

Hodge-Riemann property of Griffiths positive matrices of $(1, 1)$ forms

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Abstract

The classical Hard Lefschetz theorem (HLT), Hodge-Riemann bilinear relation theorem (HRR) and Lefschetz decomposition theorem (LD) are stated for a power of a Kähler class on a compact Kähler manifold. These theorems are not true for an arbitrary class, even if it contains a smooth strictly positive representative. Explicit counterexamples of bidegree $(2, 2)$ classes in dimension 4 can be found in Timorin (1998) and Berndtsson-Sibony (2002).

Gromov (1990), Timorin (1998), Dinh-Nguyên (2006, 2013) proved the mixed HLT, HRR, LD for a product of arbitrary Kähler classes. Instead of products, Dinh-Nguyên (2013) conjectured that determinants of Griffiths positive $k \times k$ matrices with $(1, 1)$ form entries in \mathbb{C}^n satisfies these theorems in the linear case.

This paper proves that Dinh-Nguyên's conjecture holds for $k = 2$ and $n = 2, 3$. Moreover, assume that the matrix only has diagonalized entries, for $k = 2$ and $n \geq 4$, the determinant satisfies HLT for bidegrees $(n - 2, 0)$, $(n - 3, 1)$, $(1, n - 3)$ and $(0, n - 2)$. In particular, Dinh-Nguyên's conjecture is true for $k = 2$ and $n = 4, 5$ with this extra assumption.

Keywords: Griffiths positivity, Hard Lefschetz theorem, Lefschetz decomposition theorem, Hodge-Riemann bilinear relation theorem, Hodge index theorem, compact Kähler manifold.

1 Introduction

Let X be a compact Kähler manifold of dimension n and let ω be a Kähler form. The cohomology of X satisfies the Hodge-decomposition:

$$H^d(X, \mathbb{C}) = \bigoplus_{p+q=d} H^{p,q}(X, \mathbb{C}), \quad (0 \leq d \leq n); \quad \overline{H^{p,q}(X, \mathbb{C})} = H^{q,p}(X, \mathbb{C});$$

where $H^{p,q}(X, \mathbb{C})$ is the Hodge cohomology group of bidegree (p, q) of X with the convention that $H^{p,q}(X, \mathbb{C}) = 0$ unless $0 \leq p, q \leq n$. When $p, q \geq 0$ and $p + q \leq n$, let $k = n - p - q$ and define $\Omega := \omega^{n-p-q}$ a (k, k) form on X .

Recall the classical hard Lefschetz theorem (HLT), the Hodge-Riemann bilinear relation theorem (HRR) and the Lefschetz decomposition theorem (LD). One may refer to BDIP [2], Griffiths and Harris [10] and Voisin [22].

Theorem 1.1 (HLT). *The linear map*

$$\begin{aligned} H^{p,q}(X, \mathbb{C}) &\rightarrow H^{n-q, n-p}(X, \mathbb{C}) \\ \{\alpha\} &\mapsto \{\alpha\} \smile \{\Omega\} \end{aligned}$$

is an isomorphism, where \smile denotes the cup-product on the cohomology ring $\oplus H^*(X, \mathbb{C})$.

Define the primitive subspace $P^{p,q}(X, \mathbb{C})$ of $H^{p,q}(X, \mathbb{C})$ by

$$P^{p,q}(X, \mathbb{C}) := \left\{ \{\alpha\} \in H^{p,q}(X, \mathbb{C}), \{\alpha\} \smile \{\Omega\} \smile \{\omega\} = 0 \right\}.$$

Note that this primitive subspace depends on the class of Ω . Define $Q = Q_\Omega$ a Hermitian form on $H^{p,q}(X, \mathbb{C})$ by

$$Q(\{\alpha\}, \{\beta\}) := (\sqrt{-1})^{p-q} (-1)^{\frac{(p+q)(p+q-1)}{2}} \int_X \alpha \wedge \bar{\beta} \wedge \Omega$$

for smooth closed (p, q) forms α and β on X . The integral depends only on the cohomology classes $\{\alpha\}, \{\beta\}$ of α, β in $H^{p,q}(X, \mathbb{C})$.

Theorem 1.2 (HRR: Hodge-Riemann bilinear relations). *The Hermitian form Q is positive-definite on $P^{p,q}(X, \mathbb{C})$.*

Theorem 1.3 (LD: Lefschetz decomposition theorem). *The decomposition*

$$H^{p,q}(X, \mathbb{C}) = \left(\{\omega\} \smile H^{p-1,q-1}(X, \mathbb{C}) \right) \oplus P^{p,q}(X, \mathbb{C})$$

is orthogonal with respect to the Hermitian form Q .

Thus we get the signature of Q in terms of the Hodge numbers $h^{p,q} := \dim H^{p,q}(X, \mathbb{C})$. For example when $p = q = 1$ we obtain

Corollary 1.4 (Hodge index theorem). *The signature of Q on $H^{1,1}(X, \mathbb{C})$ is $(h^{1,1} - 1, 1)$.*

The above theorems are not true if we replace $\{\Omega = \omega^{n-p-q}\}$ with an arbitrary class in $H^{n-p-q, n-p-q}(X, \mathbb{R})$, even when the class contains a strictly positive form, see for example Berndtsson and Sibony [1, Sect. 9].

The sufficient conditions on $\{\Omega\}$ for these theorems has been studied by different groups of mathematicians. We recall some positive results. Let $\omega_1, \dots, \omega_{n-p-q+1}$ be arbitrary Kähler forms on X . Let $\Omega = \omega_1 \wedge \omega_2 \cdots \wedge \omega_{n-p-q}$. Define the primitive space $P^{p,q}(X, \mathbb{C})$ and the Hermitian form $Q = Q_\Omega$ with respect to this Ω . More precisely

$$P^{p,q}(X, \mathbb{C}) := \left\{ \{\alpha\} \in H^{p,q}(X, \mathbb{C}), \{\alpha\} \smile \{\Omega\} \smile \{\omega_{n-p-q+1}\} = 0 \right\}.$$

and

$$Q(\{\alpha\}, \{\beta\}) := (\sqrt{-1})^{p-q} (-1)^{\frac{(p+q)(p+q-1)}{2}} \int_X \alpha \wedge \bar{\beta} \wedge \Omega.$$

Theorem 1.5 (mixed HLT, HRR, LD). *For all $p+q \leq n$, the class of $\Omega = \omega_1 \wedge \cdots \wedge \omega_{n-p-q}$ satisfies the hard Lefschetz theorem, the Hodge-Riemann theorem and the Lefschetz decomposition theorem for the bidegree (p, q) .*

Gromov in [11] stated that Q is positive semi-definite when $p = q$. He gave a complete proof for the case $p = q = 1$. Later, Timorin in [21] proved the mixed HRR in the linear case, i.e. when X is a complex torus of dimension n , see also [18, 19]. Let V be a complex vector space of dimension n and \bar{V} be its complex conjugate. Let $V^{p,q} = \Lambda^p V \otimes \Lambda^q \bar{V}$ with the convention that $V^{p,q} = 0$ unless $0 \leq p, q \leq n$. A form $\omega \in V^{1,1}$ is a *Kähler form* if $\omega = \sum_{l=1}^n \frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l$ in some complex coordinates (z_1, \dots, z_n) where $z_i \otimes \bar{z}_j$ is identified with $dz_i \wedge d\bar{z}_j$. Timorin proved the following.

Theorem 1.6 (Linear mixed HLT, HRR, LD). *Let $\omega_1, \dots, \omega_{n-p-q+1} \in V^{1,1}$ be Kähler forms. Let $\Omega = \omega_1 \wedge \dots \wedge \omega_{n-p-q}$. Then $\Lambda\Omega : V^{p,q} \rightarrow V^{n-q, n-p}$ is an isomorphism. Define a Hermitian form on $V^{p,q}$*

$$Q(\alpha, \beta) = (\sqrt{-1})^{p-q} (-1)^{\frac{(p+q)(p+q-1)}{2}} \star (\alpha \wedge \bar{\beta} \wedge \Omega),$$

where \star is the Hodge-star operator, and define the mixed primitive space

$$P^{p,q} = \{\alpha \in V^{p,q} \mid \alpha \wedge \Omega \wedge \omega_{n-p-q+1} = 0\}$$

Then Q is positive definite on $P^{p,q}$. Moreover, the decomposition

$$V^{p,q} = \left(\omega_{n-p-q+1} \wedge V^{p-1, q-1} \right) \oplus P^{p,q}$$

is orthogonal with respect to Q .

Dinh-Nguyễn in [6] proved Theorem 1.5 for general compact Kähler manifolds, see also Cattani [3] for a proof using the theory of variations of Hodge structures. Later, in [7] Dinh-Nguyễn posed a point-wise condition and proved a stronger version.

We will introduce in the next section the notion of Hodge-Riemann cone in the exterior product $V^{k,k} := \Lambda^k V \otimes \Lambda^k \bar{V}$ with $0 \leq k \leq n$. In practice, V is the complex cotangent space at an arbitrary point x of X and we define Hodge-Riemann cone point-wisely on X .

Theorem 1.7 (Dinh-Nguyễn 2013). *Let Ω be a closed smooth form of bidegree $(n-p-q, n-p-q)$ on X . Assume that Ω takes values only in the Hodge-Riemann cone associated with X point-wisely. Then $\{\Omega\}$ satisfies HLT, HRR and LD for all bidegrees (p, q) .*

Roughly speaking, taking values in the Hodge-Riemann cone means Ω can be continuously deformed to ω^{n-p-q} in a nice way that some hard Lefschetz properties are preserved. Such deformation does not need to depend on x continuously. Moreover, it does not need to preserve the closedness nor smoothness of the form. The key is to check hard Lefschetz properties point-wisely.

Dinh-Nguyễn asked in [7] whether the Griffiths cone is contained in the Hodge-Riemann cone. Let $M = (\alpha_{i,j}) \in M_{k,k}(\Lambda^{1,1}V)$ be a $k \times k$ matrix of constant coefficient $(1,1)$ forms. We will define the Griffiths positivity of M in the next section. The Griffiths cone is the collection of (k,k) forms which are determinants of Griffiths positive matrices.

Fix complex coordinates (z_1, \dots, z_n) of \mathbb{C}^n . In this paper, to simplify computations, we treat matrices with diagonalized entries, i.e.

$$\alpha_{i,j} = \sum_{l=1}^n b_{i,j}^{(l)} \frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l.$$

That is to say, there are no terms like $dz_1 \wedge d\bar{z}_2$.

Theorem 1.8 (Main theorem 1). *Let M be a 2×2 matrix of constant coefficient $(1,1)$ forms on \mathbb{C}^n with $n = 2, 3$. Suppose M is Griffiths positive. Let $\Omega = \det(M)$. Then Ω satisfies HLT, HRR and LD for all bidegrees.*

Theorem 1.9 (Main theorem 2). *Let M be a $k \times k$ matrix of constant coefficient $(1,1)$ forms. Suppose M is Griffiths positive and has only diagonalized entries. Let $\Omega = \det(M)$. Then*

1. when $n \leq 5$, $k = 2$, the form Ω satisfies HLT, HRR and LD for all bidegrees (p, q) with $p + q = n - k$;

2. when $n \geq 6$, $k = 2$, the form Ω satisfies HLT for bidegrees $(n-2, 0)$, $(n-3, 1)$, $(1, n-3)$ and $(0, n-2)$.

HRR has applications in mixed-volumes inequalities, see Khovanskii [16, 17], Teissier [19, 20] and Dinh-Nguyên [6]. The reader will find some related results and applications in Cattani [3], de Cataldo and Migliorini [4], Gromov [11], Dinh and Sibony [5, 8] and Keum, Oguiso and Zhang [15, 24]. A generalization of mixed HRR and m -positivity is proved by Xiao in [23].

This paper is organized as follows. In Section 2 we recall the notion of Hodge-Riemann forms and the Griffiths cone defined by Dinh-Nguyen. In Section 3 we establish a normal form of Griffiths positive matrices under two group actions: $GL_k(\mathbb{C})$ action and $GL_n(\mathbb{C})$ action. In Section 4 we prove Theorem 1.8. In Section 5, we add the assumption that all entries of M are diagonalized, and we prove Theorem 1.9.

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2 Hodge-Riemann forms

In this section we recall the Hodge-Riemann form in the linear setting, defined by Dinh-Nguyên in [7].

Let V be an n -dimensional complex vector space and \bar{V} be its conjugate space. Denote by

$$V^{p,q} = \Lambda^p V \otimes \Lambda^q \bar{V}$$

the space of constant coefficient (p, q) forms with the convention that $V^{p,q} = 0$ unless $0 \leq p, q \leq n$. It is a complex vector space of dimension $\binom{n}{p} \binom{n}{q}$. A form $\omega \in V^{1,1}$ is a *Kähler form* if

$$\omega = \frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1 + \cdots + \frac{\sqrt{-1}}{2} dz_n \wedge d\bar{z}_n$$

in certain complex coordinates (z_1, \dots, z_n) , where $z_i \otimes \bar{z}_j$ is identified with $dz_i \wedge d\bar{z}_j$.

A form $\Omega \in V^{k,k}$ with $0 \leq k \leq n$ is *real* if $\Omega = \bar{\Omega}$. Let $V_{\mathbb{R}}^{k,k}$ be the space of all real (k, k) forms. A form Ω is *positive*¹ if it is a combination with positive coefficients of forms of type $(\sqrt{-1})^{k^2} \alpha \wedge \bar{\alpha}$ with $\alpha \in V^{k,0}$. A positive (k, k) form is *strictly positive* if its restriction on any k dimensional subspace is non-zero. Fix a Kähler form ω as above.

Definition 2.1 (Lefschetz forms). A (k, k) form $\Omega \in V^{k,k}$ is said to be a *Lefschetz form for the bidegree (p, q)* if $k = n - p - q$ and the map $\alpha \mapsto \alpha \wedge \Omega$ is an isomorphism between $V^{p,q}$ and $V^{n-q, n-p}$.

Definition 2.2 (Hodge-Riemann forms). A real (k, k) form $\Omega \in V_{\mathbb{R}}^{k,k}$ is said to be a *Hodge-Riemann form for the bidegree (p, q)* if there is a continuous deformation $\Omega_t \in V_{\mathbb{R}}^{k,k}$ with $0 \leq t \leq 1$, $\Omega_0 = \Omega$ and $\Omega_1 = \omega^k$ such that

$$(*) \quad \Omega_t \wedge \omega^{2r} \text{ is a Lefschetz form for the bidegree } (p-r, q-r)$$

for every $0 \leq r \leq \min\{p, q\}$ and $0 \leq t \leq 1$. The cone of all such forms is called the *Hodge-Riemann cone for bidegree (p, q)* . We say Ω is *Hodge-Riemann* if it is a Hodge-Riemann form for any bidegree (p, q) with $p + q = n - k$.

¹In [13] this is called strongly positive.

Note that a priori the definition of Hodge-Riemann forms depends on the choice of ω . Due to Dinh-Nguyen's results in [7], a Hodge-Riemann (k, k) form satisfies HLT, HRR and Lefschetz decomposition for all bidegree (p, q) such that $p + q = n - k$. It is proper to use the name "Hodge-Riemann" here.

According to the classical HLT in the linear case, $(*)$ holds for $t = 1$. Moreover, due to Timorin's results in [21], for any Kähler forms $\omega_1, \dots, \omega_k$, the product $\Omega := \omega_1 \wedge \dots \wedge \omega_k$ is a Hodge-Riemann form. In this paper we assume $2 \leq k \leq n$ to avoid the trivial case $k = 1$.

Let $M = (\alpha_{i,j})$ be a $k \times k$ matrix with entries in $V^{1,1}$. Assume that M is Hermitian, i.e. $\alpha_{i,j} = \overline{\alpha_{j,i}}$ for all i, j . We say that M is *Griffiths positive* if for all $\theta = (\theta_1, \dots, \theta_k) \in \mathbb{C}^k \setminus \{0\}$, $\theta \cdot M \cdot \bar{\theta}^t$ is a Kähler form. We call *Griffiths cone* the set of (k, k) forms $\Omega := \det(M)$ with M Griffiths positive.

Dinh-Nguyen conjectured in [7]

Conjecture 2.3. *Let M be a Griffiths positive $k \times k$ matrix. Then $\det(M)$ is a Hodge-Riemann form.*

They also explained the relation between this conjecture and

Conjecture 2.4. *Let M be a Griffiths positive $k \times k$ matrix. Then $\det(M)$ is a Lefschetz form for all bidegree (p, q) such that $p + q = n - k$.*

in the following sense:

Conjecture 2.3 holds for k and for some $\omega \Rightarrow$ Conjecture 2.4 holds for k .

Conjecture 2.4 holds for $k + 2r$, $\forall 0 \leq r \leq (n - k)/2 \Rightarrow$ Conjecture 2.3 holds for k and for any ω .

Proof. (relation between the two conjectures) Conjecture 2.3 implies Conjecture 2.4 for the same k due to Dinh-Nguyen's Theorem 1.7. Suppose Conjecture 2.4 holds for all $k + 2r$, $\forall 0 \leq r \leq (n - k)/2$. Then for any Griffiths positive $k \times k$ matrix and for any $t \in [0, 1]$, $(1 - t)M + t\omega Id_k$ is again a Griffiths positive $k \times k$ matrix. Let $\Omega_t := \det((1 - t)M + t\omega Id_k)$. Then Ω_t is a continuous family such that $\Omega_0 = \Omega = \det(M)$ and $\Omega_1 = \omega^k$. Moreover, the block matrix

$$\begin{pmatrix} (1 - t)M + t\omega Id_k & 0 \\ 0 & \omega Id_{2r} \end{pmatrix}$$

is a Griffiths positive $(k + 2r) \times (k + 2r)$ matrix. So its determinant, $\Omega_t \wedge \omega^{2r}$ is Lefschetz for all suitable p, q, r . By definition, $\Omega = \det(M)$ is Hodge-Riemann. \square

In this paper we treat Conjecture 2.4, whose statement does not depend on the choice of ω .

3 Griffiths positive matrices

In this section we prove some properties of Griffiths positive $k \times k$ matrices.

3.1 Two group actions on Griffiths positive matrices

$(GL_k(\mathbb{C})$ action) Let M be a Griffiths positive $k \times k$ matrices and let $C = (c_{i,j}) \in GL_k(\mathbb{C})$. Then $C \cdot M \cdot C^H$ is also Griffiths positive. Thus we define a $GL_k(\mathbb{C})$ action on Griffiths positive matrices. Moreover, the (k, k) form $\det(C \cdot M \cdot C^H) = \underbrace{|\det(C)|^2}_{>0} \det(M)$, is Lefschetz for the

bidegree (p, q) if and only if $\det(M)$ is Lefschetz for the same bidegree. It suffices to study conjecture 2.4 for one representative in each $GL_k(\mathbb{C})$ -orbit.

$(GL_n(\mathbb{C}) \text{ action})$ Write $M = (\alpha_{i,j})$ where each entry $\alpha_{i,j}$ is a $(1, 1)$ form. The Griffiths positivity of M implies that each $\alpha_{i,i}$ is a Kähler form. In complex coordinates (z_1, \dots, z_n) , each $\alpha_{i,j}$ can be expressed by a $n \times n$ matrix $(a_{i,j}^{(u,v)})$ such that

$$\alpha_{i,j} = \sum_{1 \leq u, v \leq n} a_{i,j}^{(u,v)} \frac{\sqrt{-1}}{2} dz_u \wedge d\bar{z}_v.$$

The $(1, 1)$ form $\alpha_{i,j}$ is real (resp. Kähler) if and only if the matrix $(a_{i,j}^{(u,v)})$ indexed by (u, v) is Hermitian (resp. positive definite). Follows from the Griffiths positivity, $(a_{i,i}^{(u,v)})$ is positive definite for each $1 \leq i \leq k$.

Let (w_1, \dots, w_n) be another complex coordinates and let $P \in GL_n(\mathbb{C})$ be the transition matrix between the two coordinates systems, i.e.

$$\begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix} = P \cdot \begin{pmatrix} w_1 \\ \dots \\ w_n \end{pmatrix}$$

Then each $(1, 1)$ form $\alpha_{i,j}$, in new coordinates, can be expressed as

$$\alpha_{i,j} = \sum_{1 \leq u, v \leq n} \tilde{a}_{i,j}^{(u,v)} \frac{\sqrt{-1}}{2} dw_u \wedge d\bar{w}_v,$$

where the matrix $(\tilde{a}_{i,j}^{(u,v)}) = P \cdot (a_{i,j}^{(u,v)}) P^H$.

Clearly, neither the Griffiths positivity nor the Lefschetz condition depends on the choice of complex coordinates. So it suffices to study conjecture 2.4 for one representative in each $GL_n(\mathbb{C})$ -orbit.

We will use these $GL_k(\mathbb{C})$ and $GL_n(\mathbb{C})$ actions to simplify the expression of M . Thus we can compute $\det(M)$ and verify its Lefschetz property efficiently.

3.2 $GL_n(\mathbb{C})$ reduction: Diagonalizability of $(1, 1)$ forms

Now we distinguish three notions for a matrix $M = (\alpha_{i,j})$ of $(1, 1)$ forms.

- **Diagonal entries** of M are the $(1, 1)$ forms $\alpha_{i,i}$, $1 \leq i \leq k$.
- **Diagonalizable entries** of M are the $(1, 1)$ forms $\alpha_{i,j}$ such that in some complex coordinates (z_1, \dots, z_n) ,

$$(**) \quad \alpha_{i,j} = \sum_{l=1}^n b_{i,j}^{(l)} \frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l.$$

- Fix a complex coordinate system (z_1, \dots, z_n) . **Diagonalized entries** of M are the $(1, 1)$ forms $\alpha_{i,j}$ of the form $(**)$.

Recall that the Griffiths positivity of M implies that all diagonal entries $\alpha_{i,i}$ are Kähler. Thus there exists complex coordinates (z_1, \dots, z_n) such that

$$\alpha_{1,1} = \omega = \sum_{l=1}^n \frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l.$$

In new coordiantes (w_1, \dots, w_n) , $a_{1,1}$ has the same expression

$$\alpha_{1,1} = \omega = \sum_{l=1}^n \frac{\sqrt{-1}}{2} dw_l \wedge d\bar{w}_l$$

if and only if the transition matrix P is *unitary*, i.e. $P \cdot P^H = Id_n$.

Theorem 3.1 (Schur's unitary triangularization theorem). *(See [14] p79 Thm 2.3.1) Given an arbitrary $n \times n$ matrix A with complex coefficients. There is a unitary matrix U such that $U \cdot A \cdot U^H$ is upper triangular.*

Corollary 3.2. *Given an arbitrary (n, n) Hermitian positive definite matrix A , there is a unitary matrix U such that $U \cdot A \cdot U^H = \text{diag}\{\lambda^{(1)}, \dots, \lambda^{(n)}\}$ with $\lambda^{(1)}, \dots, \lambda^{(n)} > 0$.*

Thus in M we can diagonalize at least two diagonal entries after a $GL_n(\mathbb{C})$ action. We conclude the following.

Corollary 3.3. *Let $M = (a_{i,j})$ be a Griffiths positive matrix of $(1, 1)$ forms. Then there exists complex coordinates (z_1, \dots, z_n) such that*

$$\begin{aligned} \alpha_{1,1} = \omega &= \sum_{l=1}^n \frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l, \\ \alpha_{2,2} &= \sum_{l=1}^n \lambda^{(l)} \frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l, \end{aligned}$$

for some $\lambda^{(1)}, \dots, \lambda^{(n)} > 0$.

3.3 $GL_k(\mathbb{C})$ reduction: Lefschetz decomposition

Now we assume $\alpha_{1,1} = \omega = \sum_{l=1}^n \frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l$. According to the Lefschetz decomposition in the linear case

$$V^{1,1} = \mathbb{C}\omega \oplus P^{1,1}$$

where $P^{1,1} = \{\alpha \in V^{1,1} | \alpha \wedge \omega^{n-1} = 0\}$. Thus we may write

$$M = (\alpha_{i,j}) = \begin{pmatrix} \omega & \lambda_{1,2}\omega + \rho_{1,2} & \cdots & \lambda_{1,k}\omega + \rho_{1,k} \\ \overline{\lambda_{1,2}\omega + \rho_{1,2}} & \lambda_{2,2}\omega + \rho_{2,2} & \cdots & \lambda_{2,k}\omega + \rho_{2,k} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{\lambda_{1,k}\omega + \rho_{1,k}} & \overline{\lambda_{2,k}\omega + \rho_{2,k}} & \cdots & \lambda_{k,k}\omega + \rho_{k,k} \end{pmatrix}$$

where $\lambda_{i,j} \in \mathbb{C}$, $\lambda_{j,j} \in \mathbb{R}_{>0}$ and $\rho_{i,j} \in P^{1,1}$. The fact $\lambda_{j,j} \in \mathbb{R}_{>0}$ follows from the Griffiths positivity of M .

After a $GL_k(\mathbb{C})$ reduction we may assume that $\lambda_{1,j} = 0$ for all $2 \leq j \leq k$. More precisely, let

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\overline{\lambda_{1,2}} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -\overline{\lambda_{1,k}} & 0 & \cdots & 1 \end{pmatrix} \in GL_k(\mathbb{C})$$

then

$$C \cdot M \cdot C^H = \begin{pmatrix} \omega & \rho_{1,2} & \cdots & \rho_{1,k} \\ \overline{\rho_{1,2}} & \lambda'_{2,2}\omega + \rho'_{2,2} & \cdots & \lambda'_{2,k}\omega + \rho'_{2,k} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{\rho_{1,k}} & \overline{\lambda'_{2,k}\omega + \rho'_{2,k}} & \cdots & \lambda'_{k,k}\omega + \rho'_{k,k} \end{pmatrix}$$

where $\lambda'_{i,j} \in \mathbb{C}$, $\rho'_{i,j} \in P^{1,1}$ is also Griffiths positive and shares the same determinant with M . By the Griffiths positivity $\lambda'_{j,j} > 0$.

Furthermore, after a $GL_k(\mathbb{C})$ action with

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -\frac{\lambda'_{2,3}}{\lambda'_{2,2}} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -\frac{\lambda'_{2,k}}{\lambda'_{2,2}} & 0 & \cdots & 1 \end{pmatrix} \in GL_k(\mathbb{C})$$

we may assume that $\lambda'_{2,j} = 0$ for $3 \leq j \leq k$. After finitely many such $GL_k(\mathbb{C})$ reductions and a positive dilation on the diagonal we may assume that $\lambda'_{i,j} = 0$ for $i \neq j$ and $\lambda'_{j,j} = 1$ for $2 \leq j \leq k$. To conclude

Proposition 3.4. *Let M be a Griffiths positive matrix with $\alpha_{1,1} = \omega$. Then there is $C \in GL_k(\mathbb{C})$ such that*

$$C \cdot M \cdot C^H = \begin{pmatrix} \omega & \rho_{1,2} & \cdots & \rho_{1,k} \\ \overline{\rho_{1,2}} & \omega + \rho_{2,2} & \cdots & \rho_{2,k} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{\rho_{1,k}} & \overline{\rho_{2,k}} & \cdots & \omega + \rho_{k,k} \end{pmatrix}$$

where $\rho_{i,j} \in P^{1,1}$. Moreover, by Corollary 3.3 we may assume that $\rho_{2,2}$ is a diagonalized entry, i.e. $\rho_{2,2} = \sum_{l=1}^n b_{2,2}^{(l)} \frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l$ with $\sum_{l=1}^n b_{2,2}^{(l)} = 0$ and $b_{2,2}^{(l)} > -1$ for each $1 \leq l \leq n$.

Proof. The fact $\sum_{l=1}^n b_{2,2}^{(l)} = 0$ follows from $\rho_{2,2} \in P^{1,1}$. By Griffiths positivity, $\omega + \rho_{2,2}$ is a Kähler form. Thus $b_{2,2}^{(l)} > -1$ for each $1 \leq l \leq n$. \square

Thus to treat Conjecture 2.4, it suffices to study Griffiths positive matrices of the normalized form in Proposition 3.4.

4 Proof for $k = 2$ and $n = 2, 3$

According to the appendix of Griffiths [9], when $k = 2$ the determinat Ω is a weakly positive $(2, 2)$ -form. In particular when $n \leq 3$, Ω is strictly positive, hence Lefschetz for suitable bidegrees and hence Hodge-Riemann. In this section we reproduce the proof in the linear case.

4.1 Trivial case $2 = k = n$

Theorem 4.1. *For $n = 2$, let M be a Griffiths positive 2×2 matrix. Then $\Omega = \det(M)$ is a Lefschetz form for the bidegree $(0, 0)$.*

Proof. In \mathbb{C}^2 , let

$$M = \begin{pmatrix} \omega & \rho_{1,2} \\ \overline{\rho_{1,2}} & \omega + \rho_{2,2} \end{pmatrix}$$

where $\rho_{1,2}, \rho_{2,2} \in P^{1,1}$. Then

$$\Omega = \det(M) = \omega^2 + \underbrace{\omega \wedge \rho_{2,2} - \rho_{1,2} \wedge \overline{\rho_{1,2}}}_{=0}$$

By definition of the primitive space, $\omega \wedge \rho_{2,2} = 0$. By linear HRR, $-\rho_{1,2} \wedge \overline{\rho_{1,2}} = \lambda \cdot \text{Vol}$ where $\lambda > 0$ and $\text{Vol} = \omega^2$ is the euclidean volumn form on \mathbb{C}^2 . Thus $\Omega = (1 + \lambda) \cdot \text{Vol}$ defines an isomorphism $V^{0,0} = \mathbb{C} \rightarrow V^{2,2} = \mathbb{C}\text{Vol}$. \square

Corollary 4.2. *For $n = 2$, let M be a Griffiths positive 2×2 matrix. Then $\det(M)$ is a Hodge-Riemann form.*

4.2 Case $k = 2, n = 3$

Theorem 4.3. *For $n = 3$, let M be a Griffiths positive 2×2 matrix. Let $\Omega = \det(M)$ be a real $(2, 2)$ form. Define a Hermitian form on $V^{1,0}$ by*

$$Q(\alpha, \beta) = \sqrt{-1} \star (\alpha \wedge \overline{\beta} \wedge \Omega)$$

Then Q is positive definite on $V^{1,0}$.

Proof. Let

$$M = \begin{pmatrix} \omega & \rho_{1,2} \\ \overline{\rho_{1,2}} & \omega' \end{pmatrix}$$

where $\rho_{1,2} \in P^{1,1}$, i.e. $\rho_{1,2} \wedge \omega^2 = 0$, and ω' is a Kähler form. Then $\Omega = \omega \wedge \omega' - \rho_{1,2} \wedge \overline{\rho_{1,2}}$.

For any $\alpha \in V^{1,0}$, $\alpha \neq 0$, after a $U_n(\mathbb{C})$ action on coordinates we may assume that $\alpha = \lambda dz_1$ for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$, while ω and the subspace $P^{1,1}$ stays unchanged. However, ω' may no longer be diagonalized. Under these coordinates, each $(1, 1)$ form can be expressed by a 3×3 matrix:

$$\sum_{1 \leq i, j \leq 3} a_{i,j} \frac{\sqrt{-1}}{2} dz_i \wedge d\overline{z_j} \longleftrightarrow (a_{i,j})$$

In particular

$$\begin{aligned} \omega &\longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \omega' &\longleftrightarrow \begin{pmatrix} * & * & * \\ * & a & b \\ * & \overline{b} & \frac{|b|^2 + t}{a} \end{pmatrix}, \\ \rho_{1,2} &\longleftrightarrow \begin{pmatrix} * & * & * \\ * & c_{2,2} & c_{2,3} \\ * & c_{3,2} & c_{3,3} \end{pmatrix}, & \overline{\rho_{1,2}} &\longleftrightarrow \begin{pmatrix} * & * & * \\ * & \overline{c_{2,2}} & \overline{c_{3,2}} \\ * & \overline{c_{2,3}} & \overline{c_{3,3}} \end{pmatrix}, \end{aligned}$$

where $a > 0, t > 0, b \in \mathbb{C}, c_{i,j} \in \mathbb{C}$.

We calculate

$$\begin{aligned} Q(\alpha, \alpha) &= 2|\lambda|^2 \star (V_1 \wedge \omega \wedge \omega') - 2|\lambda|^2 \star (V_1 \wedge \rho_{1,2} \wedge \overline{\rho_{1,2}}) \\ &= 2|\lambda|^2 \left(a + \frac{|b|^2 + t}{a} \right) - 2|\lambda|^2 (c_{2,2}\overline{c_{3,3}} + c_{3,3}\overline{c_{2,2}} - |c_{2,3}|^2 - |c_{3,2}|^2) \\ &= 2|\lambda|^2 \left(a + \frac{|b|^2 + t}{a} - c_{2,2}\overline{c_{3,3}} - c_{3,3}\overline{c_{2,2}} + |c_{2,3}|^2 + |c_{3,2}|^2 \right). \end{aligned}$$

It suffices to analyse that $a + \frac{|b|^2+t}{a} - c_{2,2}\overline{c_{3,3}} - c_{3,3}\overline{c_{2,2}} > 0$, which is the job of Griffiths positivity.

Let $\theta = (1, z)$ for an arbitrary $z \in \mathbb{C}$. By Griffiths positivity, the form

$$\theta \cdot M \cdot \overline{\theta}^t = \omega + z\rho_{1,2} + \overline{z}\overline{\rho_{1,2}} + |z|^2\omega'$$

is a Kähler form, i.e. the matrix

$$\begin{pmatrix} * & * & * \\ * & 1 + zc_{2,2} + \overline{z}\overline{c_{2,2}} + |z|^2a & * \\ * & * & 1 + zc_{3,3} + \overline{z}\overline{c_{3,3}} + |z|^2(\frac{|b|^2+t}{a}) \end{pmatrix}$$

is positive definite. So for any $z \in \mathbb{C}$

$$\begin{cases} 1 + zc_{2,2} + \overline{z}\overline{c_{2,2}} + |z|^2a > 0 \\ 1 + zc_{3,3} + \overline{z}\overline{c_{3,3}} + |z|^2(\frac{|b|^2+t}{a}) > 0. \end{cases}$$

Let $f(z, \overline{z}) := 1 + zc_{2,2} + \overline{z}\overline{c_{2,2}} + |z|^2a$. The minimum of this quadratic is achieved when $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \overline{z}} = 0$, i.e. when $z = \frac{-\overline{c_{2,2}}}{a}$. Since $f(z, \overline{z}) > 0$ for all $z \in \mathbb{C}$, we conclude that

$$f(z, \overline{z}) \Big|_{z=\frac{-\overline{c_{2,2}}}{a}} = 1 - \frac{|c_{2,2}|^2}{a} > 0 \quad \Rightarrow |c_{2,2}|^2 < a.$$

By the same method

$$|c_{3,3}|^2 < \frac{|b|^2+t}{a}.$$

So

$$\begin{aligned} Q(\alpha, \alpha) &= 2|\lambda|^2 \left(a + \frac{|b|^2+t}{a} - c_{2,2}\overline{c_{3,3}} - c_{3,3}\overline{c_{2,2}} + |c_{2,3}|^2 + |c_{3,2}|^2 \right) \\ &> 2|\lambda|^2 \left(|c_{2,2}|^2 + |c_{3,3}|^2 - c_{2,2}\overline{c_{3,3}} - c_{3,3}\overline{c_{2,2}} + |c_{2,3}|^2 + |c_{3,2}|^2 \right) \\ &= 2|\lambda|^2 \left(|c_{2,2} - c_{3,3}|^2 + |c_{2,3}|^2 + |c_{3,2}|^2 \right) \geq 0. \quad \square \end{aligned}$$

Corollary 4.4. *For $n = 3$, let M be a Griffiths positive 2×2 matrix. Then $\Omega = \det(M)$ is a Lefschetz form for bidegree $(1, 0)$, hence a Hodge-Riemann form.*

Proof. Since Q is non-degenerate on $V^{1,0}$, we know $\wedge\Omega : V^{1,0} \rightarrow V^{3,2}$ is an injective linear map between vector spaces of the same dimension, hence an isomorphism. \square

4.3 Difficulty of the case $k = 2, n = 4$

First we explain why the proof above does not work for $n = 4$. Still let M be normalized and $\Omega = \omega \wedge (\omega + \rho_{2,2}) - \rho_{1,2} \wedge \overline{\rho_{1,2}}$. Define the Hermitian form

$$Q(\alpha, \beta) := -(\sqrt{-1})^{p-q} \star (\alpha \wedge \overline{\beta} \wedge \Omega)$$

on $V^{p,q}$ with $p + q = n - k = 2$. For any $\alpha \in V^{2,0} = P^{2,0}$, $\alpha \neq 0$, in the sum

$$Q(\alpha, \alpha) = \star (\alpha \wedge \overline{\alpha} \wedge \omega \wedge (\omega + \rho_{2,2})) - \star (\alpha \wedge \overline{\alpha} \wedge \rho_{1,2} \wedge \overline{\rho_{1,2}})$$

the first part

$$\star (\alpha \wedge \overline{\alpha} \wedge \omega \wedge (\omega + \rho_{2,2})) > 0$$

by the mixed linear HRR. However, the second part

$$-\star(\alpha \wedge \bar{\alpha} \wedge \rho_{1,2} \wedge \overline{\rho_{1,2}})$$

is not always non-negative. Although $\alpha \wedge \bar{\alpha} + \epsilon \omega^2$ is strictly positive for each $\epsilon > 0$, $\alpha \wedge \bar{\alpha} + \epsilon \omega^2$ may not satisfy HRR. For example when $\alpha = dz_1 \wedge dz_2$, the bilinear form

$$\tilde{Q}(\beta, \gamma) := -\star(\alpha \wedge \bar{\alpha} \wedge \beta \wedge \bar{\gamma}) = -4\star(V_{1,2} \wedge \beta \wedge \bar{\gamma})$$

is no longer semi-positive definite on $P^{1,1}$. For $\beta = 2V_1 - V_3 - V_4 \in P^{1,1}$,

$$\tilde{Q}(\beta, \beta) = -4\star(V_{1,2} \wedge 2V_{3,4}) < 0.$$

At this moment we cannot conclude that Q is non-degenerate on $P^{2,0}$.

5 Case all entries of M being diagonalized

In this paper, for simplifications, we write $V_j = \frac{\sqrt{-1}}{2} dz_j \wedge d\bar{z}_j$ for the euclidean volume form on the complex line of z_j , which is two times the euclidean volume form. We write $V_{j_1, j_2, \dots, j_s} = V_{j_1} \wedge \dots \wedge V_{j_s}$ for the volume form on the s -dimensional subspace spanned by those complex lines. Without extra specifications, $\omega = \sum_{l=1}^n V_l$ denotes the standard Kähler form in the linear case. We write $\text{Vol} = V_{1,2,\dots,n}$ for the volume form on \mathbb{C}^n .

Now we assume that all entries of $M = (\alpha_{i,j})$ are diagonalized, i.e. $\alpha_{i,j} = \sum_{l=1}^n b_{i,j}^{(l)} V_l$. The following lemma holds for general $k \geq 2$.

Proposition 5.1. *Let M be a $k \times k$ matrix with diagonalized entries. We can write M as a matrix valued $(1,1)$ form*

$$M = (\alpha_{i,j}) = \left(\sum_{l=1}^n b_{i,j}^{(l)} V_l \right) = \sum_{l=1}^n (b_{i,j}^{(l)}) V_l$$

Then M is Griffiths positive if and only if the matrix $B^{(l)} := (b_{i,j}^{(l)})$ is a positive definite $k \times k$ matrix for $1 \leq l \leq n$.

Proof. For any $\theta \in \mathbb{C}^k$, $\theta \neq 0$. The $(1,1)$ form

$$\theta \cdot M \cdot \bar{\theta}^t = \sum_{l=1}^n \theta \cdot B^{(l)} \cdot \bar{\theta}^t V_l.$$

Thus M is Griffiths positive if and only if $\theta \cdot M \cdot \bar{\theta}^t$ is a Kähler form if and only if $\theta \cdot B^{(l)} \cdot \bar{\theta}^t > 0$ for $1 \leq l \leq n$. \square

After a dilation we may assume that $\alpha_{1,1} = \omega = \sum_{l=1}^n V_l$, i.e. $b_{1,1}^{(l)} = 1$ for $1 \leq l \leq n$.

To calculate $\Omega = \det(M)$, we introduce the hyperdeterminant among $B^{(l)}$.

Definition 5.2. Let $B^{(1)}, \dots, B^{(k)}$ be (k, k) complex valued matrices with $B^{(l)} = (b_{i,j}^{(l)})$. We define the $k \times k \times k$ hypermatrix $\mathbf{B} = (B^{(1)}, \dots, B^{(k)})$ a 3-dim array whose layers $B^{(l)}$ are matrices. We define its hyperdeterminant by

$$\text{hdet}(\mathbf{B}) := \sum_{\sigma, \tau \in S_k} \text{sgn}(\sigma) \prod_{j=1}^k b_{j, \sigma(j)}^{(\tau(j))}$$

where S_k is the permutation group of k elements.

We remark that switching two layers does not change the hyperdeterminant. The determinant

$$\Omega = \det(M) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \Omega_{i_1, \dots, i_k} V_{i_1, \dots, i_k}$$

where $\Omega_{i_1, \dots, i_k} = \text{hdet}((B^{(i_1)}, \dots, B^{(i_k)}))$, whose positivity is proved by Zheng (MCM, AMSS).

Theorem 5.3 (Zheng 2021). *Let $B^{(1)}, \dots, B^{(k)}$ be positive semidefinite Hermitian matrices. Denote by $\mu^{(l)}$ and $\lambda^{(l)}$ the minimal and the maximal eigenvalues of $B^{(l)}$. Then*

$$k! \mu^{(1)} \dots \mu^{(k)} \leq \text{hdet}(\mathbf{B}) \leq k! \lambda^{(1)} \dots \lambda^{(k)}.$$

In particular, if $B^{(1)}, \dots, B^{(k)}$ are positive definite, then $\text{hdet}(\mathbf{B}) > 0$.

Proof. We proceed by induction on k . The case $k = 1$ is trivial.

For the general case, note that for any $k \times k$ matrix U , we have $\text{hdet}(\mathbf{B}U) = \det(U) \text{hdet}(\mathbf{B}) = \text{hdet}(U\mathbf{B})$, where $\mathbf{B}U$ and $U\mathbf{B}$ are the hypermatrices defined by layerwise multiplication $(\mathbf{B}U)^{(l)} := B^{(l)}U$ and $(U\mathbf{B})^{(l)} := UB^{(l)}$. Thus, up to replacing \mathbf{B} by $U\mathbf{B}U^H$ for a unitary matrix U , we may assume that one layer, say $B^{(1)}$, is diagonal.

Once $B^{(1)}$ is diagonal, in the definition of $\text{hdet}(\mathbf{B})$ above, if $\tau(j) = 1$, then $b_{j, \sigma(j)}^{(1)}$ is nonzero only if $\sigma(j) = j$. Thus

$$\text{hdet}(\mathbf{B}) = \sum_{j=1}^k b_{j,j}^{(1)} \text{hdet}(\mathbf{B}_j)$$

where \mathbf{B}_j is the $(k-1) \times (k-1) \times (k-1)$ hypermatrix obtained from \mathbf{B} by removing the layer $B^{(1)}$ and removing the j -th row together with the j -th column in each layer $B^{(i)}$. We conclude by the induction hypothesis. \square

Corollary 5.4. *Let M be a (k, k) Griffiths positive matrix with diagonalized entries. Then $\Omega = \det(M)$ is a strictly positive (k, k) form, hence a Lefschetz form for bidegree $(n-k, 0)$ and $(0, n-k)$.*

Corollary 5.5. *Let M be a (k, k) Griffiths positive matrix with diagonalized entries. If $k = n-1$ or $k = n$, then $\det(M)$ satisfies HLT, HRR, LD for all suitable bidegrees.*

5.1 Case $k = 2$, $n = 4$, M with diagonalized entries

When $k = 2$, each positive definite matrix $B^{(l)}$ can be written as

$$B^{(l)} = \begin{pmatrix} 1 & b_l \\ \overline{b_l} & |b_l|^2 + t_l \end{pmatrix}$$

for some $b_l \in \mathbb{C}$ and $t_l > 0$. The determinant

$$\Omega = \det(M) = \sum_{1 \leq i < j \leq 4} \Omega_{i,j} V_{i,j}$$

where

$$\Omega_{i,j} = \text{hdet}(B^{(i)}, B^{(j)}) = |b_i|^2 + t_i + |b_j|^2 + t_j - b_i \bar{b}_j - b_j \bar{b}_i = |b_i - b_j|^2 + t_i + t_j > 0.$$

For simplifications, we define $b_{i,j} := b_i - b_j \in \mathbb{C}$.

Theorem 5.6. *Let $n = 4$. Let M be a $(2, 2)$ Griffiths positive matrix with diagonalized entries. Then $\Omega = \det(M)$ is a Lefschetz form for bidegree $(2, 0)$, $(1, 1)$ and $(0, 2)$.*

Proof. It suffices to check bidegree $(2, 0)$ and $(1, 1)$.

(Trivial part) For bidegree $(2, 0)$, we take the standard basis of $V^{2,0}$ by the lexicographical order

$$\{dz_1 \wedge dz_2, dz_1 \wedge dz_3, \dots, dz_3 \wedge dz_4\}$$

and we take a basis of $V^{4,2}$ by the Hodge-star of their conjugates

$$\{dz_1 \wedge dz_2 \wedge V_{3,4}, dz_1 \wedge dz_3 \wedge V_{2,4}, \dots, dz_3 \wedge dz_4 \wedge V_{1,2}\}.$$

Under these two basis, the linear map $\wedge \Omega : V^{2,0} \rightarrow V^{4,2}$ can be expressed as a diagonal matrix $\text{diag}\{\Omega_{3,4}, \Omega_{2,4}, \dots, \Omega_{1,2}\}$ with positive entries. This map is an isomorphism.

(Non-trivial part) For bidegree $(1, 1)$, again, we take the standard basis of $V^{1,1}$ as follows

$$\underbrace{\{V_1, V_2, V_3, V_4\}}_{4 \text{ elements}}, \underbrace{\{dz_1 \wedge d\bar{z}_2, dz_1 \wedge d\bar{z}_3, \dots, dz_4 \wedge d\bar{z}_3\}}_{12 \text{ elements}}$$

and we take a basis of $V^{3,3}$ by the Hodge star of their conjugates

$$\underbrace{\{V_{2,3,4}, V_{1,3,4}, V_{1,2,4}, V_{1,2,3}\}}_{4 \text{ elements}}, \underbrace{\{dz_1 \wedge d\bar{z}_2 \wedge V_{3,4}, dz_1 \wedge d\bar{z}_3 \wedge V_{2,4}, \dots, dz_4 \wedge d\bar{z}_3 \wedge V_{1,2}\}}_{12 \text{ elements}}$$

Under these two basis, the linear map $\wedge \Omega : V^{1,1} \rightarrow V^{3,3}$ can be expressed as a blocked matrix

$$\begin{pmatrix} G & 0 \\ 0 & \text{diag}\{\Omega_{3,4}, \Omega_{2,4}, \dots, \Omega_{1,2}\} \end{pmatrix}$$

where

$$G = \begin{pmatrix} 0 & \Omega_{3,4} & \Omega_{2,4} & \Omega_{2,3} \\ \Omega_{3,4} & 0 & \Omega_{1,4} & \Omega_{1,3} \\ \Omega_{2,4} & \Omega_{1,4} & 0 & \Omega_{1,2} \\ \Omega_{2,3} & \Omega_{1,3} & \Omega_{1,2} & 0 \end{pmatrix}.$$

It suffices to verify that $\det(G) \neq 0$. Let

$$A = \sqrt{\Omega_{3,4}\Omega_{1,2}}, \quad B = \sqrt{\Omega_{2,4}\Omega_{1,3}}, \quad C = \sqrt{\Omega_{2,3}\Omega_{1,4}}.$$

Then $\det(G) = -(A + B + C)(A + B - C)(A - B + C)(-A + B + C)$ has the form of the Heron formula which calculates the area of a triangle with side length (A, B, C) . We are going

to show that the side lengths (A, B, C) actually forms a triangle. After permutations among b_j and among t_j , it suffices to verify that

$$\sqrt{\Omega_{3,4}\Omega_{1,2}} + \sqrt{\Omega_{2,4}\Omega_{1,3}} > \sqrt{\Omega_{2,3}\Omega_{1,4}},$$

i.e.

$$\Omega_{3,4}\Omega_{1,2} + \Omega_{2,4}\Omega_{1,3} + 2\sqrt{\Omega_{3,4}\Omega_{1,2}}\sqrt{\Omega_{2,4}\Omega_{1,3}} - \Omega_{2,3}\Omega_{1,4} > 0$$

The left hand side is

$$\begin{aligned} LHS = & (|b_{3,4}|^2 + t_3 + t_4)(|b_{1,2}|^2 + t_1 + t_2) + (|b_{2,4}|^2 + t_2 + t_4)(|b_{1,3}|^2 + t_1 + t_3) \\ & + 2\sqrt{(|b_{3,4}|^2 + t_3 + t_4)(|b_{1,2}|^2 + t_1 + t_2)(|b_{2,4}|^2 + t_2 + t_4)(|b_{1,3}|^2 + t_1 + t_3)} \\ & - (|b_{2,3}|^2 + t_2 + t_3)(|b_{1,4}|^2 + t_1 + t_4) \end{aligned}$$

We expand the product in the square-root and sort each summand according to the order of t_j :

$$\Omega_{3,4}\Omega_{1,2}\Omega_{2,4}\Omega_{1,3} = \underbrace{(\dots)}_{b \text{ part}} + \sum_{j=1}^4 \underbrace{(\dots)t_j}_{t_j \text{ part}} + \sum_{j=1}^4 \underbrace{(\dots)t_j^2}_{t_j^2 \text{ part}} + \sum_{1 \leq j < l \leq 4} \underbrace{(\dots)t_j t_l}_{t_j t_l \text{ part}} + O_{t_j}(3)$$

where

$$\begin{aligned} (b \text{ part}) &:= |b_{3,4}|^2 |b_{1,2}|^2 |b_{2,4}|^2 |b_{1,3}|^2 \\ (t_1^2 \text{ part}) &:= (t_1 |b_{3,4}| |b_{2,4}|)^2 \\ (t_1 \text{ part}) &:= t_1 |b_{3,4}|^2 |b_{2,4}|^2 (|b_{1,2}|^2 + |b_{1,3}|^2) \\ &\geq 2t_1 |b_{3,4}|^2 |b_{2,4}|^2 |b_{1,2}| |b_{1,3}| \\ &= 2\sqrt{(t_1^2 \text{ part})} \sqrt{(b \text{ part})} \\ (t_1 t_2 \text{ part}) &:= t_1 t_2 |b_{3,4}|^2 (|b_{1,2}|^2 + |b_{2,4}|^2 + |b_{1,3}|^2) \\ &\geq 2t_1 t_2 |b_{3,4}|^2 |b_{2,4}| |b_{1,3}| \\ &= 2\sqrt{(t_1^2 \text{ part})} \sqrt{(t_2^2 \text{ part})} \\ (t_1 t_4 \text{ part}) &:= t_1 t_4 (|b_{2,4}|^2 + |b_{3,4}|^2) (|b_{1,2}|^2 + |b_{1,3}|^2) \\ &\geq 2t_1 t_4 |b_{1,2}| |b_{1,3}| |b_{2,4}| |b_{3,4}| \\ &= 2\sqrt{(t_1^2 \text{ part})} \sqrt{(t_4^2 \text{ part})} \end{aligned}$$

In fact $(t_j \text{ part}) \geq 2\sqrt{(t_j^2 \text{ part})}\sqrt{(b \text{ part})}$ and $(t_j t_l \text{ part}) \geq 2\sqrt{(t_j^2 \text{ part})}\sqrt{(t_l^2 \text{ part})}$ for any j, l . Hence

$$\begin{aligned} & 2\sqrt{\Omega_{3,4}\Omega_{1,2}\Omega_{2,4}\Omega_{1,3}} \\ & > 2\sqrt{(b \text{ part}) + \sum_{j=1}^4 (t_j^2 \text{ part}) + \sum_{j=1}^4 2\sqrt{(t_j^2 \text{ part})}\sqrt{(b \text{ part})} + \sum_{1 \leq j < l \leq 4} 2\sqrt{(t_j^2 \text{ part})}\sqrt{(t_l^2 \text{ part})}} \\ & = 2\left(\sqrt{(b \text{ part})} + \sum_{j=1}^4 \sqrt{(t_j^2 \text{ part})}\right). \end{aligned}$$

Here the first inequality is strict because there is a term $t_1 t_2 t_3 t_4 > 0$. Thus

$$\begin{aligned}
LHS &> |b_{3,4}|^2 |b_{1,2}|^2 + |b_{2,4}|^2 |b_{1,3}|^2 + 2\sqrt{(b \text{ part})} - |b_{2,3}|^2 |b_{1,4}|^2 \\
&\quad + t_1 (|b_{3,4}|^2 + |b_{2,4}|^2 + \underbrace{2|b_{3,4}||b_{2,4}|}_{\text{comes from the square root}} - |b_{2,3}|^2) \\
&\quad + t_2 (|b_{3,4}|^2 + |b_{1,3}|^2 + 2|b_{3,4}||b_{1,3}| - |b_{1,4}|^2) \\
&\quad + t_3 (|b_{1,2}|^2 + |b_{2,4}|^2 + 2|b_{1,2}||b_{2,4}| - |b_{1,4}|^2) \\
&\quad + t_4 (|b_{1,2}|^2 + |b_{1,3}|^2 + 2|b_{1,2}||b_{1,3}| - |b_{2,3}|^2) \\
&\quad + \underbrace{(t_3 + t_4)(t_1 + t_2) + (t_2 + t_4)(t_1 + t_3) - (t_2 + t_3)(t_1 + t_4)}_{=0}.
\end{aligned}$$

The first line is $(|b_{3,4}||b_{1,2}| + |b_{2,4}||b_{1,3}|)^2 - |b_{2,3}|^2 |b_{1,4}|^2$. In fact

$$\begin{aligned}
b_{2,3}b_{1,4} &= (b_2 - b_3)(b_1 - b_4) \\
&= (b_3 - b_4)(b_2 - b_1) + (b_2 - b_4)(b_1 - b_3) = -b_{3,4}b_{1,2} + b_{2,4}b_{1,3} \\
|b_{2,3}b_{1,4}| &\leq |b_{3,4}b_{1,2}| + |b_{2,4}b_{1,3}|
\end{aligned}$$

Indeed this is Ptolemy Theorem.

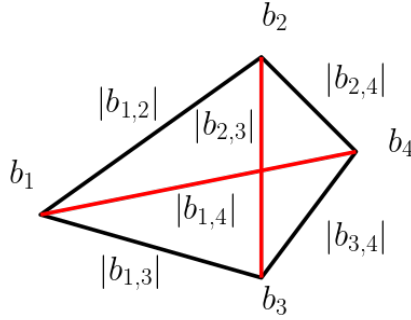


Figure 1: Ptolemy Theorem: $|b_{2,3}b_{1,4}| \leq |b_{3,4}b_{1,2}| + |b_{2,4}b_{1,3}|$

In the second line

$$t_1 (|b_{3,4}|^2 + |b_{2,4}|^2 + 2|b_{3,4}||b_{2,4}| - |b_{2,3}|^2) = t_1 ((|b_{3,4}| + |b_{2,4}|)^2 - |b_{2,3}|^2) \geq 0$$

by the triangle inequality. So $LHS > 0$ which implies $A+B-C > 0$. After permutations among b_j and among t_j , we conclude that $\det(G) < 0$ and $\wedge\Omega : V^{1,1} \rightarrow V^{3,3}$ is an isomorphism. \square

5.2 Case $k = 2$, $n \geq 4$, M with diagonalized entries

As before, write $M = \sum_{l=1}^n B^{(l)} V_l$ with

$$B^{(l)} = \begin{pmatrix} 1 & b_l \\ \frac{1}{b_l} & |b_l|^2 + t_l \end{pmatrix}$$

for some $b_l \in \mathbb{C}$ and $t_l > 0$. The determinant

$$\Omega = \det(M) = \sum_{1 \leq i < j \leq n} \Omega_{i,j} V_{i,j}$$

where

$$\Omega_{i,j} = |b_{i,j}|^2 + t_i + t_j > 0.$$

Theorem 5.7. *Let $n \geq 4$. Let M be a $(2, 2)$ Griffiths positive matrix with diagonalized entries. Then $\Omega = \det(M)$ is a Lefschetz form for bidegree $(n-2, 0)$, $(n-3, 1)$, $(1, n-3)$ and $(0, n-2)$.*

Proof. The technique is the same as in Theorem 5.6. We only need to choose basis carefully.

For bidegree $(n-2, 0)$, take the lexicographical ordered basis

$$\{dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n-2}, \dots, dz_3 \wedge \cdots \wedge dz_n\}$$

of $V^{n-2,0}$ and take the Hodge-star of their conjugates as basis of $V^{n,2}$. Then the matrix form of the linear map $\wedge\Omega : V^{n-2,0} \rightarrow V^{n,2}$ is $\text{diag}(\Omega_{n-1,n}, \Omega_{n-2,n}, \dots, \Omega_{1,2})$ where the indices are in the reversed lexicographical order. Each $\Omega_{i,j} > 0$ implies that Ω is a Lefschetz form for bidegree $(n-2, 0)$.

For bidegree $(n-3, 1)$, take basis of $V^{n-3,1}$ as follows

$$\left\{ \begin{array}{ll} dz_1 \wedge \cdots \wedge dz_{n-4} \wedge V_{n-3}, & dz_1 \wedge \cdots \wedge dz_{n-4} \wedge V_{n-2}, \\ dz_1 \wedge \cdots \wedge dz_{n-4} \wedge V_{n-1}, & dz_1 \wedge \cdots \wedge dz_{n-4} \wedge V_n, \\ dz_1 \wedge \cdots \wedge dz_{n-4} \wedge dz_{n-3} \wedge \overline{dz_{n-2}}, & \dots, dz_1 \wedge \cdots \wedge dz_{n-4} \wedge dz_n \wedge \overline{dz_{n-1}}, \\ dz_1 \wedge \cdots \wedge dz_{n-5} \wedge dz_{n-3} \wedge V_{n-4}, & dz_1 \wedge \cdots \wedge dz_{n-5} \wedge dz_{n-3} \wedge V_{n-2}, \\ dz_1 \wedge \cdots \wedge dz_{n-5} \wedge dz_{n-3} \wedge V_{n-1}, & dz_1 \wedge \cdots \wedge dz_{n-5} \wedge dz_{n-3} \wedge V_n, \\ dz_1 \wedge \cdots \wedge dz_{n-5} \wedge dz_{n-3} \wedge dz_{n-4} \wedge \overline{dz_{n-2}}, & \dots, dz_1 \wedge \cdots \wedge dz_{n-5} \wedge dz_{n-3} \wedge dz_n \wedge \overline{dz_{n-1}}, \\ & \dots, \\ dz_5 \wedge \cdots \wedge dz_n \wedge V_1, & dz_5 \wedge \cdots \wedge dz_n \wedge V_2, \\ dz_5 \wedge \cdots \wedge dz_n \wedge V_3, & dz_5 \wedge \cdots \wedge dz_n \wedge V_4, \\ dz_5 \wedge \cdots \wedge dz_n \wedge dz_1 \wedge \overline{dz_2}, & \dots, dz_5 \wedge \cdots \wedge dz_n \wedge dz_4 \wedge \overline{dz_3}, \end{array} \right\}$$

and take basis of $V^{n-1,3}$ by the Hodge-star of their conjugates. Then the matrix form of the linear map $\wedge\Omega : V^{n-3,1} \rightarrow V^{n-1,3}$ is

$$\begin{pmatrix} G_{n-3,n-2,n-1,n} & 0 & \cdots & 0 & 0 \\ 0 & \text{diag}\{\Omega_{n-1,n}, \dots, \Omega_{n-3,n-2}\} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & G_{1,2,3,4} & 0 \\ 0 & 0 & \cdots & 0 & \text{diag}\{\Omega_{3,4}, \dots, \Omega_{1,2}\} \end{pmatrix}$$

where each

$$G_{i_1,i_2,i_3,i_4} = \begin{pmatrix} 0 & \Omega_{i_3,i_4} & \Omega_{i_2,i_4} & \Omega_{i_2,i_3} \\ \Omega_{i_3,i_4} & 0 & \Omega_{i_1,i_4} & \Omega_{i_1,i_3} \\ \Omega_{i_2,i_4} & \Omega_{i_1,i_4} & 0 & \Omega_{i_1,i_2} \\ \Omega_{i_2,i_3} & \Omega_{i_1,i_3} & \Omega_{i_1,i_2} & 0 \end{pmatrix}$$

is invertible. So Ω is a Lefschetz form for bidegree $(n-3, 1)$. \square

Corollary 5.8. *Let $n = 4, 5$. Let M be a $(2, 2)$ Griffiths positive matrix with diagonalized entries. Then $\det(M)$ is a Hodge-Riemann form.*

5.3 Difficulty of the case $k = 2$, $n = 6$ and M with diagonalized entries

We already checked that Ω is a Lefschetz form for bidegree $(4, 0)$, $(3, 1)$, $(1, 3)$ and $(0, 4)$. The only thing left is the bidegree $(2, 2)$. It amounts to prove that the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega[5, 6] & \Omega[4, 6] & \Omega[4, 5] & \Omega[3, 6] & \Omega[3, 5] & \Omega[3, 4] \\ 0 & 0 & 0 & 0 & 0 & 0 & \Omega[5, 6] & \Omega[4, 6] & \Omega[4, 5] & 0 & 0 & 0 & \Omega[2, 6] & \Omega[2, 5] & \Omega[2, 4] \\ 0 & 0 & 0 & 0 & 0 & \Omega[5, 6] & 0 & \Omega[3, 6] & \Omega[3, 5] & 0 & \Omega[2, 6] & \Omega[2, 5] & 0 & 0 & \Omega[2, 3] \\ 0 & 0 & 0 & 0 & 0 & \Omega[4, 6] & \Omega[3, 6] & 0 & \Omega[3, 4] & \Omega[2, 6] & 0 & \Omega[2, 4] & 0 & \Omega[2, 3] & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega[4, 5] & \Omega[3, 5] & \Omega[3, 4] & 0 & \Omega[2, 5] & \Omega[2, 4] & 0 & \Omega[2, 3] & 0 & 0 \\ 0 & 0 & \Omega[5, 6] & \Omega[4, 6] & \Omega[4, 5] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega[1, 6] & \Omega[1, 5] & \Omega[1, 4] \\ 0 & \Omega[5, 6] & 0 & \Omega[3, 6] & \Omega[3, 5] & 0 & 0 & 0 & 0 & 0 & \Omega[1, 6] & \Omega[1, 5] & 0 & 0 & \Omega[1, 3] \\ 0 & \Omega[4, 6] & \Omega[3, 6] & 0 & \Omega[3, 4] & 0 & 0 & 0 & 0 & \Omega[1, 6] & 0 & \Omega[1, 4] & 0 & \Omega[1, 3] & 0 \\ 0 & \Omega[4, 5] & \Omega[3, 5] & \Omega[3, 4] & 0 & 0 & 0 & 0 & 0 & \Omega[1, 5] & \Omega[1, 4] & 0 & \Omega[1, 3] & 0 & 0 \\ \Omega[5, 6] & 0 & 0 & \Omega[2, 6] & \Omega[2, 5] & 0 & 0 & \Omega[1, 6] & \Omega[1, 5] & 0 & 0 & 0 & 0 & 0 & \Omega[1, 2] \\ \Omega[4, 6] & 0 & \Omega[2, 6] & 0 & \Omega[2, 4] & 0 & \Omega[1, 6] & 0 & \Omega[1, 4] & 0 & 0 & 0 & 0 & \Omega[1, 2] & 0 \\ \Omega[4, 5] & 0 & \Omega[2, 5] & \Omega[2, 4] & 0 & 0 & \Omega[1, 5] & \Omega[1, 4] & 0 & 0 & 0 & 0 & \Omega[1, 2] & 0 & 0 \\ \Omega[3, 6] & \Omega[2, 6] & 0 & 0 & \Omega[2, 3] & \Omega[1, 6] & 0 & 0 & \Omega[1, 3] & 0 & 0 & \Omega[1, 2] & 0 & 0 & 0 \\ \Omega[3, 5] & \Omega[2, 5] & 0 & \Omega[2, 3] & 0 & \Omega[1, 5] & 0 & \Omega[1, 3] & 0 & 0 & \Omega[1, 2] & 0 & 0 & 0 & 0 \\ \Omega[3, 4] & \Omega[2, 4] & \Omega[2, 3] & 0 & 0 & \Omega[1, 4] & \Omega[1, 3] & 0 & 0 & \Omega[1, 2] & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is invertible. However, unlike the case before, the determinant is irreducible in $\mathbb{C}[\sqrt{\Omega_{i,j}}]$. Indeed, by a Mathematica program, the determinant is irreducible on $\mathbb{C}[\Omega_{i,j}^{1/t}]$ for $t = 1, 2, \dots, 15$. It is difficult to prove that this determinant is non-zero by the techniques, analogues of Heron formula, as before.

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