Geometric Properties of Functions Containing Derivatives of Bessel Functions

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Abstract In this paper our aim is to find the radii of γ-Spirallike of order α and convex γ-Spirallike of order α for three different kinds of normalizations of the function $N_\nu(z) = az^2 J_\nu''(z) + bz J_\nu'(z) + c J_\nu(z)$, where $J_\nu(z)$ is the Bessel function of the first kind of order ν. Moreover, the $\mathcal{S}^*(\varphi)-$radii and $\mathcal{C}(\varphi)-$radii of these normalized functions are investigated. The tables are created and visual verification with graphs are made by giving special values to the real numbers a, b and c in the obtained results.

1 Introduction

Let $\mathcal{A}$ be the class of analytic functions normalized by the condition $f(0) = 0 = f'(0) - 1$ in the unit disk $\mathbb{D} := \mathbb{D}_1$, where $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. We say that a function $f \in \mathcal{A}$ is γ-Spirallike of order α if and only if

$$\text{Re}\left(e^{-i\gamma} \frac{z f'(z)}{f(z)}\right) > \alpha \cos \gamma,$$

where $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $0 \leq \alpha < 1$. The class of such functions, we denote by $\mathcal{S}_p^\gamma(\alpha)$. In view of well known Alexender's relation, let $\mathcal{CS}_p^\gamma(\alpha)$ be the class of

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convex γ-spirallike functions of order α, which is defined below

$$\text{Re}\left(e^{-i\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \alpha\cos\gamma.$$

Spacek [30] introduced and studied the class $\mathcal{S}_p^\gamma(0)$. Each function in $\mathcal{CS}_p^\gamma(\alpha)$ is univalent in $\mathbb{D}$, but they do not necessarily be starlike. Further, it is worth to mention that for general values of $\gamma(|\gamma| < \pi/2)$, a function in $\mathcal{CS}_p^\gamma(0)$ need not be univalent in $\mathbb{D}$. For example: $f(z) = i(1 - z)^i - i \in \mathcal{CS}_p^{\pi/4}(0)$, but not univalent. Indeed, $f \in \mathcal{CS}_p^\gamma(0)$ is univalent if $0 < \cos\gamma < 1/2$, see Robertson [28] and Pfaltzgraff [27]. Note that for $\gamma = 0$, the classes $\mathcal{S}_p^\gamma(\alpha)$ and $\mathcal{CS}_p^\gamma(\alpha)$ reduce to the classes of starlike and convex functions of order α, given by

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{and} \quad \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,$$

which we denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$, respectively.

In the recent past, connections between the special functions and their geometrical properties have been established in terms of radius problems [1, 2, 3, 5, 8, 9, 10, 15]. In particular, the $\mathcal{S}^*(\alpha)$-radius of a normalized function f is to find

$$\sup\{r \in \mathbb{R}^+ : \text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \mathbb{D}_r\}$$

and similarly, we can define $\mathcal{C}(\alpha)$-radius. In this direction, for $\mathcal{S}^*(\alpha)$-radius and more, we refer to Bessel functions [1, 3, 5] (see Watson's treatise [33] for more on Bessel function), Struve functions [1, 2], Wright functions [9], Lommel functions [1, 2] and Ramanujan type entire functions [15]. For their generalization to Ma-Minda classes [24] of starlike and convex functions, we refer to see [16, 21].

However, with the best of our knowledge, $\mathcal{S}_p^\gamma(\alpha)$-radius and $\mathcal{CS}_p^\gamma(\alpha)$-radius for special functions are not handled till date. Therefore, we define $\mathcal{S}_p^\gamma(\alpha)$-radius and its convex analog here below:

Definition 1 Let f in $\mathcal{A}$ be a special function. Then the radius of γ-Spirallike of order α is to find

$$r_{sp}^*(\alpha, \gamma; f) = \sup\left\{r \in \mathbb{R}^+ : \text{Re}\left(e^{-i\gamma}\frac{zf'(z)}{f(z)}\right) > \alpha\cos\gamma, z \in \mathbb{D}_r\right\}$$

and the radius of convex γ-Spirallike of order α is given by

$$r_{sp}^c(\alpha, \gamma; f) = \sup\left\{r \in \mathbb{R}^+ : \text{Re}\left(e^{-i\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \alpha\cos\gamma, z \in \mathbb{D}_r\right\}.$$

In the present investigation, we consider our special normalized functions to be the derivatives of Bessel functions. For this to proceed further, recall that the Bessel function of the first kind of order ν is defined by [26, p.217]

$$J_\nu(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n + \nu + 1)}\left(\frac{z}{2}\right)^{2n+\nu} \quad (z \in \mathbb{C}). \tag{1.1}$$

We know that it has all its zeros real for $\nu > -1$. Here now we consider mainly the general function

$$N_\nu(z) = az^2 J_\nu''(z) + bz J_\nu'(z) + c J_\nu(z)$$

studied by Mercer [25]. Here, as in [25], $q = b - a$ and

$$(c = 0 \text{ and } q \neq 0) \text{ or } (c > 0 \text{ and } q > 0).$$

From (1.1), we have the power series representation

$$N_\nu(z) = \sum_{n=0}^{\infty} \frac{Q(2n+\nu)\,(-1)^n}{n!\,\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu} \quad (z \in \mathbb{C}) \tag{1.2}$$

where $Q(\nu) = a\nu(\nu-1) + b\nu + c \ (a, b, c \in \mathbb{R})$. There are three important works on the function N_ν. Firstly, Mercer's paper [25] in which it has been proved that the kth positive zero of N_ν increases with ν in $\nu > 0$. Secondly, Ismail and Muldoon [17] showed that under the conditions $a, b, c \in \mathbb{R}$ such that $c = 0$ and $b \neq a$ or $c > 0$ and $b > a$;

(i) For $\nu > 0$, the zeros of $N_\nu(z)$ are either real or purely imaginary.
(ii) For $\nu \geq \max\{0, \nu_0\}$, where ν_0 is the largest real root of the quadratic $Q(\nu) = a\nu(\nu-1) + b\nu + c$, the the zeros of $N_\nu(z)$ are real.
(iii) If $\nu > 0$, $\left(a\nu^2 + (b-a)\nu + c\right) \big/ (b-a) > 0$ and $a \big/ (b-a) < 0$, the zeros of $N_\nu(z)$ are all real except for a single pair which are conjugate purely imaginary.

Baricz et al. [6] obtained sufficient and necessary conditions for the starlikeness of a normalized form of N_ν by using results of Mercer [25], Ismail and Muldoon [17] and Shah and Trimble [29].

Note that N_ν do not belong to $\mathcal{A}$. To prove our main results, we consider the following normalizations of the function N_ν given by:

$$f_\nu(z) = \left[\frac{2^\nu \Gamma(\nu+1)}{Q(\nu)} N_\nu(z)\right]^{\frac{1}{\nu}}, \tag{1.3}$$

$$g_\nu(z) = \frac{2^\nu \Gamma(\nu+1)z^{1-\nu}}{Q(\nu)} N_\nu(z), \tag{1.4}$$

$$h_\nu(z) = \frac{2^\nu \Gamma(\nu+1)z^{1-\frac{\nu}{2}}}{Q(\nu)} N_\nu(\sqrt{z}). \tag{1.5}$$

In the rest of this paper, for the quadratic $Q(\nu) = a\nu(\nu-1) + b\nu + c$, we will always assume that $a, b, c \in \mathbb{R}$ $(c = 0 \text{ and } a \neq b)$ or $(c > 0 \text{ and } a < b)$. Moreover, ν_0 is the largest real root of the quadratic $Q(\nu)$ defined according to the above conditions.

One can see that the functions $N_\nu(z)$ and $N_\nu'(z)$ are entire functions of order zero, then they have infinitely many zeros. According to Hadamard factorization theorem [22]

$$N_\nu(z) = \frac{Q(\nu)z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n\geq 1} \left(1 - \frac{z^2}{\lambda_{\nu,n}^2}\right) \tag{1.6}$$

and

$$N'_\nu(z) = \frac{Q(\nu)z^{\nu-1}\nu}{2^\nu \Gamma(\nu+1)} \prod_{n\geq 1}\left(1 - \frac{z^2}{\lambda'^2_{\nu,n}}\right) \tag{1.7}$$

where $\lambda_{\nu,n}$ and $\lambda'_{\nu,n}$ denote the nth positive zero of $N_\nu(z)$ and $N'_\nu(z)$, respectively. Recently, for studies on the Geometric properties of Bessel functions, see [3, 4, 5, 6, 7, 11, 12, 13, 14, 20, 23, 31].

Recall that Ma and Minda [24] subclasses of starlike and convex functions, respectively, are given by:

$$\mathcal{S}^*(\varphi) = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z)\right\}$$

and

$$\mathcal{C}(\varphi) = \left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)\right\},$$

where φ, a Ma-Minda function, is analytic and univalent with $\operatorname{Re}\varphi(z) > 0$, $\varphi'(0) > 0$ and $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0) = 1$ and symmetric about real axis. Note that $\varphi \in \mathcal{P}$, the class of normalized Carathéodory functions. We also recall that

Definition 2 Let $f \in \mathcal{A}$ be a special function. Then $\mathcal{S}^*(\varphi)-$radius and $\mathcal{C}(\varphi)-$radius of f are defined as follows:

$$r^*_\varphi(f) = \sup\left\{r \in \mathbb{R}^+ : \frac{zf'(z)}{f(z)} \in \varphi(\mathbb{D}),\ z \in \mathbb{D}_r\right\}$$

and

$$r^c_\varphi(f) = \sup\left\{r \in \mathbb{R}^+ : 1 + \frac{zf''(z)}{f'(z)} \in \varphi(\mathbb{D}),\ z \in \mathbb{D}_r\right\},$$

respectively.

In this paper, we deal with the radius of γ-Spirallike of order α and the radius of convex γ-Spirallike of order α for the functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$ defined by (1.3), (1.4) and (1.5) in the case when $\nu \geq \max\{0, \nu_0\}$. Also, we determine the $\mathcal{S}^*(\varphi)-$radii and $\mathcal{C}(\varphi)-$radii of these functions. The key tools in their proofs are some new Mittag-Leffler expansions for quotients of the function N_ν, special properties of the zeros of the function N_ν and their derivatives.

2 Zeros of hyperbolic polynomials and the Laguerre–Pólya class of entire functions

In this section, we recall some necessary information about polynomials and entire functions with real zeros. An algebraic polynomial is called hyperbolic if all its zeros are real. We formulate the following specific statement that we shall need, see [14] for more details.

By definition, a real entire function ψ belongs to the Laguerre–Pólya class $\mathcal{LP}$ if it can be represented in the form

$$\psi(x) = cx^m e^{-ax^2 + \beta x} \prod_{k \geq 1} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}},$$

with $c, \beta, x_k \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N} \cup \{0\}$ and $\sum x_k^{-2} < \infty$. Similarly, ϕ is said to be of type $\mathcal{I}$ in the Laguerre-Pólya class, written $\varphi \in \mathcal{LPI}$, if $\phi(x)$ or $\phi(-x)$ can be represented as

$$\phi(x) = cx^m e^{\sigma x} \prod_{k \geq 1} \left(1 + \frac{x}{x_k}\right),$$

with $c \in \mathbb{R}$, $\sigma \geq 0$, $m \in \mathbb{N} \cup \{0\}$, $x_k > 0$ and $\sum x_k^{-1} < \infty$. The class $\mathcal{LP}$ is the complement of the space of hyperbolic polynomials in the topology induced by the uniform convergence on the compact sets of the complex plane while $\mathcal{LPI}$ is the complement of the hyperbolic polynomials whose zeros possess a preassigned constant sign. Given an entire function φ with the Maclaurin expansion

$$\varphi(x) = \sum_{k \geq 0} \mu_k \frac{x^k}{k!},$$

its Jensen polynomials are defined by

$$P_m(\varphi; x) = P_m(x) = \sum_{k=0}^{m} \binom{m}{k} \mu_k x^k.$$

The next result of Jensen [18] is a well-known characterization of functions belonging to $\mathcal{LP}$.

Lemma 1 *The function φ belongs to $\mathcal{LP}$ ($\mathcal{LPI}$, respectively) if and only if all the polynomials $P_m(\varphi; x)$, $m = 1, 2, ...$, are hyperbolic (hyperbolic with zeros of equal sign). Moreover, the sequence $P_m(\varphi; z/n)$ converges locally uniformly to $\varphi(z)$.*

The following result is a key tool in the proof of main results.

Lemma 2 *[19] If $\nu \geq \max\{0, \nu_0\}$ then the functions $z \longmapsto \Psi_\nu(z) = \frac{2^\nu \Gamma(\nu+1)}{Q(\nu)z^\nu} N_\nu(z)$ has infinitely many zeros and all of them are positive. Denoting by $\lambda_{\nu,n}$ the nth positive zero of $\Psi_\nu(z)$, under the same conditions the Weierstrassian decomposition*

$$\Psi_\nu(z) = \prod_{n \geq 1} \left(1 - \frac{z^2}{\lambda_{\nu,n}^2}\right)$$

is valid, and this product is uniformly convergent on compact subsets of the complex plane. Moreover, if we denote by $\lambda'_{\nu,n}$ the nth positive zero of $\Phi'_\nu(z)$, where $\Phi_\nu(z) = z^\nu \Psi_\nu(z)$, then the positive zeros of $\Psi_\nu(z)$ are interlaced with those of $\Phi'_\nu(z)$. In the other words, the zeros satisfy the chain of inequalities

$$\lambda'_{\nu,1} < \lambda_{\nu,1} < \lambda'_{\nu,2} < \lambda_{\nu,2} < \lambda'_{\nu,3} < \lambda_{\nu,3} < \cdots .$$

3 Main Results

3.1 Radii of $\gamma-$Spirallike and convex $\gamma-$Spirallike of The Functions f_ν, g_ν and h_ν

The first principal result we established concerns the radii of $\gamma-$Spirallike of order α and reads as follows.

Theorem 1 *Let $\alpha \in [0,1)$ and $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The following statements hold:*

a) *If $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$ then the radius of $\gamma-$Spirallike of order α of the function f_ν is the smallest positive root of the equation*

$$\frac{1}{\nu} \frac{ar^3 J_\nu'''(r) + (2a+b)\, r^2 J_\nu''(r) + (b+c)\, rJ_\nu'(r)}{ar^2 J_\nu''(r) + brJ_\nu'(r) + cJ_\nu(r)} = 1 - (1-\alpha)\cos\gamma.$$

b) *If $\nu \geq \max\{0, \nu_0\}$, then the radius of $\gamma-$Spirallike of order α of the function g_ν is the smallest positive root of the equation*

$$\frac{ar^3 J_\nu'''(r) + (2a+b)\, r^2 J_\nu''(r) + (b+c)\, rJ_\nu'(r)}{ar^2 J_\nu''(r) + brJ_\nu'(r) + cJ_\nu(r)} = \nu - (1-\alpha)\cos\gamma.$$

c) *If $\nu \geq \max\{0, \nu_0\}$, then the radius of $\gamma-$Spirallike of order α of the function h_ν is the smallest positive root of the equation*

$$\frac{ar\sqrt{r}J_\nu'''(\sqrt{r}) + (2a+b)\, rJ_\nu''(\sqrt{r}) + (b+c)\, \sqrt{r}J_\nu'(\sqrt{r})}{arJ_\nu''(\sqrt{r}) + b\sqrt{r}J_\nu'(\sqrt{r}) + cJ_\nu(\sqrt{r})} = \nu - 2\,(1-\alpha)\cos\gamma.$$

Proof Firstly, we prove part **a** for $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$ and **b** and **c** for $\nu \geq \max\{0, \nu_0\}$. We need to show that the following inequalities for $\alpha \in [0,1)$ and $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\operatorname{Re}\left(e^{-i\gamma}\frac{zf_\nu'(z)}{f_\nu(z)}\right) > \alpha\cos\gamma, \quad \operatorname{Re}\left(e^{-i\gamma}\frac{zg_\nu'(z)}{g_\nu(z)}\right) > \alpha\cos\gamma \text{ and } \operatorname{Re}\left(e^{-i\gamma}\frac{zh_\nu'(z)}{h_\nu(z)}\right) > \alpha\cos\gamma \tag{3.1}$$

are valid for $z \in \mathbb{D}_{r_{sp}^*(\alpha,\gamma;f_\nu)}$, $z \in \mathbb{D}_{r_{sp}^*(\alpha,\gamma;g_\nu)}$ and $z \in \mathbb{D}_{r_{sp}^*(\alpha,\gamma;h_\nu)}$ respectively, and each of the above inequalities does not hold in larger disks.
a): When we write the equation (1.3) in definition of the function $f_\nu(z)$, we get by using logarithmic derivation

$$\frac{zf_\nu'(z)}{f_\nu(z)} = \frac{1}{\nu}\frac{zN_\nu'(z)}{N_\nu(z)} = 1 - \frac{1}{\nu}\sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}, \quad (\nu \geq \max\{0,\nu_0\}, \ \nu \neq 0), \tag{3.2}$$

It is known [13] that if $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ are such that $|z| \leq r < \lambda$, then

$$\operatorname{Re}\left(\frac{z}{\lambda - z}\right) \leq \left|\frac{z}{\lambda - z}\right| \leq \frac{|z|}{\lambda - |z|}. \tag{3.3}$$

Then the inequality

$$\operatorname{Re}\left(\frac{z^2}{\lambda_{\nu,n}^2 - z^2}\right) \leq \left|\frac{z^2}{\lambda_{\nu,n}^2 - z^2}\right| \leq \frac{|z|^2}{\lambda_{\nu,n}^2 - |z|^2}$$

holds for every $|z| < \lambda_{\nu,1}$. Therefore, from (3.2) and (3.3), we have

$$
\mathrm{Re}\left(e^{-i\gamma}\frac{zf_\nu'(z)}{f_\nu(z)}\right) = \mathrm{Re}\left(e^{-i\gamma}\right) - \frac{1}{\nu}\,\mathrm{Re}\left(e^{-i\gamma}\sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}\right)
$$

$$
\geq \cos\gamma - \frac{1}{\nu}\left|e^{-i\gamma}\sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}\right| \geq \cos\gamma - \frac{1}{\nu}\sum_{n\geq 1}\frac{2\,|z|^2}{\lambda_{\nu,n}^2 - |z|^2}
$$

$$
= \frac{|z|\,f_\nu'(|z|)}{f_\nu(|z|)} + \cos\gamma - 1,
$$

with equality when $z = |z| = r$. Thus, for $r \in (0, \lambda_{\nu,1})$ it follows that

$$
\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(e^{-i\gamma}\frac{zf_\nu'(z)}{f_\nu(z)} - \alpha\cos\gamma\right)\right\} = \frac{|z|\,f_\nu'(|z|)}{f_\nu(|z|)} + (1-\alpha)\cos\gamma - 1.
$$

Now, the mapping $\Theta_\nu : (0, \lambda_{\nu,1}) \longrightarrow \mathbb{R}$ defined by

$$
\Theta_\nu(r) = \frac{rf_\nu'(r)}{f_\nu(r)} + (1-\alpha)\cos\gamma - 1 = (1-\alpha)\cos\gamma - \frac{1}{\nu}\sum_{n\geq 1}\left(\frac{2r^2}{\lambda_{\nu,n}^2 - r^2}\right).
$$

is strictly decreasing since

$$
\Theta_\nu'(r) = -\frac{1}{\nu}\sum_{n\geq 1}\left(\frac{4r\lambda_{\nu,n}^2}{\left(\lambda_{\nu,n}^2 - r^2\right)^2}\right) < 0
$$

for all $\nu \geq \max\{0, \nu_0\}, \nu \neq 0$. On the other hand, since

$$
\lim_{r\searrow 0}\Theta_\nu(r) = (1-\alpha)\cos\gamma > 0 \quad\text{and}\quad \lim_{r\nearrow\lambda_{\nu,1}}\Theta_\nu(r) = -\infty,
$$

in view of the minimum principle for harmonic functions imply that the corresponding inequality in (3.1) for $\nu \geq \max\{0, \nu_0\}, \nu \neq 0$ holds if and only if $z \in \mathbb{D}_{r_f}$, where r_f is the smallest positive root of equation

$$
\frac{rf_\nu'(r)}{f_\nu(r)} = 1 - (1-\alpha)\cos\gamma
$$

which is equivalent to

$$
\frac{rN_\nu'(r)}{\nu N_\nu(r)} = 1 - (1-\alpha)\cos\gamma,
$$

situated in $(0, \lambda_{\nu,1})$.
b): Since,

$$
\frac{zg_\nu'(z)}{g_\nu(z)} = (1-\nu) + \frac{zN_\nu'(z)}{N_\nu(z)} = 1 - \sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}, \quad (\nu \geq \max\{0, \nu_0\}). \quad (3.4)
$$

By using the inequality (3.3), for all $z \in \mathbb{D}_{\lambda_{\nu,1}}$ we get the inequality

$$
\begin{aligned}
\mathrm{Re}\left(e^{-i\gamma}\frac{zg_\nu'(z)}{g_\nu(z)}\right) &= \mathrm{Re}\left(e^{-i\gamma}\right) - \mathrm{Re}\left(e^{-i\gamma}\sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}\right) \\
&\geq \cos\gamma - \left|e^{-i\gamma}\sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}\right| \geq \cos\gamma - \sum_{n\geq 1}\frac{2\left|z\right|^2}{\lambda_{\nu,n}^2 - \left|z\right|^2} \\
&= \frac{rg_\nu'(r)}{g_\nu(r)} + \cos\gamma - 1,
\end{aligned}
$$

where $|z| = r$. Thus, for $r \in (0, \lambda_{\nu,1})$ we obtain

$$
\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(e^{-i\gamma}\frac{zg_\nu'(z)}{g_\nu(z)} - \alpha\cos\gamma\right)\right\} = \frac{rg_\nu'(r)}{g_\nu(r)} + (1-\alpha)\cos\gamma - 1.
$$

The function $\Phi_\nu : (0, \lambda_{\nu,1}) \longrightarrow \mathbb{R}$ defined by

$$
\Phi_\nu(r) = \frac{rg_\nu'(r)}{g_\nu(r)} + (1-\alpha)\cos\gamma - 1 = (1-\alpha)\cos\gamma - \sum_{n\geq 1}\left(\frac{2r^2}{\lambda_{\nu,n}^2 - r^2}\right).
$$

is strictly decreasing and $\lim_{r\searrow 0}\Phi_\nu(r) = (1-\alpha)\cos\gamma > 0$ and $\lim_{r\nearrow\lambda_{\nu,1}}\Phi_\nu(r) = -\infty$.

Consequently, in view of the minimum principle for harmonic functions for $z \in \mathbb{D}_{r_g}$, we have that

$$
\mathrm{Re}\left(e^{-i\gamma}\frac{zg_\nu'(z)}{g_\nu(z)}\right) > \alpha\cos\gamma
$$

if and only if r_g is the smallest positive root of equation

$$
\frac{rg_\nu'(r)}{g_\nu(r)} = 1 - (1-\alpha)\cos\gamma \ \text{ or } \ \frac{rN_\nu'(r)}{N_\nu(r)} = \nu - (1-\alpha)\cos\gamma
$$

situated in $(0, \lambda_{\nu,1})$.

c): Observe that

$$
\frac{zh_\nu'(z)}{h_\nu(z)} = (1 - \frac{\nu}{2}) + \frac{1}{2}\frac{\sqrt{z}N_\nu'(\sqrt{z})}{N_\nu(\sqrt{z})} = 1 - \sum_{n\geq 1}\frac{z}{\lambda_{\nu,n}^2 - z}, \quad (\nu \geq \max\{0, \nu_0\}),
\tag{3.5}
$$

By using the inequality (3.2), for all $z \in \mathbb{D}_{\lambda_{\nu,1}^2}$ we obtain

$$
\begin{aligned}
\mathrm{Re}\left(e^{-i\gamma}\frac{zh_\nu'(z)}{h_\nu(z)}\right) &= \mathrm{Re}\left(e^{-i\gamma}\right) - \mathrm{Re}\left(e^{-i\gamma}\sum_{n\geq 1}\frac{z}{\lambda_{\nu,n}^2 - z}\right) \\
&\geq \cos\gamma - \left|e^{-i\gamma}\sum_{n\geq 1}\frac{z}{\lambda_{\nu,n}^2 - z}\right| \geq \cos\gamma - \sum_{n\geq 1}\frac{2\left|z\right|}{\lambda_{\nu,n}^2 - \left|z\right|} \\
&= \frac{rh_\nu'(r)}{h_\nu(r)} + \cos\gamma - 1,
\end{aligned}
$$

Equality holds for $z = |z| = r$. As a result, for $r \in (0, \lambda_{\nu,1})$ it follows that

$$\inf_{z \in \mathbb{D}_r} \left\{ \operatorname{Re} \left(e^{-i\gamma} \frac{z h'_\nu(z)}{h_\nu(z)} - \alpha \cos \gamma \right) \right\} = \frac{r h'_\nu(r)}{h_\nu(r)} + (1 - \alpha) \cos \gamma - 1.$$

The fucntion $\Psi_\nu : \left(0, \lambda^2_{\nu,1}\right) \longrightarrow \mathbb{R}$ defined by

$$\Psi_\nu(r) = \frac{r h'_\nu(r)}{h_\nu(r)} + (1 - \alpha) \cos \gamma - 1 = (1 - \alpha) \cos \gamma - \sum_{n \geq 1} \left(\frac{2r}{\lambda^2_{\nu,n} - r} \right).$$

is strictly decreasing, since

$$\Psi'_\nu(r) = - \sum_{n \geq 1} \left(\frac{\lambda^2_{\nu,n}}{\left(\lambda^2_{\nu,n} - r\right)^2} \right) < 0$$

for all $\nu \geq \max\{0, \nu_0\}$. On the other hand, since

$$\lim_{r \searrow 0} \Psi_\nu(r) = (1 - \alpha) \cos \gamma > 0 \quad \text{and} \quad \lim_{r \nearrow \lambda^2_{\nu,1}} \Psi_\nu(r) = -\infty$$

in view of the minimum principle for harmonic functions imply that the corresponding inequality in (3.1) holds if and only if $z \in \mathbb{D}_{r_h}$, where r_h is the smallest positive root of equation

$$\frac{r h'_\nu(r)}{h_\nu(r)} = 1 - (1 - \alpha) \cos \gamma$$

which is equivalent to

$$\frac{\sqrt{r} N'_\nu(\sqrt{r})}{N_\nu(\sqrt{r})} = 1 - (1 - \alpha) \cos \gamma$$

and situated in $\left(0, \lambda^2_{\nu,1}\right)$. This completes the proof of part **a** when $\nu \geq \max\{0, \nu_0\}, \nu \neq 0$, and parts **b** and **c** when $\nu \geq \max\{0, \nu_0\}$. $\square$

Remark 1 In Theorem 1, the choice of $\gamma = 0$ yields the result for the class of starlike functions of order α.

Example 1 Indeed, we can write the function $N_\nu(z)$ in terms of elementary trigonometric functions for the value $\nu = 1/2$ as follows:

$$N_{1/2}(z) = \frac{4(b-a)z \cos z + \left[a\left(3 - 4z^2\right) - 2b + 4c\right] \sin z}{2\sqrt{2\pi}\sqrt{z}}.$$

Thus, we have

$$f_{1/2}(z) = \frac{\left[4(a-b)z \cos z + \left(4az^2 - 3a + 2b - 4c\right) \sin z\right]^2}{(a - 2b - 4c)^2 z},$$

$$g_{1/2}(z) = \frac{4(a-b)z \cos z + \left(4az^2 - 3a + 2b - 4c\right) \sin z}{a - 2b - 4c}$$

and

$$h_{1/2}(z) = \frac{4(a-b)z \cos \sqrt{z} + \left(4az - 3a + 2b - 4c\right)\sqrt{z} \sin \sqrt{z}}{a - 2b - 4c}.$$

	$b = 3$ and $c = 0$			$a = 1$ and $c = 0$			$a = 1$ and $b = 2$		
	$a = 2$	$a = 3$	$a = 4$	$b = 2$	$b = 3$	$b = 4$	$c = 2$	$c = 3$	$c = 4$
$R\left(f_{\frac{1}{2}}\right)$	0.1539	0.1190	0.0886	0.1746	0.1990	0.2130	0.3038	0.3401	0.3680
$R\left(g_{\frac{1}{2}}\right)$	0.2121	0.1639	0.1220	0.2409	0.2747	0.2942	0.4217	0.4730	0.5126
$R\left(g_{\frac{1}{2}}\right)$	0.0817	0.0486	0.0268	0.1056	0.1377	0.1582	0.3309	0.4193	0.4953

Table 1 $R(--)$: Radii of $\frac{\pi}{3}-$Spirallike of order $1/2$ for $f_{1/2}$, $g_{1/2}$ and $h_{1/2}$.

Taking $\nu = 1/2$, $a = 1$, $b = 2$, $c = 0$, $\alpha = 1/2$ and $\gamma = \pi/3$ in Theorem 1, we have following results.

Corollary 1 *The following statements are true.*

a) *The radius of $\frac{\pi}{3}-$Spirallike of order $1/2$ of the function $f_{1/2}$ is the smallest positive root of the equation*

$$\frac{4x\,(8x+1)\cos x + \left(68x^2 - 7\right)\sin x}{4\left(-4x\cos x + \left(4x^2 + 1\right)\sin x\right)} = 0$$
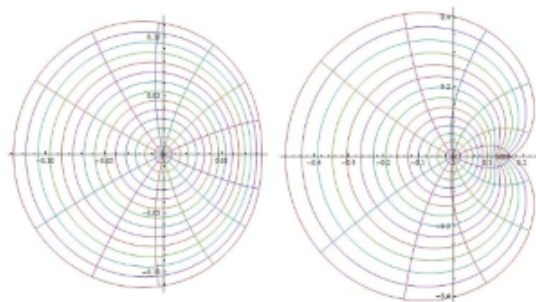
b) *The radius of $\frac{\pi}{3}-$Spirallike of order $1/2$ of the function $g_{1/2}$ is the smallest positive root of the equation*

$$\frac{16x^3\cos x + 3\left(12x^2 - 1\right)\sin x}{4\left(-4x\cos x + \left(4x^2 + 1\right)\sin x\right)} = 0$$

c) *The radius of $\frac{\pi}{3}-$Spirallike of order $1/2$ of the function $h_{1/2}$ is the smallest positive root of the equation*

$$\frac{2\sqrt{x}\,(4x-1)\cos\sqrt{x} + (20x-1)\sin\sqrt{x}}{2\left(-4\sqrt{x}\cos\sqrt{x} + (4x+1)\sin\sqrt{x}\right)} = 0.$$

The figure 1 was created by taking the special values in the Corollary 1.

Fig. 1 Images of function $h_{1/2}(z)$ for $r = 0.1056$ and $r = 0.3$, respectively as per Corollary 1.

Remark 2 When $\nu = 0.5$, considering the special values of a, b and c real numbers, radii of $\gamma-$Spirallike of order α of the function f_ν is seen from the table above. In addition, according to the increasing values of a, b and c reel numbers, it is clear that radii of $\gamma-$Spirallike of order α of the function f_ν is monotone.

In the following, we deal with convex analogue of the class of γ-spiralllike functions of order α.

Theorem 2 *Let $\alpha \in [0,1)$ and $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The following statements hold:*

a) If $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$ then the radius of the convex $\gamma-$Spirallike of order α of the function f_ν is the smallest positive root of the equation

$$\frac{rN_\nu''(r)}{N_\nu'(r)} + \left(\frac{1}{\nu} - 1\right)\frac{rN_\nu'(r)}{N_\nu(r)} = (\alpha - 1)\cos\gamma.$$

b) If $\nu \geq \max\{0, \nu_0\}$, then the radius of the convex $\gamma-$Spirallike of order α of the function g_ν is the smallest positive root of the equation

$$\frac{r^2 N_\nu''(r) + (2 - 2\nu)\, rN_\nu'(r) + \left(\nu^2 - \nu\right) N_\nu(r)}{rN_\nu'(r) + (1 - \nu)\, N_\nu(r)} = (\alpha - 1)\cos\gamma.$$

c) If $\nu \geq \max\{0, \nu_0\}$, then the radius of the convex $\gamma-$Spirallike of order α of the function h_ν is the smallest positive root of the equation

$$\frac{rN_\nu''(\sqrt{r}) + (3 - 2\nu)\sqrt{r}N_\nu'(\sqrt{r}) + \left(\nu^2 - 2\nu\right) N_\nu(\sqrt{r})}{2\sqrt{r}N_\nu'(\sqrt{r}) + 2\,(2 - \nu)\, N_\nu(\sqrt{r})} = (\alpha - 1)\cos\gamma.$$

Proof We first prove the part (a). From (1.3), we have

$$1 + \frac{zf_\nu''(z)}{f_\nu'(z)} = 1 + \frac{zN_\nu''(z)}{N_\nu'(z)} + \left(\frac{1}{\nu} - 1\right)\frac{zN_\nu'(z)}{N_\nu(z)}$$

and by means of (1.6) and (1.7), we obtain

$$1 + \frac{zf_\nu''(z)}{f_\nu'(z)} = 1 - \left(\frac{1}{\nu} - 1\right)\sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2} - \sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}'^2 - z^2}.$$

For $1 \geq \nu > \max\{0, \nu_0\}$ by using (3.3), we get

$$\mathrm{Re}\left(e^{-i\gamma}\left(1 + \frac{zf_\nu''(z)}{f_\nu'(z)}\right)\right)$$

$$= \mathrm{Re}(e^{-i\gamma}) - \mathrm{Re}\left(e^{-i\gamma}\left(\sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}'^2 - z^2} + \left(\frac{1}{\nu} - 1\right)\sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}\right)\right)$$

$$\geq \cos\gamma - \left(\frac{1}{\nu} - 1\right)\sum_{n\geq 1}\frac{2r^2}{\lambda_{\nu,n}^2 - r^2} - \sum_{n\geq 1}\frac{2r^2}{\lambda_{\nu,n}'^2 - r^2}$$

$$\geq \cos\gamma + \frac{rf_\nu''(r)}{f_\nu'(r)} \tag{3.6}$$

	$b = 3$ and $c = 0$			$a = 1$ and $c = 0$			$a = 1$ and $b = 2$		
	$a = 2$	$a = 3$	$a = 4$	$b = 2$	$b = 3$	$b = 4$	$c = 2$	$c = 3$	$c = 4$
$R^c\left(f_{\frac{1}{2}}\right)$	0.0875	0.0677	0.0504	0.0993	0.1131	0.1210	0.1722	0.1925	0.2082
$R^c\left(g_{\frac{1}{2}}\right)$	0.1221	0.0944	0.0703	0.1386	0.1579	0.1691	0.2411	0.2699	0.2921
$R^c\left(h_{\frac{1}{2}}\right)$	0.0406	0.0242	0.0134	0.0524	0.0682	0.0783	0.1622	0.2046	0.2408

Table 2 Radii of the convex $\frac{\pi}{3}-$Spirallike of order $1/2$ for $f_{1/2}$, $g_{1/2}$ and $h_{1/2}$

where $|z| = r$. Moreover, observe that if we use the inequality [4, Lemma 2.1]

$$\mu \operatorname{Re}\left(\frac{z}{a - z}\right) - \operatorname{Re}\left(\frac{z}{b - z}\right) \geq \mu \frac{|z|}{a - |z|} - \frac{|z|}{b - |z|}$$

where $a > b > 0$, $\mu \in [0, 1]$ and $z \in \mathbb{C}$ such that $|z| < b$, then we get that the inequality (3.6) is also valid when $\nu \geq 1$. Here we used that the zeros of N_ν and N'_ν are interlacing according to Lemma 2. The above inequality implies for $r \in (0, \lambda'_{\nu,1})$

$$\inf_{z \in \mathbb{D}_r}\left\{\operatorname{Re}\left(e^{-i\gamma}\left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)}\right) - \alpha \cos\gamma\right)\right\} = (1 - \alpha)\cos\gamma + \frac{rf''_\nu(r)}{f'_\nu(r)}.$$

Now, the proof of part (a) follows on similar lines as of Theorem 1.

For the other parts, note that the functions g_ν and h_ν belong to the Laguerre-Pólya class $\mathcal{LP}$, which is closed under differentiation, their derivatives g'_ν and h'_ν also belong to $\mathcal{LP}$ and the zeros are real. Thus assuming $\delta_{\nu,n}$ and $\gamma_{\nu,n}$ are the positive zeros of g'_ν and h'_ν, respectively, we have the following representations:

$$g'_\nu(z) = \prod_{n \geq 1}\left(1 - \frac{z^2}{\delta^2_{\nu,n}}\right) \quad \text{and} \quad h'_\nu(z) = \prod_{n \geq 1}\left(1 - \frac{z}{\gamma_{\nu,n}}\right),$$

which yield

$$1 + \frac{zg''_\nu(z)}{g'_\nu(z)} = 1 - \sum_{n \geq 1}\frac{2z^2}{\delta^2_{\nu,n} - z^2} \quad \text{and} \quad 1 + \frac{zh''_\nu(z)}{h'_\nu(z)} = 1 - \sum_{n \geq 1}\frac{z}{\gamma_{\nu,n} - z}.$$

Further, reasoning along the same lines as in Theorem 1, the result follows at once. □

Remark 3 In Theorem 2, the choice of $\gamma = 0$ yields the result for the class of convex functions of order α.

3.2 Radii of Ma-Minda Starlikeness and Convexity of the Functions f_ν, g_ν and h_ν

In the following, now we establish the result for the complete family of Ma-Minda class of starlike functions.

Theorem 3 *Let $\nu \geq \max\{0, \nu_0\}$. The $\mathcal{S}^*(\varphi)$-radii of the functions f_ν, g_ν and h_ν are respectively, given by the smallest positive root of the equations*

a) $rN'_\nu(r) - (1 - \beta)\nu N_\nu(r) = 0$ *with $\nu \neq 0$;*
b) $rN'_\nu(r) - (\nu - \beta)N_\nu(r) = 0$;
c) $\sqrt{r}N'_\nu(\sqrt{r}) - (\nu - 2\beta)N_\nu(\sqrt{r}) = 0$

in $|z| < (0, \lambda_{\nu,1})$, $(0, \lambda_{\nu,1})$ and $(0, \lambda_{\nu,1}^2)$, where $\varphi(-1) = 1 - \beta$ and β is the radius of the largest disk $\{w : |w - 1| < \beta\} \subseteq \varphi(\mathbb{D})$.

Proof We first prove part $a)$: from equation (3.2), we have

$$\frac{zf'_\nu(z)}{f_\nu(z)} = \frac{1}{\nu}\frac{zN'_\nu(z)}{N_\nu(z)} = 1 - \frac{1}{\nu}\sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}, \quad (\nu \geq \max\{0, \nu_0\}, \; \nu \neq 0). \quad (3.7)$$

Let us consider the continuous function $T_{f_\nu} : (0, \lambda_{\nu,1}) \to \mathbb{R}$

$$T_{f_\nu}(r) = \frac{1}{\nu}\sum_{n\geq 1}\frac{2r^2}{\lambda_{\nu,n}^2 - r^2} - \beta. \quad (3.8)$$

Then

$$T'_{f_\nu}(r) = \frac{1}{\nu}\sum_{n\geq 1}\left(\frac{4r\lambda_{\nu,n}^2}{(\lambda_{\nu,n}^2 - r^2)^2}\right) > 0$$

for all $\nu \geq \max\{0, \nu_0\} \neq 0$ and for $r < \lambda_{\nu,1}$. Also, $T_{f_\nu}(0) = -\beta < 0$ and $\lim_{r \nearrow \lambda_{\nu,1}} T_{f_\nu}(r) = \infty$. This implies that there exist a unique positive root, say $R_\varphi(f_\nu)$ of the equation $T_{f_\nu}(r) = 0$ in $(0, \lambda_{\nu,1})$. Now let $\{w : |w-1| < \beta\} \subseteq \varphi(\mathbb{D})$ such that $\varphi(-1) = 1 - \beta$. Therefore, using the Lemma [13] that if $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ are such that $|z| \leq r < \lambda$, then

$$\text{Re}\left(\frac{z}{\lambda - z}\right) \leq \left|\frac{z}{\lambda - z}\right| \leq \frac{|z|}{\lambda - |z|},$$

in (3.7), we get that

$$\left|\frac{zf'_\nu(z)}{f_\nu(z)} - 1\right| = \left|\frac{1}{\nu}\sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}\right| \leq \frac{1}{\nu}\sum_{n\geq 1}\frac{2r^2}{\lambda_{\nu,n}^2 - r^2} \leq \beta \quad (3.9)$$

implies that $f_\nu \in \mathcal{S}^*(\varphi)$ in $|z| < R_\varphi(f_\nu)$. Further, taking $z \in \mathbb{D}$ such that $z = r = -R_\varphi(f_\nu)$, from (3.8) and (3.9), it follows that

$$\left|\frac{zf'_\nu(z)}{f_\nu(z)} - 1\right| = \beta,$$

which implies that for $zf'_\nu(z)/f_\nu(z) \notin \varphi(\mathbb{D})$ for all $|z| = r \geq R_\varphi(f_\nu)$, which proves the sharpness part. Reasoning along the same lines, proofs of the parts $b)$ and $c)$ follow. $\qquad\square$

| | $b = 3$ and $c = 0$ | | | $a = 1$ and $c = 0$ | | | $a = 1$ and $b = 2$ | | |
	$a = 2$	$a = 3$	$a = 4$	$b = 2$	$b = 3$	$b = 4$	$c = 2$	$c = 3$	$c = 4$
$r_{e^z}^*\left(f_{\frac{1}{2}}\right)$	0.2354	0.1818	0.1352	0.2674	0.3050	0.3268	0.4696	0.5272	0.5718
$r_{e^z}^*\left(g_{\frac{1}{2}}\right)$	0.3140	0.2421	0.1798	0.3571	0.4078	0.4372	0.6350	0.7160	0.7792
$r_{e^z}^*\left(h_{\frac{1}{2}}\right)$	0.1611	0.0953	0.0523	0.2090	0.2736	0.3150	0.6845	0.8819	1.0559

Table 3 Radii of $\mathcal{S}^*(e^z)$ for the functions $f_{1/2}$, $g_{1/2}$ and $h_{1/2}$

The figure 2 was created by taking $a = 1$, $b = 2$, $c = 0$ and $\varphi(z) = e^z$ in the Theorem 3.
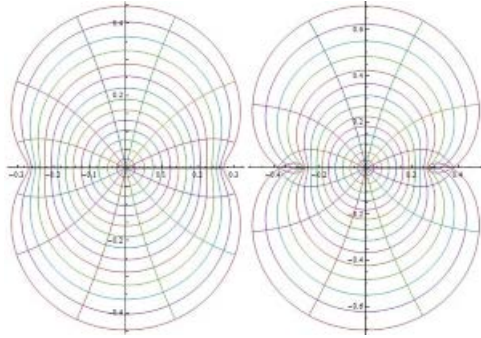
Fig. 2 Images of function $g_{1/2}(z)$ for $r = 0.3571$ and $r = 0.48$, respectively as per Theorem 3.

Remark 4 Theorem 3 with $\varphi(z) = (1 + (1 - 2\beta)z)/(1 - z)$ coincide with Theorem 1 with $\alpha = \beta$ and $\gamma = 0$. In particular, choosing $\varphi(z) = (1 + z)/(1 - z)$ and $\beta = 1$ in Theorem 3 reduces to Theorem 1 with $\alpha = 0$ and $\gamma = 0$.

As a consequence of Theorem 3, we get the sufficient conditions for the functions to belong in the class $\mathcal{S}^*(\varphi)$.

Corollary 2 *Let $\nu \geq \max\{0, \nu_0\}$. Further, $\varphi(-1) = 1 - \beta$ and β is the radius of the largest disk $\{w : |w - 1| < \beta\} \subseteq \varphi(\mathbb{D})$. Then*

(i) *if $1 - \frac{1}{\nu}\frac{N'_\nu(1)}{N_\nu(1)} < \beta$ and $\nu \neq 0$, then the function f_ν belongs to $\mathcal{S}^*(\varphi)$.*

(ii) *if $\nu - \frac{N'_\nu(1)}{N_\nu(1)} < \beta$, then the function g_ν belongs to $\mathcal{S}^*(\varphi)$.*

(iii) *if $\frac{\nu}{2} - \frac{N'_\nu(1)}{2N_\nu(1)} < \beta$, then the function h_ν belongs to $\mathcal{S}^*(\varphi)$.*

In the following, we prove the result for the Ma-Minda class of convex functions.

Theorem 4 *The $\mathcal{C}(\varphi)$-radii of the functions f_ν, g_ν and h_ν are respectively, given by the smallest positive root of the equations*

a) $r f_\nu''(r) + \beta \nu f_\nu'(r) = 0$, where $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$,
b) $r g_\nu''(r) + \beta g_\nu'(r) = 0$, where $\nu \geq \max\{0, \nu_0\}$,
c) $\sqrt{r} h_\nu''(\sqrt{r}) + \beta h_\nu'(\sqrt{r}) = 0$, where $\nu \geq \max\{0, \nu_0\}$

in $|z| < (0, \lambda_{\nu,1})$, $(0, \lambda_{\nu,1})$ and $(0, \lambda_{\nu,1}^2)$, where $\varphi(-1) = 1 - \beta$ and β is the radius of the largest disk $\{w : |w - 1| < \beta\} \subseteq \varphi(\mathbb{D})$.

Proof From (3.7), it follows that

$$\frac{z f_\nu''(z)}{f_\nu'(z)} = \frac{z N_\nu''(z)}{N_\nu'(z)} + \left(\frac{1}{\nu} - 1\right) \frac{z N_\nu'(z)}{N_\nu(z)}. \tag{3.10}$$

Now using the Weierstrass decompositions of the functions N_ν and N_ν', which are given below

$$N_\nu(z) = \frac{Q(\nu) z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n \geq 1} \left(1 - \frac{z^2}{\lambda_{\nu,n}^2}\right)$$

and

$$N_\nu'(z) = \frac{Q(\nu) \nu z^{\nu-1}}{2^\nu \Gamma(\nu+1)} \prod_{n \geq 1} \left(1 - \frac{z^2}{\tilde{\lambda}_{\nu,n}^2}\right),$$

where $\lambda_{\nu,n}$ and $\tilde{\lambda}_{\nu,n}$ are the nth positive roots of N_ν and N_ν', we have from (3.10) that

$$1 + \frac{z f_\nu''(z)}{f_\nu'(z)} = 1 - \left(\frac{1}{\nu} - 1\right) \sum_{n \geq 1} \frac{2 z^2}{\lambda_{\nu,n}^2 - z^2} - \sum_{n \geq 1} \frac{2 z^2}{\tilde{\lambda}_{\nu,n}^2 - z^2}. \tag{3.11}$$

Using the Lemma [13] in (3.11) that if $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ are such that $|z| \leq r < \lambda$, then

$$\text{Re}\left(\frac{z}{\lambda - z}\right) \leq \left|\frac{z}{\lambda - z}\right| \leq \frac{|z|}{\lambda - |z|},$$

it follows that the inequalities

$$\left|\frac{z f_\nu''(z)}{f_\nu'(z)}\right| \leq \left(\frac{1}{\nu} - 1\right) \sum \frac{2 r^2}{\lambda_{\nu,n}^2 - r^2} + \sum_{n \geq 1} \frac{2 r^2}{\tilde{\lambda}_{\nu,n}^2 - r^2} = -\frac{r f_\nu''(r)}{f_\nu'(r)}$$

hold in $|z| = r < \tilde{\lambda}_{\nu,1}$ for $\nu \leq 1$. Observe that the inequality $|z f_\nu''(z)/f_\nu'(z)| \leq -r f_\nu''(r)/f_\nu'(r)$ also holds for $\nu > 1$ in $|z| = r < \tilde{\lambda}_{\nu,1}$. Let $R_\varphi^c(f_\nu)$ be the smallest positive root of the equation

$$\frac{r f_\nu''(r)}{f_\nu'(r)} + \beta = 0.$$

Now let $\{w : |w - 1| < \beta\} \subseteq \varphi(\mathbb{D})$ such that $\varphi(-1) = 1 - \beta$. Then it follows that

$$\left|\frac{z f_\nu''(z)}{f_\nu'(z)}\right| \leq -\frac{r f_\nu''(r)}{f_\nu'(r)} \leq \beta$$

	$b = 3$ and $c = 0$			$a = 1$ and $c = 0$			$a = 1$ and $b = 2$		
	$a = 2$	$a = 3$	$a = 4$	$b = 2$	$b = 3$	$b = 4$	$c = 2$	$c = 3$	$c = 4$
$r_\varphi^c(f_{\frac{1}{2}})$	0.1271	0.0983	0.0732	0.1443	0.1644	0.1761	0.2512	0.2812	0.3044
$r_\varphi^c(g_{\frac{1}{2}})$	0.1749	0.1352	0.1005	0.1987	0.2266	0.2427	0.3482	0.3906	0.4234
$r_\varphi^c(h_{\frac{1}{2}})$	0.0759	0.0451	0.0248	0.0983	0.1283	0.1474	0.3124	0.3979	0.4719

Table 4 Radii of $\mathcal{C}(\varphi)$ for the functions $f_{1/2}$, $g_{1/2}$ and $h_{1/2}$ with $\varphi(z) = z + \sqrt{1 + z^2}$

hold in $|z| = r < R_\varphi^c(f_{f_\nu})$ for $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$, which implies that the function $f_\nu \in \mathcal{C}(\varphi)$ in $|z| = r < R_\varphi^c(f_{f_\nu})$ for $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$. For the sharpness, let us consider $z = -R_\varphi^c(f_\nu)$. Then

$$\left| \frac{z f_\nu''(z)}{f_\nu'(z)} \right| = -\frac{r f_\nu''(r)}{f_\nu'(r)} = \beta$$

such that $1 + z f_\nu''(z)/f_\nu'(z) \notin \varphi(\mathbb{D})$ for all $|z| = r \geq R_\varphi^c(f_\nu)$, which proves the sharpness part. Reasoning along the same lines, proofs of the parts b) and c) follow. $\qquad\square$

The following corollary yields sufficient conditions on parameters for functions to be in $\mathcal{C}(\varphi)$.

Corollary 3 *Let $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$. Let $\varphi(-1) = 1 - \beta$ and β is the radius of the largest disk $\{w : |w - 1| < \beta\} \subseteq \varphi(\mathbb{D})$. Then*

(i) The function f_ν belongs to $\mathcal{C}(\varphi)$, if

$$\left(1 - \frac{1}{\nu}\right) \frac{N_\nu'(1)}{N_\nu(1)} - \frac{N_\nu''(1)}{N_\nu'(1)} < \beta \quad and \quad 1 \geq \nu \geq \max\{0, \nu_0\},\ \nu \neq 0 \quad and \quad \nu > 1.$$

(ii) The function g_ν belongs to $\mathcal{C}(\varphi)$, if

$$\frac{N_\nu''(1) + 2(1 - \nu)N_\nu'(1) + (\nu^2 + \nu)N_\nu(1)}{(1 - \nu)N_\nu(1) - N_\nu'(1)} < \beta$$

(iii) The function h_ν belongs to $\mathcal{C}(\varphi)$, if

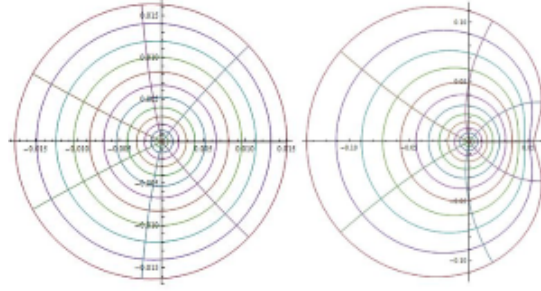$$\frac{1}{2} \frac{N_\nu''(1) + (3 - 2\nu)N_\nu'(1) + (\nu^2 - 2\nu)N_\nu(1)}{(\nu - 2)N_\nu(1) - N_\nu'(1)} < \beta.$$

The figure 3 was created by taking $\nu = 1/2$, $a = 2$, $b = 3$, $c = 0$ and $\varphi(z) = z + \sqrt{1 + z^2}$ in Theorem 4.

Remark 5 Let β be the radius of the lagest disk $\{w : |w - 1| < \beta\}$ inside $\varphi(\mathbb{D})$. Then Theorem 3 and Theorem 4 hold for each of the following cases and the radii are sharp, where

(i) $\beta = \min\left\{\left|1 - \frac{1+A}{1+B}\right|, \left|1 - \frac{1-A}{1-B}\right|\right\} = \frac{A-B}{1+|B|}$ when $\varphi(z) = \frac{1+Az}{1+Bz}$, where $-1 \leq B < A \leq 1$;

Fig. 3 Images of function $h_{1/2}(z)$ for $r = 0.1271$ and $r = 0.3$, respectively as per Theorem 4 and Remark 5.

(ii) $\beta = \sqrt{2 - 2\sqrt{2} + \sqrt{-2 + 2\sqrt{2}}}$ when $\varphi(z) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$;

(iii) $\beta = \sqrt{2} - 1$ when $\varphi(z) = \sqrt{1+z}$;

(iv) $\beta = 1 - 1/e$ when $\varphi(z) = e^z$;

(v) $\beta = 2 - \sqrt{2}$ when $\varphi(z) = z + \sqrt{1+z^2}$;

(vi) $\beta = \frac{e-1}{e+1}$ when $\varphi(z) = \frac{2}{1+e^{-z}}$;

(vii) $\beta = \sin 1$ when $\varphi(z) = 1 + \sin z$;

$(viii)$ $\beta = 1 - e^{e^{-1}-1}$ when $\varphi(z) = e^{e^z - 1}$;

(ix) for the domains bounded by the conic sections $\Omega_\kappa := \{w = u + iv : u^2 > \kappa^2(u-1)^2 + \kappa^2 v^2; \kappa \in [0, \infty)\}$, we have

$$\beta = \frac{1}{\kappa + 1},$$

where the boundary curve of Ω_κ for fixed κ is represented by the imaginary axis ($\kappa = 0$), the right branch of a hyperbola ($0 < \kappa < 1$), a parabola ($\kappa = 1$) and an ellipse ($\kappa > 1$). The univalent Carathéodory functions mapping $\mathbb{D}$ onto Ω_κ is given by

$$\varphi(z) := \varphi_\kappa(z) = \begin{cases} \frac{1+z}{1-z} & \text{for } \kappa = 0; \\ 1 + \frac{2}{1-\kappa^2} \sinh^2(A(\kappa) \, arctanh\sqrt{z}) & \text{for } \kappa \in (0,1); \\ 1 + \frac{2}{\pi^2} \log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}} & \text{for } \kappa = 1; \\ 1 + \frac{2}{\kappa^2 - 1} \sin^2\left(\frac{\pi}{2K(t)} F\left(\frac{\sqrt{z}}{\sqrt{t}}, t\right)\right) & \text{for } \kappa > 1, \end{cases}$$

where $A(\kappa) = (2/\pi)\arccos(\kappa)$, $F(w,t) = \int_0^w \frac{dx}{\sqrt{(1-x^2)(1-t^2 x^2)}}$ is the Legender elliptic integral of the first kind, $K(t) = F(1,t)$ and $t \in (0,1)$ is choosen such that $\kappa = \cosh(\pi K'(t)/2K(t))$.

Statements and Declarations

– **Conflict of interest**: The authors declare that they have no conflict of interest
– **Availability of data and materials** : NA
– **Code availability**: NA
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