1. (a)

$$\nabla^2 f(x) = \begin{pmatrix} e^{x_1} & 0 & 0 \\ 0 & 4e^{2x_2} & 0 \\ 0 & 0 & 4e^{2x_3} \end{pmatrix}, \quad \nabla f(x) = \begin{pmatrix} e^{x_1} \\ 2e^{2x_2} \\ 2e^{2x_3} \end{pmatrix}$$

$$A = (1, 1, 1)$$

$$\begin{pmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} d \\ \lambda \end{pmatrix} = \begin{pmatrix} -\nabla f(x) \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} d_1 e^{x_1} + \lambda = -e^{x_1} \\ 4 d_2 e^{2x_2} + \lambda = -2e^{2x_2} \\ 4 d_3 e^{2x_3} + \lambda = -2e^{2x_3} \\ d_1 + d_2 + d_3 = 0 \end{cases}$$

Let $a = e^{x_1}$, $b = e^{2x_2}$, $c = e^{2x_3}$

$$\text{Then} \begin{cases} \lambda = -2\left(\frac{1}{a} + \frac{1}{4b} + \frac{1}{4c}\right)^{-1} \\ d_1 = 2\left(1 + \frac{a}{4b} + \frac{a}{4c}\right)^{-1} - 1 \\ d_2 = \left(\frac{2b}{a} + \frac{1}{2} + \frac{b}{2c}\right)^{-1} - \frac{1}{2} \\ d_3 = \left(\frac{2c}{a} + \frac{c}{2b} + \frac{1}{2}\right)^{-1} - \frac{1}{2} \end{cases}$$

(b) The output is

```
iteration 0: [0. 1. 0.]
iteration 1: [ 0.55783402  0.55270748 -0.1105415 ]
iteration 2: [0.74171111 0.22388047 0.03440841]
iteration 3: [0.83735858 0.09139269 0.07124873]
iteration 4: [0.8464719  0.07685719 0.07667091]
iteration 5: [0.84657358 0.07671322 0.0767132 ]
iteration 6: [0.84657359 0.0767132  0.0767132 ]
iteration 7: [0.84657359 0.0767132  0.0767132 ]
optimal value: 4.663287963194248
```

2. (a) Use the log barrier function.

The approximating equality constrained problem is

$$\min_{x \in \mathbb{R}^n} \; c^T x - \frac{1}{t} \sum_{i=1}^{n} \log x_i$$

$$\text{s.t.} \quad Ax = b$$

(b) Let $f(x)$ be the objective function

$$\nabla f(x) = c - \frac{1}{t} \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n} \right)^T$$

$$\nabla^2 f(x) = \frac{1}{t} \operatorname{diag} \left\{ \frac{1}{x_1^2}, \frac{1}{x_2^2}, \cdots, \frac{1}{x_n^2} \right\}$$

(d) The standard form can be

$$\min_{x, s} \quad -x_1 - 3x_2$$

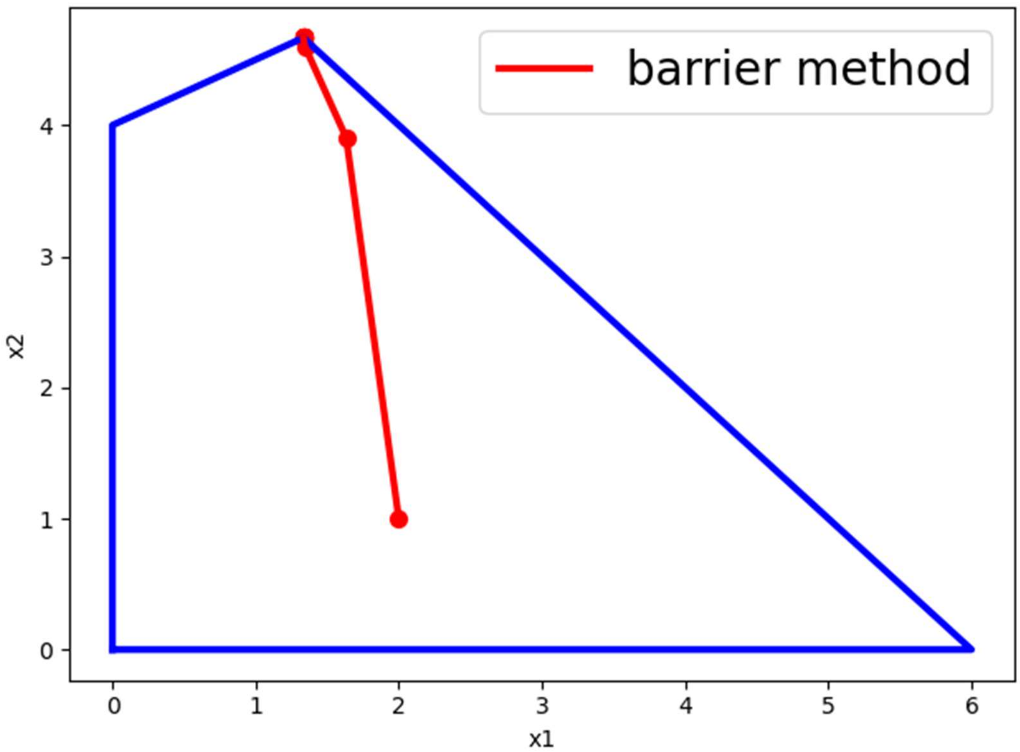
$$\text{s.t.} \quad x_1 + x_2 + s_1 = 6$$

$$-x_1 + 2x_2 + s_2 = 8$$

$$x_1, x_2, s_1, s_2 \geq 0$$

The output is

```
iteration 0: [2. 1. 3. 8.]
iteration 1: [1.63307563 3.90402064 0.46290373 1.82503435]
iteration 2: [1.34600004 4.59597677 0.05802319 0.1540465 ]
iteration 3: [1.3343604  4.65966009 0.00597951 0.01504022]
iteration 4: [1.33343360e+00 4.66596660e+00 5.99794362e-04 1.50040182e-03]
iteration 5: [1.33334334e+00 4.66659667e+00 5.99979425e-05 1.50004017e-04]
iteration 6: [1.33333433e+00 4.66665967e+00 5.99997937e-06 1.50000400e-05]
iteration 7: [1.33333343e+00 4.66666597e+00 5.99969651e-07 1.49992506e-06]
iteration 8: [1.33333334e+00 4.66666660e+00 5.99939572e-08 1.49984903e-07]
iteration 9: [1.33333333e+00 4.66666666e+00 5.39865034e-09 1.34966269e-08]
iteration 10: [1.33333333e+00 4.66666667e+00 4.85082592e-10 1.21270654e-09]
optimal value: -15.333333331716402
```



barrier method
x2
x1
0
1
2
3
4
5
6

3. (a)

$$c = (-1,-3)^T, \quad G = \begin{pmatrix} -1 & -1 \\ 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h = (-6,-8,0,0)^T$$

The dual LP is

$$\max_{\mu \in \mathbb{R}^4} \ \mu^T h \qquad \text{i.e.} \qquad \max_{\mu \in \mathbb{R}^4} \ -6\mu_1 - 8\mu_2$$

$$\text{s.t.} \ G^T \mu = c \qquad\qquad \text{s.t.} \ -\mu_1 + \mu_2 + \mu_3 = -1$$

$$\mu \geq \vec{0} \qquad\qquad -\mu_1 - 2\mu_2 + \mu_4 = -3$$

$$\mu \geq \vec{0}$$

(b) The symmetric dual LP is

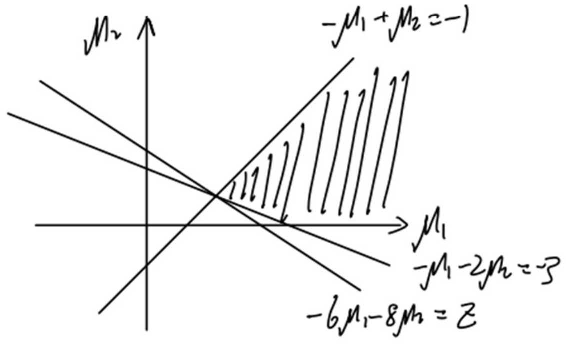
$$\max_{\mu \in \mathbb{R}^2} \ -6\mu_1 - 8\mu_2$$

$$\text{s.t.} \ -\mu_1 + \mu_2 \leq -1$$

$$-\mu_1 - 2\mu_2 \leq -3$$

$$\mu \geq \vec{0}$$

(c)



From the graph we know that $\mu^* = \left(\frac{5}{3}, \frac{2}{3}\right)$.

The dual optimal value is $-\frac{46}{3}$, which is the same as the primal optimal value.

(d) The output is

```
iteration 0: [4. 1. 2. 3.]
iteration 1: [2.1602764  0.54793489 0.61234151 0.25614619]
iteration 2: [1.72344886 0.64915465 0.0742942  0.02175816]
iteration 3: [1.67237818 0.66488395 0.00749423 0.00214608]
iteration 4: [1.66723807e+00 6.66488125e-01 7.49943598e-04 2.14317864e-04]
iteration 5: [1.66672381e+00 6.66648810e-01 7.49994366e-05 2.14288927e-05]
iteration 6: [1.66667238e+00 6.66664881e-01 7.49961753e-06 2.14275278e-06]
iteration 7: [1.66666724e+00 6.66666488e-01 7.49924434e-07 2.14264174e-07]
iteration 8: [1.66666672e+00 6.66666649e-01 7.42466209e-08 2.12133217e-08]
iteration 9: [1.66666667e+00 6.66666665e-01 6.66555703e-09 1.90444503e-09]
iteration 10: [1.66666667e+00 6.66666667e-01 1.75886088e-10 5.02528551e-11]
dual optimal value: -15.333333333802369
```