Achievement Check

16. Consider the plane that contains points A(2, 3, 1), B(−11, 1, 2), and C(−7, −3, −6).

a) Find two vectors parallel to the plane.

b) Find two vectors perpendicular to the plane.

c) Write a vector equation of the plane.

d) Write the scalar equation of the plane.

e) Determine if the point D(9, 5, 2) is also on the plane.

f) Write an equation of the line through the x- and y-intercepts of the plane.

17. Since the normal vector to a plane determines the orientation of the plane, the angle between two planes can be defined as the angle between their two normal vectors. Determine the angle between the planes in each pair.

a) π_1: $3x + 2y + 5z = 5$
π_2: $4x - 3y - 2z = 2$

b) π_1: $2x + 3z - 10 = 0$
π_2: $-x + 6y + 3z - 6 = 0$

c) π_1: $4x + y + -2z = 8$
π_2: $3x - 2y + 5z = 1$

18. Consider the planes $2x + 6y - 2z - 5 = 0$ and $5x + 15y + kz - 7 = 0$.

a) For what value of k are the planes parallel?

b) For what value of k are the planes perpendicular?

19. Explain why there is more than one plane parallel to the z-axis with x-intercept $(4, 0, 0)$.

20. A plane is defined by the equation $3x - 4y + 6z = 18$.

a) Determine three non-collinear points, A, B, and C, on this plane.

b) Determine the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$.

c) Find each result.

i) $\overrightarrow{AB} + \overrightarrow{AC}$
ii) $\overrightarrow{AC} + \overrightarrow{BC}$
iii) $2\overrightarrow{AC} - 4\overrightarrow{AB}$
iv) $3\overrightarrow{AB} + 5\overrightarrow{BC}$

d) Use the normal vector to the plane to show that all of the vectors from part c) are parallel to the plane.

21. **Use Technology** Write an equation of the plane through P(3, 1, −1) and perpendicular to the planes $2x - 3y = 10$ and $x + 2y - 2z = 8$. Verify your answer using 3-D graphing technology.

C Extend and Challenge

22. Determine the scalar equation of the plane through A(2, 1, −5), perpendicular to both $3x - 2y + z = 8$ and $4x + 6y - 5z = 10$.

23. Two parallel sides of a cubic box are defined by the equations $x + y = 0$ and $x + y = k$. The bottom of the box is defined by the xy-plane.

a) Write the equations of the planes that form the remaining sides.

b) Determine the lengths of the sides of the box in terms of k.

24. Write equations for three non-parallel planes that all intersect at A(3, −1, −2).

25. Determine the equations of the planes that make up a tetrahedron with one vertex at the origin and the other three vertices at (5, 0, 0), (0, −6, 0), and (0, 0, 2).

26. Given the equation of the plane $Ax + By + Cz + D = 0$, find the conditions on A, B, C, and D such that each statement is true.

a) The plane is perpendicular to the x-axis.

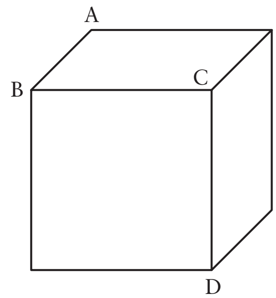
b) The plane has x-intercept at (3, 0, 0), y-intercept at (0, 5, 0), and z-intercept at (0, 0, 6).

c) The plane is parallel to the z-axis.

d) The plane is perpendicular to the plane $x + 4y - 7z = -4$.

27. Show that the normal vectors to the planes $3x + 4z = 12$ and $4x - 5z = 40$ define a family of planes perpendicular to the y-axis.

28. **Math Contest** This cube has side length 12 cm. One bug starts at corner A and travels to corner B. Another bug starts at corner C and travels to corner D. The two bugs leave at the same time and travel at the same rate of speed. What is the minimum distance between the bugs?



29. **Math Contest** If $\log_9 x = \log_3 y + 1$, determine y as a function of x.

8.4 Intersections of Lines in Two-Space and Three-Space

When sailboats are racing, they sometimes get into a tacking duel. This occurs when the boats cross paths as they are travelling into the wind toward the same mark. A collision would be very expensive! GPS and RADAR technology can be used to predict the paths of the boats and determine if they will collide.

Investigate A Linear systems in two-space

Tools

- grid paper
- graphing calculator

1. Solve each system of linear equations algebraically.

a) $2x + 3y = 18$ ①
$5x - 3y = 3$ ②

b) $x - 4y - 6 = 0$ ③
$3x - 12y = 18$ ④

2. Graph each system of linear equations from step 1. Compare the graphs to the algebraic solutions.

3. Solve each system of linear equations algebraically.

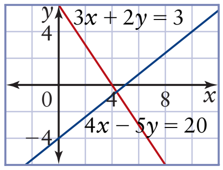
a) $5x + 2y = 10$ ①
$5x + 2y = 8$ ②

b) $7x - 3y + 4 = 0$ ③
$14x - 6y + 6 = 0$ ④

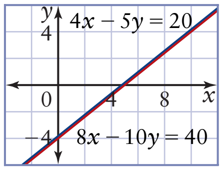
4. Predict what the graph of each system from step 3 will look like. Graph each system to check your predictions.

5. **Reflect** How can you use the algebraic solution to a system of linear equations to predict what the graph of the system will look like?

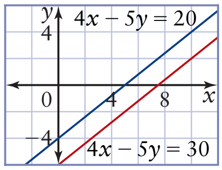
If the algebraic solution to a system of linear equations yields a unique pair of numbers, the lines intersect and there is exactly one solution or intersection point.



If the algebraic solution to a linear system gives an equation such as $0x = 0$ or $0y = 0$, all (x, y) that satisfy the original equations will also satisfy these equations and the lines are coincident. Coincident lines lie on top of one another, so there are an infinite number of solutions.



If the algebraic solution to a linear system yields an impossible equation such as $0x = 4$ or $0y = 4$, the lines are parallel and there are no solutions.



Example 1 Determine the Number of Solutions for Linear Systems in Two-Space

Classify each system as having zero, one, or infinitely many solutions.

a) ℓ_1: $4x - 6y = -10$
ℓ_2: $6x - 9y = -15$

b) ℓ_1: $[x, y] = [1, 5] + s[-6, 8]$
ℓ_2: $[x, y] = [2, 1] + t[9, -12]$

Solution

a) Method 1: ***Use algebraic thinking.***

$$\begin{aligned} 4x - 6y &= -10 \quad ① \\ 6x - 9y &= -15 \quad ② \end{aligned}$$

$$\begin{aligned} 12x - 18y &= -30 \quad 3① \qquad \text{Eliminate } x. \\ 12x - 18y &= -30 \quad 2② \\ \hline 0y &= 0 \quad 3① - 2② \end{aligned}$$

This equation is true for all values of y, so there are infinitely many solutions, all of the points (x, y) satisfying the original equations.

Method 2: ***Use geometric thinking.***

Examine the normals to the lines.

$\vec{n}_1 = [4, -6]$ and $\vec{n}_2 = [6, -9]$.

$$\vec{n}_1 = \frac{2}{3}\vec{n}_2$$

The normals are parallel, so the lines are either parallel or coincident.

If ℓ_2 is multiplied by $\frac{2}{3}$, the result is ℓ_1. The lines are identical. They intersect at infinitely many points.

b) **Method 1:** ***Use algebraic thinking.***

$[x, y] = [1, 5] + s[-6, 8]$ ①
$[x, y] = [2, 1] + t[9, -12]$ ②

Write the equations in parametric form.

① $\begin{cases} x = 1 - 6s \\ y = 5 + 8s \end{cases}$ ② $\begin{cases} x = 2 + 9t \\ y = 1 - 12t \end{cases}$

Equate the x- and y-variables.

$1 - 6s = 2 + 9t$ $5 + 8s = 1 - 12t$
$-1 = 6s + 9t$ ③ $4 = -8s - 12t$ ④

Use equations ③ and ④ to solve for s and t:

$$\begin{array}{rl} -4 = 24s + 36t & 4③ \\ \underline{12 = -24s - 36t} & 3④ \\ 8 = 0s & 4③ + 3④ \end{array}$$

This equation is true for no values of s. There are zero solutions to the system.

Method 2: ***Use geometric thinking.***

The direction vectors of the lines are $\vec{m}_1 = [-6, 8]$ and $\vec{m}_2 = [9, -12]$.

Since $\vec{m}_1 = \frac{2}{3}\vec{m}_2$, the direction vectors are parallel.

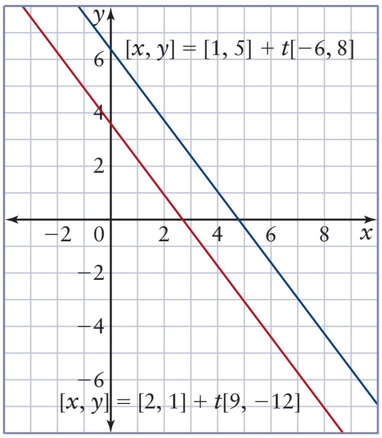
The lines are either parallel or coincident.

Check if (1, 5) lies on line ②.

$1 = 2 + 9t$ $5 = 1 - 12t$

$t = -\frac{1}{9}$ $t = -\frac{1}{3}$

Since the values for t are different, (1, 5) is not on ②. The lines are parallel and distinct. There are zero solutions to the system.

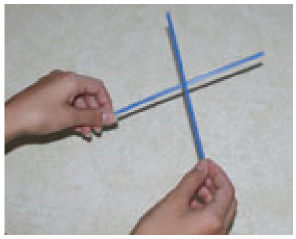


Investigate B Linear systems in three-space

Tools

- 2 straws

1. Use straws to represent lines in three-space.



   a) How could the straws be oriented to produce an intersection set with exactly one solution?

   b) How could the straws be oriented to produce an intersection set with an infinite number of solutions?

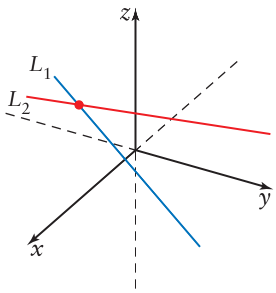
   c) How could the straws be oriented to produce an intersection set with no solutions?

2. **Reflect** For each situation in step 1, describe how the direction vectors of the lines might be related to one another.

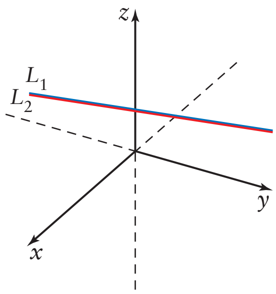
3. **Reflect** Compare each of the situations in step 1 with those in two-space.

Given two lines in three-space, there are four possibilities for the intersection of the lines.

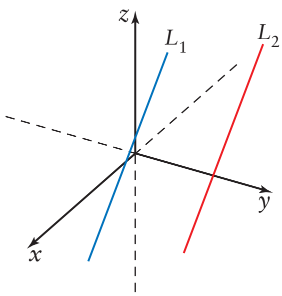
- The lines are distinct and intersect at a point, so there is exactly one solution.



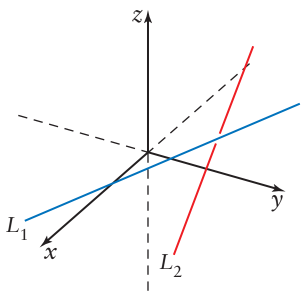
- The lines are coincident, so there are an infinite number of solutions.



- The lines are parallel and distinct, so there is no solution.



- The lines are distinct but not parallel, and they do not intersect. These are **skew lines**. There is no solution.



CONNECTIONS

Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.4 to access dynamic versions of these diagrams.

Example 2 Determine If Two Lines in Three-Space Intersect

Determine if these lines intersect. If they do, find the coordinates of the point of intersection.

ℓ_1: $[x, y, z] = [5, 11, 2] + s[1, 5, -2]$
ℓ_2: $[x, y, z] = [1, -9, 9] + t[1, 5, -2]$

Solution

From the equations, the direction vectors are the same, so the two lines are parallel. Determine if the lines are parallel and distinct or coincident.

Method 1: ***Verify a Point***

If the lines are coincident, then the coordinates of the point (5, 11, 2) from ℓ_1 will satisfy the equation for ℓ_2. Write the parametric equations for ℓ_2.

$x = 1 + t$
$y = -9 + 5t$
$z = 9 - 2t$

Substitute the coordinates of (5, 11, 2) and solve each equation for t.

$5 = 1 + t$	$11 = -9 + 5t$	$2 = 9 - 2t$
$t = 4$	$t = 4$	$t = 3.5$

Since the t-values are not identical, the point (5, 11, 12) does not lie on line ℓ_2. The lines are parallel and distinct. Since the values of t are close to one another, the two lines are likely to be relatively close to each other.

Method 2: ***Solve for an Intersection Point***

Write each equation in parametric form.

$$\ell_1: \begin{cases} x = 5 + s \\ y = 11 + 5s \\ z = 2 - 2s \end{cases} \qquad \ell_2: \begin{cases} x = 1 + t \\ y = -9 + 5t \\ z = 9 - 2t \end{cases}$$

Equate like coordinates.

$5 + s = 1 + t$	$11 + 5s = -9 + 5t$	$2 - 2s = 9 - 2t$
$s - t = -4$ ①	$5s - 5t = -20$ ②	$2s - 2t = -7$ ③

Rearrange equation ① to isolate s.

$s = t - 4$

Substitute $s = t - 4$ into equations ② and ③ and solve for t.

$5(t - 4) - 5t = -20$	$2(t - 4) - 2t = -7$
$5t - 20 - 5t = -20$	$2t - 8 - 2t = -7$
$0t = 0$	$0t = 1$

The equation $0t = 1$ has no solutions. The two lines are parallel and distinct.

Example 3 Intersection of Lines in Three-Space

Find the point of intersection of lines ℓ_1 and ℓ_2.

ℓ_1: $[x, y, z] = [7, 2, -6] + s[2, 1, -3]$
ℓ_2: $[x, y, z] = [3, 9, 13] + t[1, 5, 5]$

Solution

The direction vectors are not parallel.

Write the equations in parametric form.

$$\ell_1: \begin{cases} x = 7 + 2s \\ y = 2 + s \\ z = -6 - 3s \end{cases} \qquad \ell_2: \begin{cases} x = 3 + t \\ y = 9 + 5t \\ z = 13 + 5t \end{cases}$$

Equate expressions for like coordinates.

$$\begin{array}{lll} 7 + 2s = 3 + t & 2 + s = 9 + 5t & -6 - 3s = 13 + 5t \\ 2s - t = -4 \text{①} & s - 5t = 7 \text{②} & 3s + 5t = -19 \text{③} \end{array}$$

Solve equations ② and ③ for s and t.

$$\begin{aligned} s - 5t &= 7 \\ \underline{3s + 5t} &\underline{= -19} \\ 4s &= -12 \\ s &= -3 \end{aligned}$$

Substituting in ②,

$$\begin{aligned} (-3) - 5t &= 7 \\ t &= -2 \end{aligned}$$

Check that $s = -3$ and $t = -2$ satisfy ①.

$$\begin{aligned} \text{LS} &= 2(-3) - (-2) \qquad \text{RS} = -4 \\ &= -6 + 2 \\ &= -4 \\ &= \text{RS} \end{aligned}$$

Substitute $s = -3$ into ℓ_1 (or $t = -2$ into ℓ_2).

$$\begin{aligned} [x, y, z] &= [7, 2, -6] + (-3)[2, 1, -3] \\ &= [1, -1, 3] \end{aligned}$$

This system has a unique solution at $(1, -1, 3)$.

CONNECTIONS

Solutions to Examples 2, 3, and 4 can be verified using 3D graphing software

Example 4 Skew Lines

Determine if these lines are skew.

ℓ_1: $[x, y, z] = [5, -4, -2] + s[1, 2, 3]$
ℓ_2: $[x, y, z] = [2, 0, 1] + t[2, -1, -1]$

Solution

The direction vectors are not parallel since $\vec{m}_1 \neq k\vec{m}_2$.

Write the equations in parametric form and equate the expressions for each variable.

$5 + s = 2 + 2t$	$-4 + 2s = 0 - t$	$-2 + 3s = 1 - t$
$s - 2t = -3$ ①	$2s + t = 4$ ②	$3s + t = 3$ ③

Solve ② and ③ for s and t.

$$\begin{aligned} 2s + t &= 4 && ② \\ \underline{3s + t} &\underline{= 3} && ③ \\ -s &= 1 && ② - ③ \\ s &= -1 \end{aligned}$$

Substituting,

$$\begin{aligned} 2(-1) + t &= 4 \\ t &= 6 \end{aligned}$$

Check $s = -1$ and $t = 6$ in equation ①.

$$\begin{aligned} \text{LS} &= (-1) - 2(6) & \text{RS} &= -3 \\ &= -13 \\ &\neq \text{RS} \end{aligned}$$

Thus the lines do not intersect.

Since the direction vectors are not parallel, the lines are not parallel.

These are skew lines.

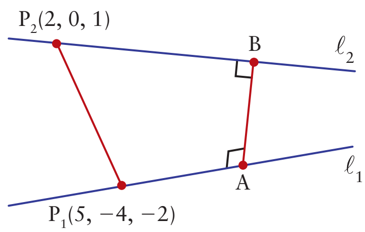
Example 5 The Distance Between Two Skew Lines

Determine the distance between the skew lines.

ℓ_1: $[x, y, z] = [5, -4, -2] + s[1, 2, 3]$
ℓ_2: $[x, y, z] = [2, 0, 1] + t[2, -1, 1]$

Solution

The distance required is the shortest distance.



It can be shown that the shortest distance between skew lines is the length of the common perpendicular.

Recall our work with projection vectors in Chapter 7.

Since $\angle P_1AB = 90°$ and $\angle P_2BA = 90°$,
$proj_{\vec{n}}\, \overrightarrow{P_1P_2} = \overrightarrow{AB}$ if $\vec{n}$ is any vector in the direction of $\overrightarrow{AB}$.
Thus $|proj_{\vec{n}}\, \overrightarrow{P_1P_2}| = |\overrightarrow{AB}|$, which is the distance we are looking for.

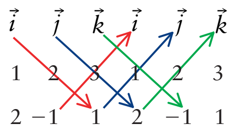
$$|proj_{\vec{n}}\, \overrightarrow{P_1P_2}| = \left|\frac{\overrightarrow{P_1P_2} \cdot \vec{n}}{\vec{n}}\right|$$

Use position vectors to determine $\overrightarrow{P_1P_2}$.

$$\begin{aligned}\overrightarrow{P_1P_2} &= [2, 0, 1] - [5, -4, -2]\\ &= [-3, 4, 3]\end{aligned}$$

Use the direction vectors for both lines to calculate the normal vector to the lines.

$$\begin{aligned}\vec{n} &= \vec{m}_1 \times \vec{m}_2\\ &= [1, 2, 3] \times [2, -1, 1]\\ &= [2(1) - 3(-1),\ 3(2) - 1(1),\ 1(-1) - 2(2)]\\ &= [5, 5, -5]\end{aligned}$$



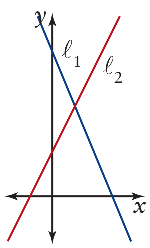
A vector normal to both lines is $\vec{n} = [1, 1, -1]$. Divide by 5 for convenience.

$$\begin{aligned}|\overrightarrow{AB}| &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{n}|}{|\vec{n}|}\\ &= \frac{|[-3, 4, 3] \cdot [1, 1, -1]|}{|[1, 1, -1]|}\\ &= \frac{|-3(1) + 4(1) + 3(-1)|}{\sqrt{1^2 + 1^2 + (-1)^2}}\\ &= \frac{|-2|}{\sqrt{3}}\\ &\doteq 1.15\end{aligned}$$

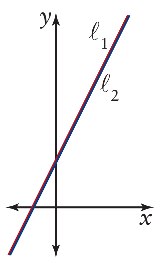
The perpendicular distance between these lines is approximately 1.15 units.

KEY CONCEPTS

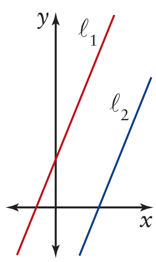
- In two-space, there are three possibilities for the intersection of two lines and the related system of equations.



The lines intersect at a point, so there is exactly one solution.

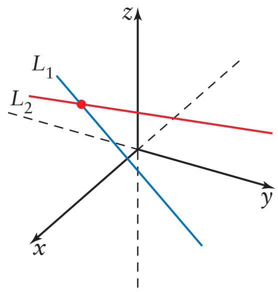


The lines are coincident, so there are infinitely many solutions.

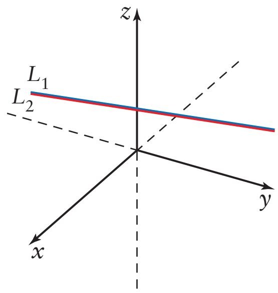


The lines are parallel, so there is no solution.

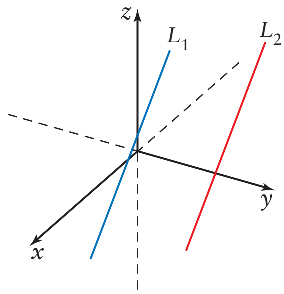
- In three-space, there are four possibilities for the intersection of two lines and the related system of equations.



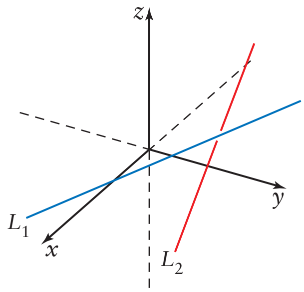
The lines intersect at a point, so there is exactly one solution.



The lines are coincident, so there are infinitely many solutions.



The lines are parallel, so there is no solution.



The lines are skew—that is, they are not parallel and they do not intersect. There is no solution.

- The distance between two skew lines can be calculated using the formula $d = \left|\frac{\overrightarrow{P_1P_2} \cdot \vec{n}}{\vec{n}}\right|$, where P_1 and P_2 are any points on each line and $\vec{n} = \vec{m}_1 \times \vec{m}_2$ is a normal common to both lines.

Communicate Your Understanding

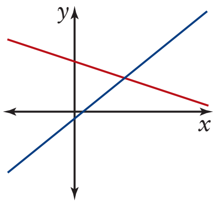
C1 Given a system of linear equations in two-space, how many types of solutions are possible? Explain.

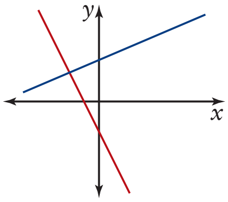
C2 The direction vectors of two lines in three-space are not parallel. Does this indicate that the lines intersect? Explain.

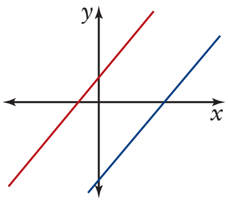
C3 How can you tell if two lines in three-space are skew? Use examples to explain.

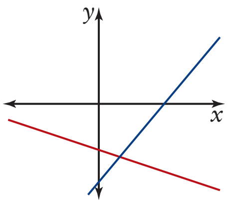
A Practise

1. How many solutions does each linear system have? Explain.

a)


b)


c)


d)


2. Solve each linear system in two-space.

a) $x + y = 8$
$x - y = 13$

b) $3.5x - 2.1y = 14$
$1.5x - 0.3y = 8$

c) $[x, y] = [-12, -7] + s[8, -5]$
$[x, y] = [2, -1] + t[3, -2]$

d) $[x, y] = [16, 1] + s[5, 1]$
$[x, y] = [-7, 12] + t[-7, 3]$

3. Determine the number of solutions for each system without solving.

a) $3x + 12y = -27$
$4x + 2y = 34$

b) $12x - 21y = 9$
$8x - 14y = 3$

c) $[x, y] = [1, 6] + s[3, -2]$
$[x, y] = [4, 4] + t[-6, 4]$

d) $[x, y] = [-17, -7] + t[8, -3]$
$8x - 3y = 11$

e) $-6x + 45y = 33$
$10x - 75y = -55$

f) $[x, y] = [11, 12] + s[2, 7]$
$[x, y] = [2, 3] + t[1, 4]$

4. Determine if the parallel lines in each pair are distinct or coincident.

a) $[x, y, z] = [5, 1, 3] + s[2, 1, 7]$
$[x, y, z] = [2, 3, 9] + t[2, 1, 7]$

b) $[x, y, z] = [4, 1, 0] + s[3, -5, 6]$
$[x, y, z] = [13, -14, 18] + t[-3, 5, -6]$

c) $[x, y, z] = [5, 1, 3] + s[1, 4, -1]$
$[x, y, z] = [3, -7, 5] + t[-2, -8, 2]$

d) $[x, y, z] = [4, -8, 0] + s[7, 21, -14]$
$[x, y, z] = [25, 55, -42] + t[-8, -24, 16]$

5. Determine if the lines in each pair intersect. If so, find the coordinates of the point of intersection.

a) $[x, y, z] = [6, 5, -14] + s[-1, 1, 3]$
$[x, y, z] = [11, 0, -17] + t[4, -1, -6]$

b) $[x, y, z] = [3, -2, 2] + s[-1, -2, 0]$
$[x, y, z] = [1, 0, -1] + t[0, 2, -3]$

c) $[x, y, z] = [7, 0, -15] + s[2, 1, -5]$
$[x, y, z] = [-7, -7, 20] + t[2, 1, -5]$

d) $[x, y, z] = [8, -1, 8] + s[2, -3, 0]$
$[x, y, z] = [1, 20, 0] + t[1, -5, 3]$

6. Determine if the non-parallel lines in each pair are skew.

a) $[x, y, z] = [4, 7, -1] + s[-2, 1, 2]$
$[x, y, z] = [1, 3, -1] + t[4, -1, 2]$

b) $[x, y, z] = [6, 2, 1] + s[6, 18, -6]$
$[x, y, z] = [7, 13, 1] + t[6, -1, -2]$

c) $[x, y, z] = [2, 4, 2] + s[2, 1, -5]$
$[x, y, z] = [4, 3, 7] + t[-2, 1, -5]$

d) $[x, y, z] = [-6, 12, 8] + s[2, 1, -5]$
$[x, y, z] = [8, -9, -7] + t[-2, 1, -5]$

e) $[x, y, z] = [5, -4, 1] + s[6, 4, -2]$
$[x, y, z] = [2, -3, 4] + t[1, 2, -3]$

f) $[x, y, z] = [1, -1, 0] + s[2, -1, 3]$
$[x, y, z] = [1, 2, 3] + t[3, 2, 4]$

B Connect and Apply

7. Verify your solutions to questions 4, 5, and 6 using 3D graphing technology.

8. The parametric equations of three lines are given. Do these define three different lines, two different lines, or only one line? Explain.

$$\ell_1: \begin{cases} x = 2 + 3s \\ y = -8 + 4s \\ z = 1 - 2s \end{cases} \qquad \ell_2: \begin{cases} x = 4 + 9s \\ y = -16 + 12s \\ z = 2 - 6s \end{cases}$$

$$\ell_3: \begin{cases} x = 3 + 9s \\ y = 7 + 12s \\ z = 2 + 6s \end{cases}$$

9. Determine the distance between the skew lines in each pair.

a) ℓ_1: $[x, y, z] = [3, 1, 0] + s[1, 8, 2]$
ℓ_2: $[x, y, z] = [-4, 2, 1] + t[-1, -2, 1]$

b) ℓ_1: $[x, y, z] = [1, -5, 6] + s[3, 1, -4]$
ℓ_2: $[x, y, z] = [0, 7, 2] + t[2, -1, 5]$

c) ℓ_1: $[x, y, z] = [2, 0, 8] + s[0, 3, 2]$
ℓ_2: $[x, y, z] = [1, 1, 1] + t[4, 0, -1]$

d) ℓ_1: $[x, y, z] = [5, 2, -3] + s[5, 5, 1]$
ℓ_2: $[x, y, z] = [-1, -4, -4] + t[7, -2, -2]$

10. These equations represent the sides of a triangle.

ℓ_1: $[x, y] = [-1, -1] + r[5, -1]$
ℓ_2: $[x, y] = [7, -10] + s[3, -8]$
ℓ_3: $[x, y] = [3, 13] + t[2, 7]$

a) Determine the intersection of each pair of lines.

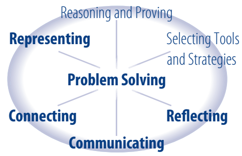
b) Find the perimeter of the triangle.

CONNECTIONS

Recall the formula for the distance between two points:
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

11. In three-space, there are four possibilities for the intersection of two lines. If one line is the y-axis, give a possible equation for the second line in each of the four cases.

12. The Port Huron to Mackinac yachting race has been run annually for more than 80 years. For almost 30 years the boats rounded the Cove Island Buoy at the mouth of Georgian Bay. The current course rounds NOAA Weather Buoy 45003.



a) The path from the start to Cove Island is represented by $[x, y] = [-82.2, 43] + s[0.8, 2.2]$, and the path from Cove Island to the finish line is represented by $[x, y] = [-84.4, 45.5] + t[3.0, -0.3]$. Determine the position of the Cove Island mark for the old course.

b) The path from the start to the weather buoy is represented by $[x, y] = [-82.2, 43] + s[0.2, 2.0]$ and the path from the weather buoy to the finish line is represented by $[x, y] = [-84.4, 44.5] + t[2.0, -0.5]$. Determine the position of the weather buoy for the new course

c) A boat mistakenly goes to the Cove Island mark. In what direction would the boat have to point to be heading for the weather buoy at that instant?

13. Can the distance formula of Example 5 be used to find the distance between two parallel lines in three-space? Explain by giving an example.

14. **Chapter Problem** As the games become more complex, so does the mathematics. In the previous game, only simple motions had to be analysed. For a flight simulator game, three-dimensional analysis must be done. In this particular game, a passenger jet is taking off from the airport and flying on a path given by $[x, y, z] = s[8, 4, 1]$. A private jet is flying by the airport, waiting to land, and is on the path given by $[x, y, z] = [60, -20, 22] + t[2, 6, -1]$, where s and t represent time in minutes.

a) Assume that both jets continue on their paths. Will their paths meet?

b) If the paths do meet, find the location. If they do not meet, find the least distance between the paths.

c) If the paths meet, does it necessarily mean there is a collision? Explain.

15. a) Determine if these lines are parallel.

ℓ_1: $[x, y, z] = [7, 7, -3] + s[1, 2, -3]$
ℓ_2: $[x, y, z] = [10, 7, 0] + t[2, 2, -1]$

b) Rewrite the equation of each line in parametric form. Show that the lines do not intersect.

c) Determine the least distance between the lines.

16. Write equations of two non-parallel lines in three-space that intersect at each point.

a) $(1, -7, 1)$ **b)** $(2, 4, -3)$

C Extend and Challenge

17. A median of a triangle is the line from a vertex to the midpoint of the opposite side. The point of intersection of the medians of a triangle is called the centroid.

a) The vertices of a triangle are at A(2, 6), B(10, 9), and C(9, 3). Find the centroid of the triangle.

b) Plot the points to confirm your answer to part a).

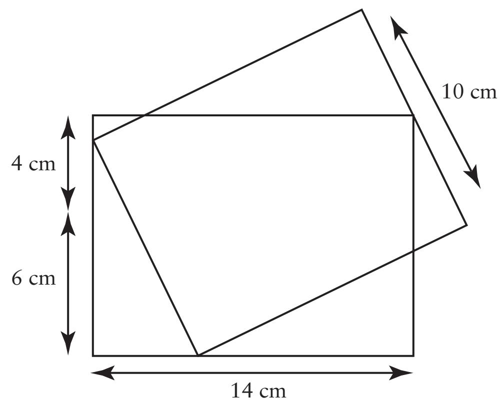
c) Find the centroid of a triangle with vertices at D(2, −6, 8), E(9, 0, 2), and F(−1, 3, −2).

18. Develop a formula for the solution to each system of equations.

a) $ax + by = c$
$dx + ey = f$

b) $[x, y] = [a, b] + s[c, d]$
$[x, y] = [e, f] + t[g, h]$

19. **Math Contest** The two overlapping rectangles shown have the same width, but different lengths. Determine the length of the second rectangle. (The diagram is not to scale.)



20. **Math Contest** The angle between the planes $x + y + 2z = 11$ and $2x - y + kz = 99$ is 60°. Determine all possible values of k.

8.5 Intersections of Lines and Planes

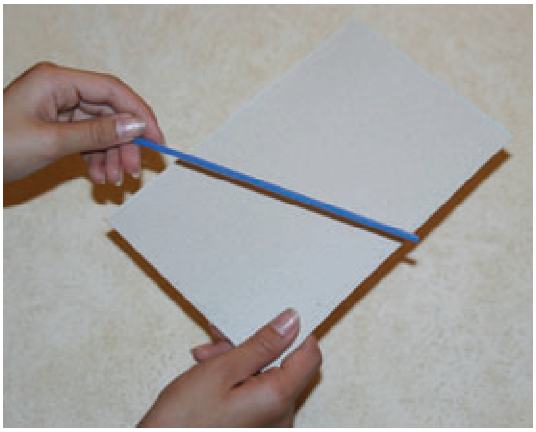
A jet is approaching a busy airport. Although the pilot may not physically see the airport yet, the jet's path is a straight line aimed at the flat surface of the runway. Electronic navigation aids (GPS, Radar, etc) help the pilot and the air traffic controller to guide the jet to a safe landing within a small window of time. Being able to predict whether the flight path will land the jet on the runway at the correct time is of vital importance to the safety of the passengers and the efficiency of the airport.

Investigate Intersections of lines and planes in three-space

Tools

- 1 straw
- cardboard

1. Use a straw and a piece of cardboard to represent a line and a plane in three-space.



a) How could the line and plane be oriented to produce a system with exactly one solution?

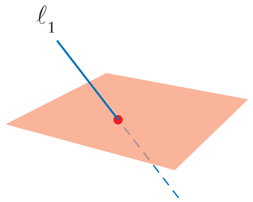
b) How could the line and plane be oriented to produce a system with an infinite number of solutions?

c) How could the line and plane be oriented to produce a system with no solution?

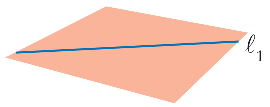
2. **Reflect** How would the direction vectors of the lines and the normal vectors of the planes be related to each other in each of the situations in step 1?

Given a line and a plane in three-space, there are three possibilities for the intersection of the line with the plane.

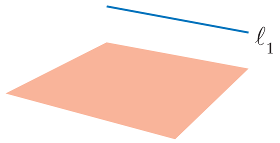
- The line and the plane intersect at a point. There is exactly one solution.



- The line lies on the plane, so every point on the line intersects the plane. There are an infinite number of solutions.



- The line is parallel to the plane. The line and the plane do not intersect. There are no solutions.



Example 1 Intersection of a Line and a Plane

In each case, determine if the line and the plane intersect. If so, determine the solution.

a) $\pi_1: 9x + 13y - 2z = 29$

$$\ell_1: \begin{cases} x = 5 + 2t \\ y = -5 - 5t \\ z = 2 + 3t \end{cases}$$

b) $\pi_2: x + 3y - 4z = 10$

$$\ell_2: \begin{cases} x = 4 + 6t \\ y = -7 + 2t \\ z = 1 + 3t \end{cases}$$

c) $\pi_3: 4x - y + 11z = -1$
$\ell_3: [x, y, z] = [-2, 4, 1] + t[3, 1, -1]$

Solution

a) Substitute the parametric equations into the scalar equation of the plane.

$$\begin{aligned} 9(5 + 2t) + 13(-5 - 5t) - 2(2 + 3t) &= 29 && \text{Expand and solve for } t. \\ 45 + 18t - 65 - 65t - 4 - 6t &= 29 \\ -53t &= 53 \\ t &= -1 \end{aligned}$$

Since a single value of t was found, the line and the plane intersect at a single point. Substitute $t = 1$ in the parametric equations.

$$x = 5 + 2(-1) = 3$$
$$y = -5 - 5(-1) = 0$$
$$z = 2 + 3(-1) = -1$$

The line and the plane intersect at a single point $(3, 0, -1)$.

b) Substitute the parametric equations into the scalar equation of the plane.

$$\begin{aligned}(4 + 6t) + 3(-7 + 2t) - 4(1 + 3t) &= 10\\ 4 + 6t - 21 + 6t - 4 - 12t &= 10\\ 0t &= 31\end{aligned}$$

Since there are no values of t that make the equation true, the plane and the line do not intersect. The line is parallel to and distinct from the plane.

c) Substitute the parametric equations into the scalar equation of the plane.

$$\begin{aligned}4(-2 + 3t) - (4 + t) + 11(1 - t) &= -1\\ -8 + 12t - 4 - t + 11 - 11t &= -1\\ 0t - 1 &= -1\\ 0t &= 0\end{aligned}$$

This equation is true for all values of t. Any point on the line is a solution. The line lies completely on the plane.

CONNECTIONS

Solutions to Examples 1 and 2 can be verified using 3D graphing software

Example 2 Use Vectors to Determine If a Line Intersects a Plane

Without solving, determine if each line intersects the plane.

a) ℓ_1: $\vec{r} = [2, -5, 3] + s[3, 2, 1]$
π_1: $3x - y + z = -6$

b) ℓ_2: $\vec{r} = [1, 0, 1] + t[-2, 1, -4]$
π_2: $4x - 2z = 11$

Solution

a) The direction vector of the line is $\vec{m} = [3, 2, 1]$.

The normal vector of the plane is $\vec{n} = [3, -1, 1]$.

Examine the dot product.

$$\vec{m} \cdot \vec{n} = 3(3) + 2(-1) + 1(1) = 8$$

Since $\vec{m} \cdot \vec{n} \neq 0$, $\vec{m}$ and $\vec{n}$ are not perpendicular.

Therefore, the line and the plane are not parallel and so they must intersect.

b) The direction vector of the line is $\vec{m} = [-2, 1, -4]$.

The normal vector of the plane is $\vec{n} = [4, 0, -2]$.

Examine the dot product.

$\vec{m} \cdot \vec{n} = -2(4) + 1(0) + -4(-2) = 0$

Since $\vec{m} \cdot \vec{n} = 0$, $\vec{m} \perp \vec{n}$.

The line and plane are parallel and can be either coincident or distinct.

Test point (1, 0, 1), which lies on the line, to determine if it also lies on the plane.

$$\begin{aligned} LS &= 4(1) - 2(1) \qquad RS = 11 \\ &= 2 \\ &\neq RS \end{aligned}$$

The point (1, 0, 1) does not lie on the plane. Therefore, the line and the plane are parallel and distinct.

If a line and a plane are parallel and do not intersect, it is reasonable to ask how far apart they are. The shortest distance is the perpendicular distance. Since the line and plane are parallel we need to find the perpendicular distance from any point on the line to the plane. The method needed is a slight modification of the technique used in Section 8.4 to find the distance between skew lines.

Example 3 The Distance From a Point to a Plane

Find the distance between the plane $4x + 2y + z - 16 = 0$ and each point.

a) P(10, 3, −8) **b)** B(2, 2, 4)

Solution

a) First check if the point P(10, 3, −8) lies on the plane.

$$\begin{aligned} LS &= 4(10) + 2(3) + (-8) \qquad RS = 16 \\ &= 38 \\ &\neq RS \end{aligned}$$

The point P(10, 3, −8) does not lie on the plane.

A normal vector to the plane is $\vec{n} = [4, 2, 1]$.

Choose any point, Q, on the plane by selecting arbitrary values for x and y.

If $x = 4$ and $y = 0$, then $z = 0$. Q(4, 0, 0) is a point on the plane.

The projection of $\overrightarrow{PQ}$ onto $\vec{n}$ gives the distance, d, we require.

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= [4, 0, 0] - [10, 3, -8] \\ &= [-6, -3, 8]\end{aligned}$$

$$\begin{aligned}d &= \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{|\vec{n}|} \\ &= \frac{|[4, 2, 1] \cdot [-6, -3, 8]|}{\sqrt{4^2 + 2^2 + 1^2}} \\ &= \frac{|-24 - 6 + 8|}{\sqrt{21}} \\ &= \frac{22}{\sqrt{21}} \\ &\doteq 4.80\end{aligned}$$

The distance from point P to the plane is approximately 4.8 units.

b) Check the coordinates of point B(2, 2, 4) in the equation of the plane.

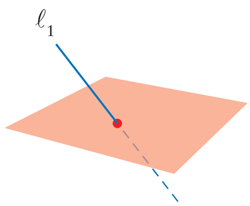
$$\begin{aligned}LS &= 4(2) + 2(2) + (4) \\ &= 16 \\ &= RS\end{aligned} \qquad RS = 16$$

Point B is on the plane, so it is 0 units from the plane.

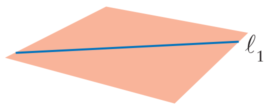
To find the distance between a plane and a parallel line, choose any point on the line, P, and follow the procedure outlined in Example 3.

KEY CONCEPTS

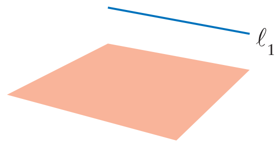
- In three-space, there are three possibilities for the intersection of a line and a plane and the related system of equations.



The line intersects the plane at a point. The system has a unique solution.



The line lies on the plane. The system has infinitely many solutions.



The line is parallel to the plane. The system has no solutions.

- The distance d between a point P and a plane is $d = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{|\vec{n}|}$, where Q is any point on the plane and $\vec{n}$ is a normal vector of the plane.

Communicate Your Understanding

C1 How does the dot product of the normal of a plane with the direction vector of a line help identify whether the plane and line intersect?

C2 When algebraically determining the intersection of a plane and a line, the result is $0 = 0$. What does this tell you about the plane and line? Justify your answer.

C3 Consider the lines defined by these vector equations.

$[x, y, z] = [1, 3, -4] + s[5, 1, 6]$
$[x, y, z] = [7, -3, 0] + t[2, -9, 2]$

Will the plane defined by the two direction vectors and the point (1, 3, 4) be different from the plane defined by the direction vectors and the point (7, −3, 0)? Explain.

A Practise

1. Determine the coordinates of the point of intersection of the line defined by the parametric equations and the plane defined by the vector equation.

$$\ell: \begin{cases} x = 4 + t \\ y = 2 - 2t \\ z = 6 + 3t \end{cases}$$

$$\pi: [x, y, z] = [-1, 1, 2] + m[-1, -1, 3] + n[2, 4, 1]$$

2. In each case, verify that the plane and line are parallel, then determine if they are distinct or coincident.

a) $3x + 5y + z - 5 = 0$
$[x, y, z] = [1, 2, -8] + t[2, -1, -1]$

b) $4x - y + 6z - 12 = 0$
$[x, y, z] = [4, 3, 10] + t[7, -14, -7]$

c) $3y + 10z + 1 = 0$
$[x, y, z] = [7, 1, -9] + t[2, -10, 3]$

d) $x + 2y - 5z + 4 = 0$
$[x, y, z] = [10, 3, 4] + t[1, 2, 1]$

3. In each case, determine if the plane and the line intersect. If so, state the solution.

a) $3x - y + 4z - 8 = 0$
$[x, y, z] = [3, 0, 5] + t[7, -11, -8]$

b) $-2x + 6y + 4z - 4 = 0$
$[x, y, z] = [5, -1, 4] + t[1, -2, 3]$

c) $5x + 3y + 4z - 20 = 0$
$[x, y, z] = [4, 1, 5] + t[1, 2, 3]$

d) $5x - 3y + 7z + 7 = 0$
$[x, y, z] = [10, -5, 0] + t[2, 1, -2]$

e) $9x - 6y + 12z - 24 = 0$
$[x, y, z] = [4, 0, -1] + t[2, 1, -1]$

f) $6x - 2y + 3z + 6 = 0$
$[x, y, z] = [4, 12, -19] + t[2, -3, 5]$

4. Use direction vectors to determine if each line intersects the plane $3x - 2y + 4z = 5$.

a) $\vec{r} = [-3, 2, 7] + t[3, 6, 2]$

b) $\vec{r} = [-3, -5, 1] + t[-2, 1, 2]$

c) $\vec{r} = [0, 1, 2] + t[4, 4, -1]$

B Connect and Apply

5. Does each line intersect the plane $[x, y, z] = [4, -15, -8] + s[1, -3, 1] + t[2, 3, 1]$? If so, how many solutions are there?

a) $[x, y, z] = [5, -9, 3] + k[1, -12, 2]$

b) $[x, y, z] = [-2, 9, -21] + k[2, -5, 4]$

c) $[x, y, z] = [3, -2, 1] + k[1, 4, -2]$

d) $[x, y, z] = [4, 6, 2] + k[2, -1, 1]$

e) $[x, y, z] = [2, -3, 0] + k[-1, 3, -1]$

f) $[x, y, z] = [9, 4, 1] + k[-2, 2, 4]$

6. Find the distance between the parallel line and plane.

a) $\ell: \vec{r} = [2, 0, 1] + t[1, 4, 1]$
$\pi: 2x - y + 2z = 4$

b) $\ell: [x, y, z] = [2, -1, -1] + s[2, 2, 0]$
$\pi: x - y + z = 4$

c) $\ell: \vec{r} = [0, -1, 1] + k[6, 4, -7]$
$\pi: 2x - 3y = 2$

d) $\ell: [x, y, z] = [1, 5, 1] + d[1, 2, -7]$
$\pi: 11x - 24y - 5z = 4$

e) $\ell: [x, y, z] = [2, -1, 0] + g[4, 2, -2]$
$\pi: -14y - 14z = 1$

f) $\ell: [x, y, z] = [3, 8, 1] + s[-1, 3, -2]$
$\pi: 8x - 6y - 13z = 12$

7. Find the distance between the planes.

a) $\pi_1: 2x - y - z - 1 = 0$
$\pi_2: 2x - y - z - 4 = 0$

b) $\pi_1: x + 3y - 2z = 3$
$\pi_2: x + 3y - 2z = 1$

c) $\pi_1: 2x - 3y + z = 6$
$\pi_2: 4x - 6y + 2z = 8$

d) $\pi_1: 2x + 4y - 6z = 8$
$\pi_2: 3x + 6y - 8z - 12 = 0$

e) $\pi_1: 3x - y - 12z + 2 = 0$
$\pi_2: 6x - 2y - 24z - 7 = 0$

f) $\pi_1: x - 6y - 3z + 4 = 0$
$\pi_2: -2x + 12y + 6z + 3 = 0$

8. Determine the distance between each point and the given plane.

a) P(2, 1, 6)
$3x + 9y - z - 1 = 0$

b) P(−5, 0, 1)
$x + 2y + 6z - 10 = 0$

c) P(1, 4, −9)
$4x - 2y - 7z - 21 = 0$

d) P(−2, 3, −3)
$2x - 5y + 3z + 6 = 0$

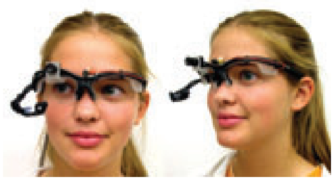
e) P(5, −3, 2)
$2x - y + 5z + 4 = 0$

f) P(−4, −5, 3)
$-3x - 3y + 5z - 9 = 0$

9. Write an equation of the plane that contains the point P(2, −3, 6) and is parallel to the line $[x, y, z] = [3, 3, -2] + t[1, 2, -3]$.

10. Does the line through A(2, 3, 2) and B(4, 0, 2) intersect the plane $2x + y - 3z + 4 = 0$? Explain.

11. An eye-tracking device is worn to determine where a subject is looking. Using two micro-lens video cameras, one pointed at the eye, the other at the target, the device can calculate exactly where the eye is looking. Scientists at Yale University used this device to compare the visual patterns of people with and without autism during conversation. People with autism looked at the speaker's mouth, whereas those without autism watched the speaker's mouth and eyes. It is believed that people with autism often miss visual cues during conversation. Research using 3D mathematics helps to understand autism better and may lead to helping those with autism function better with their disability.



In a particular test, the eye of a person wearing the device was located at a point A(2, 2.5, 1.3). The subject was looking at a screen defined by the equation $x = 0$. The eye-tracking device determined that the subject's line of sight passed through the point B(1.95, 2.48, 1.2). Determine the place on the screen that the person was looking at that instant.

12. Write the equations of a line and a plane that intersect at the point $(1, 4, -1)$.

13. Is each situation possible? Explain.

   a) Two skew lines lie in parallel planes.

   b) Two skew lines lie in non-parallel planes.

14. Consider these lines.

$$\ell_1: \begin{cases} x = 3s \\ y = 2 + s \\ z = 1 + s \end{cases} \quad \text{and} \quad \ell_2: \begin{cases} x = 1 + 2t \\ y = -3 - t \\ z = t \end{cases}$$

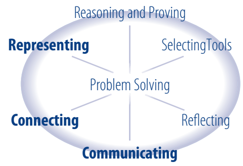
   a) Determine if lines ℓ_1 and ℓ_2 are skew.

   b) Write the equations of parallel planes that each contain one of ℓ_1 and ℓ_2.

15. Write the equations of two skew lines such that one line lies on each of these parallel planes.

π_1: $x - 3y + 2z - 5 = 0$
π_2: $x - 3y + 2z + 8 = 0$

16. Lines and planes can intersect in a variety of ways. Usually, the type of intersection can be predicted by analysing the parametric equations of the line and the scalar equation of the plane. For each situation, draw a sketch of the geometric figures involved and explain how the coefficients of the equations indicate this type of intersection.



   a) A line and a plane do not intersect.

   b) A line and a plane are parallel and distinct.

   c) Two planes are parallel and have no intersection points.

C Extend and Challenge

17. Consider these lines.
ℓ_1: $[x, y, z] = [1, -2, 4] + s[1, 1, -3]$ and
ℓ_2: $[x, y, z] = [4, -2, k] + t[2, 3, 1]$,

   a) Determine an equation of the plane that contains ℓ_1 and is parallel to ℓ_2.

   b) Determine a value of k so ℓ_2 lies in the plane.

   c) Determine a different value of k so ℓ_2 is 10 units away from the plane.

18. Find the distance between the parallel lines in each pair.

   a) ℓ_1: $[x, y, z] = [1, 3, -4] + s[2, -5, 2]$
   ℓ_2: $[x, y, z] = [4, 0, 2] + t[2, -5, 2]$

   b) ℓ_1: $[x, y, z] = [6, 1, 2] + s[4, -1, 5]$
   ℓ_2: $[x, y, z] = [-2, 8, 7] + t[4, -1, 5]$

19. In each case, determine the distance between point P and the line.

   a) $P(1, 3, -4)$
   $[x, y, z] = [4, 1, 4] + t[2, 4, -3]$

   b) $P(5, -2, 0)$
   $[x, y, z] = [-1, 6, -2] + t[1, -6, 6]$

20. Develop a formula for the distance from a point $P_1(x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$.

21. **Math Contest** Determine the value of k if $[3, 7, -2] \times [a, 3, b] = [-1, k, -5]$.

22. **Math Contest** Given that $k = \log_{10} 2$ and $m = \log_{10} 3$, determine the integer value of n such that $\log_{10} n = 3 - 2k + m$.

8.6 Intersections of Planes

Spatial relationships in art and architecture are important in terms of both strength of structures and aesthetics. Because large sculptures and structures are built in three dimensions, understanding how various parts of these structures intersect is important for the artist, architect, or engineer.

Investigate How can sets of planes intersect?

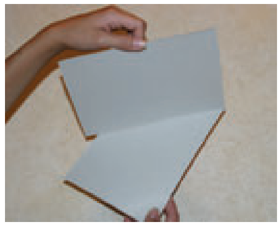
Tools

- cardboard
- templates for intersecting planes

1. Consider two planes. In how many different ways can the planes intersect?

   a) How could the planes be oriented to form a system with no solution?

   b) How could the planes be oriented to form a system with infinitely many solutions?

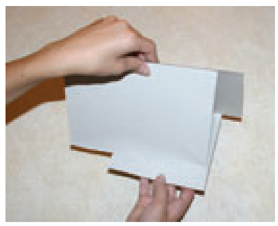


2. To help envision one of the ways, use templates (page 1) to create the intersection of two planes. Which of the situations in step 1 will this produce?
3. **Reflect** How would you describe the intersection of two planes? How would you compare the two normal vectors in this case?
4. **Reflect** Is it possible for two planes to intersect at exactly one point? Explain your answer.
5. Consider three planes. In how many different ways can the three planes intersect?

   a) How could the planes be oriented to form a system with no solution?

   b) How could the planes be oriented to form a system with infinitely many solutions?

   c) How could the planes be oriented to form a system with exactly one solution?



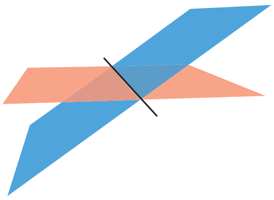
6. To help envision some of the ways, use templates (pages 2 to 5) to create the intersection of three planes. Which of the situations in step 5 will these produce?

7. **Reflect** Describe ways in which three planes can intersect at one point. How do the normal vectors of the planes compare to one another in each situation? Which of these ways form consistent dependent or independent systems? Summarize your results in table form.

8. **Reflect** Describe ways in which three planes can form a system that has no common points of intersection. How do the normal vectors of these planes compare in each situation? Summarize your results, including as many different configurations as possible.

Given two planes in three-space, there are three possible geometric models for the intersection of the planes.

If two distinct planes intersect, the solution is the set of points that lie on the line of intersection.



If the planes are coincident, every point on the plane is a solution.

Parallel planes do not intersect, so there is no solution.

You can verify these situations algebraically by solving linear systems of equations representing the planes.

Example 1 Intersection of Two Planes

Describe how the planes in each pair intersect.

a) π_1: $2x - y + z - 1 = 0$
π_2: $x + y + z - 6 = 0$

b) π_3: $2x - 6y + 4z - 7 = 0$
π_4: $3x - 9y + 6z - 2 = 0$

c) π_5: $x + y - 2z + 2 = 0$
π_6: $2x + 2y - 4z + 4 = 0$

Solution

a) The normal for the plane π_1 is $\vec{n}_1 = [2, -1, 1]$ and the normal for the plane π_2 is $\vec{n}_2 = [1, 1, 1]$. The normals are not parallel, so the planes must intersect.

There are three variables but only two equations. This is not enough information to find a unique solution. Use the technique of elimination to solve for the intersection points. We expect the result to be a line of intersection, that is, a solution involving one parameter.

$$\begin{array}{rll} 2x - y + z - 1 = 0 & ① & \\ \underline{x + y + z - 6 = 0} & ② & \\ x - 2y \quad + 5 = 0 & ③ & ① - ② \end{array}$$

Assign a variable to be the parameter.

Let $y = t$.

③ becomes $x = -5 + 2t$
② becomes $(-5 + 2t) + (t) + z = 6$
$z = 11 - 3t$

The parametric equations for the line of intersection, ℓ, are

$x = -5 + 2t$
$y = t$
$z = 11 - 3t$

A vector equation of the line of intersection is

$[x, y, z] = [-5, 0, 11] + t[2, 1, -3]$.

b) The normal for π_3 is $\vec{n}_3 = [2, -6, 4]$ and the normal for π_4 is $\vec{n}_4 = [3, -9, 6]$.

Since $\vec{n}_4 = 1.5\vec{n}_3$, the normals are parallel. The planes may be distinct or coincident.

$$\begin{array}{rl} 2x - 6y + 4z - 7 = 0 & ① \\ 3x - 9y + 6z - 2 = 0 & ② \end{array}$$

Eliminate one of the variables, say x, as follows.

$$\begin{array}{rl} 6x - 18y + 12z - 21 = 0 & 3① \\ \underline{6x - 18y + 12z - 4 \ = 0} & 2② \\ -17 = 0 & 3① - 2② \end{array}$$

This equation can never be true. There is no solution to this system, so the planes are parallel and distinct.

c) Notice that $\vec{n}_5 = [1, 1, -2]$, $\vec{n}_6 = [2, 2, -4]$, and $\vec{n}_5 = \frac{1}{2}\vec{n}_6$.

Therefore the normals are parallel and the planes are again either parallel and distinct or coincident.

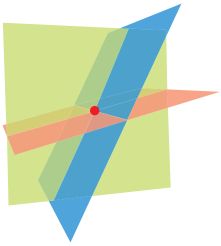
However, ② can be simplified by dividing every term by 2; the result is the same as ①. In this case, the planes are coincident and the intersection set is every point on the plane.

CONNECTIONS

Solutions to the Examples in this section can be verified using 3D graphing software. Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.1 for instructions on how to access 3-D Grapher software.

A system of three planes is **consistent** if it has one or more solutions.

The planes intersect at a point. There is exactly one solution.



The normals are not parallel and not coplanar.

The planes intersect in a line. There are an infinite number of solutions.

The normals are coplanar, but not parallel.

The planes are coincident. There are an infinite number of solutions.

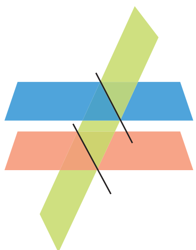
The normals are parallel.

A system of three planes is **inconsistent** if it has no solution.

The three planes are parallel and at least two are distinct.

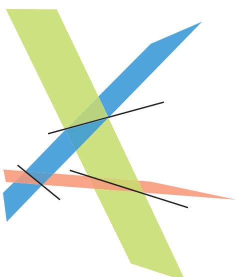
The normals are parallel.

Two planes are parallel and distinct. The third plane is not parallel.



Two of the normals are parallel.

The planes intersect in pairs. Pairs of planes intersect in lines that are parallel and distinct.



The normals are coplanar but not parallel.

CONNECTIONS

When three or more planes intersect at a line, this is sometimes referred to as a **sheaf** of planes. For a better view of a sheaf of planes, go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.6.

It is easy to check if normals are parallel; each one is a scalar multiple of the others.

To check if normals are coplanar, use the triple scalar product, $\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3$. Recall that this product gives the volume of a parallelepiped defined by the three vectors. If the product is zero, the volume is zero and the vectors must be coplanar. If the product is not zero, the vectors are not coplanar.

Example 2 Solving Systems With Three Planes

For each set of planes, describe the number of solutions and how the planes intersect.

a) $\pi_1: 2x + y + 6z - 7 = 0$
$\pi_2: 3x + 4y + 3z + 8 = 0$
$\pi_3: x - 2y - 4z - 9 = 0$

b) $\pi_4: x - 5y + 2z - 10 = 0$
$\pi_5: x + 7y - 2z + 6 = 0$
$\pi_6: 8x + 5y + z - 20 = 0$

Solution

a) The normals of the planes are

$$\vec{n}_1 = [2, 1, 6]$$
$$\vec{n}_2 = [3, 4, 3]$$
$$\vec{n}_3 = [1, -2, -4]$$

None of the normals are parallel. Therefore none of the planes are parallel.

Now check if the normals are coplanar.

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [2, 1, 6] \cdot [3, 4, 3] \times [1, -2, -4] \\ &= [2, 1, 6] \cdot [-10, 15, -10] \\ &= 65\end{aligned}$$

Since $\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 \neq 0$, the normals are not coplanar and the planes intersect in a single point.

Verify this algebraically.

Work with pairs of equations to eliminate one of the variables.

$$\begin{array}{rll} 8x + 4y + 24z = 28 & 4① & \\ 3x + 4y + 3z = -8 & ② & \\ \hline 5x \quad + \quad 21z = 36 & ④ & 4① - ② \end{array}$$

$$\begin{array}{rll} 4x + 2y + 12z = 14 & 2① & \\ x - 2y - 4z = 9 & ③ & \\ \hline 5x \quad + \quad 8z = 23 & ⑤ & 2① + ③ \end{array}$$

Solve equations ④ and ⑤ for x and z.

$$\begin{array}{rll} 5x + 21z = 36 & ④ & \\ 5x + 8z = 23 & ⑤ & \\ \hline 13z = 13 & & ④ - ⑤ \\ z = 1 & & \end{array}$$

Substitute $z = 1$ into equation ④.

$$\begin{aligned} 5x + 21(1) &= 36 \\ x &= 3 \\ 2(3) + y + 6(1) &= 7 \qquad \text{Substitute } x = 3 \text{ and } z = 1 \text{ into ①.} \\ y &= -5 \end{aligned}$$

The three planes form a consistent system. The point of intersection is $(3, -5, 1)$.

b) The normals for the planes are

$\vec{n}_4 = [1, -5, 2]$
$\vec{n}_5 = [1, 7, -2]$
$\vec{n}_6 = [8, 5, 1]$

None of the normals are parallel. Therefore none of the planes are parallel.

Now check if the normals are coplanar.

$$\begin{aligned} \vec{n}_4 \cdot \vec{n}_5 \times \vec{n}_6 &= [1, -5, 2] \cdot [1, 7, -2] \times [8, 5, 1] \\ &= [1, -5, 2] \cdot [17, -17, -51] \\ &= 0 \end{aligned}$$

Since $\vec{n}_4 \cdot \vec{n}_5 \times \vec{n}_6 = 0$, the normals are coplanar and the planes intersect either in a line or not at all. Verify this algebraically.

Work with pairs of equations to eliminate one of the variables.

$$\begin{array}{rll} x - 5y + 2z = 10 & ④ & \\ x + 7y - 2z = -6 & ⑤ & \\ \hline -12y + 4z = 16 & ⑦ & ④ - ⑤ \end{array}$$

$$\begin{array}{rll} 8x + 56y - 16z = -48 & 8⑤ & \\ 8x + 5y + z = 20 & ⑥ & \\ \hline 51y - 17z = -68 & ⑧ & \text{Subtract.} \end{array}$$

Now eliminate y from ⑦ and ⑧

$$\begin{array}{rll} -204y + 68z = 272 & 17⑦ & \\ 204y - 68z = -272 & 4⑧ & \\ \hline 0z = 0 & & \text{Add.} \end{array}$$

This equation is true for all values of y and z. These three planes intersect in a line. Determine parametric equations of the line. Let z a parameter.

Let $z = t$.

$$\begin{aligned} -12y + 4t &= 16 \qquad \text{Substitute } z = t \text{ into ⑦.} \\ y &= \frac{1}{3}t - \frac{4}{3} \end{aligned}$$

$$x + 7\left(\frac{1}{3}t - \frac{4}{3}\right) - 2(t) = -6$$

Substitute $z = t$ and $y = \frac{1}{3}t - \frac{4}{3}$ into ⑤.

$$x + \frac{7}{3}t - \frac{28}{3} - 2t = -6$$
$$3x + 7t - 28 - 6t = -18$$
$$x = \frac{10}{3} - \frac{1}{3}t$$

This gives the parametric equations of the line:

$$x = -\frac{1}{3}t + \frac{10}{3}$$
$$y = \frac{1}{3}t - \frac{4}{3}$$
$$z = t$$

So the three planes form a consistent system. The planes all intersect in the line $[x, y, z] = \left[\frac{10}{3}, -\frac{4}{3}, 0\right] + t\left[-\frac{1}{3}, \frac{1}{3}, 1\right]$.

CONNECTIONS

To see an interactive sketch of Example 2b, go to *www.mcgrawhill.ca/links/calculus12* and follow the links to 8.6 Example 2.htm.

Example 3 Analysing Inconsistent Solutions

Determine if each system can be solved; then solve the system, or describe it.

a) $3x + y - 2z = 12$
$x - 5y + z = 8$
$12x + 4y - 8z = -4$

b) $x + 3y - z = -10$
$2x + y + z = 8$
$x - 2y + 2z = -4$

c) $4x - 2y + 6z = 35$
$-10x + 5y - 15z = 20$
$6x - 3y + 9z = -50$

Solution

a) The normals of the planes are

$\vec{n}_1 = [3, 1, -2]$
$\vec{n}_2 = [1, -5, 1]$
$\vec{n}_3 = [12, 4, -8]$

By inspection, π_1 and π_3 are parallel since $4\vec{n}_1 = \vec{n}_3$, but π_2 is not parallel to π_1 or π_3. π_3 is not a scalar multiple of π_1, so this is an inconsistent system with two distinct but parallel planes that are intersected by a third plane.

b) The normals of the planes are

$\vec{n}_1 = [1, 3, -1]$
$\vec{n}_2 = [2, 1, 1]$
$\vec{n}_3 = [1, -2, 2]$

By inspection, none of the normals are parallel.

Now check if the normals are coplanar.

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [1, 3, -1] \cdot [2, 1, 1] \times [1, -2, 2]\\ &= [1, 3, -1] \cdot [4, -3, -5]\\ &= 0\end{aligned}$$

Since $\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 = 0$, the normals are coplanar and the planes intersect either in a line or not at all.

Solve the system algebraically to determine if there is a solution.

$$\begin{aligned} x + 3y - z &= -10 && ① \\ 2x + y + z &= 8 && ② \\ x - 2y + 2z &= -4 && ③ \end{aligned}$$

$$\begin{aligned} 2x + 6y - 2z &= -20 && 2① \\ 2x + y + z &= 8 && ② \\ \hline 5y - 3z &= -28 && ④ \quad 2① - ② \end{aligned}$$

$$\begin{aligned} x + 3y - z &= -10 && ① \\ x - 2y + 2z &= -4 && ③ \\ \hline 5y - 3z &= -6 && ⑤ \quad ① - ③ \end{aligned}$$

$$\begin{aligned} 5y - 3z &= -28 && ④ \\ 5y - 3z &= -6 && ⑤ \\ \hline 0 &= -22 && ④ - ⑤ \end{aligned}$$

This system has no solution. Since none of the planes are parallel, these planes intersect in pairs.

c) The normals of the planes are

$\vec{n}_1 = [4, -2, 6]$
$\vec{n}_2 = [-10, 5, -15]$
$\vec{n}_3 = [6, -3, 9]$

The normals are all parallel, since $15\vec{n}_1 = -6\vec{n}_2 = 10\vec{n}_3$.

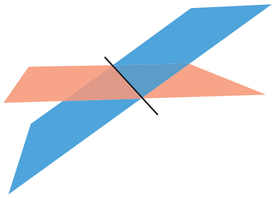
Examine the equations to determine if the planes are coplanar coincident or parallel and distinct.

$$\begin{aligned} 2x - y + 3z &= \frac{35}{2} && ① \div 2 \\ 2x - y + 3z &= -4 && ② \div (-5) \\ 2x - y + 3z &= -\frac{50}{3} && ③ \div 3 \end{aligned}$$

The equations are not scalar multiples of one another. The planes are parallel but distinct.

KEY CONCEPTS

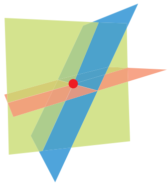
- There are three possibilities for the intersection of two planes.
  - The planes intersect in a line. There are an infinite number of solutions.



  - The planes are coincident. There are an infinite number of solutions.

  - The planes are parallel and distinct. There is no solution.

- There are three ways that three planes can intersect.
  - The planes intersect at a point. There is exactly one solution.



The normals are not parallel and not coplanar.

  - The planes intersect in a line. There are an infinite number of solutions.

The normals are coplanar, but not parallel.

  - The planes are coincident. There are an infinite number of solutions.

The normals are parallel.

- There are three possibilities for a system of three planes with no solution.
  - The planes are parallel and at least two are distinct.
  - Two planes are parallel and the third intersects both of the parallel planes.
  - The planes intersect in pairs.
- Normals can be analysed to determine the type and number of solutions to a system of three planes.

Communicate Your Understanding

C1 In Example 3a), when you see that two normals are parallel and the third is not, how do you know that this situation is not two coincident planes and one non-parallel plane?

C2 How can normal vectors be used to determine the nature of the solutions to a system of three linear equations?

C3 Explain what the term *consistent* means in the context of a system of three planes.

C4 Create a table summarizing all the different intersections of three planes. It should have an example of each type of system and how the normals are related in each case. Complete the table as you work through the following exercise questions.

A Practise

1. Determine the line through which the planes in each pair intersect

a) $x + 5y - 3z - 8 = 0$
$y + 2z - 4 = 0$

b) $5x - 4y + z - 3 = 0$
$x + 3y - 9 = 0$

c) $2x - y + z - 22 = 0$
$x - 11y + 2z - 8 = 0$

d) $3x + y \;\; - 5z - 7 = 0$
$2x - y - 21z + 33 = 0$

2. Which pairs of planes are parallel and distinct and which are coincident?

a) $2x + 3y - 7z - 2 = 0$
$4x + 6y - 14z - 8 = 0$

b) $3x + 9y - 6z - 24 = 0$
$4x + 12y - 8z - 32 = 0$

c) $4x - 12y - 16z - 52 = 0$
$-6x + 18y + 24z + 78 = 0$

d) $x - 2y + 2.5z - 1 = 0$
$3x - 6y + 7.5z - 3 = 0$

3. Show that the line
$[x, y, z] = [10, 5, 16] + t[3, 1, 5]$
is contained in each of these planes.

a) $x + 2y - z - 4 = 0$

b) $9x - 2y - 5z = 0$

4. For each system of equations, determine the point of intersection.

a) $x + y + z - 7 = 0$
$2x + y + 3z - 17 = 0$
$2x - y - 2z + 5 = 0$

b) $2x + y + 4z = 15$
$2x + 3y + z = -6$
$2x - y + 2z = 12$

c) $5x - 2y - 7z - 19 = 0$
$x - y + z - 8 = 0$
$3x + 4y + z - 1 = 0$

d) $2x - 5y - z = 9$
$x + 2y + 2z = -13$
$2x + 8y + 3z = -19$

5. Determine the line of intersection of each system of equations.

a) $2x + y + z = 7$
$4x + 3y - 3z = 13$
$4x + 2y + 2z = 14$

b) $x + 3y - z = 4$
$3x + 8y - 4z = 4$
$x + 2y - 2z = -4$

c) $x + 9y + 3z = 23$
$x + 15y + 3z = 29$
$4x - 13y + 12z = 43$

d) $x - 6y + z = -1$
$x - y = 5$
$2x - 12y + 2z = -2$

6. In each system, at least one pair of planes are parallel. Describe each system.

a) $3x + 15y - 9z = 12$
$6x + 30y - 18z = 24$
$5x + 25y - 15z = 10$

b) $2x - y + 4z = 5$
$6x - 3y + 12z = 15$
$4x - 2y + 8z = 10$

c) $3x + 2y - z = 8$
$12x + 8y - 4z = 20$
$18x + 12y - 6z = -3$

d) $8x + 4y + 6z = 7$
$12x + 6y + 9z = 1$
$3x + 2y + 4z = 1$

7. Determine if each system of planes is consistent or inconsistent. If possible, solve the system.

a) $3x + y - 2z = 18$
$6x - 4y + 10z = -10$
$3x - 5y + 10z = 10$

b) $2x + 5y - 3z = 12$
$3x - 2y + 3z = 5$
$4x + 10y - 6z = -10$

c) $2x - 3y + 2z = 10$
$5x - 15y + 5z = 25$
$-4x + 6y - 4z = -4$

d) $3x - 5y - 5z = 1$
$-6x + 10y + 10z = -2$
$15x - 25y - 25z = 20$

B Connect and Apply

8. **Use Technology** Use 3-D graphing technology to verify any of your answers to questions 1 to 7.

9. Planes that are perpendicular to one another are said to be **orthogonal**. Determine which of the systems from question 4 contain orthogonal planes.

10. For each system of planes in question 5, find the triple scalar product of the normal vectors. Compare the answers. What does this say about the normal vectors of planes that intersect in a line?

CONNECTIONS

The triple scalar product is $\vec{a} \cdot (\vec{b} \times \vec{c})$.

11. Describe each system of planes. If possible, solve the system.

a) $3x + 2y - z = -2$
$2x + y - 2z = 7$
$2x - 3y + 4z = -3$

b) $3x - 4y + 2z = 1$
$6x - 8y + 4z = 10$
$15x - 20y + 10z = -3$

c) $2x + y + 6z = 5$
$5x + y - 3z = 1$
$3x + 2y + 15z = 9$

d) $x - y + z = 20$
$x + y + 3z = -4$
$2x - 5y - z = -6$

e) $4x + 8y + 4z = 7$
$5x + 10y + 5z = -10$
$3x - y - 4z = 6$

f) $2x + 2y + z = 10$
$5x + 4y - 4z = 13$
$3x + 5y - 2z = 6$

g) $2x + 5y = 1$
$4y + z = 1$
$7x - 4z = 1$

h) $3x - 2y - z = 0$
$x - z = 0$
$3x + 2y - 5z = 0$

i) $y + 2z = 11$
$4y - 4z = -16$
$3y - 3z = -12$

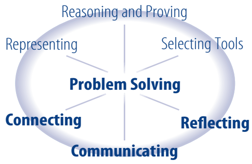
12. A student solved some systems of planes and got these algebraic solutions. Interpret how the planes intersect in each case as exactly as you can.

a) $x = 5$
$y = 10$
$z = -3$

b) $x - y + 2z = 4$
$y - 3z = 8$
$0 = 0$

c) $x + 4y + 8z = 1$
$2y - 4z = 10$
$0 = 12$

13. In each case, describe all the ways in which three planes could intersect.



a) The normals are not parallel.

b) Two of the three normals are parallel.

c) All three of the normals are parallel.

14. Show that the planes in each set are mutually perpendicular and have a unique solution.

a) $5x + y + 4z = 18$
$x + 3y - 2z = -16$
$x - y - z = 0$

b) $2x - 6y + z = -16$
$7x + 3y + 4z = 41$
$27x + y - 48z = -11$

✓ Achievement Check

15. You are given the following two planes:
$x + 4y - 3z - 12 = 0$
$x + 6y - 2z - 22 = 0$

a) Determine if the planes are parallel.

b) Find the line of intersection of the two planes.

c) Use 3-D graphing technology to check your answer to part a).

d) Use the two original equations to determine two other equations that have the same solution as the original two.

e) Verify your answers in part d) by graphing.

f) Find a third equation that will form a unique solution with the original two equations.

C Extend and Challenge

16. A **dependent** system of equations is one whose solution requires a parameter to express it. Change one of the coefficients in the following system of planes so that the solution is consistent and dependent.

a) $x - 3z = -3$
$2x - z = 4$
$3x + 5z = 3$

b) $2x + 8y = -6$
$y + 5z = 20$
$3x + 12y + 6z = -9$

17. Determine the equations of three planes that

a) are all parallel but distinct

b) intersect at the point P(3, 1, −9)

c) intersect in a sheaf of planes

d) intersect in pairs

e) intersect in the line $[x, y, z] = [1, 3, -4] + t[4, 1, 9]$

f) intersect in pairs and are all parallel to the y-axis

g) intersect at the point (−2, 4, −4) and are all perpendicular to each other

h) intersect along the z-axis

18. Find the volume of the figure bounded by the following planes.

$x + z = -3$
$10x - 3z = 22$
$4x - 9z = -38$
$y = -4$
$y = 10$

19. Solve the following system of equations.

$2w + x + y + 2z = -9$
$w - x + y + 2z = 1$
$2w + x + 2y - 3z = 18$
$3w + 2x + 3y + z = 0$

20. Create a four-dimensional system that has a solution (3, 1, −4, 6).

21. **Math Contest** A canoeist is crossing a river that is 77 m wide. She is paddling at 7 m/s. The current is 7 m/s (downstream). The canoeist heads out at a 77° angle (upstream). How far down the opposite shore will the canoeist be when she gets to the other side of the river?

22. **Math Contest** The parallelepiped formed by the vectors $\vec{a} = [1, 2, -3]$, $\vec{b} = [1, k, -3]$, and $\vec{c} = [2, k, 1]$ has volume 180 units^3. Determine the possible values of k.

Extension

Solve Systems of Equations Using Matrices

Mathematicians use algorithms, short cuts, and special notations to help streamline calculations. When solving systems of equations, it is useful to use a matrix. A **matrix** is a rectangular array of terms called elements. The rows and columns of numbers are enclosed in square brackets.

Matrices are used to represent a system of linear equations without using variables.

The coefficients of the variables in this system can be written as a **coefficient matrix**.

$$\begin{aligned} x + 9y - 2z &= 10 \\ 5x - 8y + 2z &= -4 \\ 7x + 4y - z &= 11 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 9 & -2 \\ 5 & -8 & 2 \\ 7 & 4 & -1 \end{bmatrix}$$

This system can also be written as an **augmented matrix**.

$$\begin{aligned} x + 9y - 2z &= 10 \\ 5x - 8y + 2z &= -4 \\ 7x + 4y - z &= 11 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 9 & -2 & 10 \\ 5 & -8 & 2 & -4 \\ 7 & 4 & -1 & 11 \end{array}\right]$$

the constant terms
the coefficients of *z*
the coefficients of *y*
the coefficients of *x*

Each horizontal row of the matrix represents the coefficients from a single equation. The vertical line separates the coefficients from the constant terms.

The positions of the rows are not important; they can be interchanged. The positions of the columns are important; they are always written in the same order.

Example 1 Converting To and From Matrix Form

a) Write this system in matrix form.

$$\begin{aligned} -3x + 2y - 5z &= 6 \\ 4x - 7y \phantom{{}-5z} &= 8 \\ -4y + z &= 10 \end{aligned}$$

b) Write the scalar equations that correspond to this matrix.

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 1 \\ -6 & 11 & -1 & 12 \\ 1 & -8 & 0 & -3 \end{array}\right]$$

Solution

a) Where an equation has a missing variable, the coefficient is understood to be 0.

$$\left[\begin{array}{rrr|r} -3 & 2 & -5 & 6 \\ 4 & -7 & 0 & 8 \\ 0 & -4 & 1 & 10 \end{array}\right]$$

b) Assume the variables involved are x, y, and z.

The system of equations that corresponds to this matrix is

$$\begin{aligned} 2x + 3z &= 1 \\ -6x + 11y - z &= 12 \\ x - 8y \quad &= -3 \end{aligned}$$

Elementary row operations can be performed on a matrix. These operations correspond to the actions taken when solving a system of equations by elimination. Elementary row operations change the appearance of a matrix but do not change the solution set to the corresponding system of equations.

Elementary Row Operations

- Any row can be multiplied or divided by a non-zero scalar value.
- Any two rows can be exchanged with each other.
- A row can be replaced by the sum of that row and a multiple of another row.

A matrix can be changed from its original form into **row reduced echelon form** (**RREF**) by performing a series of elementary row operations. This row reduction process allows the solution to be read directly from the reduced matrix.

Row Reduced Echelon Form

- The first non-zero element in a row is a 1, and is called a leading 1.
- Each leading 1 is strictly to the right of the leading 1 in any preceding row.
- In a column that contains a leading 1, all other elements are 0.
- Any row containing only zeros appears last.

$$\left[\begin{array}{rrr|r} 2 & -3 & 4 & -8 \\ 3 & 4 & 2 & 13 \\ 5 & 2 & -3 & 25 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Example 2 Write a Matrix in Row Reduced Echelon Form

Use a matrix to solve this system.

$$\begin{aligned} x + 3y - z &= -9 \\ x - y + z &= 11 \\ x + 2y + 4z &= 5 \end{aligned}$$

Solution

Method 1: ***Use Paper and Pencil***

Write the system in matrix form.

$$\begin{aligned} x + 3y - z &= -9 \quad ① \\ x - y + z &= 11 \quad ② \\ x + 2y + 4z &= 5 \quad ③ \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & -9 \\ 1 & -1 & 1 & 11 \\ 1 & 2 & 4 & 5 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

With matrices, refer to row numbers rather than equation numbers.

Apply elementary row operations to get 0 entries in rows 2 and 3 of column 1.

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & -9 \\ 0 & 4 & -2 & -20 \\ 0 & 1 & -5 & -14 \end{array}\right] \begin{array}{l} R_1 \\ R_1 - R_2 \\ R_1 - R_3 \end{array}$$

When "R" is used to describe the algebraic operations, it refers to the row of the previous matrix only.

Now, apply elementary row operations to get 1 in row 2, column 2, and 0s in rows 1 and 3 of column 2.

$$\left[\begin{array}{ccc|c} 1 & 0 & 14 & 33 \\ 0 & 1 & -5 & -14 \\ 0 & 0 & 18 & 36 \end{array}\right] \begin{array}{l} R_1 - 3R_3 \\ R_3 \\ R_2 - 4R_3 \end{array}$$

Apply elementary row operations to get 1 in row 3, column 3, and 0s in rows 1 and 2 of column 3.

$$\left[\begin{array}{ccc|c} 18 & 0 & 0 & 90 \\ 0 & 18 & 0 & -72 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} 18R_1 - 14R_3 \\ 18R_2 + 5R_3 \\ R_3 \div 18 \end{array}$$

Finally, multiply or divide each row by a scalar so the leading non-zero value is 1.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 \div 18 \\ R_2 \div 18 \\ \end{array}$$

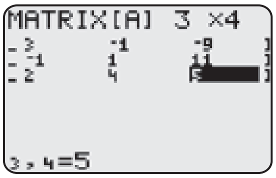
The matrix is now in RREF.

The solution to this system is $(x, y, z) = (5, -4, 2)$.

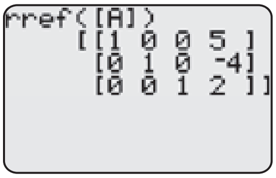
Method 2: ***Use a Graphing Calculator***

- Set up the matrix in the calculator.

  Press 2ND [MATRIX] ▶ ▶ ENTER 3 ENTER 4 ENTER.
- Next enter the matrix elements, pressing ENTER after each element. Press 2ND [QUIT].



- To convert the matrix to row reduced echelon form, press 2ND [MATRIX] ▶ ▲. Use the arrow keys to select **B:rref(**. Press ENTER. Press 2ND [MATRIX] 1) ENTER.



The matrix is now in RREF.

The solution to this system is $(x, y, z) = (5, -4, 2)$.

Example 3 Solve Dependent or Inconsistent Systems

Solve each linear system.

a) $x - y - 2z = 5$
$2x + 2y + z = 1$
$x + 3y + 3z = 10$

b) $x - y - z = -4$
$2x - z = 3$
$x - 7y - 4z = -37$

Solution

a) Write the system in matrix form.

$$\begin{array}{c} \left[\begin{array}{rrr|r} 1 & -1 & -2 & 5 \\ 2 & 2 & 1 & 1 \\ 1 & 3 & 3 & 10 \end{array}\right] \end{array}\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Method 1: ***Use Paper and Pencil***

$$\left[\begin{array}{rrr|r} 1 & -1 & -2 & 5 \\ 0 & -4 & -5 & 9 \\ 0 & -4 & -5 & -5 \end{array}\right]\begin{array}{l} \\ 2R_1 - R_2 \\ R_1 - R_3 \end{array}$$

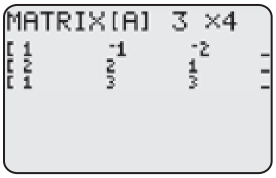
Then,

$$\begin{array}{l}\left[\begin{array}{rrr|r} 4 & 0 & -3 & 11 \\ 0 & -4 & -5 & 9 \\ 0 & 0 & 0 & -14 \end{array}\right]\end{array}\begin{array}{l} 4R_1 - R_2 \\ \\ R_3 - R_2 \end{array}$$

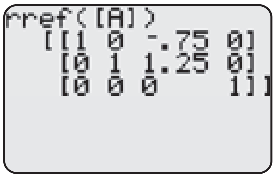
We can stop the process here since the last row of the matrix represents $0 = -14$, which has no solutions. The linear system is inconsistent.

Method 2: ***Use a Graphing Calculator***

1. Enter the matrix in the calculator, as in Example 1.



2. Convert the matrix to RREF.



Note that the paper-and-pencil solution is different from the calculator solution. This is because we stopped as soon as we determined the system was inconsistent, whereas the calculator continued to reduce the matrix into RREF. We interpret the result in the same way; there is no solution.

b) Write the system in matrix form.

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & -4 \\ 2 & 0 & -1 & 3 \\ 1 & -7 & -4 & -37 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Method 1: ***Use Paper and Pencil***

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & -4 \\ 0 & -2 & -1 & -11 \\ 0 & 6 & 3 & 33 \end{array}\right]\begin{array}{l} \\ 2R_1 - R_2 \\ R_1 - R_3 \end{array}$$

$$\left[\begin{array}{rrr|r} 2 & 0 & -1 & 3 \\ 0 & -2 & -1 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} 2R_1 - R_2 \\ R_2 \\ 3R_2 + R_3 \end{array}$$

The last row is all zeros, so $0 = 0$ and the solution is consistent and dependent.

Determine an equation of the line of intersection.

From R_2, $-2y - z = -11$. Let $y = t$ be a parameter.

$z = 11 - 2t$

Now substitute $z = 11 - 2t$ and $y = t$ into R_1, $2x - z = 3$.

$$2x - (11 - 2t) = 3$$
$$x = 7 - t$$

Parametric equations of the line are

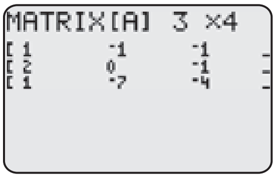
$x = 7 - t$
$y = t$
$z = 11 - 2t$

A vector equation of the line is

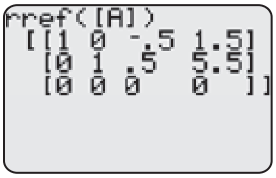
$[x, y, z] = [7, 0, 11] + t[-1, 1, -2]$

Method 2: ***Use a Graphing Calculator***

1. Enter the matrix in the calculator.



2. Convert the matrix to RREF.



This system is consistent and dependent.

In this case, from R_2, we can set $z = t$ to be a parameter.

$$y + 0.5t = 5.5$$
$$y = 5.5 - 0.5t$$

Now substitute y and z into R_1.

$$x - 0.5(t) = 1.5$$
$$x = 1.5 + 0.5t$$

Thus the parametric equations are

$x = 1.5 + 0.5t$
$y = 5.5 - 0.5t$
$z = t$

A vector equation is $[x, y, z] = [1.5, 5.5, 0] + t[0.5, -0.5, 1]$, which looks different than the result from Method 1. By inspection, the two direction vectors are parallel since $[-1, 1, -2] = -2[0.5, -0.5, 1]$, and it can be verified that the coordinates of either position vector satisfy the other equation.

KEY CONCEPTS

- The linear system

$$\begin{aligned} ax + by + cz &= d \\ ex + fy &= gz = h \\ ix + jy + kz &= l \end{aligned} \quad \text{can be written in matrix form:} \quad \left[\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ i & j & h & l \end{array}\right]$$

- Any system of linear equations can be solved using a matrix and elementary row operations.
- A matrix is in row reduced echelon form (RREF) if
  - The first non-zero element in a row is a 1, which is called a leading 1.
  - Each leading 1 is strictly to the right of the leading 1 in any preceding row.
  - In a column that contains a leading 1, all other elements are 0.
  - Any row containing only 0 appears last.

Communicate Your Understanding

C1 What are the advantages of using a matrix method compared to an algebraic method for solving systems of equations?

C2 Can you use a matrix to determine how lines intersect in two-space?

C3 Will an augmented matrix always have one more column than it has rows?

A Practise

1. Write each system of equations in matrix form. Do not solve.

a) $3x + 2y + 5z = 8$
$x - 8y + 4z = 1$
$-2x + 10y + 6z = 7$

b) $x + y + z = 14$
$3y - z = 10$
$5x + 2y = 1$

c) $x = 3$
$y = 7$
$z = -9$

d) $8y + 4x + 24z - 28 = 0$
$-8 = 3x + 4y + 3z$
$x - 5y = 2z - 10$

2. Write the system of equations that corresponds to each matrix. Do not solve.

a) $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 8 & -1 & 4 & -9 \\ 2 & -3 & 7 & -6 \end{array}\right]$

b) $\left[\begin{array}{ccc|c} -2 & 0 & 1 & 10 \\ 0 & 0 & 6 & 3 \\ 3 & -1 & 0 & -10 \end{array}\right]$

c) $\left[\begin{array}{ccc|c} 3 & -2 & -1 & 6 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 5 & 15 \end{array}\right]$

d) $\left[\begin{array}{ccc|c} 4 & -1 & -7 & 9 \\ 0 & 5 & 0 & 12 \\ 0 & 0 & 8 & 2 \end{array}\right]$

3. Solve each system.

a) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 3 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right]$

b) $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 3 & 2 & 24 \\ 0 & 1 & -3 & -20 \end{array}\right]$

c) $\begin{bmatrix} 4 & 0 & 3 & -6 \\ 0 & 1 & -4 & 12 \\ 0 & 0 & 6 & -12 \end{bmatrix}$

d) $\begin{bmatrix} 2 & -3 & 4 & -8 \\ 3 & 4 & 2 & 13 \\ 5 & 2 & -3 & 25 \end{bmatrix}$

4. Solve each system.

a) $x + y + z = 4$
$x - 2y + 3z = 15$
$x + 3y - z = -4$

b) $x + y + 3z = 10$
$2x - y + 2z = 10$
$3x + 2y + z = 20$

c) $3x + 4y - 2z = 13$
$2x - 3y + z = 4$
$4x + y - 5z = 22$

5. Use a matrix to solve each system. Then interpret the solution.

a) $4x + 12y - 8z = 1$
$3x + 5y + 6z = -1$
$6x + 18y - 12z = -15$

b) $x + 8y + 5z = 1$
$3x - 2y + z = 6$
$5x + y + 4z = -10$

c) $x - 6y + 2z = 3$
$x - 3y + z = 1$
$2x - 12y + 4z = 6$

d) $4x - 5y + 2z = 7$
$3x - 2y + z = 6$
$x - z = 4$

e) $2x + y - 3z = 10$
$10x + 5y - 15z = -3$
$6x + 3y - 9z = 4$

f) $3x + 2y + 2z = 15$
$x + 5y - 6z = 46$
$4x + 2y + z = 9$

B Connect and Apply

6. Describe each solution as specifically as possible based on the reduced matrix.

a) $\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & -7 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 9 & 3 & 0 \\ 0 & 1 & 2 & 15 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 3 & 5 & -3 & 1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 12 \end{bmatrix}$

7. **Use Technology** Use a graphing calculator to solve any of questions 1 to 6.

C Extend and Challenge

8. Use a matrix to solve this system of equations.

$w - 3x - 2y + 3z = 10$
$4w + 2x + 3y + z = 2$
$2w + x + y + 5z = 11$
$-2w - 4x + y + 10z = -8$

9. **Math Contest** What is the best way to measure out 11 L of water with only a 3-L jug and a 7-L jug? (For example, you want to get exactly 11 L of water into a pail.)

10. **Math Contest** There are three glasses on the table; their capacities are 3, 5, and 8 ounces. The first two are empty and the last contains 8 ounces of water. By pouring water from one glass to another, make at least one of them contain exactly 4 ounces of water.

Chapter 8 REVIEW

8.1 Equations of Lines in Two-Space and Three-Space

1. Write the vector and parametric equations of each line.

a) $\vec{m} = [1, 2]$, P$(-3, 2)$

b) $\vec{m} = [6, 5, 1]$, P$(-9, 0, 4)$

c) parallel to the x-axis with z-intercept 7

d) perpendicular to the xy-plane and through $(3, 0, -4)$

2. Given each scalar equation, write a vector equation.

a) $5x - 2y = 9$ **b)** $x + 7y = 10$

c) $x = 8$ **d)** $x - 4y = 0$

3. Write the scalar equation for each line.

a) $[x, y] = [1, 4] + t[2, 7]$

b) $[x, y] = [10, -3] + t[5, -7]$

4. A line is defined by the equation $[x, y, z] = [1, -1, 5] + t[3, 4, 7]$.

a) Write the parametric equations for the line.

b) Does the point (13, 15, 23) lie on the line?

5. The vertices of a parallelogram are the origin and points A$(-1, 4)$, B$(3, 6)$, and C$(7, 2)$. Write the vector equations of the lines that make up the sides of the parallelogram.

6. A line has the same x-intercept as $[x, y, z] = [-21, 8, 14] + t[-12, 4, 7]$ and the same y-intercept as $[x, y, z] = [6, -8, 12] + s[2, -5, 4]$. Write the parametric equations of the line.

8.2 Equations of Planes

7. Find three points on each plane.

a) $[x, y, z] = [3, 4, -1] + s[1, 1, -4] + t[2, -5, 3]$

b) $x + 2y - z + 12 = 0$

c) $x = 3k + 4p$
$y = -5 - 2k + p$
$z = 2 + 3k - 2p$

8. A plane contains the line $[x, y, z] = [2, -9, 10] + t[3, -8, 7]$ and the point P$(5, 1, 3)$. Write the vector and parametric equations of the plane.

9. Does P$(-3, 4, -5)$ lie on each plane?

a) $[x, y, z] = [1, -5, 6] + s[2, 1, 3] + t[1, 7, 1]$

b) $4x + y - 2z - 2 = 0$

10. Do the points A$(2, 1, 5)$, B$(-1, -1, 10)$, and C$(8, 5, -5)$ define a plane? Explain why or why not.

11. A plane is defined by the equation $x - 4y + 2z = 16$.

a) Find two vectors parallel to the plane.

b) Determine the x-, y-, and z-intercepts.

c) Write the vector and parametric equations of the plane.

8.3 Properties of Planes

12. Write the scalar equation of the plane with $\vec{n} = [1, 2, -9]$ that contains P$(3, -4, 0)$.

13. Write the scalar equation of this plane. $[x, y, z] = [5, 4, -7] + s[0, 1, 0] + t[0, 0, 1]$

14. Write the scalar equation of each plane.

a) parallel to the yz-plane with x-intercept 4

b) parallel to the vector $\vec{a} = [3, -7, 1]$ and to the y-axis, and through $(1, 2, 4)$

8.4 Intersections of Lines in Two-Space and Three-Space

15. Determine the number of solutions for each linear system in two-space. If possible, solve each system.

a) $2x - 5y = 6$
$$\begin{cases} x = -9 + 7t \\ y = -4 + 3t \end{cases}$$

b) $[x, y] = [9, 4] + s[1,1]$
$[x, y] = [0, 9] + t[3, -4]$

16. Write two other equations that have the same solution as this system of equations.
$3x - 4y = -14$
$-x + 3y = 18$

17. Determine if the lines in each pair intersect. If so, find the coordinates of the point of intersection.

a) $[x, y, z] = [1, 5, -2] + s[1, 7, -3]$
$[x, y, z] = [-3, -23, 10] + t[1, 7, -3]$

b) $[x, y, z] = [15, 2, -1] + s[4, 1, -1]$
$[x, y, z] = [13, -5, -4] + t[-5, 2, 3]$

18. Find the distance between these skew lines.
$[x, y, z] = [1, 0, -1] + s[2, 3, -4]$
$[x, y, z] = [8, 1, 3] + t[4, -5, 1]$

8.5 Intersections of Lines and Planes

19. Determine if each line intersects the plane. If so, state the solution.

a) $5x - 2y + 4z = 23$
$[x, y, z] = [-17, 7, -6] + t[4, 1, -3]$

b) $x + 4y + 3z = 11$
$[x, y, z] = [-1, -9, 16] + t[3, 3, -5]$

20. Find the distance between point P(3, −2, 0) and the plane $4x - y + 8z = 2$.

8.6 Intersections of Planes

21. Find the line of intersection for these two planes.
$3x + y + z = 10$
$5x + 4y - 2z = 31$

22. How do the planes in each system intersect?

a) $2x + 5y + 2z = 3$
$x + 2y - 3z = -11$
$2x + y + 5z = 8$

b) $x + 3y + 2z = 10$
$3x - 5y + z = 1$
$6x + 4y + 7z = -5$

c) $x + 3y - z = -2$
$3x + y + z = 14$
$5x + 7y + z = 10$

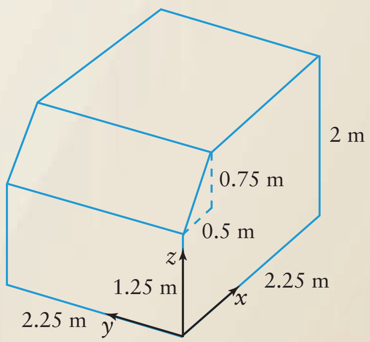
23. Use the normal vectors of the planes to describe each system.

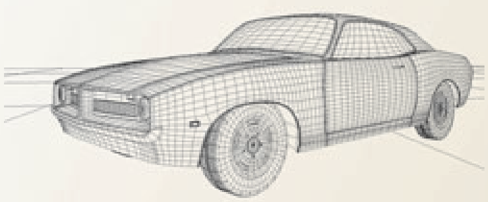
a) $2x + 5y + 3z = 0$
$x - 3y + 6z = 19$
$3x + 2y + 9z = -7$

b) $8x + 20y + 16z = 3$
$3x + 15y + 12z = 10$
$2x - 5y + 4z = 2$

CHAPTER PROBLEM WRAP-UP

When 3-D artists create objects that will eventually be animated, they start with a wire frame model of the object. Programmers then work to develop shaders that will be applied as materials for the models. Meanwhile, programmers and specialized technical artists create rigs, which the animator will use to control the motion of the models. A vehicle for your game is modelled with the wire frame shown in the diagram. Write the equations for all of the outside surfaces and edges.





Chapter 8 PRACTICE TEST

For questions 1 to 10, choose the best answer.

1. A line in two-space has scalar equation $3x + 5y + 9 = 0$. Which vector is parallel to the line?

A $[3, 5]$ **B** $[3, -5]$

C $[5, 3]$ **D** $[-5, 3]$

2. Which line is the same as $[x, y] = [1, -8] + t[4, -3]$?

A $[x, y] = [2, 4] + t[-4, 3]$

B $[x, y] = [13, -17] + t[8, -6]$

C $[x, y] = [-10, 20] + t[12, -9]$

D $[x, y] = [-11, -4] + t[4, -3]$

3. Which plane does not contain the point P(10, −3, 5)?

A $3x + 6y - 2z - 2 = 0$

B $x + y - z - 12 = 0$

C $2x - 2y - 3z - 11 = 0$

D $4x + 5y + z - 30 = 0$

4. Which does not exist for a line in three-space?

A vector equation

B scalar equation

C parametric equation

D equation in slope–intercept form

5. The equation $2x + 5y = 9$ can represent

A a scalar equation of a line in two-space

B a scalar equation of a plane in three-space

C neither A nor B

D both A and B

6. Which vector is not normal to the plane $x + 2y - 3z - 4 = 0$?

A $[2, -3, 4]$ **B** $[-1, -2, 3]$

C $[2, 4, -6]$ **D** $[3, 6, -9]$

7. Which point is not a solution to the equation $3x - 4y + z - 12 = 0$?

A $(-8, -9, 0)$ **B** $(4, 1, 4)$

C $(16, 10, 4)$ **D** $(18, 12, 2)$

8. How many solutions does this system have?

$10x - 7y = 10$
$4x + 5y = 12$

A 0

B 1

C 12

D infinitely many

9. Which word best describes the solution to this system of equations?

$[x, y, z] = [4, -2, 3] + s[5, -3, 2]$
$[x, y, z] = [1, -3, 1] + t[5, -3, 2]$

A consistent

B coincident

C inconsistent

D skew

10. Which statement is never true for two planes?

A they intersect at a point

B they intersect at a line

C they are parallel and distinct

D they are coincident

11. A line passes through points A(1, 5, −4) and B(2, −9, 0).

a) Write vector and parametric equations of the line.

b) Determine two other points on the line.

12. Write the parametric equations of a line perpendicular to $4x + 8y + 7 = 0$ with the same x-intercept as $[x, y] = [2, 7] + t[-10, 3]$.

13. Find the parametric equations of the line through the point P(−6, 4, 3), that is perpendicular to both the lines with equations $[x, y, z] = [0, -10, -2] + s[4, 6, -3]$ and $[x, y, z] = [5, 5, -5] + t[3, 2, 4]$.

14. Write the scalar equation of a plane that is parallel to the xz-plane and contains the line $[x, y, z] = [3, -1, 5] + t[4, 0, -1]$.

15. Determine if the lines in each pair intersect. If they intersect, find the intersection point.

a) $6x + 2y = 5$
$[x, y] = [4, -7] + t[1, -3]$

b) $2x + 3y = 21$
$4x - y = 7$

c) $[x, y, z] = [-2, 4, -1] + s[3, -3, 1]$
$[x, y, z] = [7, 10, 4] + t[1, 4, 3]$

d) $[x, y, z] = [3, 4, -6] + s[5, 2, -2]$
$[x, y, z] = [1, -4, 4] + t[10, 4, -4]$

16. Consider these lines.
$[x, y, z] = [2, 6, 1] + s[5, 1, 3]$
$[x, y, z] = [1, 5, -3] + t[-2, 4, 1]$

a) Show that these lines are skew.

b) Find the distance between the lines.

17. Determine whether the line with equation $[x, y, z] = [-3, -6, -11] + k[22, 1, -11]$ lies in the plane that contains the points A(2, 5, 6), B(−7, 1, 4), and C(6, −2, −9).

18. The direction cosines can be used as the direction vector of a line. Explain why this can be done. Provide an example to support your explanation.

19. Determine if the planes in each set intersect. If so, describe how they intersect.

a) $2x + 5y - 7z = 31$
$x - 2y - 5z = -9$

b) $x + y - z = 11$
$2x + 3y + 4z = 0$
$2x - 2y + z = 4$

c) $2x + y - 3z = 2$
$x - 4y + 2z = 5$
$4x + 2y - 6z = -12$

d) $2x - 3y - 4z = -16$
$11x - 3y + 5z = 47$
$5x + y + 7z = 45$

20. A plane passes through the points A(1, 13, 2), B(−2, −6, 5), and C(−1, −1, −3).

a) Write the vector and parametric equations of the plane.

b) Write the scalar equation of the plane.

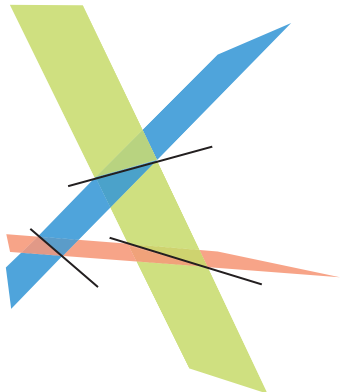
c) Determine two other points on the plane.

21. The plane π has vector equation
$\vec{r} = [2, 1, 3] + s[1, 1, 1] + t[2, 0, 2]$, $s, t \in \mathbb{R}$.

a) Verify that π does not pass through the origin.

b) Find the distance from the origin to π.

22. A plane π_1 has equation $2x - 3y + 5z = 1$. Two other planes, π_2 and π_3, intersect the plane, and each other, as illustrated in the diagram. State possible equations for π_2 and π_3 and justify your reasoning.



23. Given the plane $2x + 3y + 4z = 24$ and the lines
$[x, y, z] = [0, 0, 6] + r[3, 2, -3]$
$[x, y, z] = [14, -12, 8] + s[1, -2, 1]$
$[x, y, z] = [9, -10, 9] + t[1, 6, -5]$

a) Show that each line is contained in the plane.

b) Find the vertices of the triangle formed by the lines.

c) Find the perimeter of the triangle formed by the lines.

d) Find the area of the triangle formed by the lines.

Chapters 6 to 8 REVIEW

Chapter 6

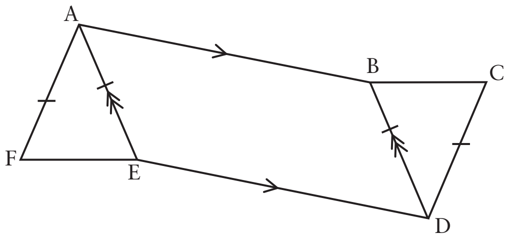
1. Convert each quadrant bearing to its equivalent true bearing.

   a) N20°W b) S36°E c) S80°W

2. Convert each true bearing to its equivalent quadrant bearing.

   a) 130° b) 280° c) 94°

3. Consider this diagram.



   a) State a vector opposite to $\overrightarrow{AE}$.

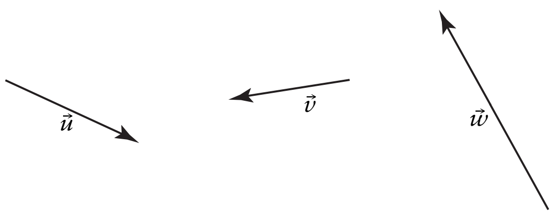
   b) Is $\overrightarrow{AF} = \overrightarrow{CD}$? Explain.

   c) State the conditions under which $\overrightarrow{FE} = \overrightarrow{BC}$.

   d) Express $\overrightarrow{AB}$ as the difference of two other vectors.

4. Three forces, each of magnitude 50 N, act in eastward, westward, and northwest directions. Draw a scale diagram showing the sum of these forces. Determine the resultant force.

5. Use vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ to draw each combination of vectors.



   a) $2\vec{u}$ b) $-\vec{v}$ c) $\vec{u} + \vec{v}$

   d) $\vec{u} + \vec{v} + \vec{w}$ e) $2\vec{u} - 3\vec{v} + 5\vec{w}$

6. Refer to the vectors from question 5. For $k = 3$, show that $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$.

7. Describe a scenario that could be represented by each scalar multiplication.

   a) $2\vec{a}$, given $|\vec{a}| = 9.8$ m/s^2

   b) $-10\vec{v}$, given $|\vec{v}| = 100$ N

8. Determine each resultant.

   a) 10 km/h north followed by 8 km/h west

   b) 80 m/s upward and 12 m/s horizontally

   c) 25 N at 10° to the horizontal and 30 N horizontally

9. A 75-N load rests on a ramp inclined at 20° to the horizontal. Resolve the weight of the load into the rectangular components keeping it at rest.

Chapter 7

10. Consider $\vec{u} = [2, 3]$ and $\vec{v} = [-4, -6]$.

    a) Find two unit vectors parallel to $\vec{u}$.

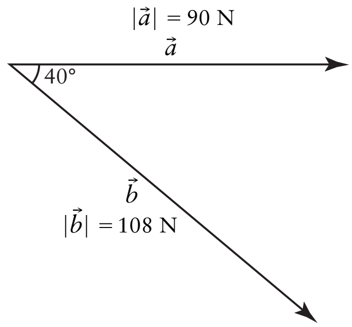
    b) Determine the magnitude of $\vec{u}$.

    c) $\overrightarrow{PQ}$ is parallel to $\vec{v}$, and has initial point P$(-7, 5)$. Determine the coordinates of Q.

11. Use an example to show that $(k + m)\vec{u} = k\vec{u} + m\vec{u}$.

12. A boat is travelling at 18 knots on a heading of 234°. The current is 8 knots, flowing from a bearing of 124°. Determine the resultant velocity of the boat.

13. Calculate the dot product of $\vec{a}$ and $\vec{b}$.



14. Calculate the dot product of each pair of vectors.

    a) $\vec{m} = [-2, -8]$, $\vec{n} = [9, 0]$

    b) $\vec{p} = [4, 5, -1]$, $\vec{q} = [6, -2, 7]$

15. Determine a vector perpendicular to $\vec{u} = [6, -5]$.

16. Calculate the angle between $\vec{u} = [5, 7, -1]$ and $\vec{v} = [8, 7, 8]$.

17. Tyler applies a force of 80 N at 10° to the horizontal to pull a cart 20 m horizontally. How much mechanical work does Tyler do?

18. Determine the vector $\vec{v} = \overrightarrow{AB}$ and its magnitude, given A(3, 7, −2) and B(9, 1, 5).

19. Given $\vec{a} = [2, -4, 5]$, $\vec{b} = [7, 3, 4]$, and $\vec{c} = [-3, 7, 1]$, simplify each expression.

a) $\vec{b} \cdot \vec{a} \times \vec{c}$ **b)** $\vec{a} \times \vec{b} - \vec{b} \times \vec{c}$

20. Is $\vec{u} \times \vec{v} = \vec{v} \times \vec{u}$? Explain and illustrate with a diagram.

21. Determine two vectors orthogonal to both $\vec{c} = [6, 3, -2]$ and $\vec{d} = [4, 5, -7]$.

22. A parallelogram has side lengths 12 cm and 10 cm and area 95 cm^2. Determine the measures of the interior angles.

23. A crank is pushed by a 40-N force. The shaft of the crank is 18 cm long. Find the magnitude of the torque vector, in newton-metres, about the pivot point.

Chapter 8

24. Write a vector equation and parametric equations of the line through each pair of points.

a) P(3, 5) and Q(−4, 7)

b) A(6, −1, 5) and B(−2, −3, 6)

25. Write the scalar equation of the line with parametric equations $x = 6 - 7t$, $y = -2 + 3t$.

26. Determine the scalar and vector equations of a line through P(−4, 6) with normal vector $\vec{n} = [5, 1]$.

27. Determine an equation for each plane.

a) parallel to both the x- and y-axes and contains the point A(2, 5, −1)

b) contains the points A(6, 2, 3), B(5, 1, −3), and C(5, 7, −2)

c) parallel to $2x + 6y + 4z = 1$ and contains the point P(3, 2, 1)

28. Explain why three direction vectors of a plane cannot be mutually perpendicular.

29. Write the scalar equation of each plane.

a) $[x, y, z] = [2, 3, 5] + s[-1, 3, 4] + t[6, 1, -2]$

b) contains the point P(6, −2, 3) and has normal vector $\vec{n} = [1, 2, -5]$

30. Determine the angle between the planes defined by $2x + 2y + 7z = 8$ and $3x - 4y + 4z = 5$.

31. Determine if the lines intersect. If so, find the coordinates of the point of intersection.
$[x, y, z] = [2, 1, -4] + s[-5, 3, 1]$
$[x, y, z] = [-11, 7, 2] + t[2, -3, 3]$

32. Determine the distance between these skew lines.
$[x, y, z] = [4, 2, -3] + s[-1, -2, 2]$
$[x, y, z] = [-6, -2, 3] + t[-2, 2, 1]$

33. Determine the intersection of each line and plane. Interpret the solution.

a) $2x + 3y - 5z = 3$
$[x, y, z] = [4, 6, -1] + s[3, 4, -2]$

b) $x + 3y - 2z = 6$
$[x, y, z] = [8, -2, -2] + s[4, -2, -1]$

34. Determine the distance between P(3, −2, 5) and the plane $2x + 4y - z = 2$.

35. Determine if the planes in each pair are parallel and distinct or coincident.

a) $5x + y - 2z = 2$
$5x + y - 2z = -8$

b) $7x + 3y - 4z = 9$
$14x + 6y - 8z = 18$

36. Solve each system of equations and interpret the results geometrically.

a) $2x + 3y - 5z = 9$
$5x - y + 2z = -3$
$-x + 7y - 12z = 21$

b) $2x + 3y + z = 5$
$x + 4y - 2z = 10$
$7x + 6y + 5z = 7$

c) $3x + y - z = 4$
$4x - 2y - 3z = 5$
$8x + 6y - z = 7$

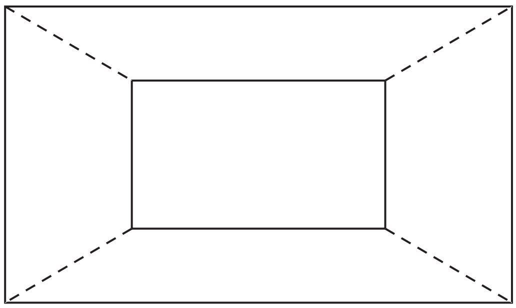
TASK

Simulating 3-D Motion on a Television Screen

During television programs or advertisements, images often move on the screen so it appears that they are moving toward or away from the viewer. This effect can be created by using different sizes of the same image, while simultaneously changing the horizontal and vertical position of the image on the screen. For instance, to make an image appear as if it is moving closer to the viewer, start with a smaller image on the left side of the screen and increase its size while moving it toward the centre.

A flat screen television has a width of 24 inches and a height of 14 inches. An advertising company has designed a rectangular picture that will initially appear at the top left corner of the screen. The image will slide to the centre of the screen and increase in size. In its final position, the dimensions of the image are double its initial dimensions.

a) If the height and width of the original image are 2 inches and 3 inches, respectively, determine vector equations of the lines that would describe the transformation of the vertices.



b) How would your answer change if you were to triple the dimensions at the centre, instead of doubling them? Explain your reasoning.

c) How would your answer change if you wanted the image to be smaller by a factor of $\frac{1}{k}$? Explain your reasoning.

d) Consider that the transformation lines are meant to represent an orthogonal transformation of a plane, with the magnitudes of the transformation vectors representing the distances transformed. If the original rectangle lies in the plane with equation $[x, y, z] = [2, 3, 0] + s[4, 1, 5] + t[1, 2, 6]$ for some coordinate system, describe the image planes in parts a) and c) using multiple representations.

Chapters 1 to 8 COURSE REVIEW

Chapter 1 Rates of Change

1. Consider the function $f(x) = 2x^2 - 3x + 4$.

a) Determine the average rate of change between the point where $x = 1$ and the point determined by each x-value.

i) $x = 4$ **ii)** $x = 1.5$ **iii)** $x = 1.1$

b) Describe the trend in the average slopes. Predict the slope of the tangent at $x = 1$.

2. State whether each situation represents average or instantaneous rate of change. Explain your reasoning.

a) Ali calculated his speed as 95 km/h when he drove from Toronto to Windsor.

b) The speedometer on Eric's car read 87 km/h.

c) The temperature dropped by 2°C/h on a cold winter night.

d) At a particular moment, oil was leaking from a container at 20 L/s.

3. a) Expand and simplify each expression, then evaluate for $a = 2$ and $h = 0.01$.

i) $\dfrac{3(a+h)^2 - 3a^2}{h}$ **ii)** $\dfrac{(a+h)^3 - a^3}{h}$

b) What does each answer represent?

4. A ball was tossed into the air. Its height, in metres, is given by $h(t) = -4.9t^2 + 6t + 1$, where t is time, in seconds.

a) Write an expression that represents the average rate of change over the interval $2 \le t \le 2 + h$.

b) Find the instantaneous rate of change of the height of the ball after 2 s.

c) Sketch the curve and the rate of change.

5. Determine the limit of each infinite sequence. If the limit does not exist, explain.

a) 2, 2.1, 2.11, 2.111, … **b)** $5, 5\frac{1}{2}, 5\frac{1}{3}, 5\frac{1}{4}, \ldots$

c) $2, -2, 2, -2, 2, \ldots$ **d)** $450, -90, 18, \ldots$

6. Evaluate each limit. If it does not exist, provide evidence.

a) $\lim\limits_{x\to 2}(3x^2 - 4x + 1)$ **b)** $\lim\limits_{x\to -8}\dfrac{5x+40}{x+8}$

c) $\lim\limits_{x\to 6^+}\sqrt{x-6}$ **d)** $\lim\limits_{x\to 3}\dfrac{2x-9}{x-3}$

7. a) Sketch a fully labelled graph of

$$f(x) = \begin{cases} 2x+1, & x < 0 \\ 3, & x = 0 \\ 2x^2 - 1, & x > 0. \end{cases}$$

b) Evaluate each limit. If it does not exist, explain why.

i) $\lim\limits_{x\to 3} f(x)$ **ii)** $\lim\limits_{x\to 0} f(x)$

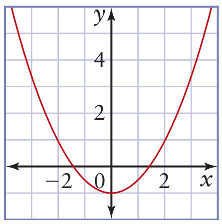
8. Use first principles to determine the derivative of each function. Then, determine an equation of the tangent at the point where $x = 2$.

a) $y = 4x^2 - 3$ **b)** $f(x) = x^3 - 2x^2$

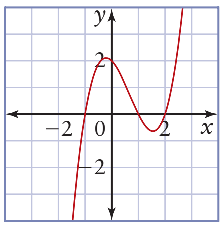
c) $g(x) = \dfrac{3}{x}$ **d)** $h(x) = 2\sqrt{x}$

9. Given the graph of $f(x)$, sketch a graph that represents its rate of change.

a)



b)



Chapter 2 Derivatives

10. Differentiate each function.

a) $y = -3x^2 + 4x - 5$

b) $f(x) = 6x^{-1} - 5x^{-2}$

c) $f(x) = 4\sqrt{x}$

d) $y = (3x^2 - 4x)(\sqrt{x} - 1)$

e) $y = \dfrac{x^2 + 4}{3x}$

11. Show that each statement is false.

a) $(f(x) \cdot g(x))' = f'(x) \cdot g'(x)$

b) $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)}{g'(x)}$

12. Determine the equation of the tangent to each function at the given value of x.

a) $f(x) = 2x^2 - 1$ at $x = -2$

b) $g(x) = \sqrt{x} + 5$ at $x = 4$

13. The position of a particle, s metres from a starting point, after t seconds, is given by the function $s(t) = 2t^3 - 7t^2 + 4t$.

a) Determine its velocity at time t.

b) Determine the velocity after 5 s.

14. Determine the points at which the slope of the tangent to $f(x) = x^3 - 2x^2 - 4x + 4$ is zero.

15. A fireworks shell is shot upward with an initial velocity of 28 m/s from a height of 2.5 m.

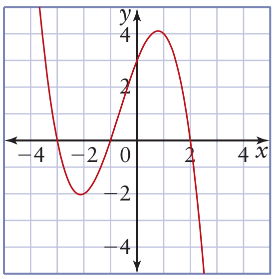
a) State an equation to represent the height of the shell at time t, in seconds.

b) Determine equations for the velocity and acceleration of the shell.

c) State the height, velocity, and acceleration after 2 s.

d) After how many seconds will the shell have the same speed, but be falling downward?

16. Copy this graph of the position function of an object. Draw graphs to represent the velocity and acceleration.



17. Determine the derivative of each function.

a) $y = 2(3x - x^{-1})^2$

b) $g(x) = \sqrt{2x + 5}$

c) $y = \dfrac{1}{\sqrt{3x - 1}}$

d) $\dfrac{-1}{\sqrt[3]{x^2 + 3x}}$

18. Determine an equation of the tangent to $f(x) = x^2(x^3 - 3x)^3$ at the point $(-1, 8)$.

19. The cost, in dollars, of water consumed by a factory is given by the function $C(w) = 15 + 0.1\sqrt{w}$, where w is the water consumption, in litres. Determine the cost and the rate of change of the cost when the consumption is 2000 L.

20. A fast-food restaurant sells 425 large orders of fries per week at a price of \$2.75 each. A market survey indicates that for each \$0.10 decrease in price, sales will increase by 20 orders of fries.

a) Determine the demand function.

b) Determine the revenue function.

c) Determine the marginal revenue function.

d) With what price will the marginal revenue be zero? Interpret the meaning of this value.

21. A car has constant deceleration of 10 km/h/s until it stops. If the car's initial velocity is 120 km/h, determine its stopping distance.

Chapter 3 Curve Sketching

22. Evaluate each limit.

a) $\lim\limits_{x \to \infty} (2x^3 - 5x^2 + 9x - 8)$

b) $\lim\limits_{x \to \infty} \dfrac{x + 1}{x - 1}$

c) $\lim\limits_{x \to -\infty} \dfrac{x^2 - 3x + 1}{x^2 + 4x + 8}$

23. Sketch the graph of $f(x)$ based on the information in the table.

x	$(-\infty, -2)$	-2	$(-2, -1)$	-1	$(-1, 0)$	0	$(0, \infty)$
$f(x)$		-5		-1		3	
$f'(x)$	$-$	0	$+$	$+$	$+$	0	$-$
$f''(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$

24. For each function, determine the coordinates of the local extrema, the points of inflection, the intervals of increase and decrease, and the concavity.

a) $f(x) = x^3 + 2x^2 - 4x + 1$

b) $f(x) = \dfrac{3}{4}x^4 - x^3 - x^2 + 5x - 3$

c) $f(x) = \dfrac{3}{x^2 + 1}$

25. Analyse and sketch each function.

a) $f(x) = 3x^4 - 8x^3 + 6x^2$

b) $y = -x^3 + x^2 + 8x - 3$

c) $f(x) = \dfrac{x}{x^2 + 1}$

26. The power, in amps, transmitted by a belt drive from a motor is given by the function $P = 100v - \dfrac{3}{16}v^3$ where v is the linear velocity of the belt, in metres per second.

a) For what value of v is the power at a maximum value?

b) What is the maximum power?

27. A ship is sailing due north at 12 km/h while another ship is observed 15 km ahead, travelling due east at 9 km/h. What is the closest distance of approach of the two ships?

28. The Perfect Pizza Parlor estimates the average daily cost per pizza, in dollars, to be $C(x) = \dfrac{0.00025x^2 + 8x + 10}{x}$, where x is the number of pizzas made in a day.

a) Determine the marginal cost at a production level of 50 pizzas a day.

b) Determine the production level that would minimize the average daily cost per pizza.

c) What is the minimum average daily cost?

Chapter 4 Derivatives of Sinusoidal Functions

29. Differentiate each function.

a) $y = \cos^3 x$

b) $y = \sin(x^3)$

c) $f(x) = \cos(5x - 3)$

d) $f(x) = \sin^2 x \cdot \cos\left(\dfrac{x}{2}\right)$

e) $f(x) = \cos^2(4x^2)$

f) $g(x) = \dfrac{\cos x}{\cos x - \sin x}$

30. Determine the equation of the tangent to $y = 2 + \cos 2x$ at $x = \dfrac{5\pi}{6}$.

31. Use the derivatives of $\sin x$ and $\cos x$ to develop the derivatives of $\sec x$, $\csc x$, and $\tan x$.

32. Find the local maxima, local minima, and inflection points of the function $y = \sin^2 x - \dfrac{x}{2}$. Use technology to verify your findings and to sketch the graph.

33. The height above the ground of a rider on a large Ferris wheel can be modelled by $h(t) = 10\sin\left(\dfrac{2\pi}{30}t\right) + 12$, where h is the height above the ground, in metres, and t is time, in seconds. What is the maximum height reached by the rider, and when does this first occur?

34. A weight is oscillating up and down on a spring. Its displacement, from rest is given by the function $d(t) = \sin 6t - 4\cos 6t$, where d is in centimetres and t is time, in seconds.

a) What is the rate of change of the displacement after 1 s?

b) Determine the maximum and minimum displacements and when they first occur.

Chapter 5 Exponential and Logarithmic Functions

35. a) Compare the graphs of $y = e^x$ and $y = 2^x$.

b) Compare the graphs of the rates of change of $y = e^x$ and $y = 2^x$.

c) Compare the graphs of $y = \ln x$ and $y = e^x$, and their rates of change.

36. Evaluate, accurate to two decimal places.

a) $\ln 5$ b) $\ln e^2$ c) $(\ln e)^2$

37. Simplify.

a) $\ln(e^x)$ b) $e^{\ln x}$ c) $D^{\ln h^z}$

38. Determine the derivative of each function.

a) $y = -2e^x$ b) $g(x) = 5 \cdot 10^x$

c) $h(x) = \cos(e^x)$ d) $f(x) = xe^{-x}$

39. Determine the equation of the line perpendicular to $f(x) = \frac{1}{2}e^{x+1}$ at its y-intercept.

40. Radium decays at a rate that is proportional to its mass, and has a half-life of 1590 years. If 20 g of radium is present initially, how long will it take for 90% of this mass to decay?

41. Determine all critical points of $f(x) = x^2e^x$. Sketch a graph of the function.

42. The power supply, in watts, of a satellite is given by the function $P(t) = 200e^{-0.001t}$, where t is the time, in days, after launch. Determine the rate of change of power

a) after t days **b)** after 200 days

43. The St. Louis Gateway Arch is in the shape of a catenary defined by the function $y = -20.96\left(\frac{e^{0.0329x} + e^{-0.0329x}}{2} - 10.06\right)$, with all measurements in metres.

a) Determine an equation for the slope of the arch at any point x m from its centre.

b) What is the slope of the arch at a point 2 m horizontally from the centre?

c) Determine the width and the maximum height of the arch.

44. Show how the function $y = e^{-x}\sin x$ could represent dampened oscillation.

a) Determine the maximum and minimum points for a sequence of wavelengths.

b) Show that the maximum points represent exponential decay.

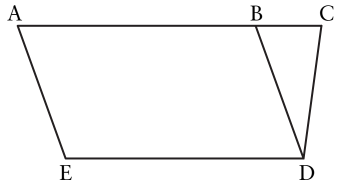
Chapter 6 Geometric Vectors

45. Use an appropriate scale to draw each vector.

a) displacement of 10 km at a bearing of 15°

b) velocity of 6 m/s upward

46. ACDE is a trapezoid, such that $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AE} = \vec{v}$, BD || AE, and AB = 5BC. Express each vector in terms of $\vec{u}$ and $\vec{v}$.



a) $\overrightarrow{DE}$ **b)** $\overrightarrow{DB}$ **c)** $\overrightarrow{BC}$

d) $\overrightarrow{AD}$ **e)** $\overrightarrow{CD}$ **f)** $\overrightarrow{CE}$

47. Simplify algebraically.

a) $3\vec{u} + 5\vec{u} - 7\vec{u} - 6\vec{u}$

b) $-5(\vec{c} + \vec{d}) - 8(\vec{c} - \vec{d})$

48. Determine the resultant of each vector sum.

a) 12 km north followed by 15 km east

b) a force of 60 N upward with a horizontal force of 40 N

49. A rocket is propelled vertically at 450 km/h. A horizontal wind is blowing at 15 km/h. What is the ground velocity of the rocket?

50. Two forces act on an object at 20° to each other. One force has a magnitude of 200 N, and the resultant has a magnitude of 340 N.

a) Draw a diagram illustrating this situation.

b) Determine the magnitude of the second force and direction it makes with the resultant.

51. An electronic scoreboard of mass 500 kg is suspended from a ceiling by four cables, each making an angle of 70° with the ceiling. The weight is evenly distributed. Determine the tensions in the cables, in newtons.

52. A ball is thrown with a force that has a horizontal component of 40 N. The resultant force has a magnitude of 58 N. Determine the vertical component of the force.

53. Resolve the velocity of 120 km/h at a bearing of 130° into its rectangular components.

54. A 100-N box is resting on a ramp inclined at 42° to the horizontal. Resolve the weight into the rectangular components keeping it at rest.

Chapter 7 Cartesian Vectors

55. Express each vector in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

a) $[5, 6, -4]$ **b)** $[0, -8, 7]$

56. Given points P(2, 4), Q(−6, 3), and R(4, −10), determine each of the following.

a) $|\overrightarrow{PQ}|$ **b)** $\overrightarrow{PR}$

c) $3\overrightarrow{PQ} - 2\overrightarrow{PR}$ **d)** $\overrightarrow{PQ} \cdot \overrightarrow{PR}$

57. Write each of the following as a Cartesian vector.

a) 30 m/s at a heading of 20°

b) 40 N at 80° to the horizontal

58. A ship's course is set at a heading of 214°, with a speed of 20 knots. A current is flowing from a bearing of 93°, at 11 knots. Use Cartesian vectors to determine the resultant velocity of the ship, to the nearest knot.

59. Use examples to explain these properties.

a) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

b) $k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$

60. Calculate the dot product for each pair of vectors.

a) $|\vec{u}| = 12$, $|\vec{v}| = 21$, $\theta = 20°$

b) $|\vec{s}| = 115$, $|\vec{t}| = 150$, $\theta = 42°$

61. Given $\vec{u} = [3, -4]$, $\vec{v} = [6, 1]$, and $\vec{w} = [-9, 6]$, evaluate each of the following, if possible. If it is not possible, explain why.

a) $\vec{u} \cdot (\vec{v} + \vec{w})$ **b)** $\vec{u} \cdot (\vec{v} \cdot \vec{w})$

c) $\vec{u}(\vec{v} \cdot \vec{w})$ **d)** $(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v})$

62. Determine the value of k so that $\vec{u} = [-3, 7]$ and $\vec{v} = [6, k]$ are perpendicular.

63. Determine whether or not the triangle with vertices A(3, 5), B(−1, 4), and C(6, 2) is a right triangle.

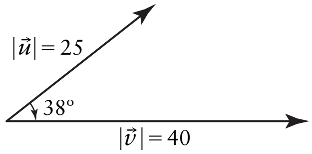
64. Determine the angle between the vectors $\vec{u} = [4, 10, -2]$ and $\vec{v} = [1, 7, -1]$, accurate to the nearest degree.

65. Determine the work done by each force $\vec{F}$, in newtons, for an object moving along the vector $\vec{d}$, in metres.

a) $\vec{F} = [1, 4]$, $\vec{d} = [6, 3]$

b) $\vec{F} = [320, 145]$, $\vec{d} = [32, 15]$

66. Determine the projection of $\vec{u}$ on $\vec{v}$.



67. Roni applies a force at 10° to the horizontal to move a heavy box 3 m horizontally. He does 100 J of mechanical work. What is the magnitude of the force he applies?

68. Graph each position vector.

a) $[4, 2, 5]$ **b)** $[-1, 0, -4]$ **c)** $[2, -2, -2]$

69. The initial point of vector $\overrightarrow{AB} = [6, 3, -2]$ is A(1, 4, 5). Determine the coordinates of the terminal point B.

70. Consider the vectors $\vec{a} = [7, 2, 4]$, $\vec{b} = [-6, 3, 0]$, and $\vec{c} = [4, 8, 6]$. Determine the angle between each pair of vectors.

71. Prove for any three vectors $\vec{u} = [u_1, u_2, u_3]$, $\vec{v} = [v_1, v_2, v_3]$, and $\vec{w} = [w_1, w_2, w_3]$, that $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

72. Determine two possible vectors that are orthogonal to each vector.

a) $\vec{u} = [-1, 5, 4]$ **b)** $\vec{v} = [5, 6, 2]$

73. A small airplane takes off at an airspeed of 180 km/h, at an angle of inclination of 14°, toward the east. A 15-km/h wind is blowing from the southwest. Determine the resultant ground velocity.

74. Determine $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$. Confirm that the cross product is orthogonal to each vector. $\vec{a} = [4, -3, 5]$, $\vec{b} = [2, 7, 2]$

75. Determine the area of the triangle with vertices P(3, 4, 8), Q(−2, 5, 7), and R(−5, −1, 6).

76. Use an example to verify that
$k\vec{u} \times \vec{v} = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$.

77. Determine two vectors that are orthogonal to both $\vec{u} = [-1, 6, 5]$ and $\vec{v} = [4, 9, 10]$.

78. Find the volume of the parallelepiped defined by the vectors $\vec{u} = [-2, 3, 6]$, $\vec{v} = [6, 7, -4]$, and $\vec{w} = [5, 0, 1]$.

79. When a wrench is rotated, the magnitude of the torque is 26 N·m. A 95-N force is applied 35 cm from the fulcrum. At what angle to the wrench is the force applied?

Chapter 8 Lines and Planes

80. Write vector and parametric equations of the line through.

A(−2, −1, −7) and B(5, 0, 10).

81. Graph each line.

a) $[x, y] = [2, 3] + t[-2, 5]$

b) $x = -3 + 5t$
$y = 4 + 3t$

82. Write the scalar equation of each line, given the normal vector $\vec{n}$ and point P_0.

a) $\vec{n} = [4, 8]$, $P_0(3, -1)$

b) $\vec{n} = [-6, -7]$, $P_0(-5, 10)$

83. Determine the vector equation of each line.

a) parallel to the y-axis and through P(6, −3)

b) perpendicular to $2x + 7y = 5$ and through A(1, 6)

84. Determine the coordinates of two points on the plane with equation $5x + 4y - 3z = 6$.

85. Write the parametric and scalar equations of the plane given its vector equation $[x, y, z] = [4, 3, -5] + s[2, 1, 4] + t[6, 6, -3]$.

86. Determine the intercepts of each plane.

a) $[x, y, z] = [1, 8, 6] + s[1, -12, -12] + t[2, 4, -3]$

b) $2x - 6y + 9z = 18$

87. Determine an equation for each plane. Verify your answers using 3-D graphing technology.

a) containing the points A(5, 2, 8), B(−9, 10, 3), and C(−2, −6, 5)

b) parallel to both the x-axis and y-axis and through the point P(−4, 5, 6)

88. Write the equation of the line perpendicular to the plane $7x - 8y + 5z = 1$ and passing through the point P(6, 1, −2).

89. Determine the angle between the planes with equations $8x + 10y + 3z = 4$ and $2x - 4y + 6z = 3$.

90. Determine whether the lines intersect. If they do, find the coordinates of the point of intersection.
$[x, y, z] = [2, -3, 2] + s[3, 4, -10]$
$[x, y, z] = [3, 4, -2] + t[-4, 5, 3]$

91. Determine the distance between these skew lines.
$[x, y, z] = [2, -7, 3] + s[3, -10, 1]$
$[x, y, z] = [3, 4, -2] + t[1, 6, 1]$

92. Determine the distance between A(2, 5, −7) and the plane $3x - 5y + 6z = 9$.

93. Does the line $r = \vec{r}[-5, 1, -2] + k[1, 6, 5]$ intersect the plane with equation $[x, y, z] = [2, 3, -1] + s[1, 3, 4] + t[-5, 4, 7]$? If so, how many solutions are there?

94. State equations of three planes that

a) are parallel

b) are coincident

c) intersect in a line

d) intersect in a point

95. Solve each system of equations.

a)
$$2x - 4y + 5z = -4$$
$$3x + 2y - z = 1$$
$$4x + 3y - 3z = -1$$

b)
$$5x + 4y + 2z = 7$$
$$3x + y - 3z = 2$$
$$7x + 7y + 7z = 12$$