1. (a) $x_1 = 1 - 2x_2$, so $f(x_1, x_2) = f(x_2) = x_2^2 - 2x_2 - \frac{1}{2}$

$x_2^* = 1$, so $x_1^* = -1$

(b)

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \succeq 0,$$ so $f(x_1, x_2)$ is convex.

$$F(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda (x_1 + 2x_2 - 1)$$

$\nabla F(x_1, x_2, \lambda) = \vec{0}$, i.e.

$$\begin{cases} x_1 + x_2 - 1 + \lambda = 0 \\ x_1 + 2x_2 - 3 + 2\lambda = 0 \\ x_1 + 2x_2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -1 \\ x_2 = 1 \\ \lambda = 1 \end{cases}$$

so $x_1^* = -1$, $x_2^* = 1$

2. (a) $\begin{cases} \nabla f(x) + A^T\lambda = Qx + g + A^T\lambda = 0 & ① \\ Ax - b = 0 & ② \end{cases}$

(b) $Q \succ 0$, so Q is invertible and $Q^{-1} \succ 0$.

rank $A = k$, so $\forall x \neq 0$, $A^T x \neq 0$

Thus, $\forall x \neq 0$, $x^T A Q^{-1} A^T x = (A^T x)^T Q^{-1} (A^T x) > 0$

i.e. $AQ^{-1}A^T \succ 0$, so $AQ^{-1}A^T$ is invertible.

From ①: $x = -Q^{-1}(g + A^T\lambda)$

Plug it into ②: $\lambda = -(AQ^{-1}A^T)^{-1}(AQ^{-1}g + b)$

Thus $\begin{cases} x^* = Q^{-1}\left[A^T(AQ^{-1}A^T)^{-1}(AQ^{-1}g + b) - g\right] \\ \lambda^* = -(AQ^{-1}A^T)^{-1}(AQ^{-1}g + b) \end{cases}$

(c) $Q = I$, $g = x_0$

so $x^* = A^T(AA^T)^{-1}(Ax_0 + b) - x_0$

When $x_0 = \vec{0}$, $x^* = A^T(AA^T)^{-1} b$

(d) $x^* = w(w^T w)^{-1}(w^T x_0 + b) - x_0$

$\text{dist}(x_0, P) = \|x_0 - x^*\| = \dfrac{|w^T x_0 + b|}{\|w\|^2} \cdot \|w\|$

$= \dfrac{|w^T x_0 + b|}{\|w\|}$

3. By Cauchy inequaily,

$(a^2+b^2)(c^2+d^2) \geq (ac+bd)^2$

Let $a = x_1$, $b = 2x_2$, $c = 1$, $d = -1$. Then

$(x_1^2+4x_2^2)(1+1) \geq (x_1-2x_2)^2 \geq -8x_1x_2$

Thus, $x_1x_2 \geq -\frac{1}{4}$

The equality holds iff $\begin{cases} -x_1 = 2x_2 & \text{(Cauchy)} \\ x_1 = -2x_2 & \text{(Last inequality)} \\ x_1^2+4x_2^2 = 1 \end{cases}$

so $(x_1^*, x_2^*) = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4})$ or $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4})$

Or by Lagrange condition:

$$\begin{cases} 2\lambda x_1 + x_2 = 0 \\ x_1 + 8\lambda x_2 = 0 \\ x_1^2 + 4x_2^2 - 1 = 0 \end{cases}$$

Let $g(x) = x_1^2+4x_2^2-1$. Then $\nabla g(x) = (2x_1, 8x_2)^T \neq \vec{0}$

so $g(x)$ has no critical points.

$\lambda^* = \pm\frac{1}{4}$. $\lambda^* = -\frac{1}{4}$ is the global maximum.

$\lambda^* = \frac{1}{4}$ is the global minimum (计算过程略)

4. (a) $g(x) = \|x\|^2 - 1$ has no critical points.

The Lagrange condition is

$$\begin{cases} Ax + 2\mu x = 0 & ① \\ \|x\|^2 = 1 & ② \end{cases}$$

$\mu \in \mathbb{R}$, so from ① we know that x^* is an eigenvector of A. Assume x^* is associated to λ_i. Then the optimal value $x^{*T} A x^* = \lambda_i \|x^*\|^2 = \lambda_i$, so λ_i is the smallest eigenvalue λ_1.

(b) (i) The lagrange condition is

$$\begin{cases} Ax + 2\mu_0 x + \mu_1 v_1 = 0 & ① \\ \|x\|^2 = 1 & ② \\ v_1^T x = 0 & ③ \end{cases}$$

Critical points satisfy that $x \parallel v_1$. $x \neq \vec{0}$, $v_1 \neq \vec{0}$, so there is no critical point.

From ① we know that $c_0 = -2\mu_0$, $c_1 = -\mu_1$.

(ii) $v_1^T(Ax + 2\mu_0 x + \mu_1 v_1)$

$= (Av_1)^T x + \mu_1 \|v_1\|^2 = \lambda_1 v_1^T x + \mu_1 \|v_1\|^2$

$= \mu_1 \|v_1\|^2 = 0$

$\|v_1\|^2 \neq 0$, so $\mu_1 = 0$, i.e. $c_1 = 0$

(iii) $Ax = c_0 x$, so x^* is an eigenvector of A.

$v_1^T x = 0$, so x^* is associated to λ_i $(i \neq 1)$

The optimal value $x^{*T} A x^* = \lambda_i \|x^*\|^2 = \lambda_i$, so λ_i is the smallest possible eigenvalue λ_2.