

LOEWNER THEORY FOR BERNSTEIN FUNCTIONS II: APPLICATIONS TO INHOMOGENEOUS CONTINUOUS-STATE BRANCHING PROCESSES

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ABSTRACT. The main purpose of this paper is to study time-inhomogeneous one-dimensional branching processes (mainly on a continuous but also on a discrete state space) with the help of recent achievements in Loewner Theory dealing with evolution families of holomorphic self-maps in simply connected domains of the complex plane. Under a suitable stochastic continuity condition, we show that the families of the Laplace exponents of branching processes on $[0, \infty]$ can be characterized as topological (i.e. depending continuously on the time parameters) reverse evolution families whose elements are Bernstein functions. For the case of a stronger regularity w.r.t. time, we establish a Loewner–Kufarev type ODE for the Laplace exponents and characterize branching processes with finite mean in terms of the vector field driving this ODE. Similar results are obtained for families of probability generating functions for branching processes on the discrete state space $\{0, 1, 2, \dots\} \cup \{\infty\}$. In addition, we find necessary and sufficient conditions for “spatial” embeddability of such branching processes into branching processes on $[0, \infty]$. Finally, we give some probabilistic interpretations of the Denjoy–Wolff point at 0 and at ∞ .

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1. INTRODUCTION

Branching processes are certain Markov processes with various applications including biological population growth e.g. in epidemic models [8]. The case of discrete time and one-dimensional discrete state (i.e. the set of non-negative integers) was rather classical and it goes back to the work of Bienaymé, Galton and Watson in the 19th century; see [2, 27, 31]. The continuous time setting — mostly in the *time-homogeneous* case — also has been intensively studied in the literature; we refer

the reader to [2, 27] for the discrete space case and to [32] for the (one-dimensional) continuous state case.

In the continuous state setting, a crucial tool for analyzing branching processes is one-parameter (compositional) semigroups of Laplace exponents of transition kernels. The Laplace exponents in this case are Bernstein functions, which are known to appear as the Laplace exponents of subordinators too. Hence, one-parameter semigroups of Bernstein functions seem to play an important role in this context.

Time-inhomogeneous branching processes have been also investigated; see e.g. [27, 30] and references therein, but this area seems to be considerably less explored compared with the time-homogeneous case. To some extent, this situation is perhaps because, from the mathematical aspect, one needs to go beyond the well developed theory of one-parameter semigroups and deal with the so-called reverse evolution families. One of the difficulties arising in this case is related to the regularity w.r.t. time: for one-parameter compositional semigroups, continuity in time often implies differentiability or even real analyticity, but such is not the case for evolution families.

The main purpose of this paper is to apply to the study of time-inhomogeneous one-dimensional branching processes recent achievements on a Carathéodory-type ODE for (reverse) evolution families of holomorphic functions developed in the frames of Loewner Theory since around 2010; see e.g. [5, 7, 12]. We mainly work with the continuous state space $[0, \infty]$ but also discuss the discrete states $\{0, 1, 2, \dots\} \cup \{\infty\}$. In fact, the idea of using reverse evolution families to analyze time-inhomogeneous branching processes appeared for the case of discrete state space in Goryainov [21, 22], see also [23, p. 1012]. We will more systematically develop this idea, especially for the continuous state space $[0, \infty]$. In particular, we are aiming at finding interpretations of some relevant notions in Complex Analysis from the viewpoint of Probability Theory and vice versa. Worth mentioning that this adds another application of Loewner Theory in Probabilities; other previously known applications are mostly limited to SLE, see e.g. [33], and certain aspects of non-commutative probability [3, 17, 29].

The work has been divided into two parts. In the first part [25] we have prepared complex-analytic tools related to reverse evolution families, which are applied in the second part, i.e. in the present paper, to time-inhomogeneous branching processes. Here is a brief summary of the most important results included in the second part:

- (1) a one-to-one correspondence is established between transition kernels of branching processes and topological reverse evolution families consisting of Bernstein functions (Theorem 2.5),
- (2) a one-to-one correspondence is established between absolutely continuous reverse evolution families consisting of Bernstein functions and a certain class of Herglotz vector fields (Theorem 2.13),
- (3) a branching process has finite first moments at any time if and only if the point 0 is a boundary regular fixed point of the Laplace exponents (Theorem 2.16),
- (4) similar results are obtained for branching processes on the discrete space (Theorems 3.3, 3.4, 3.7),
- (5) a branching process on the discrete space can be embedded into a branching process on the continuous space if and only if the given branching process is “non-decreasing” in a certain sense, see Theorem 3.9 for the precise statement.

Some closely related results have been recently obtained by Feng and Li [16]. They constructed a wide class of time-inhomogeneous branching processes on $[0, \infty]$ by solving an integral equation for the Laplace exponents. Moreover, they wrote a stochastic differential equation that the corresponding stochastic process satisfies. The integral equation considered in [16] is of rather general

form and its solutions do not even have to be continuous w.r.t. the time parameter. In the present paper, we mainly work in a less general setting, with the advantage of being able to obtain stronger results, see Remarks 2.6 and 2.15 for a more detailed discussion.

In the remaining part of this section, some preparatory facts in probability theory and complex analysis (mostly taken from the literature) are collected.

1.1. One-parameter semigroups of holomorphic self-maps. For a domain $D \subset \mathbb{C}$ and a non-empty set $E \subset \mathbb{C}$, denote by $\text{Hol}(D, E)$ the set of all holomorphic maps of D into E . We make $\text{Hol}(D, E)$ a topological space by endowing it with locally uniform convergence in D .

A *semigroup of holomorphic self-maps* of a domain $D \subset \mathbb{C}$ is a subset $\mathfrak{U} \subset \text{Hol}(D, D)$ containing the identity map id_D and such that $f \circ g \in \mathfrak{U}$ for any $f, g \in \mathfrak{U}$. We say that such a semigroup \mathfrak{U} is *topologically closed* if \mathfrak{U} is a closed subset of $\text{Hol}(D, D)$.

Let \mathfrak{U} be a semigroup of holomorphic self-maps of a domain $D \subset \mathbb{C}$. A family $(v_t)_{t \geq 0} \subset \mathfrak{U}$ is said to be a *one-parameter semigroup* in \mathfrak{U} , if the map $t \mapsto v_t$ is a continuous semigroup homomorphism from the semigroup $([0, +\infty), +)$ to the semigroup (\mathfrak{U}, \circ) .

Convention. From now on we suppose that D is conformally equivalent to the unit disk $\mathbb{D} := \{z : |z| < 1\}$. In all other cases, the theory of one-parameter semigroups is either trivial or can be reduced to the case $D = \mathbb{D}$, see [25, Section 2.1] for references and more details.

According to a result of Berkson and Porta [4], a one-parameter semigroup $(v_t)_{t \geq 0}$ in $\text{Hol}(D, D)$ is differentiable in $t \geq 0$ and can be obtained by a unique holomorphic function (called the *infinitesimal generator*) $\phi : D \rightarrow \mathbb{C}$ via the ODE

$$(1.1) \quad \frac{d}{dt} v_t(z) = -\phi(v_t(z)), \quad t \geq 0, z \in D.$$

For a semigroup \mathfrak{U} of holomorphic self-maps of D , we will denote by $\mathcal{G}(\mathfrak{U})$ the set of all infinitesimal generators of one-parameter semigroups in \mathfrak{U} . According to [25, Theorem 1], if \mathfrak{U} is topologically closed, then $\mathcal{G}(\mathfrak{U})$ is a closed real convex cone in $\text{Hol}(D, \mathbb{C})$.

1.2. Bernstein functions. A function $f : (0, \infty) \rightarrow \mathbb{R}$ is called a *Bernstein function* if it is C^∞ , $f \geq 0$ and $(-1)^{n-1} f^{(n)} \geq 0$ for all $n \in \mathbb{N}$. Thanks to monotonicity of $f^{(n)}$, there exist (finite or infinite) limits

$$(1.2) \quad f^{(n)}(0) := \lim_{\theta \rightarrow 0^+} f^{(n)}(\theta) \quad \text{and} \quad f^{(n)}(\infty) := \lim_{\theta \rightarrow \infty} f^{(n)}(\theta), \quad n \in \mathbb{N} \cup \{0\}.$$

Note that the value $f(0)$ is finite because f is non-decreasing and non-negative.

Below we mention some useful basic facts on Bernstein functions. Proofs and further details can be found, e.g. in [40]. In particular, it is known that a function $f : (0, \infty) \rightarrow \mathbb{R}$ is a Bernstein function if and only if it admits the following representation:

$$(1.3) \quad f(\theta) = \alpha + \beta\theta + \int_{(0, \infty)} (1 - e^{-\theta x}) \tau(dx), \quad \theta > 0,$$

where

$$(1.4) \quad \begin{aligned} &\alpha, \beta \geq 0 \text{ and } \tau \text{ is a non-negative Borel measure on } (0, \infty) \\ &\text{with } \int_0^\infty \min\{1, x\} \tau(dx) < \infty. \end{aligned}$$

For each Bernstein function f , the triplet (α, β, τ) is unique.

From this integral representation it easily follows that every Bernstein function extends to a holomorphic map from the right half-plane $\mathbb{H} := \{\zeta \in \mathbb{C} : \text{Re}(\zeta) > 0\}$ into its closure $\overline{\mathbb{H}}$. From now on, a holomorphic function in $\text{Hol}(\mathbb{H}, \overline{\mathbb{H}})$ whose restriction to $(0, \infty)$ is a Bernstein function will be also referred to as a Bernstein function.

Remark 1.1. Applying the maximum principle to the harmonic function $-\operatorname{Re} f$, it is easy to show that if a Bernstein function f is not identically zero, then it is a self-map of \mathbb{H} . Therefore, the set \mathfrak{BF} of all Bernstein functions $f \not\equiv 0$ is a subset of $\operatorname{Hol}(\mathbb{H}, \mathbb{H})$. Moreover, \mathfrak{BF} is a topologically closed semigroup in \mathbb{H} .

The class \mathfrak{BF} has been a basic tool for studying time-homogeneous branching processes; see [24, 32, 41]. Among others, Silverstein [41, Theorem 4] identified $\mathcal{G}(\mathfrak{BF})$, the set of all infinitesimal generators of the one-parameter semigroups in \mathfrak{BF} , with the set of functions

$$(1.5) \quad \phi(\zeta) = -q + a\zeta + b\zeta^2 + \int_0^\infty (e^{-\zeta x} - 1 + \zeta x \mathbf{1}_{(0,1)}(x)) \pi(dx), \quad \zeta \in \mathbb{H},$$

where $a \in \mathbb{R}$, $q, b \geq 0$, and π is a *Lévy measure measure*, i.e. a non-negative Borel measure on $(0, \infty)$ satisfying

$$(1.6) \quad \int_0^\infty \min\{1, x^2\} \pi(dx) < \infty.$$

Note that the quadruple (q, a, b, π) is unique for each ϕ and hence this quadruple completely parameterizes the set $\mathcal{G}(\mathfrak{BF})$. Other proofs of the representation (1.5) are available in [9] and [25, Section 3.3].

Example 1.2. Let $\phi(\zeta) := a\zeta + b\zeta^2$ with $b > 0$ and $a \in \mathbb{R}$. For $a \neq 0$, solving the differential equation $\frac{d}{dt}v_t(\zeta) = -av_t(\zeta) - bv_t(\zeta)^2$, $v_0 = \operatorname{id}_{\mathbb{H}}$ yields an explicit formula for the one-parameter semigroup in \mathfrak{BF} generated by ϕ ,

$$v_t(\zeta) = \frac{e^{-at}\zeta}{1 + \kappa(t)\zeta} = \frac{e^{-at}}{\kappa(t)^2} \int_0^\infty (1 - e^{-x\zeta}) e^{-x/\kappa(t)} dx, \quad \kappa(t) := \frac{b}{a}(1 - e^{-at}), \quad t > 0.$$

The case $a = 0$ can be covered by taking the limit:

$$v_t(\zeta) = \frac{\zeta}{1 + bt\zeta} = \int_0^\infty (1 - e^{-x\zeta}) \frac{e^{-\frac{x}{bt}}}{(bt)^2} dx, \quad t > 0.$$

One of the main achievements in this paper is that we are able to extend Silverstein's theorem to the time-inhomogeneous setting, see Section 2.2. This requires complex-analytic tools extending the theory of one-parameter semigroups to an “non-autonomous” setting, as we describe below.

1.3. Reverse evolution families. A non-autonomous version of the ODE (1.1), i.e. an equation of the same form but with the vector field ϕ depending explicitly on t , has been intensively studied in Complex Analysis. One-parameter semigroups in this setting are replaced by more general (reverse) evolution families of holomorphic self-maps.

For later use, let Δ be the set $\{(s, t) : s, t \in [0, \infty), s \leq t\}$, AC_{loc} be the set of the (real- or complex-valued) functions on $[0, \infty)$ which are absolutely continuous on each compact subinterval of $[0, \infty)$, and L_{loc}^1 be the set of all measurable (real- or complex-valued) functions on $[0, \infty)$ which are integrable with respect to the Lebesgue measure on each compact subinterval of $[0, \infty)$.

The following definitions are slight modifications of the definitions given in [5, 12].

Definition 1.3. Let $D \subsetneq \mathbb{C}$ be a simply connected domain. A function $\phi : D \times [0, \infty) \rightarrow \mathbb{C}$ is said to be a *Herlotz vector field* in D if it satisfies the following three conditions:

- (HVF1) for any $t \geq 0$, $\phi(\cdot, t)$ is an infinitesimal generator in D ;
- (HVF2) for any $z \in D$, $\phi(z, \cdot)$ is measurable on $[0, \infty)$;
- (HVF3) for any compact set $K \subset D$, there is a non-negative function $M_K \in L_{\text{loc}}^1$ such that

$$\max_{z \in K} |\phi(z, t)| \leq M_K(t) \quad \text{a.e. } t \in [0, \infty).$$

As usual we identify two Herglotz vector fields $\phi, \psi: D \times [0, \infty) \rightarrow \mathbb{C}$ if $\phi(\cdot, t) = \psi(\cdot, t)$ for a.e. $t \in [0, \infty)$.

The notion of reverse evolution families provides a way to describe solutions to the ODE driven by a Herglotz vector field.

Definition 1.4. An *absolutely continuous reverse evolution family* in a simply connected domain $D \subsetneq \mathbb{C}$ is a two-parameter family $(v_{s,t})_{(s,t) \in \Delta} \subset \text{Hol}(D, D)$ satisfying the following conditions:

- (REF1) $v_{s,s} = \text{id}_D$ for any $s \geq 0$,
- (REF2) $v_{s,u} = v_{s,t} \circ v_{t,u}$ for any $0 \leq s \leq t \leq u$,
- (REF3) For each $z \in D$ there exists a non-negative function $f_z \in L^1_{\text{loc}}$ such that

$$|v_{s,u}(z) - v_{s,t}(z)| \leq \int_t^u f_z(r) dr$$

for any $0 \leq s \leq t \leq u$.

Remark 1.5. From condition (REF2) it follows that $v_{s,t} = v_{0,s}^{-1} \circ v_{0,t}$ for any $(s, t) \in \Delta$. Therefore, the information about the whole reverse evolution family $(v_{s,t})$ is contained in its one-parameter subfamily $(v_{0,t})_{t \geq 0}$, which is the decreasing Loewner chains associated to $(v_{s,t})$, see [12, Definition 1.6 and Theorem 4.1]. A remarkable geometric property of this family is that $v_{0,t}(D) \subset v_{0,s}(D)$ for any $(s, t) \in \Delta$. Loewner chains play an important role in Loewner Theory. However, in the frames of the present paper we do not use this notion explicitly.

Furthermore, a two-parameter family $(v_{s,t})_{(s,t) \in \Delta} \subset \text{Hol}(D, D)$ is called a *topological reverse evolution family* if it satisfies (REF1), (REF2) and

- (REF3') $\Delta \ni (s, t) \mapsto v_{s,t} \in \text{Hol}(D, D)$ is continuous.

Remark 1.6. Due to normality of the family formed by all holomorphic self-mappings of D , see e.g. [18, §II.7], condition (REF3') is equivalent to the following seemingly weaker one:

- (REF3'') For each $z \in D$ the map $\Delta \ni (s, t) \mapsto v_{s,t}(z) \in D$ is continuous.

Any absolutely continuous reverse evolution family satisfies (REF3') and hence it is also a topological reverse evolution family; this is a consequence of [5, Proposition 3.5] and [12, Proposition 4.3].

From the viewpoint of dynamics in the complex plane, it is more natural to consider the condition $v_{t,u} \circ v_{s,t} = v_{s,u}$ ($0 \leq s \leq t \leq u$) rather than (REF2); such a family is called an (absolutely continuous / topological) *evolution family*. On the other hand, in the existing applications to (non-commutative) probability theory, reverse evolution families have been more natural; see [17, 33]. In the present work where branching processes are analyzed, again reverse evolution families turn out to be the natural objects.

Some of the main results of [12] can be summarized and stated in our notation as follows.

Theorem A. *Let $D \subsetneq \mathbb{C}$ be a simply connected domain and $(v_{s,t})_{(s,t) \in \Delta}$ an absolutely continuous reverse evolution family in D . Then there exists a unique Herglotz vector field $\phi: D \times [0, \infty) \rightarrow \mathbb{C}$ such that for any $z \in D$ and any $t \in (0, \infty)$, the map $[0, t] \ni s \mapsto v(s) := v_{s,t}(z) \in D$ is a solution to the initial value problem*

$$(1.7) \quad \frac{dv}{ds}(s) = \phi(v(s), s), \quad \text{a.e. } s \in [0, t]; \quad v(t) = z.$$

Conversely, let $\phi: D \times [0, \infty) \rightarrow \mathbb{C}$ be a Herglotz vector field. Then for any $z \in D$ and any $t \in (0, \infty)$ the initial value problem (1.7) has a unique solution $s \mapsto v(s) = v(s; z, t)$ defined on

some interval containing the set $[0, t]$. Setting $v_{s,t}(z) := v(s; z, t)$ for all $z \in D$ and all $(s, t) \in \Delta$ one obtains an absolutely continuous reverse evolution family $(v_{s,t})_{(s,t) \in \Delta}$.

As clear from the definition of a Herglotz vector field, the r.h.s. of the equation (1.7) is holomorphic and hence locally Lipschitz in the “spatial” complex variable z , but it is not supposed to be even continuous in the temporal variable s . This equation should be, therefore, understood as a Carathéodory-type ODE; see e.g. [11, Section 2] for the basic theory of such ODEs.

Remark 1.7. The ODE (1.7) implies the following PDE (in the sense of [12, Definition 2.1]) for each fixed $s \geq 0$:

$$(1.8) \quad \frac{\partial v_{s,t}(z)}{\partial t} + \phi(z, t) \frac{\partial v_{s,t}(z)}{\partial z} = 0, \quad \text{a.e. } t \in [s, \infty), \quad z \in D; \quad v_{s,s} = \text{id}_D.$$

In Loewner Theory, equations (1.7) and (1.8) are known as the (generalized) Loewner–Kufarev differential equations. In the context of branching processes, see Theorems 2.13 and 3.4, equations (1.7) and (1.8) correspond to *Kolmogorov’s backward and forward equations*, respectively.

In fact, (1.7) can be replaced with (1.8), see [25, Remark 2.12] for details. In particular, if $(v_{s,t})_{(s,t) \in \Delta}$ is an absolutely continuous reverse evolution family in D , then there exists a unique Herglotz vector ϕ in D such that (1.8) holds for every $s \in [0, \infty)$. Conversely, given a Herglotz vector ϕ in D , then for each fixed $s \in [0, \infty)$, the initial value problem (1.8) has a unique solution $(z, t) \mapsto v_{s,t}(z)$ and the functions $v_{s,t}$ form an absolutely continuous reverse evolution family $(v_{s,t})_{(s,t) \in \Delta}$.

According to Theorem A, there is a one-to-one correspondence between absolutely continuous reverse evolution families and Herglotz vector fields. In later applications to branching processes, we would like to restrict the maps $v_{s,t}$ to be Bernstein functions. Then a natural question is to describe the class of Herglotz vector fields that corresponds to the absolutely continuous reverse evolution families consisting of Bernstein functions. In [25] we answer a more general question.

Theorem B ([25, Theorem 2]). *Let $D \subsetneq \mathbb{C}$ be a simply connected domain and \mathfrak{U} a topologically closed semigroup of holomorphic self-maps of D . Let $(v_{s,t})_{(s,t) \in \Delta}$ be an absolutely continuous reverse evolution family in D with associated Herglotz vector field ϕ . Then the following two conditions are equivalent:*

- (i) $v_{s,t} \in \mathfrak{U}$ for all $(s, t) \in \Delta$;
- (ii) $\phi(\cdot, t) \in \mathcal{G}(\mathfrak{U})$ for a.e. $t \in [0, \infty)$.

We will say that an absolutely continuous (or topological) reverse evolution family $(v_{s,t})_{(s,t) \in \Delta}$ is contained in \mathfrak{U} if condition (i) of Theorem B is satisfied.

1.4. Fixed points of holomorphic self-maps. To make this paper more self-contained we cite here a few definitions and the most basic results gathered in [25, Section 2.3]. The proofs can be found, e.g., in the monographs [1] and [6], where this topic is covered in depth.

Fix a holomorphic self-map $v \in \mathbb{D} \rightarrow \mathbb{D}$ different from $\text{id}_{\mathbb{D}}$. It is known that

- (i) either v is an elliptic automorphism of \mathbb{D} , i.e. a Möbius transformation mapping \mathbb{D} onto itself and having one fixed point τ in \mathbb{D} ,
- (ii) or there exists a (unique) point τ in the closure $\overline{\mathbb{D}}$ of \mathbb{D} such that

$$v^{on} := \underbrace{v \circ \dots \circ v}_{n \text{ times}} \rightarrow \tau$$

locally uniformly in \mathbb{D} as $n \rightarrow +\infty$.

In both cases, the point τ is referred to as the *Denjoy–Wolff point* of v , or in short, the *DW-point*.

If the DW-point τ of $v \in \text{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\text{id}_{\mathbb{D}}\}$ lies inside \mathbb{D} , then it is the unique fixed point of v in \mathbb{D} and $|v'(\tau)| \leq 1$ with the equality in case (i) and strict inequality in case (ii). If $\tau \in \partial\mathbb{D}$, then v has no fixed points in \mathbb{D} . In this case, τ can be still considered a fixed point in the following sense. A *boundary fixed point* of v is a point $\sigma \in \partial\mathbb{D}$ such that $\angle \lim_{z \rightarrow \sigma} v(z) = \sigma$, where $\angle \lim$ denotes the so-called angular or non-tangential limit, see e.g. [6, Section 1.5]. For any boundary fixed point σ , the limit

$$v'(\sigma) := \angle \lim_{z \rightarrow \sigma} \frac{v(z) - \sigma}{z - \sigma},$$

known as the *angular derivative* of v at σ , exists finite or infinite. If $v'(\sigma) \neq \infty$, then σ is said to be a *boundary regular fixed point* of v , or *BRFP* for short.

If the DW-point $\tau \in \partial\mathbb{D}$, then τ is a BRFP and $v'(\tau) \in (0, 1]$. Conversely, if σ is BRFP but not the DW-point of v , then $v'(\sigma) \in (1, \infty)$.

All mentioned above can be extended to holomorphic self-maps of \mathbb{H} with the help of the conformal mapping

$$\mathbb{D} \ni \zeta \mapsto \frac{1 + \zeta}{1 - \zeta} \in \mathbb{H}.$$

The only significant difference is that for the boundary point $\sigma = \infty$ the angular derivative is defined by

$$v'(\infty) = \angle \lim_{z \rightarrow \infty} \frac{v(z)}{z}.$$

By the Wolff Lemma (also known as the half-plane version of the Julia Lemma), the above limit exists and it is a non-negative real number. The point ∞ is a BRFP of $v \in \text{Hol}(\mathbb{H}, \mathbb{H}) \setminus \{\text{id}_{\mathbb{H}}\}$ if and only if $v'(\infty) \neq 0$. Similarly, ∞ is the DW-point if and only if $v'(\infty) \geq 1$.

Remark 1.8. If (v_t) is a one-parameter semigroup in $\text{Hol}(\mathbb{D}, \mathbb{D})$ or in $\text{Hol}(\mathbb{H}, \mathbb{H})$, then all the elements of (v_t) different from the identity map have the same DW-point and the same boundary (regular) fixed points.

Remark 1.9. The most interesting case for the purposes of this paper is when $v : \mathbb{H} \rightarrow \mathbb{H}$ is a Bernstein function. Since in such a case $(0, \infty)$ is mapped into itself, the DW-point of v belongs to $[0, \infty]$. Notice also that existence of the angular limit trivially implies the existence of the corresponding radial limit. On the other hand, by Lindelöf's Theorem, see e.g. [6, Theorem 1.5.7 on p. 27], if $v \in \text{Hol}(\mathbb{H}, \mathbb{H})$ has a radial limit at some boundary point, then it has an angular limit at the same point. It follows that, using (1.2) as a definition of $v(\sigma)$ and $v'(\sigma)$ for $\sigma \in \{0, \infty\}$, we have that σ is a boundary fixed point of $v \in \mathfrak{BF}$ if and only if $v(\sigma) = \sigma$ and that in such a case the angular derivative of v at σ coincides with $v'(\sigma)$ as defined by (1.2).

2. TIME-INHOMOGENEOUS BRANCHING PROCESSES

In this section, we will establish our main results concerning time-inhomogeneous branching processes on a continuous one-dimensional state space.

2.1. Branching processes and topological reverse evolution families. Let S be a Polish space. Denote by $\mathcal{B}(S)$ the σ -field of all Borel sets in S and by $\mathbf{P}(S)$ the set of all Borel probability measures on S . A *transition kernel* k on S is a map $k : S \times \mathcal{B}(S) \rightarrow [0, 1]$ such that $k(x, \cdot) \in \mathbf{P}(S)$ for each $x \in S$ and $x \mapsto k(x, B)$ is Borel measurable for each $B \in \mathcal{B}(S)$. For two transition kernels k, l on S their *composition* is defined by

$$(k \circ l)(x, B) = \int_S l(y, B) k(x, dy), \quad x \in S, \quad B \in \mathcal{B}(S).$$

Note that for all $x \in S$ and all non-negative Borel measurable functions $f: S \rightarrow [0, \infty]$ we have

$$(2.1) \quad \int_S f(z) (k \circ l)(x, dz) = \int_S \left[\int_S f(z) l(y, dz) \right] k(x, dy).$$

Indeed, this equality holds obviously for the indicators of Borel subsets of S and hence for all real-valued simple functions f . To recover the general case, it is sufficient to approximate f by simple functions and apply the monotone convergence theorem.

Let $(k_{s,t})_{(s,t) \in \Delta}$ be a family of transition kernels such that

$$(K1) \quad k_{s,s}(x, \cdot) = \delta_x(\cdot) \text{ for every } s \geq 0 \text{ and } x \in S,$$

$$(K2) \quad k_{s,t} \circ k_{t,u} = k_{s,u} \text{ for every } 0 \leq s \leq t \leq u. \quad (\text{Chapman-Kolmogorov equation})$$

Then a *Markov family* on S with transition kernels $(k_{s,t})_{(s,t) \in \Delta}$ is a stochastic process $(Z_t)_{t \geq 0}$ defined on a measure space (Ω, \mathcal{F}) equipped with a family $\{\mathcal{F}_I : I \text{ is a closed interval of } [0, \infty)\}$ of sub- σ -fields of \mathcal{F} and probability measures $\mathbb{P}^{(s,x)}$ on $(\Omega, \mathcal{F}_{[s,\infty)})$ for each $s \geq 0$ and $x \in S$ such that

$$(2.2) \quad I \subseteq J \text{ implies } \mathcal{F}_I \subseteq \mathcal{F}_J,$$

$$(2.3) \quad Z_t \text{ is } \mathcal{F}_I\text{-measurable whenever } t \in I,$$

$$(2.4) \quad \mathbb{P}^{(s,x)}[Z_s = x] = 1 \quad \text{for all } s \geq 0, x \in S \quad \text{and}$$

$$(2.5) \quad \mathbb{E}^{(s,x)}[f(Z_u) | \mathcal{F}_{[s,t]}] = \int_S f(y) k_{t,u}(Z_t, dy) \quad \mathbb{P}^{(s,x)}\text{-a.s.} \quad \text{for all } 0 \leq s \leq t \leq u, x \in S,$$

and all bounded measurable functions $f: (S, \mathcal{B}(S)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

When $k_{s,t} = k_{0,t-s}$ for all $(s,t) \in \Delta$, the transition kernels and the corresponding Markov family are said to be *time-homogeneous*.

Formulas (2.4) and (2.5) imply

$$(2.6) \quad \mathbb{P}^{(s,x)}[(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n}) \in B] = \int_B k_{s,t_1}(x, dx_1) k_{t_1,t_2}(x_1, dx_2) \cdots k_{t_{n-1},t_n}(x_{n-1}, dx_n)$$

for all $B \in \mathcal{B}(S^n)$, $n \in \mathbb{N}$, $0 \leq s \leq t_1 \leq \dots \leq t_n$ and $x \in S$, see [42, Problem 8.11]. The l.h.s. of (2.6) is called a finite dimensional distribution of (Z_t) . For further details on Markov families, we refer the reader to [42].

As usual, we denote by $*$ the convolution of probability measures on $[0, \infty]$.

Definition 2.1. By a *time-inhomogeneous continuous-state branching process* (simply called a *branching process* below) we mean a Markov family on $S := [0, \infty]$ (regarded as a compact set) with transition kernels $(k_{s,t})_{(s,t) \in \Delta}$ satisfying (K1) and (K2) above and the following four conditions:

$$(K3) \quad \text{the map } \Delta \times [0, \infty) \ni (s, t, x) \mapsto k_{s,t}(x, \cdot) \in \mathcal{P}([0, \infty]) \text{ is weakly continuous,}$$

$$(K4) \quad k_{s,t}(x, \cdot) * k_{s,t}(y, \cdot) = k_{s,t}(x + y, \cdot) \text{ for any } (s, t) \in \Delta, x, y \in [0, \infty), \quad (\text{Branching property})$$

$$(K5) \quad k_{s,t}(0, \cdot) = \delta_0(\cdot) \text{ for every } (s, t) \in \Delta,$$

$$(K6) \quad k_{s,t}(\infty, \cdot) = \delta_\infty(\cdot) \text{ for every } (s, t) \in \Delta.$$

Such a family $(k_{s,t})_{(s,t) \in \Delta}$ on $[0, \infty]$ that satisfies (K1)–(K6) will be called a *family of transition kernels of a branching process*.

Remark 2.2. Properties (K5) and (K6) mean that 0 and ∞ are absorbing points for the process. Properties (K3), (K4), (K5) and (K6) imply that each probability measure $k_{s,t}(x, \cdot)$ is the law of a killed subordinator at “time” x , see e.g. [40, p. 48]. Therefore, see e.g. [40, Theorem 5.2], for

any $(s, t) \in \Delta$ there exists a unique Bernstein function $v_{s,t}: (0, \infty) \rightarrow [0, \infty)$, called the *Laplace exponent* of $k_{s,t}$, such that

$$(2.7) \quad \mathcal{L}[k_{s,t}(x, \cdot)](\theta) := \int_{[0, \infty)} e^{-\theta y} k_{s,t}(x, dy) = e^{-xv_{s,t}(\theta)}, \quad x, \theta \in (0, \infty).$$

Recall that any Bernstein function extends to a self-map of $[0, \infty)$; then the above formula can be extended to $\theta \in [0, \infty)$ by the monotone convergence theorem.

Remark 2.3. Property (K3) can be replaced with the stronger one:

(K3') the map $\Delta \times [0, \infty] \ni (s, t, x) \mapsto k_{s,t}(x, \cdot) \in \mathcal{P}([0, \infty])$ is weakly continuous.

In order to see that any family of transition kernels of a branching process satisfies (K3'), essentially it is enough to observe that $v_{s,t}(\theta) > 0$ for all $(s, t) \in \Delta$ and all $\theta > 0$. The latter will be established in the proof of Theorem 2.5, STEP 2.

Remark 2.4. Given a family of transition kernels $(k_{s,t})$ satisfying (K1) and (K2), one can construct a corresponding Markov family in a canonical way. Let $\Omega := S^{[0, \infty)}$ be the set of all functions from $[0, \infty)$ to S and $(Z_t)_{t \geq 0}$ be the coordinate process on Ω , i.e. $Z_t(\omega) := \omega(t)$, $t \geq 0$, $\omega \in \Omega$. Let \mathcal{F} and \mathcal{F}_I be the σ -fields generated by the sets $\{Z_t^{-1}(B): B \in \mathcal{B}(S), t \geq 0\}$ and $\{Z_t^{-1}(B): B \in \mathcal{B}(S), t \in I\}$, respectively. Then one can assign to each (s, x) a probability measure $\mathbb{P}^{(s,x)}$ on $(\Omega, \mathcal{F}_{[s, \infty)})$ in such a way that the coordinate process $(Z_t)_{t \geq 0}$ becomes a Markov family with the transition kernels $(k_{s,t})$, see [42, Theorem 8.2]. If furthermore the transition kernels $(k_{s,t})$ satisfy (K3)–(K6) then (Z_t) is a branching process by definition.

The transition kernels of branching processes are characterized by $v_{s,t}$ as follows.

Theorem 2.5. *Given a family $(k_{s,t})_{(s,t) \in \Delta}$ of transition kernels of a branching process, the Laplace exponents $v_{s,t}$ defined by (2.7) form a topological reverse evolution family $(v_{s,t})_{(s,t) \in \Delta}$ contained in \mathfrak{BF} . Conversely, given a topological reverse evolution family $(v_{s,t})_{(s,t) \in \Delta} \subset \mathfrak{BF}$, there exists a unique family of transition kernels $(k_{s,t})_{(s,t) \in \Delta}$ of a branching process such that (2.7) holds.*

Remark 2.6. The class of transition kernel families treated in [16] is different from the class of families $(k_{s,t})$ we define above using conditions (K1)–(K6). In particular, [16] allows the dependence of the Laplace exponents $v_{s,t}$ on the parameters s and t to be discontinuous; however, due to the integral equation [16, eqn. (1.3)], it has to be of locally bounded variation. In our case, as Theorem 2.5 shows, the only restriction on the regularity of $(s, t) \mapsto v_{s,t}(z)$ imposed by conditions (K1)–(K6) is the continuity for each $z \in \mathbb{H}$ fixed. In particular, locally *unbounded* variation w.r.t. s and t may occur. On the other hand, we will often assume that the reverse evolution family $(v_{s,t})$ is absolutely continuous rather than merely topological. According to Theorem 2.13, in such a case $s \mapsto v_{s,t}(z)$ is a solution to the initial value problem (2.13), which is equivalent to a special case of the integral equation [16, eqn. (1.3)]. This discussion is continued in Remark 2.15. Worth mentioning that in Section 2.4 we give certain sufficient conditions for $(v_{s,t})$ to be absolutely continuous.

Proof of Theorem 2.5. We divide the proof in three steps.

STEP 1. First we suppose that an arbitrary family $(k_{s,t})_{(s,t) \in \Delta}$ of transition kernels on $[0, \infty]$, not necessarily related to a branching process, and a family $(v_{s,t})_{(s,t) \in \Delta}$ of maps from $(0, \infty)$ to $[0, \infty)$ are given and related by (2.7), i.e. for each $x > 0$ and $(s, t) \in \Delta$, the function $\exp(-xv_{s,t}(\cdot))$ is the Laplace transform $\mathcal{L}[k_{s,t}(x, \cdot)]$ of the measure $k_{s,t}(x, \cdot)$. In particular, $v_{s,t}$ is non-decreasing and hence extending $v_{s,t}$ to $\theta = 0$ by $v_{s,t}(0) := \lim_{\theta \rightarrow 0+} v_{s,t}(\theta)$, we may assume that $v_{s,t}$ is a self-map of $[0, \infty)$ and that (2.7) holds for all $\theta \geq 0$. Under this circumstance we prove that the following three conditions:

- (C1) $v_{s,s} = \text{id}_{[0,\infty)}$ for any $s \geq 0$,
- (C2) $v_{s,u} = v_{s,t} \circ v_{t,u}$ on $[0, \infty)$ for any $0 \leq s \leq t \leq u$,
- (C3) for every $\theta \in (0, \infty)$, the map $\Delta \ni (s, t) \mapsto v_{s,t}(\theta) \in [0, \infty)$ is continuous,

are equivalent to (K1)–(K3) for $(k_{s,t})_{(s,t) \in \Delta}$. Recall that a bounded Borel measure on $[0, \infty)$ is uniquely determined by its Laplace transform, see e.g. [40, Proposition 1.2]. Hence, it is clear that (C1) is equivalent to (K1). Furthermore, (C2) is equivalent to the Chapman–Kolmogorov relation (K2), because thanks to (2.1) we have

$$\begin{aligned} e^{-xv_{s,t}(v_{t,u}(\theta))} &= \int_{[0,\infty)} e^{-zv_{t,u}(\theta)} k_{s,t}(x, dz) \\ &= \int_{[0,\infty)} \left(\int_{[0,\infty)} e^{-\theta y} k_{t,u}(z, dy) \right) k_{s,t}(x, dz) = \int_{[0,\infty)} e^{-\theta y} (k_{s,t} \circ k_{t,u})(x, dy). \end{aligned}$$

The implication (K3) \Rightarrow (C3) is rather obvious from the definition of weak convergence: indeed, the map $y \mapsto e^{-\theta y}$ can be regarded as an element of $C[0, \infty]$ vanishing at ∞ as long as $\theta > 0$, and hence, we can write $e^{-xv_{s,t}(\theta)} = \int_{[0,\infty]} e^{-\theta y} k_{s,t}(x, dy)$. For the converse implication, one can use the fact that the pointwise convergence of the Laplace transforms $\mathcal{L}[k_{s,t}(x, \cdot)]$ on $(0, +\infty)$ as $(s, t, x) \rightarrow (s_0, t_0, x_0) \in \Delta \times [0, \infty)$ implies the vague convergence $k_{s,t}(x, \cdot) \rightarrow k_{s_0,t_0}(x_0, \cdot)$ as *sub-probability measures on* $[0, \infty)$, see e.g. [40, Lemma A.9]. This further implies the weak convergence $k_{s,t}(x, \cdot) \rightarrow k_{s_0,t_0}(x_0, \cdot)$ as *probability measures on* $[0, \infty]$, because for every $f \in C[0, \infty]$,

$$\int_{[0,\infty]} f(y) k_{s,t}(x, dy) = \int_{[0,\infty]} [f(y) - f(\infty)] k_{s,t}(x, dy) + f(\infty),$$

and $f - f(\infty)$ is a continuous function on $[0, \infty]$ vanishing at ∞ , and hence the integral in the r.h.s. above converges to $\int_{[0,\infty]} [f(y) - f(\infty)] k_{s_0,t_0}(x_0, dy)$ as $(s, t, x) \rightarrow (s_0, t_0, x_0)$ by the vague convergence of $k_{s,t}(x, \cdot)$.

STEP 2. Now we suppose that $(k_{s,t})_{(s,t) \in \Delta}$ is a family of transition kernels of a branching process, i.e. the conditions (K1)–(K6) hold. Then by Remark 2.2, there exists a family $(v_{s,t})_{(s,t) \in \Delta}$ of Bernstein functions satisfying (2.7). As we have proved in Step 1, (K1)–(K3) imply that conditions (C1)–(C3) hold.

The Bernstein functions $v_{s,t}$ extend to holomorphic mappings from \mathbb{H} to $\overline{\mathbb{H}}$, see Section 1.2. Let us show that actually $v_{s,t}$ are self-maps of \mathbb{H} . Suppose to the contrary that $v_{s_0,t_0}(\mathbb{H}) \not\subset \mathbb{H}$ for some $0 \leq s_0 < t_0$. Then by Remark 1.1, $v_{s_0,t_0} \equiv 0$. Let $t_1 := \min\{t \in (s_0, t_0] : v_{s_0,t} \equiv 0\}$, where the minimum exists thanks to the continuity (C3). The relation $v_{s_0,t} \circ v_{t,t_1} = v_{s_0,t_1} \equiv 0$ on $[0, \infty)$ implies that, for every $s_0 \leq t < t_1$, either $v_{s_0,t} \equiv 0$ or $v_{t,t_1} \equiv 0$ (because if v_{t,t_1} takes a value in \mathbb{H} then $v_{s_0,t} \equiv 0$; otherwise $v_{t,t_1} \equiv 0$). By the definition of t_1 , the former is not the case, so that $v_{t,t_1} \equiv 0$ for all $s_0 \leq t < t_1$. Passing to the limit yields that $v_{t_1,t_1} \equiv 0$, a contradiction. Therefore, one concludes that all $v_{s,t}$'s are holomorphic self-maps of \mathbb{H} and hence $(v_{s,t}) \subset \mathfrak{BF}$.

Conditions (C1) and (C2) imply (REF1) and (REF2) by the identity principle for holomorphic functions. Condition (REF3') follows from (C3) due to Vitali's theorem, see e.g. [18, §I.2], applied to the functions $f_{s,t} := H \circ v_{s,t}$, where $H(z) := (z - 1)/(z + 1)$ maps \mathbb{H} conformally onto \mathbb{D} , and due to the uniqueness of the holomorphic extension. Thus, $(v_{s,t})$ is a topological reverse evolution family contained in \mathfrak{BF} .

STEP 3. Conversely, given a topological reverse evolution family $(v_{s,t})_{(s,t) \in \Delta}$ contained in \mathfrak{BF} , a unique family $(k_{s,t}(x, \cdot))_{(s,t) \in \Delta, x \in [0, \infty]}$ of probability measures on $[0, \infty]$ satisfying (2.7), (K4), (K5), and (K6) does exist thanks to the theory of subordinators [40, Theorem 5.2]. As we have proved in Step 1, conditions (REF1), (REF2), and (REF3') in the definition of a topological reverse evolution family imply conditions (K1)–(K3). The proof is now complete. \square

The construction of branching processes given in Remark 2.4 is sometimes not enough. For example, when we later discuss hitting times to 0 (or ∞), the monotonicity of events $\{Z_t = 0\}$ w.r.t. $t \geq 0$ is helpful. Below we prove the existence of a “good” branching process by means of standard technique of supermartingales.

In order to give the precise statement of the result we need some notation. For $s \in [0, \infty)$, denote by $D[s, \infty)$ be the set of all càdlàg functions $\omega: [s, \infty) \rightarrow [0, \infty]$ satisfying the following condition: if $r := \omega(t) \in \{0, \infty\}$ for some $t \geq s$ or $r := \omega(t-) \in \{0, \infty\}$ for some $t > s$, then $\omega(u) = r$ for all $u \geq t$.

Consider the coordinate process $(X_t^c)_{t \geq 0}$ defined by $X_t^c(\omega) := \omega(t)$, $\omega \in D[0, \infty)$, and let $(\mathcal{G}_I)_I$ stand for its natural filtration, i.e. $\mathcal{G}_I := \sigma(X_t^c : t \in I)$ for any closed interval $I \subset [0, \infty)$. Finally, let $\mathcal{G} := \mathcal{G}_{[0, \infty)}$.

Theorem 2.7. *Let $(k_{s,t})$ be a family of transition kernels satisfying (K1)–(K6). Then there exists a unique map $(s, x) \mapsto \mathbb{Q}^{(s,x)}$ assigning to each $s \geq 0$ and $x \in [0, \infty]$ a probability measure $\mathbb{Q}^{(s,x)}$ on the measurable space $(D[0, \infty), \mathcal{G}_{[s, \infty)})$ such that $((X_t^c)_{t \geq 0}, D[0, \infty), \mathcal{G}, (\mathcal{G}_I)_I, (\mathbb{Q}^{(s,x)})_{s \geq 0, x \in [0, \infty]})$ is a branching process on $[0, \infty]$ with the transition kernels $(k_{s,t})$.*

Proof. The uniqueness is standard. Fix an $s \geq 0$. For each cylinder set

$$A := (X_{t_1}^c)^{-1}(B_1) \cap \cdots \cap (X_{t_n}^c)^{-1}(B_n),$$

where $s \leq t_1 \leq \cdots \leq t_n$, $B_i \in \mathcal{B}([0, \infty])$, the value of $\mathbb{Q}^{(s,x)}(A)$ is determined only by $(k_{s,t})$. Indeed, in view of (2.6), we have

$$\mathbb{Q}^{(s,x)}(A) = \int_{B_1 \times \cdots \times B_n} k_{s,t_1}(x, dx_1) k_{t_1,t_2}(x_1, dx_2) \cdots k_{t_{n-1},t_n}(x_{n-1}, dx_n).$$

It only remains to recall that the cylinder sets form a π -system and that this π -system generates the whole σ -field $\mathcal{G}_{[s, \infty)}$. Therefore, every probability measure on $\mathcal{G}_{[s, \infty)}$ is uniquely determined by its evaluations on the cylinder sets, see [37, Chapter II, Corollary 4.7 on p. 93].

For the existence, let $((Z_t)_{t \geq 0}, \Omega, \mathcal{F}, (\mathcal{F}_I)_I, (\mathbb{P}^{(s,x)})_{s \geq 0, x \in [0, \infty]})$ be as in Remark 2.4. In order to define measures $\mathbb{Q}^{(s,x)}$ we first construct a càdlàg modification $(Y_{s,t})_{t \geq s}$ of $(Z_t)_{t \geq s}$.

STEP 1: CONSTRUCTION OF $Y_{s,t}$. Throughout this step, we fix an arbitrary $s \geq 0$. For a while we fix also some $N \in \mathbb{N}$ with $N > s$ and some $\theta_N \in v_{0,N}((0, \infty))$. Note that $v_{0,t}$ is injective on $(0, \infty)$ for each $t \geq 0$ and that $\{v_{0,t}((0, \infty))\}_{t \geq 0}$ is a non-increasing family of subsets of $(0, \infty)$ because of (REF2). Therefore, $\theta_N \in v_{0,t}((0, \infty))$ for all $0 \leq t < N$ and hence we can define the stochastic process

$$V_t^{(N)} := \exp(-v_{0,t}^{-1}(\theta_N)Z_t), \quad s \leq t < N,$$

where $V_t^{(N)}$ is set to be 0 if $Z_t = \infty$. The process $(V_t^{(N)})_{t \in [s, N]}$ is a martingale with respect to $((\mathcal{F}_{[s,t]})_{t \in [s, N]}, \mathbb{P}^{(s,x)})$ for any $x \in [0, \infty]$, because thanks to the Markov property (2.5) and (REF2), for any $s \leq t \leq u < N$ we have

$$\mathbb{E}^{(s,x)}[\exp(-v_{0,u}^{-1}(\theta_N)Z_u) | \mathcal{F}_{[s,t]}] = \exp(-Z_t v_{t,u}(v_{0,u}^{-1}(\theta_N))) = \exp(-v_{0,t}^{-1}(\theta_N)Z_t) \quad \mathbb{P}^{(s,x)}\text{-a.s.}$$

By Doob's Regularisation Theorem [37, Chapter II, Theorem 65.1 on p.170], the set

$$\begin{aligned} \Omega_s^{(N)} &:= \left\{ \omega \in \Omega : \lim_{q \downarrow t, q \in \mathbb{Q}} V_q^{(N)} \text{ and } \lim_{q \uparrow t, q \in \mathbb{Q}} V_q^{(N)} \text{ exist finitely for all } t \in (s, N) \right\} \\ &\quad \cap \left\{ \omega \in \Omega : \lim_{q \downarrow s, q \in \mathbb{Q}} V_q^{(N)} \text{ exists finitely} \right\} = \\ &= \left\{ \omega \in \Omega : \lim_{q \downarrow t, q \in \mathbb{Q}} Z_q \text{ and } \lim_{q \uparrow t, q \in \mathbb{Q}} Z_q \text{ exist, finite or infinite, for all } t \in (s, N) \right\} \\ &\quad \cap \left\{ \omega \in \Omega : \lim_{q \downarrow s, q \in \mathbb{Q}} Z_q \text{ exists, finite or infinite} \right\} \end{aligned}$$

belongs to $\mathcal{F}_{[s, \infty)}$, with $\mathbb{P}^{(s, x)}[\Omega_s^{(N)}] = 1$ for any $x \in [0, \infty]$; and moreover, by the same theorem, the stochastic process $(W_{s, t}^{(N)})_{t \in [s, N]}$ defined by

$$W_{s, t}^{(N)}(\omega) := \begin{cases} \lim_{q \downarrow t, q \in \mathbb{Q}} V_q^{(N)}(\omega), & \omega \in \Omega_s^{(N)}, \\ 0, & \omega \in \Omega \setminus \Omega_s^{(N)}, \end{cases}$$

has càdlàg sample paths.

To make $(W_{s, t}^{(N)})_{t \in [s, N]}$ a martingale, we modify the filtration using a standard technique. Following [37, Section II.67, p.172] we consider the usual augmentation $(\hat{\mathcal{F}}_{[s, t]}^x)_{t \geq s}$ of the filtration $(\mathcal{F}_{[s, t]})_{t \geq s}$ w.r.t. t , i.e. the σ -field generated by $\bigcap_{u > t} \mathcal{F}_{[s, u]}$ and all the $\mathbb{P}^{(s, x)}$ -null sets of $\mathcal{F}_{[s, \infty)}$, and we denote by $\hat{\mathbb{P}}^{(s, x)}$ be the unique extension of $\mathbb{P}^{(s, x)}$ to $\hat{\mathcal{F}}_{[s, \infty)}^x$. By [37, Chapter II, Proposition 67.6 on p.173], the stochastic process $(W_{s, t}^{(N)})_{t \in [s, N]}$ is a martingale on the filtered probability space $(\Omega, \hat{\mathcal{F}}_{[s, N]}^x, (\hat{\mathcal{F}}_{[s, t]}^x)_{t \in [s, N]}, \hat{\mathbb{P}}^{(s, x)})$ for every $x \in [0, \infty]$.

For $p \in \{0, 1\}$ we let $T_{s, p}^{(N)}(\omega) := \inf \{t \in [s, N) : W_{s, t}^{(N)}(\omega) = p \text{ or } W_{s, t}^{(N)}(\omega) = p\}$ and consider the event

$$\Omega_{s, p}^{(N)} := \{\omega \in \Omega : W_{s, t}^{(N)}(\omega) = p \text{ for all } t \in [T_{s, p}^{(N)}(\omega), N)\}.$$

Note that $W_{s, t}^{(N)}$ is $\mathcal{F}_{[s, \infty)}$ -measurable for all $t \in [s, N)$. Therefore, by assertion (i) of [37, Chapter II, Theorem 78.1 on p.191], $\Omega_{s, p}^{(N)} \in \mathcal{F}_{[s, \infty)}$. Moreover, in view of assertion (ii) of the same theorem, the fact that $(W_{s, t}^{(N)})_{t \in [s, N)}$ is a martingale (w.r.t. the filtered probability space indicated above) allows us to conclude that $\mathbb{P}^{(s, x)}[\Omega_{s, p}^{(N)}] = \hat{\mathbb{P}}^{(s, x)}[\Omega_{s, p}^{(N)}] = 1$.

Now for $t \in [s, \infty)$ we set

$$Y_{s, t}(\omega) := \begin{cases} \lim_{q \downarrow t, q \in \mathbb{Q}} Z_q(\omega), & \omega \in \Omega_s := \bigcap_{N \in \mathbb{N}} (\Omega_s^{(N)} \cap \Omega_{s, 0}^{(N)} \cap \Omega_{s, 1}^{(N)}), \\ 1, & \omega \in \Omega \setminus \Omega_s. \end{cases}$$

Note that $\Omega_s \in \mathcal{F}_{[s, \infty)}$, $\mathbb{P}^{(s, x)}[\Omega_s] = 1$ for any $x \in [0, \infty]$, and $Y_{s, t}(\omega) = -\frac{1}{v_{0, t}^{-1}(\theta_N)} \log W_{s, t}^{(N)}(\omega)$ for all $\omega \in \Omega_s$ and $t \in [s, N)$, where \log is naturally extended to the homeomorphism from $[0, 1]$ onto $[-\infty, 0]$. By construction, once a sample path $(Y_{s, t}(\omega))_{t \geq s}$ hits the boundary point 0 or ∞ , it stays there afterwards. Thus every sample path of $(Y_{s, t})_{t \geq s}$ belongs to the space $D[s, \infty)$.

STEP 2: $Y_{s, t}$ IS A MODIFICATION OF Z_t . We show that $\mathbb{P}^{(s, x)}[Y_{s, t} = Z_t] = 1$ for all $t \geq s$ and all $(s, x) \in [0, \infty) \times [0, \infty]$. To this end it suffices to prove that

$$(2.8) \quad \mathbb{E}^{(s, x)}[f(Y_{s, t})g(Z_t)] = \mathbb{E}^{(s, x)}[f(Z_t)g(Z_t)]$$

for any $f, g \in C[0, \infty]$, $t \in [s, \infty)$ and (s, x) . Indeed, let $\varphi(x) := e^{-x}$ for $x \in [0, \infty)$ and $\varphi(\infty) := 0$. Applying (2.8) with $f = g := \varphi$ and once again with $f := \varphi^2$, $g := 1$, we easily obtain

$$\mathbb{E}^{(s,x)}[(\varphi(Y_{s,t}) - \varphi(Z_t))^2] = 0.$$

It follows that $\mathbb{P}^{(s,x)}[\varphi(Y_{s,t}) \neq \varphi(Z_t)] = 0$, as desired.

It remains to prove formula (2.8). Denote $T_{t,q}f(x) := \int_{[0,\infty]} f(y)k_{t,q}(x, dy)$, $x \in [0, \infty]$. From condition (K1) and Remark 2.3 it follows that $T_{t,q}f \rightarrow f$ pointwise on $[0, \infty]$ as $q \downarrow t$. Taking this fact into account, we obtain

$$\begin{aligned} \mathbb{E}^{(s,x)}[f(Y_{s,t})g(Z_t)] &= \lim_{q \downarrow t, q \in \mathbb{Q}} \mathbb{E}^{(s,x)}[f(Z_q)g(Z_t)] = \lim_{q \downarrow t, q \in \mathbb{Q}} \mathbb{E}^{(s,x)}[\mathbb{E}^{(s,x)}[f(Z_q)|\mathcal{F}_{[s,t]}]g(Z_t)] \\ &= \lim_{q \downarrow t, q \in \mathbb{Q}} \mathbb{E}^{(s,x)}[T_{t,q}f(Z_t)g(Z_t)] = \mathbb{E}^{(s,x)}[f(Z_t)g(Z_t)], \end{aligned}$$

where the first and the last equalities hold by the Dominated Convergence Theorem.

The fact that $Y_{s,t}$ is a modification of Z_t for each $t \geq s$ implies that $(Y_{s,t})_{t \geq s}$ and $(Z_t)_{t \geq s}$ have the same finite dimensional distributions with respect to $\mathbb{P}^{(s,x)}$.

STEP 3: CONSTRUCTION OF $\mathbb{Q}^{(s,x)}$. Fix for a while some $s \geq 0$ and $x \in [0, \infty]$. Note that $Y_{s,t}$ is $(\mathcal{F}_{[s,\infty)}, \mathcal{B}([0, \infty]))$ -measurable. It follows that the map $\Phi_s: \Omega \rightarrow D[0, \infty)$ defined by

$$(\Phi_s(\omega))(t) := \begin{cases} Y_{s,t}(\omega), & t \geq s, \\ 1, & 0 \leq t < s, \end{cases}$$

is $(\mathcal{F}_{[s,\infty)}, \mathcal{G}_{[s,\infty)})$ -measurable. We set $\mathbb{Q}^{(s,x)}$ to be the push-forward of the measure $\mathbb{P}^{(s,x)}$ w.r.t. Φ_s . Then for every $n \in \mathbb{N}$, $s \leq t_1 \leq \dots \leq t_n$ and $B_1, \dots, B_n \in \mathcal{B}([0, \infty])$, we have

$$\begin{aligned} \mathbb{Q}^{(s,x)}[X_{t_1}^c \in B_1, \dots, X_{t_n}^c \in B_n] &= \mathbb{P}^{(s,x)}[Y_{s,t_1} \in B_1, \dots, Y_{s,t_n} \in B_n] \\ &= \mathbb{P}^{(s,x)}[Z_{t_1} \in B_1, \dots, Z_{t_n} \in B_n], \end{aligned}$$

where the second equality holds because $(Y_{s,t})$ is a modification of (Z_t) , see Step 2.

Regard the coordinate process (X_t^c) as equipped with the measures $\mathbb{Q}^{(s,x)}$. Then the above equality trivially implies (2.4) and, by standard arguments, also implies the Markov property (2.5) for (X_t^c) , see e.g. [42, §8.3.1, p. 137]. Therefore, $((X_t^c)_{t \geq 0}, D[0, \infty), \mathcal{G}, (\mathcal{G}_I)_I, (\mathbb{Q}^{(s,x)})_{s \geq 0, x \in [0, \infty]})$ is a branching process with transition kernels $(k_{s,t})$, as desired. \square

2.2. An ODE for Laplace exponents. In the following subsections we examine branching processes which correspond to absolutely continuous reverse evolution families. In this subsection, we establish a basic ODE. Supplementary results (Lemma 2.9 and Proposition 2.11) are provided first.

In the following, we employ the notation $f^{(n)}(z, t)$, $f'(z, t)$, $f''(z, t)$ and so on for the derivatives w.r.t. the complex variable z , while keeping the notation $\frac{\partial}{\partial t}f(z, t)$ or $\partial_t f(z, t)$ for the derivatives w.r.t. the real parameter t .

Definition 2.8. A family of non-negative Borel measures $(\rho_t)_{t \geq 0}$ on $[0, \infty)$ is said to be

- (1) *measurable (in t)* if the function $[0, \infty) \ni t \mapsto \rho_t(B) \in [0, \infty]$ is measurable for every Borel subset B of $[0, \infty)$;
- (2) *locally integrable (in t)* if $t \mapsto \rho_t(B)$ is in L_{loc}^1 for every Borel subset B of $[0, \infty)$, or equivalently, if it is measurable in the sense of (1) above and $t \mapsto \rho_t([0, \infty))$ is in L_{loc}^1 .

Note that in this definition we do not assume that each measure ρ_t is bounded. However, since by the very definition, an L_{loc}^1 -function is a.e. finite, if (ρ_t) is locally integrable in the above sense then clearly $\rho_t([0, \infty)) < \infty$ for a.e. $t \geq 0$.

For a non-negative Borel measure m on $[0, \infty)$ we define its Laplace transform

$$\mathcal{L}[m](\lambda) := \int_{[0, \infty)} e^{-\lambda x} m(dx),$$

whenever it is finite at every $\lambda > 0$. Note that the finiteness of $\mathcal{L}[m]$ implies that m is locally finite. We provide a proof of the following quite expected auxiliary result.

Lemma 2.9. *Suppose that m_t is a non-negative Borel measure on $[0, \infty)$ such that $\mathcal{L}[m_t](\lambda) < \infty$ for every $t \geq 0$ and $\lambda > 0$. The following are equivalent:*

- (i) *the function $t \mapsto \mathcal{L}[m_t](\lambda)$ is measurable for each $\lambda > 0$;*
- (ii) *the function $t \mapsto m_t([0, x])$ is measurable for each $x \in [0, \infty)$;*
- (iii) *m_t is measurable in t .*

Proof of Lemma 2.9. **STEP 1.** The implication (iii) \Rightarrow (i) holds because for each $\lambda > 0$, the function $I(t) := \mathcal{L}[m_t](\lambda)$ can be represented as the pointwise limit of a sequence $I_n(t) := \int_{[0, \infty)} f_n(x) m_t(dx)$, where f_n 's are suitable simple functions in $[0, \infty)$.

STEP 2: (i) \Rightarrow (ii). Suppose first that this implication is true for the special case when m_t is a finite measure on $[0, \infty)$ for each $t \geq 0$. In order to see that it extends to the general case, we introduce the measure $m_t^\epsilon(dx) = e^{-\epsilon x} m_t(dx)$, which is finite whenever $\epsilon > 0$ because $m_t^\epsilon([0, \infty)) = \mathcal{L}[m](\epsilon)$. The assumption (i) implies that $t \mapsto \mathcal{L}[m_t^\epsilon](\lambda) = \mathcal{L}[m_t](\lambda + \epsilon)$ is measurable for every $\lambda > 0$, and hence $t \mapsto m_t^\epsilon([0, x])$ is measurable for each $x \in [0, \infty)$. The monotone convergence theorem shows that $m_t([0, x]) = \lim_{\epsilon \rightarrow 0^+} m_t^\epsilon([0, x])$, so that $t \mapsto m_t([0, x])$ is measurable as well.

As the above argument shows, we may (and we will do) assume that m_t is a finite measure for each $t \geq 0$. We use the formula

$$R(\theta, t) := \int_0^\infty \mathcal{L}[m_t](\lambda) e^{-\lambda \theta} d\lambda = \int_0^\infty \frac{1}{\theta + x} m_t(dx), \quad \theta > 0.$$

The integral in the r.h.s. is finite since m_t is a finite measure. Clearly, for each $t \geq 0$, $R(\cdot, t)$ admits a holomorphic extension to $\mathbb{C} \setminus (-\infty, 0]$, which we again denote by $R(\cdot, t)$. Let h be a conformal mapping of \mathbb{D} onto $\mathbb{C} \setminus (-\infty, 0]$ such that $h((-1, 1)) = (0, \infty)$. Since $R_1(x, \cdot) := R(h(x), \cdot)$ is measurable for each $x \in (-1, 1)$, it is easy to show by induction that the functions $R_1^{(n)}(0, \cdot)$, $n \in \mathbb{N} \cup \{0\}$, are all measurable. Therefore, representing $R_1(z, t)$ by its Taylor series in powers of z , we may conclude that $R(w, \cdot)$ is measurable for each $w \in \mathbb{C} \setminus (-\infty, 0]$.

Applying the dominated convergence theorem yields

$$m_t(\{a\}) = \lim_{\epsilon \rightarrow 0^+} i\epsilon R(-a + i\epsilon, t), \quad a \geq 0,$$

which further implies that $t \mapsto m_t(\{a\})$ is measurable. Moreover, the Stieltjes inversion formula

$$\frac{1}{2}[m_t(\{a\}) + m_t(\{b\})] + m_t((a, b)) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \int_a^b \operatorname{Im}[R(-\theta - i\epsilon, t)] d\theta, \quad 0 \leq a < b < \infty,$$

shows that $t \mapsto m_t((a, b))$ is measurable. In particular, we obtain assertion (ii).

STEP 3. The implication (ii) \Rightarrow (iii) can be proved using a standard technique in Measure Theory. It is enough to focus on a bounded interval $[0, \alpha]$ for an arbitrary fixed $\alpha > 0$ since $m_t(B) = \lim_{\alpha \rightarrow \infty} m_t(B \cap [0, \alpha])$. The class $\Pi := \{[0, x] : x \in [0, \alpha]\}$ is a π -system, and the class

$$\Lambda := \{B \in \mathcal{B}([0, \alpha]) : t \mapsto m_t(B) \text{ is measurable}\}$$

is a λ -system. By the assumption (ii), we have $\Pi \subset \Lambda$. Hence, Dynkin's π - λ theorem, see e.g. [15, Lemma 1.1], implies that the σ -field generated by Π is contained in Λ . This shows that Λ is exactly the set of all Borel subsets of $[0, \alpha]$. \square

Remark 2.10. The lemma we have just proved implies the following simple corollary: *if under the hypothesis of Lemma 2.9 the family (m_t) is measurable, then for any continuous function $f : [0, \infty) \rightarrow [0, \infty)$, the family (\tilde{m}_t) , defined by $\tilde{m}_t(dx) = f(x)m_t(dx)$, is also measurable.* Notice that for each $a > 0$, the function $I_a(t) := \tilde{m}_t([0, a])$ is measurable on $[0, \infty)$ as it can be pointwisely approximated by linear combinations of functions of the form $t \mapsto m_t([0, x])$. It remains to repeat the argument in Step 3 of the proof of Lemma 2.9 with (m_t) replaced by (\tilde{m}_t) .

Recall that a Lévy measure is a non-negative Borel measure π on $(0, \infty)$ satisfying the integrability condition (1.6).

Proposition 2.11. *For each $t \geq 0$, let $a_t \in \mathbb{R}$, $q_t, b_t \in [0, \infty)$ and π_t a Lévy measure. We define a function $\phi : \mathbb{H} \times [0, \infty) \rightarrow \mathbb{C}$ by*

$$(2.9) \quad \phi(\zeta, t) := -q_t + a_t\zeta + b_t\zeta^2 + \int_0^\infty (e^{-\zeta x} - 1 + \zeta x \mathbf{1}_{(0,1)}(x)) \pi_t(dx), \quad \zeta \in \mathbb{H}, t \geq 0.$$

Then the following assertions are equivalent:

- (i) ϕ is a Herglotz vector field, i.e. for each $\zeta \in \mathbb{H}$ the function $t \mapsto \phi(\zeta, t)$ is measurable, and for each compact subset K of \mathbb{H} the function $t \mapsto \sup_{\zeta \in K} |\phi(\zeta, t)|$ is in L^1_{loc} .
- (ii) For each $\theta > 0$ the function $t \mapsto \phi(\theta, t)$ is measurable, and in addition, there exists $\theta_0 > 0$ such that $t \mapsto \phi^{(n)}(\theta_0, t)$ is in L^1_{loc} for all $n \in \{0, 1, 2\}$.
- (iii) the functions a_t, b_t, q_t of t are all in L^1_{loc} and the finite measure $\min\{x^2, 1\} \pi_t(dx)$ on $(0, \infty)$ is locally integrable in t .

Proof. STEP 1. The implication (iii) \Rightarrow (i) is a consequence of the estimate

$$\sup_{\zeta \in K} |e^{-\zeta x} - 1 + \zeta x \mathbf{1}_{(0,1)}(x)| \leq C_K \min\{1, x^2\}, \quad x > 0,$$

where K is an arbitrary compact set in \mathbb{H} and $C_K > 0$ is a constant depending on K but not on x .

STEP 2: (i) \Rightarrow (ii). This is a consequence of Cauchy's integral formula. (We can take any $\theta_0 > 0$.)

STEP 3: (ii) \Rightarrow (iii). Note first that $\phi^{(n)}(\theta, t)$ are measurable in t for all $\theta > 0$ and $n \in \mathbb{N} \cup \{0\}$, as being pointwise limits of measurable functions. Let $m_t(dx) := x^2 \pi_t(dx) + 2b_t \delta_0$; it is a non-negative Borel measure on $[0, \infty)$ for each $t \geq 0$ and its Laplace transform is finite on $(0, \infty)$. The second θ -derivative of ϕ is then given by

$$\phi''(\theta, t) = 2b_t + \int_0^\infty x^2 e^{-\theta x} \pi_t(dx) = \int_{[0, \infty)} e^{-\theta x} m_t(dx) = \mathcal{L}[m_t](\theta).$$

By Lemma 2.9, m_t is measurable in t . Then b_t and $x^2 \pi_t(dx)$ are also measurable in t . We proceed bearing in mind Remark 2.10 but without mentioning it each time explicitly. In particular, we conclude that $\min\{x^2, 1\} \pi_t(dx)$ is measurable in t . Also, since $t \mapsto \phi''(\theta_0, t)$ is in L^1_{loc} , both $t \mapsto b_t$ and $t \mapsto \int_0^\infty x^2 e^{-\theta_0 x} \pi_t(dx)$ are in L^1_{loc} . The latter implies that

$$(2.10) \quad t \mapsto \int_{(0,1)} x^2 \pi_t(dx) \quad \text{and} \quad t \mapsto \int_{[1, \infty)} x e^{-\theta_0 x} \pi_t(dx)$$

are both in L^1_{loc} . This further implies that

$$(2.11) \quad \int_0^\infty x(-e^{-\theta_0 x} + \mathbf{1}_{(0,1)}(x)) \pi_t(dx) = \int_{(0,1)} x(1 - e^{-\theta_0 x}) \pi_t(dx) - \int_{[1, \infty)} x e^{-\theta_0 x} \pi_t(dx)$$

and

$$(2.12) \quad \int_{(0,1)} (e^{-\theta_0 x} - 1 + \theta_0 x) \pi_t(dx)$$

are both in L^1_{loc} as functions of t . From the formula

$$\phi'(\theta_0, t) = a_t + 2b_t\theta_0 + \int_0^\infty x(-e^{-\theta_0 x} + \mathbf{1}_{(0,1)}(x)) \pi_t(dx)$$

and the local integrability of $t \mapsto \phi'(\theta_0, t)$, $t \mapsto b_t$ and of (2.11), it follows that $t \mapsto a_t$ is in L^1_{loc} .

Finally, from (2.9) together with the local integrability of $a_t, b_t, \phi(\theta_0, t)$ and of (2.12), it follows that the function

$$t \mapsto q_t + \int_{[1,\infty)} (1 - e^{-\theta_0 x}) \pi_t(dx)$$

is in L^1_{loc} , and hence, so are $t \mapsto q_t$ and $t \mapsto \pi_t([1, \infty))$. Recalling the local integrability of the first function of (2.10), we conclude that

$$t \mapsto \int_0^\infty \min\{x^2, 1\} \pi_t(dx)$$

is in L^1_{loc} , which completes the proof of (iii). \square

To state the main result of this section we need the following definition.

Definition 2.12. We call a family of quadruples $(q_t, a_t, b_t, \pi_t)_{t \geq 0}$ a *Lévy family* if for each $t \geq 0$, $a_t \in \mathbb{R}$, $q_t, b_t \in [0, \infty)$, π_t is a Lévy measure on $(0, \infty)$, and if the integrability conditions in Proposition 2.11 (iii) are fulfilled.

We identify two Lévy families $(q_t, a_t, b_t, \pi_t)_{t \geq 0}$ and $(\tilde{q}_t, \tilde{a}_t, \tilde{b}_t, \tilde{\pi}_t)_{t \geq 0}$ if there exists a Lebesgue null set $N \subset [0, \infty)$ such that $(q_t, a_t, b_t, \pi_t) = (\tilde{q}_t, \tilde{a}_t, \tilde{b}_t, \tilde{\pi}_t)$ for all $t \in [0, \infty) \setminus N$. This definition is compatible with the equivalence of Herglotz vector fields introduced in Definition 1.3.

As we mentioned in Section 1.2, the semigroup \mathfrak{BF} is topologically closed in $\text{Hol}(\mathbb{H}, \mathbb{H})$. Therefore, combining Proposition 2.11, Theorem A and Theorem B, we arrive at a variant of the Loewner–Kufarev ODE for the Laplace exponents of branching processes, as stated in the theorem below. Recall that, according to Theorem 2.5, an absolutely continuous reverse evolution family contained in \mathfrak{BF} exactly means a family of the Laplace exponents $v_{s,t}$ of a branching process satisfying the condition (REF3) on the regularity of $v_{s,t}$ w.r.t. the parameters s and t .

Theorem 2.13. *Let $(v_{s,t})_{(s,t) \in \Delta}$ be an absolutely continuous reverse evolution family contained in \mathfrak{BF} . Then there exists a unique Lévy family $(q_t, a_t, b_t, \pi_t)_{t \geq 0}$ such that for every $t \geq 0$ and $\zeta \in \mathbb{H}$ the map $s \mapsto v(s) := v_{s,t}(\zeta)$ is the unique solution to the initial value problem*

$$(2.13) \quad \frac{d}{ds} v(s) = \phi(v(s), s) \quad \text{a.e. } s \in [0, t]; \quad v(t) = \zeta,$$

where $\phi: \mathbb{H} \times [0, \infty) \rightarrow \mathbb{C}$ is defined by

$$(2.14) \quad \phi(\zeta, s) = -q_s + a_s \zeta + b_s \zeta^2 + \int_0^\infty (e^{-\zeta x} - 1 + \zeta x \mathbf{1}_{(0,1)}(x)) \pi_s(dx), \quad \zeta \in \mathbb{H}, \quad s \geq 0,$$

i.e. the Herglotz vector field associated with $(v_{s,t})_{(s,t) \in \Delta}$ is of the form (2.14).

Conversely, given a Lévy family $(q_t, a_t, b_t, \pi_t)_{t \geq 0}$, the function ϕ defined by (2.14) is a Herglotz vector field, and for every $\zeta \in \mathbb{H}$ and $t \geq 0$, the initial value problem (2.13) has a unique solution $s \mapsto v(s) = v(s; t, \zeta)$. Moreover, the formula $v_{s,t}(\zeta) := v(s; t, \zeta)$ defines an absolutely continuous reverse evolution family $(v_{s,t})_{(s,t) \in \Delta}$ contained in \mathfrak{BF} .

Remark 2.14. Let ϕ be as in the above theorem. Then for $s \geq 0$ fixed, $\phi''(\cdot, s) \geq 0$ in $(0, \infty)$. Hence the following limits exist for every $s \geq 0$:

$$\phi(0, s) := \lim_{\theta \rightarrow 0^+} \phi(\theta, s) \in (-\infty, \infty] \quad \text{and} \quad \phi'(0, s) := \lim_{\theta \rightarrow 0^+} \phi'(\theta, s) \in [-\infty, \infty).$$

Recall that a branching process (Z_t) is time-homogeneous if its transition kernels satisfy the relation $k_{s,t} = k_{0,t-s}$ for any $(s, t) \in \Delta$. Stated in terms of the Laplace exponents $v_{s,t}$ defined by (2.7), this relation is equivalent to the requirement that the mappings $v_t := v_{0,t} \in \mathbf{Hol}(\mathbb{H}, \mathbb{H})$ form a one-parameter semigroup. The infinitesimal generator $\phi : \mathbb{H} \rightarrow \mathbb{C}$ of the semigroup (v_t) is commonly known in the literature as the *branching mechanism* of the process (Z_t) . In this connection, Theorem 2.13 can be stated in a less formal language as follows: in the time-inhomogeneous setting the role of branching mechanisms is played, as long as we suppose that the Laplace exponents can be described with the help of an ODE, by the Herglotz vector fields $\phi : \mathbb{H} \times [0, +\infty) \rightarrow \mathbb{C}$ of the form (2.14), where $(q_t, a_t, b_t, \pi_t)_{t \geq 0}$ is a Lévy family. It is worth mentioning that according to the theory developed in [5, 12], condition (REF3) in Definition 1.4 seems to be the most general assumption one can use in order to assure that $v_{s,t}$ is a solution to a Carathéodory-type ODE (i.e. a first order ODE with a locally integrable time-dependent vector field).

Remark 2.15. It would be interesting to extend, in one or another sense, the one-to-one correspondence between the absolutely continuous reverse evolution families in \mathfrak{BF} and the Herglotz vector fields of the form (2.14), established by Theorem 2.13, to a more general setting, with the initial value problem for the Carathéodory-type ODE (2.13) replaced by a suitable integral equation. Note that in the recent work [16] one direction has been established: the authors of [16] found a class of driving measures in the integral equation [16, eqn. (1.3)] for which a unique solution exists and represents an (in general, discontinuous) reverse evolution family contained in \mathfrak{BF} . The opposite direction, i.e. proving an intrinsic characterization of reverse evolution families that satisfy the integral equation with suitable driving measures, seems a challenging problem. Analogous problem has been, however, solved in [28] for reverse evolution families in the unit disk with a common DW-point at the origin. In such a situation, the multiplier at the DW-point, i.e. the derivative of self-maps at 0, is quite a useful parameter. Making a change of variable, one may assume that the multiplier of the DW-point is positive. In such a situation, the reverse evolution family satisfies an integral equation. Unfortunately, in our setting the situation is more complicated, because for a reverse evolution family contained in \mathfrak{BF} the DW-point may move rather arbitrarily in $[0, \infty]$. In contrast to evolution families, a *reverse* evolution family does not necessarily admit a change of variables bringing the time-dependent DW-point to a chosen (time-independent) interior point.

The above remark explains one of the reasons why we specialize to the well established case of absolutely continuous reverse evolution families, which are governed by ordinary differential equations. This will be the main subject for the remaining subsections.

2.3. Branching processes with finite first moments. Let $(Z_t)_{t \geq 0}$ be a branching process whose Laplace exponents $v_{s,t}$ are characterized as in Theorem 2.13. We say $(Z_t)_{t \geq 0}$ is *conservative* if $\mathbb{P}^{(s,x)}[Z_t < \infty] = 1$ for any $x > 0$ and $(s, t) \in \Delta$. This is equivalent to $v_{0,t}(0) = 0$ for all $t > 0$ as follows from (2.17) below. According to [24, Section 2] (or see e.g. [32, Theorem 12.3]), a necessary and sufficient condition for a *time-homogeneous* branching process to be conservative is that

$$(2.15) \quad \int_0^\epsilon \frac{1}{|\phi(\theta)|} d\theta = \infty, \quad \forall \epsilon > 0,$$

where ϕ is the branching mechanism of the process (Z_t) , i.e. the infinitesimal generator of the one-parameter semigroup (v_t) formed by the Laplace exponents $v_t(\theta) = -\log \mathbb{E}^{(0,1)}[e^{-\theta Z_t}]$.

In fact, the above condition (2.15) can be easily deduced from a general result on the relationships between the boundary fixed points of a one-parameter semigroup and the boundary behaviour of its Koenigs function, see [14] or [6, §13.6]. Unfortunately, there does not seem to be known any analogous results for (reverse) evolution families, unless we restrict consideration to boundary *regular* fixed points. As a result, in the time-inhomogeneous setting, the possibility to characterize conservative branching processes by a condition similar to (2.15) is rather unclear.

Instead, we discuss the condition of finite mean, which implies the conservativeness. Finite mean is commonly assumed for time-homogeneous branching processes [35] and can be well characterized. Even in the time-inhomogeneous case, we are able to completely characterize finite mean in terms of Herglotz vector fields. With $v_{s,t}(0)$, $v'_{s,t}(0)$, $\phi(0, s)$, $\phi'(0, s)$ understood as limits for $\theta \rightarrow 0^+$, see (1.2) and Remark 2.14, our result can be stated as follows.

Theorem 2.16. *Let $(Z_t)_{t \geq 0}$ be a branching process whose Laplace exponents $(v_{s,t})_{(s,t) \in \Delta}$ are characterized by a Herglotz vector field ϕ as in Theorem 2.13. The following conditions are equivalent:*

- (1) $\mathbb{E}^{(s,x)}[Z_t] < \infty$ for all $x \in (0, \infty)$ and $(s, t) \in \Delta$.
- (2) $\mathbb{E}^{(0,1)}[Z_t] < \infty$ for all $t \geq 0$.
- (3) $v_{0,t}(0) = 0$ and $v'_{0,t}(0) < \infty$, i.e. 0 is a boundary regular fixed point of $v_{0,t}$ for all $t \geq 0$.
- (4) $v_{s,t}(0) = 0$ and $v'_{s,t}(0) < \infty$, i.e. 0 is a boundary regular fixed point of $v_{s,t}$ for all $(s, t) \in \Delta$.
- (5) $\phi(0, t) = 0$ for a.e. $t \geq 0$ and the function $t \mapsto \phi'(0, t)$ is in L^1_{loc} .
- (6) $q_t = 0$ for a.e. $t \geq 0$ and the function $t \mapsto \int_{[1, \infty)} x \pi_t(dx)$ is in L^1_{loc} .

If one (and hence all) of the above conditions hold, then

$$(2.16) \quad \mathbb{E}^{(s,x)}[Z_t] = xv'_{s,t}(0) = x \exp \left(- \int_s^t \phi'(0, r) dr \right)$$

for all $x \in (0, \infty)$ and $(s, t) \in \Delta$.

Note that Herglotz vector fields associated with reverse evolution families $(v_{s,t}) \subset \mathfrak{B}\mathfrak{F}$ satisfying condition (4) above can be characterized by a representation formula, see Remark 2.23 in Section 2.5.1.

Proof of Theorem 2.16. We begin with establishing the equivalence of (1)–(4). Passing to the limit as $\theta \rightarrow 0^+$ in the relation $\mathbb{E}^{(s,x)}[e^{-\theta Z_t} \mathbf{1}_{\{Z_t < \infty\}}] = e^{-xv_{s,t}(\theta)}$, $\theta > 0$, and its differentiated form, with the help of the monotone convergence theorem we get

$$(2.17) \quad \mathbb{P}^{(s,x)}[Z_t < \infty] = e^{-xv_{s,t}(0)} \quad \text{and} \quad \mathbb{E}^{(s,x)}[Z_t \mathbf{1}_{\{Z_t < \infty\}}] = xv'_{s,t}(0)e^{-xv_{s,t}(0)}.$$

It is then straightforward to see that (2) is equivalent to (4) and that (1) is equivalent to (3).

Obviously, (4) \Rightarrow (3). To prove the converse implication, suppose that (3) holds. If $v_{s,t}(0) > 0$ for some $(s, t) \in \Delta$, then passing to the limit as $\theta \rightarrow 0^+$ in the identity $v_{0,t}(\theta) = v_{0,s}(v_{s,t}(\theta))$ would yield $v_{0,t}(0) = v_{0,s}(v_{s,t}(0)) \neq 0$ because $v_{0,s}$ is a self-map of \mathbb{H} . Hence $v_{s,t}(0) = 0$ for all $(s, t) \in \Delta$. Note that $v'_{0,s}(0) > 0$ for any $s \geq 0$ because the angular derivative of a holomorphic self-map at a boundary fixed point $\sigma \neq \infty$ cannot not vanish, see e.g. [6, Proposition 1.9.3 on p. 54]. (Alternatively, the inequality $v'_{0,s}(0) > 0$ can be deduced from the integral representation (1.3) if taken into account that $v_{0,s}(0) = 0$ and $v_{0,s} \not\equiv 0$.) Taking the derivative in both sides and then passing to the limit we may, therefore, conclude that $v'_{s,t}(0) = v'_{0,t}(0)/v'_{0,s}(0) < \infty$.

It remains to show that (4) \Leftrightarrow (5) \Leftrightarrow (6). To begin with, taking into account the relationship between evolution families and *reverse* evolution families, see e.g. [25, Remark 2.8], the equivalence (4) \Leftrightarrow (5) follows from [7, Theorem 1.1]. Furthermore,

$$\phi'(\theta, t) = a_t + 2b_t\theta + \int_{(0,1)} x(1 - e^{-\theta x}) \pi_t(dx) - \int_{[1, \infty)} x e^{-\theta x} \pi_t(dx).$$

Hence, passing to the limit $\theta \rightarrow 0^+$ yields that

$$\phi'(0, t) = a_t - \int_{[1, \infty)} x \pi_t(dx).$$

Because $t \mapsto a_t$ is locally integrable by the definition of a Lévy family and because $q_t = \phi(0, t)$, the equivalence between (5) and (6) now follows.

Finally, the formula for the spectral function in [7, Theorem 1.1] implies that

$$v'_{s,t}(0) = \exp \left[- \int_s^t \phi'(0, r) dr \right].$$

□

Remark 2.17. Suppose that the equivalent conditions (1)–(6) in Theorem 2.16 are satisfied. Then $v_{s,t}(0) = 0$ for all $(s, t) \in \Delta$. Restating [25, Proposition 3.18] for reverse evolution families, we obtain $v''_{s,t}(0) = -v'_{s,t}(0) \int_s^t \phi''(0, r) v'_{r,t}(0) dr$. Taking into account (2.16), we obtain the following formula for the variance:

$$\mathbb{E}^{(s,x)}[Z_t^2] - (\mathbb{E}^{(s,x)}[Z_t])^2 = \mathbb{E}^{(s,x)}[Z_t] \int_s^t \phi''(0, r) v'_{r,t}(0) dt$$

for any $(s, t) \in \Delta$ and any $x > 0$. The integral can be infinite.

2.4. Criteria for absolute continuity. As we observed in Theorem 2.5, the Laplace exponents of a branching process form a topological reverse evolution family contained in \mathfrak{BF} , and vice versa. In order to get the ODE in Theorem 2.13, one also needs the absolute continuity, i.e. condition (REF3).

Various conditions assuring that a given topological (reverse) evolution family is actually absolutely continuous are known, see e.g. [25, Section 3.4]. Below we apply two of them to the Laplace exponents of continuous-state branching processes.

Proposition 2.18. *Let $(Z_t)_{t \geq 0}$ be a branching process such that $\mathbb{E}^{(0,1)}[Z_t]$ and $\mathbb{E}^{(0,1)}[Z_t^2]$ are finite for all $t \geq 0$ and the functions $t \mapsto \mathbb{E}^{(0,1)}[Z_t]$ and $t \mapsto \mathbb{E}^{(0,1)}[Z_t^2]$ are in AC_{loc} . Then the Laplace exponents $(v_{s,t})_{(s,t) \in \Delta}$ of (Z_t) form an absolutely continuous reverse evolution family in \mathbb{H} .*

Proof. It is easy to see that the assumption $\mathbb{E}^{(0,1)}[Z_t] < \infty$ implies that $v_{0,t}(0) = 0$ for all $t \geq 0$. Moreover,

$$\mathbb{E}^{(0,1)}[Z_t] = v'_{0,t}(0) \quad \text{and} \quad \mathbb{E}^{(0,1)}[Z_t^2] = [v'_{0,t}(0)]^2 - v''_{0,t}(0).$$

Therefore, the desired conclusion follows from [25, Corollary 3.17].

□

Remark 2.19. Let (Z_t) be a branching process with transition kernels $(k_{s,t})$ and associated topological reverse evolution family $(v_{s,t})$. Let $a : [0, \infty) \rightarrow (0, \infty)$ be a continuous function. It is easy to show that $(\hat{Z}_t) := (a(t)Z_t)$ is also a branching process. The transition kernels $(\hat{k}_{s,t})$ of the process (\hat{Z}_t) and the corresponding topological reverse evolution family $(\hat{v}_{s,t})$ are given by

$$\hat{k}_{s,t}(x, B) = k_{s,t}(L_s(x), L_t(B)), \quad \hat{v}_{s,t} = L_s \circ v_{s,t} \circ L_t^{-1}, \quad L_t(\zeta) := \zeta/a(t),$$

for any $(s, t) \in \Delta$, $x \in [0, \infty]$, and any Borel set $B \subset [0, \infty]$.

Proposition 2.20. *Let $(Z_t)_{t \geq 0}$ be a branching process such that $\mathbb{E}^{(0,1)}[Z_t]$ is finite for all $t \geq 0$ and the function $t \mapsto \mathbb{E}^{(0,1)}[Z_t]$ is continuous. Then there exists a strictly increasing continuous bijection $u : [0, \infty) \rightarrow [0, \infty)$ such that the branching process $(\tilde{Z}_t)_{t \geq 0} := (Z_{u(t)}/\mathbb{E}^{(0,1)}[Z_{u(t)}])_{t \geq 0}$ has Laplace exponents that form an absolutely continuous reverse evolution family in \mathbb{H} .*

Proof. As we have seen in the proof of Theorem 2.16, condition $\mathbb{E}^{(0,1)}[Z_t] < \infty$ for all $t \geq 0$ implies that the Laplace exponents $(v_{s,t})$ of (Z_t) have a BRFP at 0 with $v'_{s,t}(0) = a(s)/a(t)$, where $a(t)$ stands for the reciprocal of $\mathbb{E}^{(0,1)}[Z_t] = v'_{0,t}(0) > 0$.

By Remark 2.19, $\tau = 0$ is the DW-point of the Laplace exponents $(\hat{v}_{s,t})$ associated with the branching process $(\hat{Z}_t) := (a(t)Z_t)$. According to [25, Corollary 3.11], there exists a strictly

increasing continuous bijection $u: [0, \infty) \rightarrow [0, \infty)$ such that the reverse evolution family $(\tilde{v}_{s,t})$ defined by $\tilde{v}_{s,t} := \hat{v}_{u(s), u(t)}$, $(s, t) \in \Delta$, is absolutely continuous. It remains to mention that $(\tilde{v}_{s,t})$ are the Laplace exponents of the branching process $(\tilde{Z}_t)_{t \geq 0} := (Z_{u(t)} / \mathbb{E}^{(0,1)}[Z_{u(t)}])_{t \geq 0}$. \square

2.5. Branching processes whose Laplace exponents share the same Denjoy–Wolff point. Denote by \mathfrak{BF}_τ the class of Bernstein functions consisting of $\text{id}_\mathbb{H}$ and all $v \in \mathfrak{BF} \setminus \{\text{id}_\mathbb{H}\}$ having the DW-point at τ . The Laplace exponents v_t of a time-homogeneous branching process form a one-parameter semigroup and hence $(v_t)_{t \geq 0} \subset \mathfrak{BF}_\tau$ for some $\tau \in [0, \infty]$, see Remarks 1.8 and 1.9.

In the time-inhomogeneous case, the Laplace exponents form a reverse evolution family $(v_{s,t})$ and, in general, the position of the DW-point of $v_{s,t}$ may depend on the time parameters s and t . However, it seems to be interesting to investigate (and give a probabilistic interpretation of) the special case $(v_{s,t}) \in \mathfrak{BF}_\tau$. More specifically, we will concentrate on the case of the Denjoy–Wolff point on the boundary. This means $\tau \in \{0, \infty\}$. It is worth mentioning that in certain questions, extending consideration to BRFPs at 0 or ∞ is rather natural or, at least, does not make the picture more complicated. That is why several results in this section are established for the more general setting of BRFPs.

2.5.1. The DW-point at the origin. Recall that a holomorphic self-map $v \in \text{Hol}(\mathbb{H}, \mathbb{H})$, $v \neq \text{id}_\mathbb{H}$, has the DW-point at 0 if and only if 0 is a boundary regular fixed point of v and $v'(0) \leq 1$. Therefore, the Laplace exponents of a branching process (Z_t) are contained in \mathfrak{BF}_0 if and only if the process is conservative and “subcritical” i.e. $\mathbb{E}^{(s,x)}[Z_t] \leq x$ for all $x \in (0, \infty)$ and $(s, t) \in \Delta$; see the proof of Theorem 2.16.

The infinitesimal generators $\phi \in \mathcal{G}(\mathfrak{BF}_0)$, which are branching mechanisms of conservative subcritical (in the above sense) time-homogeneous branching processes, can be characterized by an integral representation, see e.g. [25, Corollary 3.6] and [34, Theorem 1 on p. 21]. Namely, a function $\phi: \mathbb{H} \rightarrow \mathbb{C}$ belongs to $\mathcal{G}(\mathfrak{BF}_0)$ if and only if it admits the following representation:

$$(2.18) \quad \phi(\zeta) = c\zeta + b\zeta^2 + \int_{(0,\infty)} (e^{-\zeta x} - 1 + \zeta x) \pi(dx) \quad \text{for all } \zeta \in \mathbb{H},$$

where $c, b \geq 0$ and π is a non-negative Borel measure on $(0, \infty)$ with $\int_{(0,\infty)} \min\{\lambda^2, \lambda\} \pi(d\lambda) < \infty$.

Remark 2.21. Clearly, any $\phi \in \mathcal{G}(\mathfrak{BF}_0)$ admits also Silverstein’s representation (1.5). As we mentioned in [25, right after Corollary 3.6], the coefficient b and the measure π in the representations (1.5) and (2.18) are the same. Moreover, it is easy to show that

$$\begin{aligned} -q = \phi(0) &:= \lim_{\theta \rightarrow 0^+} \phi(\theta) = 0, & 2b = \phi''(\infty) &:= \lim_{\theta \rightarrow +\infty} \phi''(\theta), \\ c = \phi'(0) &:= \lim_{\theta \rightarrow 0^+} \phi'(\theta), & a = c + \int_{[1,\infty)} \lambda \pi(d\lambda). \end{aligned}$$

This allows us to obtain an analogue of Theorem 2.13.

Theorem 2.22. *Let $(v_{s,t})_{(s,t) \in \Delta}$ be an absolutely continuous reverse evolution family contained in \mathfrak{BF}_0 . Then for every $t \geq 0$ and $\zeta \in \mathbb{H}$ the map $s \mapsto v(s) := v_{s,t}(\zeta)$ is the unique solution to the initial value problem*

$$(2.19) \quad \frac{d}{ds} v(s) = \phi(v(s), s) \quad \text{a.e. } s \in [0, t]; \quad v(t) = \zeta,$$

where the Herglotz vector field $\phi: \mathbb{H} \times [0, \infty) \rightarrow \mathbb{C}$ is defined by

$$(2.20) \quad \phi(\zeta, t) := c_t \zeta + b_t \zeta^2 + \int_0^\infty (e^{-\zeta x} - 1 + \zeta x) \pi_t(dx), \quad \zeta \in \mathbb{H}, \quad t \geq 0,$$

with some $c_t, b_t \geq 0$, and some non-negative Borel measures π_t on $(0, \infty)$ satisfying the following conditions:

- (a) $t \mapsto c_t$ and $t \mapsto b_t$ are in L^1_{loc} ;
- (b) the family of measures $\min\{\lambda^2, \lambda\} \pi_t(d\lambda)$ on $(0, \infty)$ is locally integrable in t , see Definition 2.8.

The family (c_t, b_t, π_t) is determined by $(v_{s,t})$ uniquely up to a null-set on the t -axis.

Conversely, given a family $(c_t, b_t, \pi_t)_{t \geq 0}$ satisfying the above assumptions, the function ϕ defined by (2.20) is a Herglotz vector field, and for every $\zeta \in \mathbb{H}$ and $t \geq 0$, the initial value problem (2.13) has a unique solution $s \mapsto v(s) = v(s; t, \zeta)$. Moreover, the formula $v_{s,t}(\zeta) := v(s; t, \zeta)$ defines an absolutely continuous reverse evolution family $(v_{s,t})_{(s,t) \in \Delta}$ contained in \mathfrak{BF}_0 .

Proof. Recall that \mathfrak{BF} is topologically closed in $\text{Hol}(\mathbb{H}, \mathbb{H})$. Moreover, it is well-known, see e.g. [26, Remark 2.4], that if $v \in \text{Hol}(\mathbb{H}, \mathbb{H}) \setminus \{\text{id}_{\mathbb{H}}\}$ is the limit of a sequence of self-maps with the DW-point at 0, then the DW-point of v is also at 0. It follows that \mathfrak{BF}_0 is a topologically closed semigroup in $\text{Hol}(\mathbb{H}, \mathbb{H})$. Therefore, according to Theorem B, the absolutely continuous reverse evolution family generated by a Herglotz vector field ϕ is contained in \mathfrak{BF}_0 if and only if $\phi(\cdot, t) \in \mathcal{G}(\mathfrak{BF}_0)$.

Thus, taking into account representation (2.18) and using Theorem A, we see that it is sufficient to prove the following statement: let $\phi : \mathbb{H} \times [0, \infty) \rightarrow \mathbb{C}$ be defined by (2.20) with some $c_t, b_t \geq 0$ and some non-negative Borel measures π_t on $(0, \infty)$; then ϕ is Herglotz vector field if and only if conditions (a) and (b) are satisfied.

Suppose that ϕ is a Herglotz vector field. Then by Remark 2.21 and Proposition 2.11, we have:

- (i) $t \mapsto b_t$ is in L^1_{loc} ;
- (ii) $t \mapsto c_t + \int_{[1, \infty)} \lambda \pi_t(d\lambda)$ is in L^1_{loc} ;
- (iii) the finite measure $\min\{\lambda^2, 1\} \pi_t(d\lambda)$ on $(0, \infty)$ is locally integrable in t .

In view of Remark 2.10, assertion (iii) implies that the family of measures $\min\{\lambda^2, \lambda\} \pi_t(d\lambda)$ and hence the function $I_2(t) := \int_{[1, \infty)} \lambda \pi_t(d\lambda)$ are measurable. Therefore, taking into account that $c_t \geq 0$ and using (ii), we conclude that both $t \mapsto c_t$ and I_2 are in L^1_{loc} . Finally, by (iii), $I_1(t) := \int_{(0,1)} \lambda^2 \pi_t(d\lambda)$ is also in L^1_{loc} . Now (a) and (b) follow immediately.

Conversely, suppose that $c_t, b_t \geq 0$ and non-negative Borel measures π_t on $(0, \infty)$ satisfy conditions (a) and (b). Then using Remarks 2.21 and 2.10, it is easy to show that ϕ defined by (2.20) admits also representation (2.9) with a certain Lévy family (q_t, a_t, b_t, π_t) . By Proposition 2.11, this means that ϕ is a Herglotz vector field. \square

Remark 2.23. Note that the representation (2.18) with $c \in \mathbb{R}$ of arbitrary sign is commonly used in the time-homogeneous case. It characterises branching mechanisms ϕ of stochastic processes with finite mean or, equivalently, infinitesimal generators of one-parameter semigroups in \mathfrak{BF} with a BRFP at 0. A complex-analytic proof of this fact can be found in [25, Corollary 3.6]. In the time-inhomogeneous case, one can use Theorem 2.16 to establish an analogue of the theorem proved above, which provides a representation formula for Herglotz vector fields corresponding to branching processes with finite mean. It is literally the same representation (2.20) as in Theorem 2.22, with c_t, b_t , and π_t satisfying (a) and (b), but with the condition $c_t \geq 0$ relaxed to $c_t \in \mathbb{R}$ for all $t \geq 0$. It is worth to mention here one subtle point. Suppose it is given that ϕ is a Herglotz vector field. If (2.20) holds for a.e. $s \geq 0$ with some $c_t, b_t \geq 0$ and some non-negative Borel measures π_t on $(0, \infty)$, then the integrability conditions (a) and (b) are automatically satisfied because in this case (2.20) implies that $\phi(\cdot, s) \in \mathcal{G}(\mathfrak{BF}_0)$ for a.e. $s \geq 0$, and hence by Theorem B, the reverse evolution family associated with ϕ is contained in \mathfrak{BF}_0 . This is not the case anymore, i.e. $t \mapsto c_t$ does not have to be in L^1_{loc} , if this parameter is allowed to take negative values; see the example below.

Example 2.24. Let $\lambda(t) := 1/t$ for $t \in (0, 1)$ and $\lambda(t) := 1$ for $t \in \{0\} \cup [1, \infty)$. Define

$$\phi(\zeta, t) := \exp(-\zeta\lambda(t)) - 1, \quad t \geq 0, \zeta \in \mathbb{H}.$$

Then ϕ admits representation (2.9) with $q_t = a_t = b_t = 0$ and $\pi_t := \delta_{\lambda(t)}$. Therefore, by Proposition 2.11, ϕ is a Herglotz vector field. The reverse evolution family $(v_{s,t})$ associated with ϕ is contained in \mathfrak{BF} by Theorem 2.13. Moreover, ϕ satisfies equality (2.20) with the same $\pi_t = \delta_{\lambda(t)}$, $b_t = 0$, and with $c_t := -\lambda(t)$ for all $t \geq 0$. Clearly, $t \mapsto c_t$ is not L^1_{loc} . As a result, 0 is not a BRFP of $v_{0,t}$ for any $t > 0$.

Returning to the main topic of this subsection, assume that the Laplace exponents of a branching process (Z_t) have the DW-point at 0. Under this assumption, we are able to obtain a sufficient condition for finite extinction time. Recall that by definition the *extinction time* is

$$T_0^s := \inf\{t \geq s : Z_t = 0\}.$$

Note that when discussing the extinction time, we always assume that (Z_t) is a $D[0, \infty)$ -valued random variable, see Theorem 2.7.

Remark 2.25. From the Laplace transform (2.7), it is obvious that

$$\mathbb{P}^{(s,x)}[Z_t = 0] = e^{-xv_{s,t}(\infty)} \quad \text{whenever } 0 \leq s \leq t, x > 0.$$

Because we are assuming that (Z_t) is a $D[0, \infty)$ -valued random variable, the event $\{Z_t = 0\}$ is increasing w.r.t. the parameter t . Therefore, we have

$$\{T_0^s < \infty\} = \bigcup_{n \in \mathbb{N}, n > s} \{Z_n = 0\}$$

and hence by the monotonicity of measures

$$(2.21) \quad \mathbb{P}^{(s,x)}[T_0^s < \infty] = \lim_{t \rightarrow \infty} e^{-xv_{s,t}(\infty)}.$$

The following theorem gives a sufficient condition for almost sure extinction after within a finite time, i.e. a sufficient condition for $\mathbb{P}^{(s,x)}[T_0^s < \infty] = 1$. Later we will also establish a sufficient condition for $\mathbb{P}^{(s,x)}[T_0^s < \infty] = 0$, see Corollary 2.35.

Theorem 2.26. *Let $(v_{s,t})$ be an absolutely continuous reverse evolution family in \mathfrak{BF}_0 with associated Herglotz vector field ϕ . Suppose that*

$$(2.22) \quad \int_0^{+\infty} \phi''(\infty, t) dt = +\infty.$$

Then for any $s \geq 0$,

$$\lim_{t \rightarrow \infty} v_{s,t}(\infty) = 0$$

and hence for the branching process (Z_t) associated with $(v_{s,t})$ we have $\mathbb{P}^{(s,x)}[T_0^s < \infty] = 1$ for any $x > 0$ and any $s \geq 0$.

In the proof we use the following rather standard auxiliary result, which we prefer to prove in a slightly more general setting.

Lemma 2.27. *Let $T \in (0, \infty]$, $I \subset \mathbb{R}$. Let $f : [0, T) \rightarrow I$ and $g : [0, T) \rightarrow I$ be two locally absolutely continuous functions with $f(0) = g(0)$. Suppose that there exists a function $h : I \times [0, T) \rightarrow \mathbb{R}$ such that:*

- (i) *for any $x \in I$, the function $h(x, \cdot)$ is measurable on $[0, T)$;*

(ii) *there exists a locally integrable function $p: [0, T) \rightarrow [0, \infty)$ such that*

$$|h(x, t) - h(y, t)| \leq p(t)|x - y|$$

for all $t \in [0, T)$ and all $x, y \in I$;

(iii) *$g'(t) = h(g(t), t)$ and $f'(t) \leq h(f(t), t)$ for a.e. $t \in [0, T)$.*

Then $f \leq g$ on $[0, T)$.

Proof. Suppose to the contrary that $f(t_{\#}) > g(t_{\#})$ for some $t_{\#} \in [0, T)$. Set

$$t_* := \max\{t \in [0, t_{\#}) : f(t) = g(t)\},$$

which exists because f and g are continuous and $f(0) = g(0)$. Then $f(t) > g(t)$ for all $t \in (t_*, t_{\#}]$. Replacing $t_{\#}$ with a point in (t_*, T) close enough to t_* , we may assume that $\int_{t_*}^{t_{\#}} p(s) ds < 1$. Taking into account that by a standard argument, see e.g. [39, Chapter VIII, §8], $s \mapsto h(f(s), s)$ is integrable on $[0, t_{\#}]$, for every $t \in (t_*, t_{\#}]$, we have

$$\begin{aligned} f(t) - g(t) &= \int_{t_*}^t [f'(s) - g'(s)] ds \\ &\leq \int_{t_*}^t [h(f(s), s) - h(g(s), s)] ds \\ &\leq \int_{t_*}^t p(s)[f(s) - g(s)] ds \\ &\leq \sup_{t \in [t_*, t_{\#}]} [f(t) - g(t)] \int_{t_*}^t p(s) ds. \end{aligned}$$

It follows that

$$\sup_{t \in [t_*, t_{\#}]} [f(t) - g(t)] \leq \sup_{t \in [t_*, t_{\#}]} [f(t) - g(t)] \int_{t_*}^{t_{\#}} p(s) ds < \sup_{t \in [t_*, t_{\#}]} [f(t) - g(t)],$$

which is a contradiction. \square

Proof of Theorem 2.26. According to Theorem 2.22, the Herglotz vector field for a.e. $t \geq 0$ is represented by (2.20) with $c_t, b_t \geq 0$, and some non-negative Borel measures π_t on $(0, \infty)$ satisfying the integrability conditions (a) and (b). According to Remark 2.21 $\phi''(\infty, t) = 2b_t \geq 0$ for a.e. $t \geq 0$. Therefore, the integral in (2.22) is well-defined.

Note that $e^{-\theta x} - 1 + \theta x \geq 0$ for any $\theta, x > 0$. Hence,

$$(2.23) \quad \phi(\theta, t) \geq \phi_0(\theta, t) := b_t \theta^2 \quad \text{for a.e. } t \geq 0 \text{ and all } \theta > 0.$$

It is easy to see that ϕ_0 is a Herglotz vector field. Denote by $(v_{s,t}^0)$ the reverse evolution family associated to ϕ_0 . Using the ODE (1.7), it is then easy to check that for all $z \in \mathbb{H}$ and all $(s, t) \in \Delta$,

$$v_{s,t}^0(z) = \frac{z}{1 + b(s, t)z}, \quad b(s, t) := \frac{1}{2} \int_s^t \phi''(\infty, r) dr.$$

To prove the theorem it is enough to show that

$$(2.24) \quad v_{s,t}(x) \leq v_{s,t}^0(x) \quad \text{for all } x > 0 \text{ and all } (s, t) \in \Delta.$$

Fix arbitrary $T > 0$ and $x > 0$. According to Theorem A, the functions

$$f(s) := v_{T-t, T}(x) \quad \text{and} \quad g(s) := v_{T-t, T}^0(x)$$

satisfy the ODEs $f'(t) = -\phi(f(t), T-t)$ and $g'(t) = -\phi_0(f_0(t), T-t)$ with the initial condition $f(0) = g(0) = x$. To prove (2.24), we have to show that $f(t) \leq g(t)$ for all $t \in [0, T]$. Denote

$$\beta := \max\{f(t), f_0(t) : t \in [0, T]\}.$$

Clearly $|\phi_0(y, T-t) - \phi_0(z, T-t)| \leq 2\beta b_t|y - z|$ for all $t \in [0, T]$, and all $y, z \in [0, \beta]$. Taking into account (2.23), we obtain by Lemma 2.27 the desired inequality $f \leq g$. Thus, for any $s \geq 0$, $v_{s,t}(\infty) \rightarrow 0$ as $t \rightarrow +\infty$. According to Remark 2.25, this is equivalent to $\mathbb{P}^{(s,x)}[T_0^s < \infty] = 1$. \square

2.5.2. The Denjoy–Wolff point at infinity. Recall that $f \in \text{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\text{id}_{\mathbb{H}}\}$ has the DW-point at ∞ if and only if $\angle \lim_{z \rightarrow \infty} f(z)/z \geq 1$. If $f((0, \infty)) \subset (0, \infty)$, then using Wolff's Lemma, see e.g. [25, Theorem B], we see that ∞ is the DW-point of f if and only if for some, and hence for all $\theta_0 \geq 0$ the inequality $f(\theta) \geq \theta$ holds for any $\theta > \theta_0$.

Remark 2.28. If now $f \in \mathfrak{BF}$, then $f'(\infty)$ coincides with the coefficient β in the integral representation (1.3). Therefore, $f \in \mathfrak{BF}$ belongs to \mathfrak{BF}_{∞} if and only if $\beta \geq 1$.

Theorem 2.29. *Let (Z_t) be a branching process and let $(v_{s,t})$ be the associated topological reverse evolution family. The following four conditions are equivalent:*

- (a) $(v_{s,t}) \subset \mathfrak{BF}_{\infty}$;
- (b) for any $0 \leq s \leq t \leq u$ and any $x > 0$, $\mathbb{P}^{(s,x)}[Z_t \leq Z_u] = 1$;
- (c) for any $(s, t) \in \Delta$ and any $x > 0$, $\mathbb{P}^{(s,x)}[Z_t \geq x] = 1$;
- (d) for any $(s, t) \in \Delta$ there exists $x > 0$ such that $\mathbb{P}^{(s,x)}[Z_t \geq x] = 1$.

Proof. Suppose that (a) holds. As mentioned above, condition (a) is equivalent to that $v_{s,t}(\theta) \geq \theta$ for all $\theta > 0$ and $(s, t) \in \Delta$. Let $0 < u < x$ and $\theta > 0$. Then

$$e^{-xv_{s,t}(\theta)} = \int_{[0,\infty)} e^{-\theta y} k_{s,t}(x, dy) \geq \int_{[0,u]} e^{-\theta y} k_{s,t}(x, dy) \geq e^{-\theta u} k_{s,t}(x, [0, u])$$

and hence

$$k_{s,t}(x, [0, u]) \leq e^{\theta u} e^{-xv_{s,t}(\theta)} \leq e^{\theta u} e^{-x\theta} = e^{-\theta(x-u)}.$$

Letting $\theta \rightarrow +\infty$ yields $k_{s,t}(x, [0, u]) = 0$ for all $u \in (0, x)$. It follows that $k_{s,t}(x, [0, x)) = 0$, or equivalently that

$$\mathbb{P}^{(s,x)}[Z_t < x] = 0.$$

This proves the implication (a) \Rightarrow (c). Obviously, (c) implies (d).

Now suppose that (d) holds, or equivalently that $k_{s,t}(x, [0, x)) = 0$ for any $(s, t) \in \Delta$ and some $x > 0$, which may depend on s and t . Then

$$e^{-xv_{s,t}(\theta)} = \int_{[x,\infty)} e^{-\theta y} k_{s,t}(x, dy) \leq \int_{[x,\infty)} e^{-\theta x} k_{s,t}(x, dy) \leq e^{-\theta x}.$$

Hence, we get $v_{s,t}(\theta) \geq \theta$ for any $(s, t) \in \Delta$ and any $\theta > 0$, i.e. $(v_{s,t}) \subset \mathfrak{BF}_{\infty}$.

It remains to establish the equivalence of (b) and (c). The implication (b) \Rightarrow (c) is obvious. Conversely, suppose that (c) holds, which can be written as $k_{t,u}(y, [y, \infty]) = 1$ for all $(t, u) \in \Delta$ and all $y > 0$. Denote $\Delta^{\infty} := \{(y, z) : 0 \leq y \leq z \leq \infty\}$. For any $0 \leq s \leq t \leq u < \infty$ and $n \in \mathbb{N}_0$, using (2.6) we have

$$\begin{aligned} \mathbb{P}^{(s,x)}[Z_t \leq Z_u] &= \mathbb{P}^{(s,x)}[(Z_t, Z_u) \in \Delta^{\infty}] = \int_{\Delta^{\infty}} k_{s,t}(x, dy) k_{t,u}(y, dz) \\ &= \int_{[0,\infty]} k_{s,t}(x, dy) = 1 \end{aligned}$$

as desired. \square

Remark 2.30. If $(Z_t)_{t \geq 0}$ has cadlag paths almost surely, then the equivalent conditions (a)–(d) are further equivalent to

$$(e) \quad \mathbb{P}^{(s,x)}[Z_t \leq Z_u \text{ whenever } s \leq t \leq u] = 1 \quad \text{for any } n \in \mathbb{N}_0 \text{ and any } s \geq 0,$$

i.e. the branching process has non-decreasing sample paths almost surely. Indeed, condition (e) clearly implies (b). Conversely, (b) implies that

$$\begin{aligned} \mathbb{P}^{(s,x)}[Z_t \leq Z_u \text{ whenever } s \leq t \leq u] &= \\ \mathbb{P}^{(s,x)}[Z_t \leq Z_u \text{ whenever } s \leq t \leq u \text{ and } s, t \in \mathbb{Q}] &= 1, \end{aligned}$$

where the cadlag property is needed to ensure the first equality.

In the remark below we give a probabilistic interpretation of the angular derivative at ∞ . As it is mentioned in Section 1.4, for any $v \in \text{Hol}(\mathbb{H}, \mathbb{H})$, the limit $v'(\infty) := \lim_{\theta \rightarrow \infty} v(\theta)/\theta$ exists and it is a non-negative real number.

Remark 2.31. Recall that $\mathbb{E}^{(s,x)}[e^{-\theta Z_t}] = e^{-xv_{s,t}(\theta)}$. This can be written as

$$\|e^{-Z_t}\|_{L^\theta(\mathbb{P}^{(s,x)})} = e^{-x \frac{v_{s,t}(\theta)}{\theta}}$$

and hence taking the limit $\theta \rightarrow \infty$, we get

$$\|e^{-Z_t}\|_{L^\infty(\mathbb{P}^{(s,x)})} = e^{-xv'_{s,t}(\infty)}.$$

This is equivalent to

$$\mathbb{P}^{(s,x)}\text{-ess inf } Z_t = xv'_{s,t}(\infty).$$

In particular, ∞ is a BRFP of $v_{s,t}$ if and only if $\mathbb{P}^{(s,x)}\text{-ess inf } Z_t > 0$ for some and hence all $x \geq 0$. Similarly, ∞ is a DW-point of $v_{s,t}$ if and only if $\mathbb{P}^{(s,x)}\text{-ess inf } Z_t \geq x$ for some and hence all $x \geq 0$. The latter is yet another way to express “monotonicity” of the process (Z_t) , i.e. another condition equivalent to (c)–(d) in Theorem 2.29.

In the theorem below we characterize Herglotz vector fields associated with reverse evolution families contained in \mathfrak{BF} and having a BRFP at ∞ . As a corollary, we also characterize Herglotz vector fields associated with reverse evolution families contained in \mathfrak{BF}_∞ .

Theorem 2.32. *A function $\phi : \mathbb{H} \times [0, \infty) \rightarrow \mathbb{C}$ is a Herglotz vector field of a reverse evolution family $(v_{s,t}) \subset \mathfrak{BF}$ with a common BRFP at ∞ if and only if it admits the following representation:*

$$(2.25) \quad \phi(\zeta, t) = -q_t + d_t \zeta + \int_0^\infty (e^{-\zeta x} - 1) \pi_t(dx), \quad \zeta \in \mathbb{H}, \text{ a.e. } t \geq 0,$$

with some $q_t \geq 0$, $d_t \in \mathbb{R}$ and some non-negative measures π_t on $(0, \infty)$ satisfying the following conditions:

- (a) $t \mapsto q_t$ and $t \mapsto d_t$ are in L^1_{loc} ;
- (b) $\min\{x, 1\} \pi_t(dx)$ is locally integrable in $t \geq 0$.

In such a case,

$$(2.26) \quad v'_{s,t}(\infty) = \exp\left(-\int_s^t d_r dr\right) \quad \text{for all } (s, t) \in \Delta.$$

Remark 2.33. If ϕ admits representation (2.25) above with some $q_t \geq 0$, $d_t \in \mathbb{R}$ and some non-negative measures π_t satisfying $\int_0^\infty \min\{x, 1\} \pi_t(dx) < \infty$, then ϕ admits also representation (2.9) with $a_t := d_t + \int_{(0,1)} x \pi_t(dx)$, $b_t := 0$ and with the same q_t and π_t .

Before we prove Theorem 2.32, we establish two direct corollaries. Recall that the following two conditions are equivalent for $v \in \text{Hol}(\mathbb{H}, \mathbb{H})$:

- (i) ∞ is a BRFP of v and $v'(\infty) \geq 1$;
- (ii) ∞ is the DW-point of v or $v = \text{id}_{\mathbb{H}}$.

Therefore, with (2.26) taken into account, Theorem 2.32 implies the following characterization of Herglotz vector fields of absolutely continuous reverse evolution families contained in $\mathfrak{B}\mathfrak{F}_{\infty}$.

Corollary 2.34. *A function $\phi : \mathbb{H} \times [0, \infty) \rightarrow \mathbb{C}$ is a Herglotz vector field of a reverse evolution family $(v_{s,t}) \subset \mathfrak{B}\mathfrak{F}_{\infty}$ if and only if ϕ admits representation (2.25) with some $q_t \geq 0$, $d_t \leq 0$ and some non-negative measures π_t on $(0, \infty)$ satisfying the integrability conditions (a) and (b) in Theorem 2.32.*

In view of (2.21), we have another immediate corollary of Theorem 2.32, which gives a sufficient condition for almost surely no extinction within finite time.

Corollary 2.35. *Suppose that ϕ satisfies the necessary and sufficient condition of Theorem 2.32. Then for the associated branching process (Z_t) , we have $\mathbb{P}^{(s,x)}[T_0^s < \infty] = 0$ for any $x > 0$ and any $s \geq 0$.*

Remark 2.36. An observation analogous to that in Remark 2.23 can be made for the fixed point at ∞ as well. If it is *a priori* known that ϕ is a Herglotz vector field, and if for a.e. $t \geq 0$ it admits representation (2.25) with some $q_t \geq 0$, $d_t \leq 0$ and some non-negative measure π_t on $(0, \infty)$, then $(v_{s,t}) \subset \mathfrak{B}\mathfrak{F}_{\infty}$ and the integrability conditions (a) and (b) are automatically satisfied. However, if we replace $d_t \leq 0$ with $d_t \in \mathbb{R}$, the requirement that $t \mapsto d_t$ is in L_{loc}^1 has to be added in order to guarantee that $(v_{s,t})$ has a BRFP at ∞ . This can be illustrated by the following example: $\phi(\zeta, 0) := 0$ and $\phi(\zeta, t) := t^{-1}\zeta + t^{-2}(e^{-t\zeta} - 1)$ for all $t > 0$ and all $\zeta \in \mathbb{H}$. This is a Herglotz vector field by the equivalence of (i) and (ii) in Proposition 2.11 and it is clearly of the form (2.25) for each $t \geq 0$, but $d_t = \phi'(\infty, t) = t^{-1}$ does not belong to L_{loc}^1 .

Proof of Theorem 2.32. Suppose that $\phi : \mathbb{H} \times [0, \infty) \rightarrow \mathbb{C}$ is a Herglotz vector field. Denote by $(v_{s,t})$ the reverse evolution family associated with ϕ . Combining [25, Corollary 3.7, Theorem 2, and Theorem C], one can conclude that $(v_{s,t})$ is contained in $\mathfrak{B}\mathfrak{F}$ and has a common BRFP at ∞ if and only if the following two conditions are satisfied:

- (i) for a.e. $t \geq 0$, $\phi(\cdot, t)$ is of the form (2.25) for some $q_t \geq 0$, $d_t \in \mathbb{R}$ and non-negative measures π_t such that $\int_0^\infty \min\{x, 1\} \pi_t(dx) < \infty$,
- (ii) $t \mapsto \phi'(\infty, t) = d_t$ is in L_{loc}^1 .

In view of Remark 2.33 and Proposition 2.11, condition (i) implies that $t \mapsto q_t$ is in L_{loc}^1 and that π_t is measurable in t . It further follows that conditions (i) and (ii) imply that $\min\{x, 1\} \pi_t(dx)$ is locally integrable in $t \geq 0$, because

$$\int_0^\infty (1 - e^{-x}) \pi_t(dx) = -q_t + d_t - \phi(1, t)$$

and $\min\{x, 1\} \leq e(e-1)^{-1}(1 - e^{-x})$ on $(0, \infty)$. This proves the necessity part.

As for sufficiency, it remains to notice that any function $\phi : \mathbb{H} \times [0, \infty) \rightarrow \mathbb{C}$ of the form (2.25) with q_t , d_t , and π_t satisfying integrability conditions (a) and (b) is a Herglotz vector field. Indeed, if (a) and (b) hold, then $(a_t, 0, q_t, \pi_t)$, where $a_t := d_t + \int_{(0,1)} x \pi_t(dx)$, is a Lévy family and hence the desired conclusion follows from Remark 2.33 and Proposition 2.11.

Finally, (2.26) can be obtained from the corresponding formula in [25, Theorem C] with the help of the Cayley transformation $H(z) := (1+z)/(1-z)$, which maps \mathbb{D} conformally onto \mathbb{H} and takes the boundary point 1 to ∞ . \square

3. BRANCHING PROCESSES ON DISCRETE SPACE

Several important results in Section 2 have analogs for time-inhomogeneous branching processes on $\mathbb{N}_0^* := \mathbb{N}_0 \cup \{\infty\}$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We endow \mathbb{N}_0^* with the natural topology to make it a compact space. Note then that $\{\infty\}$ is not an open subset, while for all other (finite) non-negative integers n , the singletons $\{n\}$ are open subsets.

3.1. Characterization of branching processes. To begin with, a *branching process* on \mathbb{N}_0^* is a Markov family with transition kernels $(\ell_{s,t})_{(s,t) \in \Delta}$ on \mathbb{N}_0^* satisfying the following conditions:

- (L1) $\ell_{s,s}(n, \cdot) = \delta_n(\cdot)$ for every $s \geq 0$ and $n \in \mathbb{N}_0^*$,
- (L2) $\ell_{s,t} \circ \ell_{t,u} = \ell_{s,u}$ for every $0 \leq s \leq t \leq u$,
- (L3) for every $n \in \mathbb{N}_0$ the map $\Delta \ni (s, t) \mapsto \ell_{s,t}(n, \cdot) \in \mathcal{P}(\mathbb{N}_0^*)$ is weakly continuous,
- (L4) $\ell_{s,t}(m, \cdot) * \ell_{s,t}(n, \cdot) = \ell_{s,t}(m+n, \cdot)$, $(s, t) \in \Delta$, $m, n \in \mathbb{N}_0$, (Branching property)
- (L5) $\ell_{s,t}(0, \cdot) = \delta_0(\cdot)$ for every $(s, t) \in \Delta$,
- (L6) $\ell_{s,t}(\infty, \cdot) = \delta_\infty(\cdot)$ for every $(s, t) \in \Delta$.

Remark 3.1. Condition (L4) can also be written as $\ell_{s,t}(n, \cdot) = \ell_{s,t}(1, \cdot)^{*n}$ for $n \in \mathbb{N}$. Also, (L3) is equivalent to that for every $n, m \in \mathbb{N}_0$ the map $\Delta \ni (s, t) \mapsto \ell_{s,t}(n, \{m\}) \in [0, 1]$ is continuous. It is worth mentioning that the map $(s, t) \mapsto \ell_{s,t}(n, \{\infty\})$ is not required to be continuous.

The following (sub)probability generating function plays the role that the Laplace exponent does for the continuous state case,

$$(3.1) \quad F_{s,t}(z) := \int_{\mathbb{N}_0} z^n \ell_{s,t}(1, dn) = \sum_{n \geq 0} p_{s,t}(n) z^n, \quad z \in \mathbb{D},$$

where $p_{s,t}(n) = \ell_{s,t}(1, \{n\})$. Each $F_{s,t}$, unless it is identically equal to 1, belongs to the class

$$\mathfrak{P}\mathfrak{G}\mathfrak{F} := \left\{ \sum_{n \geq 0} p_n z^n : (\forall n \in \mathbb{N}_0) p_n \geq 0, \sum_{n \geq 0} p_n \leq 1 \right\} \setminus \{F \equiv 1\},$$

which is clearly a topologically closed subsemigroup of $\text{Hol}(\mathbb{D}, \mathbb{D})$.

Remark 3.2. In the classical Galton–Watson processes, the explosion is not allowed, i.e. $\ell_{s,t}(n, \{\infty\}) = 0$ for any $n \in \mathbb{N}_0$ and any $(s, t) \in \Delta$. This corresponds to the subsemigroup \mathfrak{P}^+ of $\mathfrak{P}\mathfrak{G}\mathfrak{F}$ consisting of functions whose sum of the Taylor coefficients equals 1. Working in this setting, Goryainov [21, 22] studied reverse evolution families of probability generating functions in \mathfrak{P}^+ . An obvious convenience in considering $\mathfrak{P}\mathfrak{G}\mathfrak{F}$ is that this semigroup is topologically closed in $\text{Hol}(\mathbb{D}, \mathbb{D})$. Note that this is not the case for \mathfrak{P}^+ . In fact, $\mathfrak{P}\mathfrak{G}\mathfrak{F}$ is the topological closure of \mathfrak{P}^+ in $\text{Hol}(\mathbb{D}, \mathbb{D})$.

Similarly to the continuous-state case, there is a one-to-one correspondence between families of transition kernels of branching processes on \mathbb{N}_0^* and topological reverse evolution families contained in $\mathfrak{P}\mathfrak{G}\mathfrak{F}$.

Theorem 3.3. *Given a family $(\ell_{s,t})_{(s,t) \in \Delta}$ of transition kernels of a branching process on \mathbb{N}_0^* , $(F_{s,t})_{(s,t) \in \Delta}$ defined by (3.1) forms a topological reverse evolution family contained in $\mathfrak{P}\mathfrak{G}\mathfrak{F}$. Conversely, given a topological reverse evolution family $(F_{s,t})_{(s,t) \in \Delta}$ contained in $\mathfrak{P}\mathfrak{G}\mathfrak{F}$, there exists a unique family of transition kernels $(\ell_{s,t})_{(s,t) \in \Delta}$ of a branching process on \mathbb{N}_0^* such that (3.1) holds.*

Proof. Suppose we are given a family $(\ell_{s,t})_{(s,t) \in \Delta}$ of transition kernels of a branching process on \mathbb{N}_0^* . Then it is straightforward to check that the functions $F_{s,t}$ regarded as self-maps of $\overline{\mathbb{D}}$ satisfy the conditions (REF1) and (REF2) in Definition 1.4.

Since the Taylor coefficients $p_{s,t}(n)$ of $F_{s,t}$ satisfy $0 \leq p_{s,t}(n) \leq 1$, for any $N \in \mathbb{N}$ we have

$$\left| F_{s,t}(z) - \sum_{n=0}^{N-1} p_{s,t}(n) z^n \right| \leq |z|^N \quad \text{for all } z \in \mathbb{D}.$$

Combined with the fact that by (L3), $(s, t) \mapsto p_{s,t}(n)$ is continuous for each $n \in \mathbb{N}_0$, this implies the continuity of the map $\Delta \ni (s, t) \mapsto F_{s,t} \in \text{Hol}(\mathbb{D}, \mathbb{C})$.

Using now the continuity of $F_{s,t}$ w.r.t. the parameters and following the argument employed in the proof of Theorem 2.5, we see that $F_{s,t} \in \text{Hol}(\mathbb{D}, \mathbb{D})$ for any $(s, t) \in \Delta$. Thus, $(F_{s,t})$ is a topological reverse evolution family contained in $\mathfrak{P}\mathfrak{E}\mathfrak{F}$.

Now suppose we are given a topological reverse evolution family $(F_{s,t}) \subset \mathfrak{P}\mathfrak{E}\mathfrak{F}$. For each $(s, t) \in \Delta$, the following relations define a unique transition kernel $\ell_{s,t}$ on \mathbb{N}_0^* :

$$\begin{aligned} \ell_{s,t}(0, \cdot) &:= \delta_0(\cdot), & \ell_{s,t}(1, \{m\}) &:= F_{s,t}^{(m)}(0)/m! \quad \text{for any } m \in \mathbb{N}_0, \\ \ell_{s,t}(\infty, \cdot) &:= \delta_\infty(\cdot), & \ell_{s,t}(n, \cdot) &:= \ell(1, \cdot)^{*n} \quad \text{for any } n \in \mathbb{N}. \end{aligned}$$

Since $(s, t) \mapsto F_{s,t} \in \text{Hol}(\mathbb{D}, \mathbb{D})$ is continuous, the maps $(s, t) \mapsto F_{s,t}^{(m)}(0)$, $m \in \mathbb{N}_0$, are continuous as well. It follows that $(\ell_{s,t})$ satisfies the condition (L3), see Remark 3.1. Checking that the rest of the conditions (L1)–(L6) hold is straightforward and therefore omitted. \square

3.2. An ODE governing probability generating functions. According to [23, Theorem 30], the set $\mathcal{G}(\mathfrak{P}\mathfrak{E}\mathfrak{F})$ of infinitesimal generators of one-parameter semigroups in $\mathfrak{P}\mathfrak{E}\mathfrak{F}$ consists of functions

$$(3.2) \quad \Phi(z) = c \left[z - b_0 - \sum_{n \geq 2} b_n z^n \right], \quad z \in \mathbb{D},$$

where $c \geq 0$ and $b_0, b_n \geq 0$ such that $b_0 + \sum_{n \geq 2} b_n \leq 1$. This can be reparameterized into

$$\Phi(z) = qz + \sum_{n \in \mathbb{N}_0 \setminus \{1\}} \alpha(n)(z - z^n),$$

where $q, \alpha(n) \geq 0$ for all $n \in \mathbb{N}_0 \setminus \{1\}$ with $\sum_{n \geq 2} \alpha(n) < \infty$. The pair (q, α) , where $\alpha := \{\alpha(n)\}_{n \in \mathbb{N}_0 \setminus \{1\}}$, is called the *generating pair* of Φ .

We say that a family of generating pairs $(q_t, \alpha_t)_{t \geq 0}$ is a *generating family* if the function $t \mapsto \alpha_t(n)$ is measurable for each $n \in \mathbb{N}_0 \setminus \{1\}$ and if the functions $t \mapsto q_t$ and $t \mapsto \sum_{n \in \mathbb{N}_0 \setminus \{1\}} \alpha_t(n)$ are in L^1_{loc} .

Theorem 3.4. *Let $(F_{s,t})_{(s,t) \in \Delta}$ be an absolutely continuous reverse evolution family contained in $\mathfrak{P}\mathfrak{E}\mathfrak{F}$. Then there exists a unique generating family $(q_t, \alpha_t)_{t \geq 0}$ such that for every $t \geq 0$ and $z \in \mathbb{D}$ the map $s \mapsto F(s) := F_{s,t}(z)$ is the unique solution to the initial value problem*

$$(3.3) \quad \frac{d}{ds} F(s) = \Phi(F(s), s) \quad \text{a.e. } s \in [0, t]; \quad F(t) = z,$$

where $\Phi: \mathbb{D} \times [0, \infty) \rightarrow \mathbb{C}$ is defined by

$$(3.4) \quad \Phi(z, s) = q_s z + \sum_{n \in \mathbb{N}_0 \setminus \{1\}} \alpha_s(n)(z - z^n), \quad z \in \mathbb{D}, \quad s \geq 0,$$

i.e. the Herglotz vector field associated with $(F_{s,t})_{(s,t) \in \Delta}$ is of the form (3.4).

Conversely, given a generating family $(q_t, \alpha_t)_{t \geq 0}$, the function Φ defined by (3.4) is a Herglotz vector field, and for every $z \in \mathbb{D}$ and $t \geq 0$, the initial value problem (3.3) has a unique solution $s \mapsto F(s) = F(s; t, z)$. Moreover, the functions $F_{s,t}(z) := F(s; t, z)$ form an absolutely continuous reverse evolution family $(F_{s,t})_{(s,t) \in \Delta} \subset \mathfrak{P}\mathfrak{E}\mathfrak{F}$.

Proof. The proof is similar to that of Theorem 2.13; one may use Theorem A and Theorem B for the topologically closed semigroup $\mathfrak{P}\mathfrak{G}\mathfrak{F} \subset \text{Hol}(\mathbb{D}, \mathbb{D})$. One only needs to replace Proposition 2.11 with the following argument.

If $\Phi(z, t)$ defined by (3.4) is a Herglotz vector field, i.e. if $\Phi(z, t)$ is measurable in t for each $z \in \mathbb{D}$ and if $t \mapsto \sup_{z \in K} |\Phi(z, t)|$ is locally integrable for every compact $K \subset \mathbb{D}$, then so does $t \mapsto \Phi^{(n)}(0, t)$ for each $n \in \mathbb{N}_0$ thanks to Cauchy's integral formula. Therefore, $t \mapsto \alpha_t(n) = -\Phi^{(n)}(0, t)/n!$ is measurable for each $n \in \mathbb{N}_0 \setminus \{1\}$, and moreover, since $\Phi'(0, t) = q_t + \sum_{n \geq 0, n \neq 1} \alpha_t(n)$, the functions $t \mapsto q_t$ and $t \mapsto \sum_{n \in \mathbb{N}_0 \setminus \{1\}} \alpha_t(n)$ are locally integrable. Thus, (q_t, α_t) is a generating family.

Conversely, if $(q_t, \alpha_t)_{t \geq 0}$ is a generating family, then a straightforward argument implies that $\Phi(z, t)$ defined by (3.4) is measurable in t for each $z \in \mathbb{D}$ and that $t \mapsto \sup_{z \in \mathbb{D}} |\Phi(z, t)|$ is locally integrable. Thus, Φ is a Herglotz vector field. \square

Remark 3.5. As a byproduct of the above proof, we see that for any Herglotz vector field Φ whose (reverse) evolution family is contained in $\mathfrak{P}\mathfrak{G}\mathfrak{F}$ satisfies a stronger version of the condition (HVF3); namely, $\sup_{z \in \mathbb{D}} |\Phi(z, t)|$ is in L^1_{loc} . The same conclusion can be obtained if one notice that

$$|\Phi(z)| \leq 2\Phi'(0), \quad z \in \mathbb{D},$$

for any function Φ given by (3.2) with $c, b_0, b_n \geq 0$ and $\sum_{n \in \mathbb{N}_0 \setminus \{1\}} b_n \leq 1$.

Remark 3.6. There is a somewhat surprising result [20]: if S a countable space with the discrete topology, then for any family of transition kernels $(\ell_{s,t})_{(s,t) \in \Delta}$ on S satisfying conditions (L1)–(L3) (with \mathbb{N}_0^* replaced by S), the transition probabilities $p_{s,t}(i, j) := \ell_{s,t}(i, \{j\})$ are a.e. differentiable in the sense that for a.e. $t \geq 0$ and all $i, j \in S$ the following limit exists finitely:

$$\lim_{\substack{|s-u| \rightarrow 0 \\ s \neq u, 0 \leq s \leq t \leq u}} \frac{p_{s,u}(i, j) - \delta_{i,j}}{u - s}.$$

This conclusion holds in our setting of branching processes if there is no explosion, i.e. if $\ell_{s,t}(n, \{\infty\}) = 0$ for all (s, t, n) . It is also worth mentioning, although not directly related to our setting, that a stronger result on absolute continuity is known for the case when the state space is a finite set; see [19].

3.3. Branching processes with finite first moments. Here we establish a characterization of branching processes on \mathbb{N}_0^* with finite first moments analogous to Theorem 2.16. Before stating the result, we introduce some notation: for $n \in \mathbb{N}_0$, $F \in \mathfrak{P}\mathfrak{G}\mathfrak{F}$ and $\Phi \in \mathcal{G}(\mathfrak{P}\mathfrak{G}\mathfrak{F})$, let

$$F^{(n)}(1) := \lim_{\mathbb{R} \ni x \rightarrow 1-0} F^{(n)}(x) \in [0, \infty] \quad \text{and} \quad \Phi^{(n)}(1) := \lim_{\mathbb{R} \ni x \rightarrow 1-0} \Phi^{(n)}(x) \in [-\infty, \infty].$$

The above limits exist because of monotonicity. Note that $F(1) \in [0, 1]$ and $\Phi(1) \in [0, \infty)$.

Theorem 3.7. *Let $(N_t)_{t \geq 0}$ be a branching process whose probability generating functions $(F_{s,t})_{(s,t) \in \Delta}$ are characterized by a Herglotz vector field Φ as in Theorem 3.4. The following conditions are equivalent:*

- (1) $\mathbb{E}^{(s,n)}[N_t] < \infty$ for all $n \in \mathbb{N}$ and $(s, t) \in \Delta$.
- (2) $\mathbb{E}^{(0,1)}[N_t] < \infty$ for all $t \geq 0$.
- (3) $F_{0,t}(1) = 1$ and $F'_{0,t}(1) < \infty$, i.e. 1 is a boundary regular fixed point of $F_{0,t}$ for all $t \geq 0$.
- (4) $F_{s,t}(1) = 1$ and $F'_{s,t}(1) < \infty$, i.e. 1 is a boundary regular fixed point of $F_{s,t}$ for all $(s, t) \in \Delta$.
- (5) $\Phi(1, t) = 0$ for a.e. $t \geq 0$, and $t \mapsto \Phi'(1, t)$ is in L^1_{loc} .
- (6) $q_t = 0$ for a.e. $t \geq 0$ and $t \mapsto \sum_{n \in \mathbb{N}_0 \setminus \{1\}} (n+1)\alpha_t(n)$ is in L^1_{loc} .

If one and hence all of the above conditions hold, then

$$\mathbb{E}^{(s,n)}[N_t] = nF'_{s,t}(1) = n \exp \left(- \int_s^t \Phi'(1, r) \, dr \right)$$

for all $n \in \mathbb{N}$ and $(s, t) \in \Delta$.

Proof. The proof is very similar to that of Theorem 2.16 and hence it is omitted. We would only mention that instead of formulas (2.17) one has to use the relations $F_{s,t}(1) = \mathbb{P}^{(s,1)}[N_t < \infty]$ and $F'_{s,t}(1) = \mathbb{E}^{(s,1)}[N_t]$. \square

3.4. Spatial embeddability. The notion of “spatial embedding” — i.e. embedding of a given discrete-state branching process into a continuous-state one — was discussed in [30, Section 4] (where the term “extension” was used for spatial embedding). In our setting, this remarkable notion can be defined as follows.

Definition 3.8. A branching process on \mathbb{N}_0^* with transition kernels $(\ell_{s,t})_{(s,t) \in \Delta}$ is said to be *spatially embeddable* into a branching process on $[0, \infty]$ if there exists a branching process on $[0, \infty]$ with transition kernels $(k_{s,t})_{(s,t) \in \Delta}$ such that for all $n \in \mathbb{N}_0^*$, $(s, t) \in \Delta$,

$$(3.5) \quad k_{s,t}(n, B) = \ell_{s,t}(n, B \cap \mathbb{N}_0^*) \quad \text{for any Borel set } B \subset [0, \infty].$$

We are able to establish a fairly simple necessary and sufficient condition of spatial embeddability in terms of the generating functions, which is further equivalent to a sort of monotonicity of the process.

Theorem 3.9. *Let $(N_t)_{t \geq 0}$ be a branching process on \mathbb{N}_0^* with probability generating functions $(F_{s,t})_{(s,t) \in \Delta}$. The following conditions are equivalent:*

- (i) (N_t) is spatially embeddable into a branching process on $[0, \infty]$;
- (ii) $F_{s,t}(0) = 0$ for any $(s, t) \in \Delta$;
- (iii) $\ell_{s,t}(n, \{m\}) = 0$ for any $(s, t) \in \Delta$ and $m, n \in \mathbb{N}_0$ with $m < n$;
- (iv) $\mathbb{P}^{(s,n)}[N_t \leq N_u] = 1$ for any $0 \leq s \leq t \leq u$ and $n \in \mathbb{N}_0$.

Proof. The proof is divided into five steps.

STEP 1: (i) \Rightarrow (ii). Suppose that (N_t) is embeddable into a branching process (Z_t) on $[0, \infty]$. As before, denote by $(v_{s,t})$ the family of its Laplace exponents defined by (2.7). It is easy to see that $e^{-v_{s,t}(\theta)} = F_{s,t}(e^{-\theta})$ for all $(s, t) \in \Delta$ and any $\theta > 0$. Fix some $(s, t) \in \Delta$. Extending the last equality by holomorphicity, we have

$$(3.6) \quad F_{s,t}(e^{-\zeta}) = \exp(-v_{s,t}(\zeta)) \quad \text{for all } \zeta \in \mathbb{H}.$$

In particular, we see that

$$(3.7) \quad v_{s,t}(\theta + 2\pi i) = v_{s,t}(\theta) + 2\pi m i \quad \text{for all } \theta > 0$$

and some $m \in \mathbb{Z}$ not depending on θ . Note that $m \neq 0$ because $v_{s,t}$ is univalent in \mathbb{H} . Indeed, by Theorem 2.5, $(v_{s,t})_{(s,t) \in \Delta}$ is a topological reverse evolution family. Taking into account the relationship between evolution families and reserve evolution families, see [25, Remark 2.8], the univalence follows by [10, Proposition 2.4].

We claim that $v_{s,t}(\infty) := \lim_{\theta \rightarrow +\infty} v_{s,t}(\theta) = +\infty$. Recall that this limit exists and belongs to $(0, +\infty]$ because $v_{s,t} \in \mathfrak{BF}$ and hence it is a positive non-decreasing function on $(0, \infty)$. By Lindelöf's Theorem, see e.g. [6, Theorem 1.5.7 on p. 27] or [36, Theorem 9.3 on p. 268], $v_{s,t}$ has angular limit at ∞ . It follows that the limit of the l.h.s. of (3.7) exists and equals $v_{s,t}(\infty)$. This would give a contradiction with the fact that $m \neq 0$ unless $v_{s,t}(\infty) = +\infty$.

Now passing to the limit as $\mathbb{R} \in \zeta \rightarrow +\infty$ in (3.6), we conclude that $F_{s,t}(0) = 0$, which proves (ii).

STEP 2: (ii) \Rightarrow (i). Suppose that $F_{s,t}(0) = 0$ for any $(s, t) \in \Delta$. Since by Theorem 3.3, $(F_{s,t})$ is a topological evolution family, arguing as above we see that $F_{s,t}$'s are univalent in \mathbb{D} . It follows that $F_{s,t}(\mathbb{D}^*) \subset \mathbb{D}^* := \mathbb{D} \setminus \{0\}$. Therefore, the mappings $F_{s,t}$ can be lifted from \mathbb{D}^* to \mathbb{H} w.r.t. the covering $\mathbb{H} \ni \zeta \mapsto e^{-\zeta} \in \mathbb{D}^*$. In other words, for any $(s, t) \in \Delta$, there exists $v_{s,t} \in \text{Hol}(\mathbb{H}, \mathbb{H})$ such that (3.6) holds. Moreover, since $F((0, 1)) \subset (0, 1)$, we may assume that $v_{s,t}((0, \infty)) \subset (0, \infty)$.

The most difficult part is to show that $(v_{s,t}) \subset \mathfrak{BF}$. Since $(F_{s,t})$ is a reverse evolution family and since $v_{s,t}(\theta) = -\log F_{s,t}(e^{-\theta})$ for all $\theta > 0$ and all $(s, t) \in \Delta$, with the help of the identity principle for holomorphic functions we may conclude that $(v_{s,t})$ satisfies conditions (REF1) and (REF2) in Definition 1.4. Furthermore, recall that $F_{s,t}(0) = 0$ and $F'_{s,t}(0) > 0$ for all $(s, t) \in \Delta$, with the strict inequality taking place thanks to the univalence of $F_{s,t}$. Hence, by the Schwarz lemma, from the identity $F_{0,t} = F_{0,s} \circ F_{s,t}$, $(s, t) \in \Delta$, it follows that $\lambda(t) := F'_{0,t}(0)$ is a positive non-increasing function. Choose¹ an increasing homeomorphism u of $[0, \infty)$ onto itself such that $\lambda \circ u$ is locally absolutely continuous on $[0, \infty)$. Then according to [25, Remark 2.8] and [5, Theorem 7.3] the formula $\tilde{F}_{s,t} := F_{u(s), u(t)}$ defines an absolutely continuous reverse evolution family $(\tilde{F}_{s,t})$. Thanks to the equivalence between (i) and (ii) in [12, Proposition 4.3], it follows that the family $(\tilde{v}_{s,t})$ defined by $\tilde{v}_{s,t} := v_{u(s), u(t)}$ for all $(s, t) \in \Delta$ is an absolutely continuous reverse evolution family in \mathbb{H} .

Let Φ be the Herglotz vector field associated with the reverse evolution family $(\tilde{F}_{s,t})$. Applying Theorem 3.4 to $(\tilde{F}_{s,t})$ and using relation (3.6) we obtain the following ODE for $(\tilde{v}_{s,t})$:

$$(3.8) \quad \frac{d\tilde{v}_{s,t}(\zeta)}{ds} = \phi(\tilde{v}_{s,t}(\zeta), s) \quad \text{for all } t > 0, \text{ all } \zeta \in \mathbb{H}, \text{ and a.e. } s \in [0, t],$$

where $\phi(\zeta, s) := -e^\zeta \Phi(e^{-\zeta}, s)$. Equation (3.8) means that ϕ is the Herglotz vector field associated with $(\tilde{v}_{s,t})$.

Using representation (3.4) and taking into account that $\Phi(0, s) = 0$ because 0 is a fixed point for $(F_{s,t})$, we find that

$$(3.9) \quad \phi(\zeta, s) = -q_s - \sum_{n \geq 2} \alpha_s(n) (1 - e^{-(n-1)\zeta}) = -q_s - \int_{(0, \infty)} (1 - e^{-x\zeta}) \pi_s(dx)$$

for all $s \geq 0$ and all $\zeta \in \mathbb{H}$, where $\pi_s := \sum_{k \in \mathbb{N}} \alpha_s(k+1) \delta_k$. Since for each $s \geq 0$ the series $\sum_{n \geq 2} \alpha_s(n)$ converges, the measures π_s satisfy the integrability condition $\int_{(0, \infty)} \min\{\lambda, 1\} \pi_s(d\lambda) < \infty$. Hence, by [25, Corollary 3.7], $\phi(\cdot, s) \in \mathcal{G}(\mathfrak{BF}_\infty)$ for all $s \geq 0$. The semigroup \mathfrak{BF}_∞ is topologically closed in $\text{Hol}(\mathbb{H}, \mathbb{H})$. Therefore, by Theorem B, $(\tilde{v}_{s,t}) \subset \mathfrak{BF}_\infty$.

It immediately follows that $(v_{s,t})$ is a topological reverse evolution family contained in \mathfrak{BF}_∞ . Thus, by Theorem 2.5 and Remark 2.4, there exists a branching process (Z_t) on $[0, \infty]$ whose Laplace exponents are exactly $(v_{s,t})$. Using the relations (3.6), (3.1), (2.7) and taking into account that a bounded Borel measure on $[0, \infty)$ is uniquely determined by its Laplace transform, see e.g. [40, Proposition 1.2], we conclude that $k_{s,t}(1, B) = \ell_{s,t}(1, B \cap \mathbb{N}_0^*)$ for any $(s, t) \in \Delta$ and any Borel set $B \subset [0, \infty]$. Thanks to the branching properties (K4) and (L4), the latter immediately implies (3.5). This proves (i).

STEP 3: (ii) \Rightarrow (iii). Note that (ii) is equivalent to $\ell_{s,t}(1, \{0\}) = 0$ for any $(s, t) \in \Delta$. The branching property $\ell_{s,t}(n, \cdot) := \ell(1, \cdot)^{*n}$ implies

$$\ell_{s,t}(n, \{m\}) = \sum_{\substack{k_1 + k_2 + \dots + k_n = m \\ k_i \geq 0}} \ell_{s,t}(1, \{k_1\}) \ell_{s,t}(1, \{k_2\}) \cdots \ell_{s,t}(1, \{k_n\}).$$

¹To construct a concrete example, consider the increasing homeomorphism f of $[0, \infty)$ defined by $f(s) := s - \lambda(s) + 1$. Since $f(s_2) - f(s_1) \geq s_2 - s_1$ whenever $0 \leq s_1 \leq s_2$, the function $u := f^{-1}$ is Lipschitz continuous and so is $\lambda(u(t)) = t - u(t) + 1$.

In case $m < n$, there exists some i such that $k_i = 0$, and hence $\ell_{s,t}(n, \{m\}) = 0$.

STEP 4: (iii) \Rightarrow (iv). Denote $\Delta^* := \{(k, m) : k, m \in \mathbb{N}_0^*, k \leq m\}$. For any $0 \leq s \leq t \leq u$ and $n \in \mathbb{N}_0$, using (2.6) we have

$$\begin{aligned} \mathbb{P}^{(s,n)}[N_t \leq N_u] &= \mathbb{P}^{(s,n)}[(N_t, N_u) \in \Delta^*] = \int_{\Delta^*} \ell_{s,t}(n, dk) \ell_{t,u}(k, dm) \\ &= \sum_{(k,m) \in \Delta^*} \ell_{s,t}(n, \{k\}) \ell_{t,u}(k, \{m\}) = \sum_{k \in \mathbb{N}_0^*} \ell_{s,t}(n, \{k\}) = 1. \end{aligned}$$

STEP 5: (iv) \Rightarrow (ii). This implication is immediate. Indeed, since by (iv), $1 = \mathbb{P}^{(s,1)}[N_s \leq N_t] = \ell_{s,t}(1, \mathbb{N} \cup \{\infty\})$, we have $F_{s,t}(0) = \ell_{s,t}(1, \{0\}) = 1 - \ell_{s,t}(1, \mathbb{N} \cup \{\infty\}) = 0$ as desired. \square

We conclude this section with a few remarks concerning the above theorem.

Remark 3.10. From the dynamics point of view, it seems curious that according to Theorem 3.9, a branching process on \mathbb{N}_0^* is spatially embeddable if and only if the Denjoy–Wolff point of the probability generating functions $F_{s,t}$ is at $\tau = 0$.

Remark 3.11. An observation similar to Remark 2.30 can be made in relation to Theorem 3.9. Namely, if the branching process $(N_t)_{t \geq 0}$ has cadlag paths, then the equivalent conditions (i)–(iv) in Theorem 3.9 are further equivalent to the requirement that $(N_t)_{t \geq s}$ has non-decreasing sample paths almost surely.

Remark 3.12. Let (Z_t) be a branching process on $[0, \infty]$ such that the corresponding reverse evolution family $(v_{s,t})$ is absolutely continuous. Let ϕ be the Herglotz vector field of $(v_{s,t})$. The proof of Theorem 3.9 shows that (Z_t) admits a restriction to \mathbb{N}_0^* , i.e. there exists a branching process (N_t) on \mathbb{N}_0^* that spatially embeds into (Z_t) in the sense of Definition 3.8, if and only if

$$\phi(\zeta, t) = -q_t + \int_{(0, \infty)} (e^{-x\zeta} - 1) \pi_t(dx), \quad \pi_t := \sum_{k \in \mathbb{N}} a_k(t) \delta_k, \quad \text{for all } \zeta \in \mathbb{H} \text{ and a.e. } t \geq 0,$$

where $t \mapsto q_t$ and a_k , $k \in \mathbb{N}$, are non-negative measurable functions such that $t \mapsto q_t + \sum_{k \in \mathbb{N}} a_k(t)$ belongs to L_{loc}^1 . The Lévy family of such a Herglotz vector field ϕ is $(q_t, 0, 0, \pi_t)_{t \geq 0}$. Note also that in such a case, the Laplace exponents $v_{s,t}$ of the process (Z_t) have the Denjoy–Wolff point at ∞ .

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