1. $\forall x_1^* \in M$, $\forall x_2^* \in M$, $\forall \theta \in [0,1]$,

$\theta x_1^* + \bar{\theta} x_2^* \in S$,

$$
\begin{aligned}
f(\theta x_1^* + \bar{\theta} x_2^*) &\leq \theta f(x_1^*) + \bar{\theta} f(x_2^*) \\
&\leq \theta f(x) + \bar{\theta} f(x) \\
&= f(x), \quad \forall x \in S
\end{aligned}
$$

Thus, $\theta x_1^* + \bar{\theta} x_2^* \in M$, which means $M$ is a convex set.

2. Assume that $\exists \theta_1$, $f(\theta_1 x + \bar{\theta}_1 y) < \theta_1 f(x) + \bar{\theta}_1 f(y)$, and assume $\theta_1 \in (0, \theta_0)$.

$$\theta_0 x + \bar{\theta}_0 y = \frac{\bar{\theta}_0}{\bar{\theta}_1}(\theta_1 x + \bar{\theta}_1 y) + \frac{\bar{\theta}_1 - \bar{\theta}_0}{\bar{\theta}_1} x.$$

$\frac{\bar{\theta}_0}{\bar{\theta}_1} > 0$, $\frac{\bar{\theta}_1 - \bar{\theta}_0}{\bar{\theta}_1} > 0$, $\frac{\bar{\theta}_0}{\bar{\theta}_1} + \frac{\bar{\theta}_1 - \bar{\theta}_0}{\bar{\theta}_1} = 1$, so this is a convex combination

$$f(\theta_0 x + \bar{\theta}_0 y) \leq \frac{\bar{\theta}_0}{\bar{\theta}_1} f(\theta_1 x + \bar{\theta}_1 y) + \frac{\bar{\theta}_1 - \bar{\theta}_0}{\bar{\theta}_1} f(x)$$

$$< \frac{\bar{\theta}_0 \theta_1}{\bar{\theta}_1} f(x) + \bar{\theta}_0 f(y) + \frac{\bar{\theta}_1 - \bar{\theta}_0}{\bar{\theta}_1} f(x)$$

$$= \theta_0 f(x) + \bar{\theta}_0 f(y)$$

This is a contradiction, so $\forall \theta \in (0, \theta_0)$,

$$f(\theta x + \bar{\theta} y) = \theta f(x) + \bar{\theta} f(y)$$

Similarly, we can prove that it holds true for any $\theta \in (\theta_0, 1)$

If $\theta = 0$ or $1$, the proposition is trivial.

3. (a) $f(x) = x^T A x$,

where $A = \frac{1}{2}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$|2| > 0$, $|1| > 0$, $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} > 0$; $\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} > 0$

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

so $\nabla^2 f(x) = 2A$ is positive semidefinite.

$f(x)$ is convex.

(b) Let $g(t) = (x_1 + td_1)^{\alpha_1}(x_2 + td_2)^{\alpha_2}$. Then

$$g''(t) = (x_1+td_1)^{\alpha_1-2}(x_2+td_2)^{\alpha_2-2}\Big[\alpha_1(\alpha_1-1)d_1^2(x_2+td_2)^2 + \alpha_2(\alpha_2-1)d_2^2(x_1+td_1)^2 + 2d_1d_2\alpha_1\alpha_2(x_1+td_1)(x_2+td_2)\Big]$$

(See the last page of the file.)

Let $h(t) = \dfrac{g''(t)}{(x_1+td_1)^{\alpha_1-2}(x_2+td_2)^{\alpha_2-2}}$. Then $h(t)$ and $g''(t)$ have the same sign.

Let $\alpha_1 = \alpha_2 = -1$. Then

$$\frac{1}{2}h(t) = d_1^2(x_2+td_2)^2 + d_2^2(x_1+td_1)^2 + d_1d_2(x_1+td_1)(x_2+td_2)$$

If $d_1 d_2 \geq 0$, $h(t) \geq 0$

If $d_1 d_2 < 0$,

$$\frac{1}{2} h(t) = \left[d_1 (x_2 + t d_2) + d_2 (x_1 + t d_1)\right]^2 - d_1 d_2 (x_1 + t d_1)(x_2 + t d_2) \geq 0$$

so $g''(t) \geq 0$.

$f(x)$ is convex.

(c) Let $x_2 = 1 - x_1$. Then $f(x) = g_1(x_1) = x_1 (1 - x_1)^2$

$g_1''(x_1) = 6x_1 - 4$. $g_1(x_1)$ is neither convex nor concave on $(0,1)$, so $f(x)$ is neither convex nor concave.

(d) Let $x_1 = x_2 + 1$. Then $f(x) = g_2(x_2) = \dfrac{x_2 + 1}{\sqrt{x_2}}$

$g_2''(x_2) = x_2^{-\frac{5}{2}} \left(\frac{3}{4} - \frac{1}{4} x_2\right)$. $g_2(x_2)$ is neither convex nor concave on $(0, +\infty)$, so $f(x)$ is neither convex nor concave.

(e) Similar to (b), let $\alpha_1 + \alpha_2 = 1$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$. Then

$$h(t) = -\alpha_1 \alpha_2 \left[d_1 (x_2 + t d_2) - d_2 (x_1 + t d_1)\right]^2 \leq 0$$

so $f(x)$ is concave

4. Let $g(t) = f(x + td)$, $g_1(t) = f_1(x_1 + td_1)$, $g_2(t) = f_2(x_2 + td_2)$

Then $\forall t_1, \forall t_2, \forall \theta \in (0,1)$

$$
\begin{aligned}
g(\theta t_1 + \bar{\theta} t_2) &= g_1(\theta t_1 + \bar{\theta} t_2) + g_2(\theta t_1 + \bar{\theta} t_2) \\
&< \theta g_1(t_1) + \bar{\theta} g_1(t_2) + \theta g_2(t_1) + \bar{\theta} g_2(t_2) \\
&= \theta g(t_1) + \bar{\theta} g(t_2)
\end{aligned}
$$

so $f(x)$ is also strictly convex.

$$
\begin{aligned}
f_1(x) = x^2, \quad & f_1(y) - f_1(x) - f_1'(x)(y - x) \\
&= y^2 - x^2 - 2x(y - x) \\
&= (x - y)^2 > 0 \quad (x \neq y)
\end{aligned}
$$

$$
\begin{aligned}
f_2(x) = x^4, \quad & f_2(y) - f_2(x) - f_2'(x)(y - x) \\
&= y^4 - x^4 - 4x^3(y - x) \\
&= (x - y)^2(3x^2 + 2xy + y^2) > 0 \quad (x \neq y)
\end{aligned}
$$

so $f_1(x_1)$, $f_2(x_2)$ are strictly convex

Thus, $f(x_1, x_2)$ is strictly convex.

5. "$\Rightarrow$": $\forall x, \forall y$, $f(y) - f(x) - \nabla^T f(x)(y-x) \geq 0$

$$f(x) - f(y) - \nabla^T f(y)(x-y) \geq 0$$

Adding them up will get

$$\left(\nabla^T f(x) - \nabla^T f(y)\right)(x-y) = \langle \nabla f(x) - \nabla f(y), x-y \rangle \geq 0$$

"$\Leftarrow$": Let $g(t) = f(x+td)$. Then dom $g$ is an open interval. $g'(t) = d^T \nabla f(x+td)$

$$\begin{aligned}\left(g'(t) - g'(s)\right)(t-s) &= d^T\left(\nabla f(x+td) - \nabla f(x+sd)\right)(t-s) \\ &= \left(\nabla^T f(x+td) - \nabla^T f(x+sd)\right)(td-sd) \\ &= \langle \nabla f(x+td) - \nabla f(x+sd), td-sd \rangle \\ &\geq 0\end{aligned}$$

so $g'(t)$ is increasing.

Thus, $f(x)$ is convex.

$$(x_1+td_1)^{\alpha_1}(x_2+td_2)^{\alpha_2}$$

$$\Rightarrow d_1\alpha_1(x_1+td_1)^{\alpha_1-1}(x_2+td_2)^{\alpha_2}+d_2\alpha_2(x_2+td_2)^{\alpha_2-1}(x_1+td_1)^{\alpha_1}$$

$$=(x_1+td_1)^{\alpha_1-1}(x_2+td_2)^{\alpha_2-1}\left[d_1\alpha_1(x_2+td_2)+d_2\alpha_2(x_1+td_1)\right]$$

$$\rightarrow (x_1+td_1)^{\alpha_1-2}(x_2+td_2)^{\alpha_2-2}\left[d_1(\alpha_1-1)(x_2+td_2)+d_2(\alpha_2-1)(x_1+td_1)\right]\left[d_1\alpha_1(x_2+td_2)+d_2\alpha_2(x_1+td_1)\right]$$
$$+(x_1+td_1)^{\alpha_1-1}(x_2+td_2)^{\alpha_2-1}(\alpha_1+\alpha_2)d_1d_2$$

$$=(x_1+td_1)^{\alpha_1-2}(x_2+td_2)^{\alpha_2-2}\Big\{\left[d_1(\alpha_1-1)(x_2+td_2)+d_2(\alpha_2-1)(x_1+td_1)\right]\left[d_1\alpha_1(x_2+td_2)+d_2\alpha_2(x_1+td_1)\right]$$
$$+d_1d_2(\alpha_1+\alpha_2)(x_1+td_1)(x_2+td_2)\Big\}$$

$$=(x_1+td_1)^{\alpha_1-2}(x_2+td_2)^{\alpha_2-2}\left[\alpha_1(\alpha_1-1)d_1^2(x_2+td_2)^2+\alpha_2(\alpha_2-1)d_2^2(x_1+td_1)^2+2d_1d_2\alpha_1\alpha_2(x_1+td_1)(x_2+td_2)\right]$$