1.

$$\nabla^2 f(x) = \nabla^2 g_1(x) = \nabla^2 g_2(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \succ 0$$

so this is a strictly convex problem.

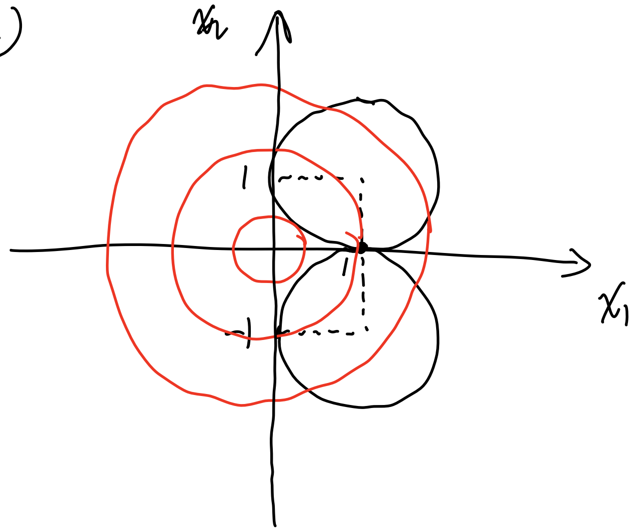
KKT conditions are

$$\begin{cases} 2x_1 + 2\mu_1(x_1-1) + 2\mu_2(x_1-1) = 0 \\ 2x_2 + 2\mu_1(x_2-1) + 2\mu_2 x_2 = 0 \\ g_i(x) \le 0, \quad i=1,2 \\ \mu_i \ge 0, \quad i=1,2 \\ \mu_i g_i(x) = 0, \quad i=1,2 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = x_2 = 1 - \frac{\sqrt{2}}{2} \\ \mu_1 = \sqrt{2}-1 \\ \mu_2 = 0 \end{cases}$$

Because this is a strictly convex problem, the solution is the only optimal point.

2. (a)



The feasible set $X = \{(1,0)\}$, so $x^* = (1,0)$ and $f^* = 1$

(2) The KKT conditions are

$$
\begin{cases}
2x_1 + 2\mu_1(x_1-1) + 2\mu_2(x_1-1) = 0 & \text{①} \\
2x_2 + 2\mu_1(x_2-1) + 2\mu_2(x_2+1) = 0 & \text{②} \\
(x_1-1)^2 + (x_2-1)^2 \le 1 & \text{③} \\
(x_1-1)^2 + (x_2-1)^2 \le 1 & \text{④} \\
\mu_i \ge 0, \quad i=1,2 & \\
\mu_1\left[(x_1-1)^2 + (x_2-1)^2 - 1\right] = 0 & \text{⑤} \\
\mu_2\left[(x_1-1)^2 + (x_2+1)^2 - 1\right] = 0 & \text{⑥}
\end{cases}
$$

③ and ④ yield $x_1 = 1$, $x_2 = 0$

Plug it into ①: $2 = 0$. Contradiction!

Thus, there are no Lagrange multipliers that satisfy the KKT conditions.

Let $g_1(x) = (x_1-1)^2 + (x_2-1)^2 - 1$

$g_2(x) = (x_1-1)^2 + (x_2+1)^2 - 1$

They are both active.

$\nabla g_1(x^*) = (0, -2)^T$

$\nabla g_2(x^*) = (0, 2)^T$

Obviously they are not linearly independent, so x^* is not regular.

3\. Let $f(x) = (x_1 - \frac{9}{4})^2 + (x_2 - 2)^2$

$g_1(x) = x_1^2 - x_2$

$g_2(x) = x_1 + x_2 - 6$

$g_3(x) = -x_1$, $g_4(x) = -x_2$

Obviously it is a convex problem.

The KKT conditions are

$$\begin{cases} 2(x_1 - \frac{9}{4}) + 2\mu_1 x_1 + \mu_2 - \mu_3 = 0 & ① \\ 2(x_2 - 2) - \mu_1 + \mu_2 - \mu_4 = 0 & ② \\ g_i(x) \leq 0, \quad i = 1,2,3,4 \\ \mu_i \geq 0, \quad i = 1,2,3,4 \\ \mu_i g_i(x) = 0, \quad i = 1,2,3,4 \end{cases}$$

For $x^{(1)}$, $g_1(x^{(1)}) = \frac{49}{16} > 0$, $x^{(1)}$ is not feasible, so $x^{(1)}$ is not an optimal solution.

For $x^{(2)}$, only $g_3(x)$ is active, so $\mu_1 = \mu_2 = \mu_4 = 0$.

$\nabla g_3(x^{(2)}) = (-1, 0)^T \neq \vec{0}$, so $x^{(2)}$ is a regular point.

① and ② yield $\mu_3 = -\frac{9}{2} < 0$, so $x^{(2)}$ is not an optimal solution.

For $x^{(3)}$, only $g_1(x)$ is active, so $\mu_2 = \mu_3 = \mu_4 = 0$.

$\nabla g_1(x^{(3)}) = (-3, 1)^T \neq \vec{0}$, so $x^{(3)}$ is a regular point

① and ② yield $\mu_1 = \frac{1}{2}$, so $x^{(3)}$ is an optimal solution.

4. (a) Obviously it is a convex problem.

The KKT conditions are

$$
\begin{cases}
x_i - z_i + \lambda y_i - \mu_i = 0, & i = 1, 2, \cdots, n \\
y^T x = 0 & \\
x \geq 0 & \\
\mu_i \geq 0, & i = 1, 2, \cdots, n \\
\mu_i x_i = 0, & i = 1, 2, \cdots, n
\end{cases}
$$

For i that satisfies $x_i > 0$, $\mu_i = 0$,

$$x_i = z_i - \lambda y_i = (z_i - \lambda y_i)^+$$

For i that satisfies $x_i = 0$, $x_i = z_i - \lambda_i y_i + \mu_i = 0$.

$\mu_i \geq 0$, so $z_i - \lambda_i y_i \leq 0$, $x_i = (z_i - \lambda y_i)^+$

Thus, $x_i^* = (z_i - \lambda y_i)^+$

$$\sum_{i=1}^{n} y_i (z_i - \lambda y_i)^+ = \sum_{i=1}^{n} y_i x_i^* = y^T x^* = 0$$

(b) The result is $x^* = (0.33, 1.33, 1.67)^T$