1.

(a)

$$f(x_1, x_2) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + 2\times\frac{1}{2}e^{-x_1-0.1}$$
$$\geq 4\sqrt[4]{e^{x_1+3x_2-0.1}\cdot e^{x_1-3x_2-0.1}\cdot\left(\frac{1}{2}e^{-x_1-0.1}\right)^2}$$
$$= 2\sqrt{2}e^{-0.1}$$

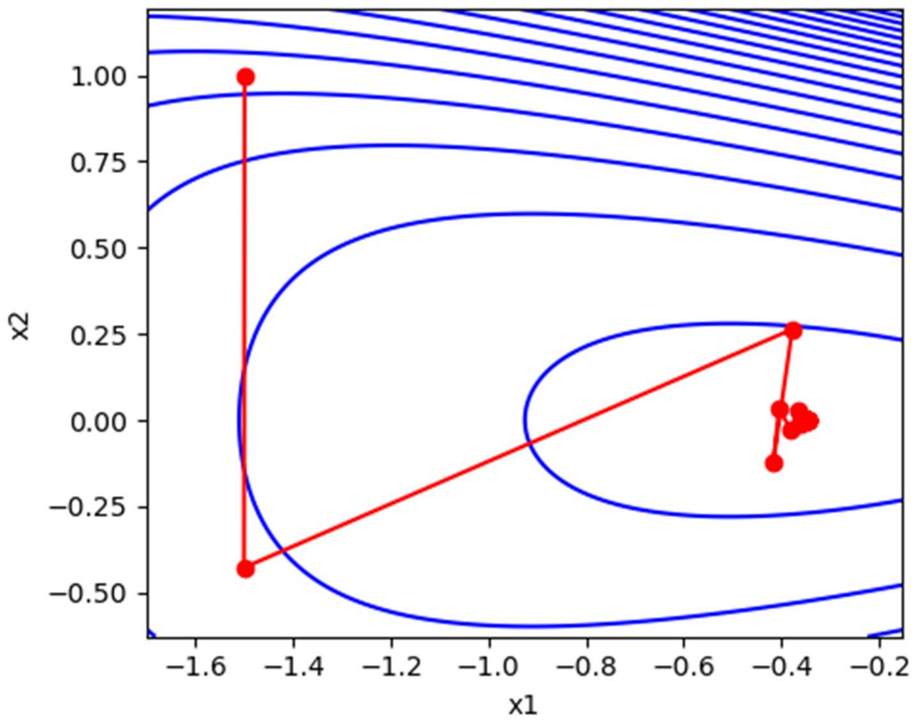
$$f(x_1, x_2) = \frac{e^{-0.1}}{\sqrt{2}} \quad \text{iff} \quad e^{x_1+3x_2-0.1} = e^{x_1-3x_2-0.1} = \frac{1}{2}e^{-x_1-0.1}$$

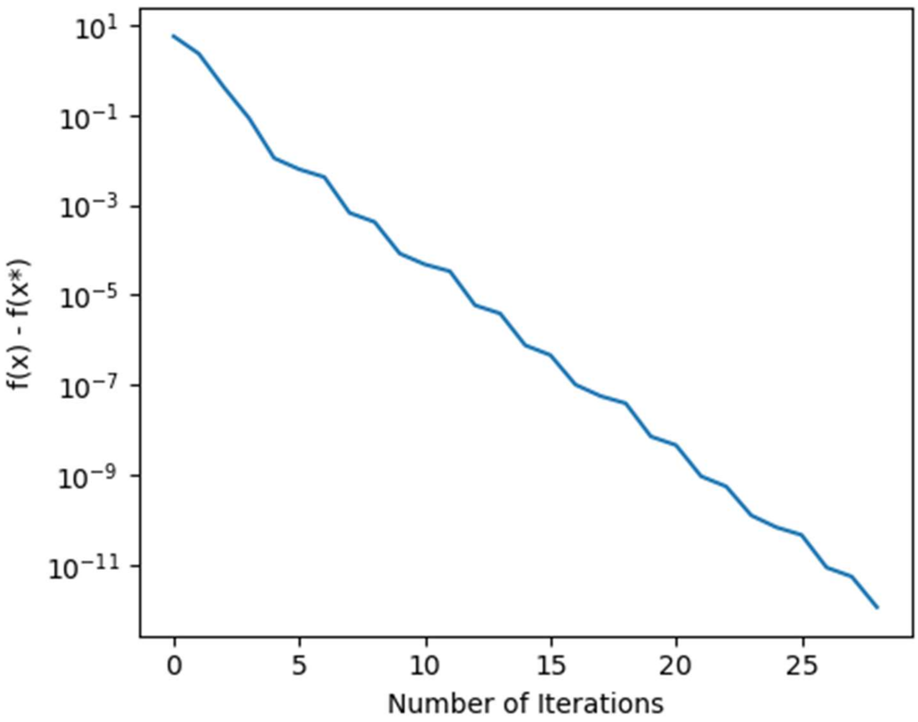
Thus, $x^* = (-\frac{\ln 2}{2}, 0)$, $f(x^*) = 2\sqrt{2}e^{-0.1}$

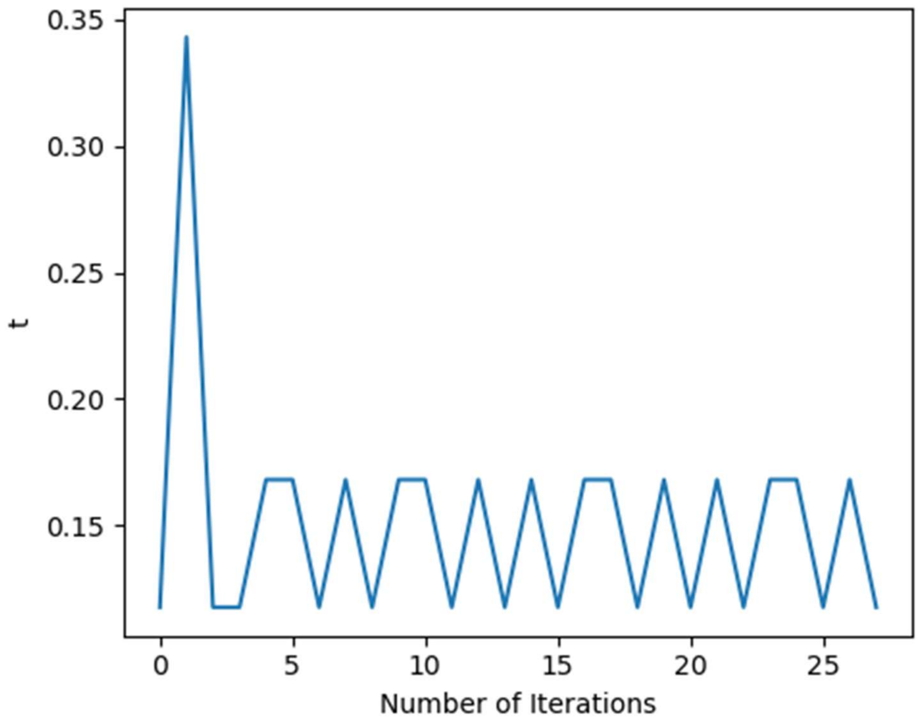
(b) Use $t_0 = 1$. The output is

```
solution: [-3.46574284e-01  3.04072749e-07]
number of iterations in the outer loop: 29
number of iterations in the inner loop: 151
```

The figures are







(c) When $t = 0.1$, the output is

```
solution: [-3.46576607e-01  3.21465960e-18]
number of iterations: 45
```

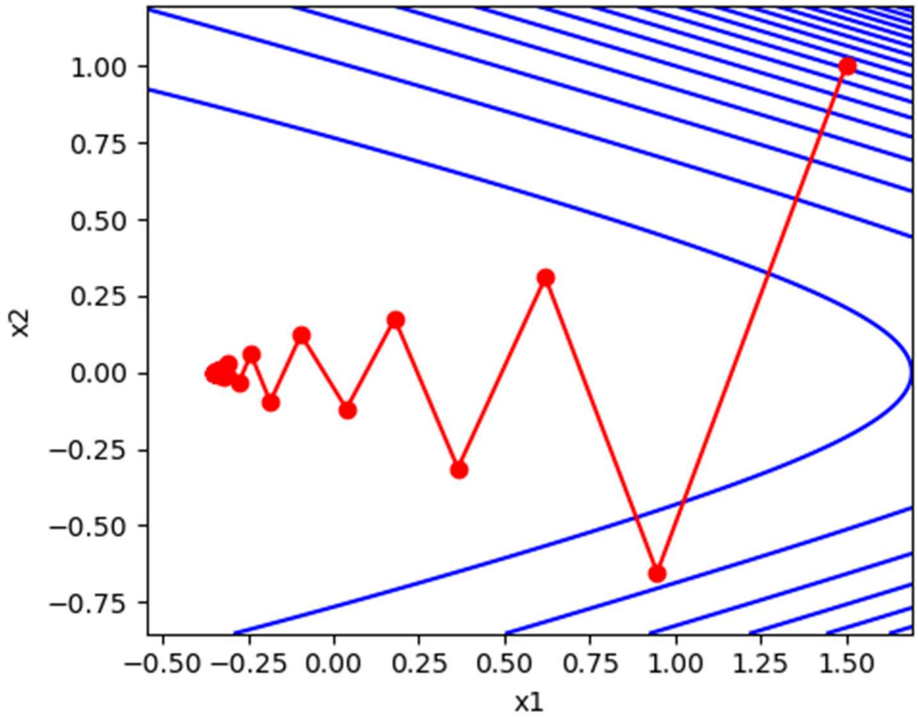
When $t = 0.01$, the output is

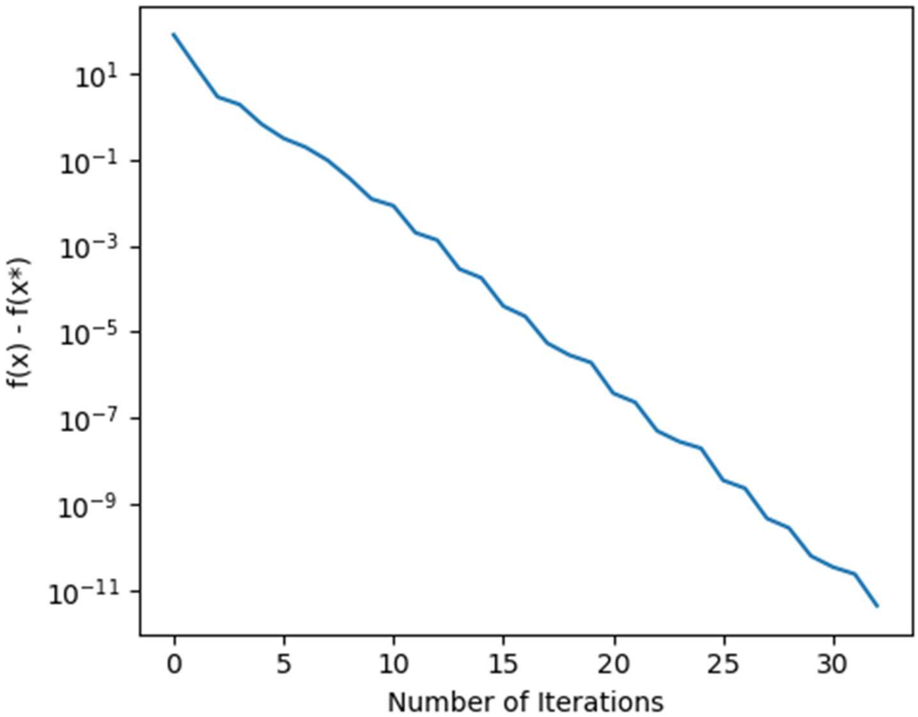
```
solution: [-3.46577419e-01  8.65140907e-18]
number of iterations: 490
```

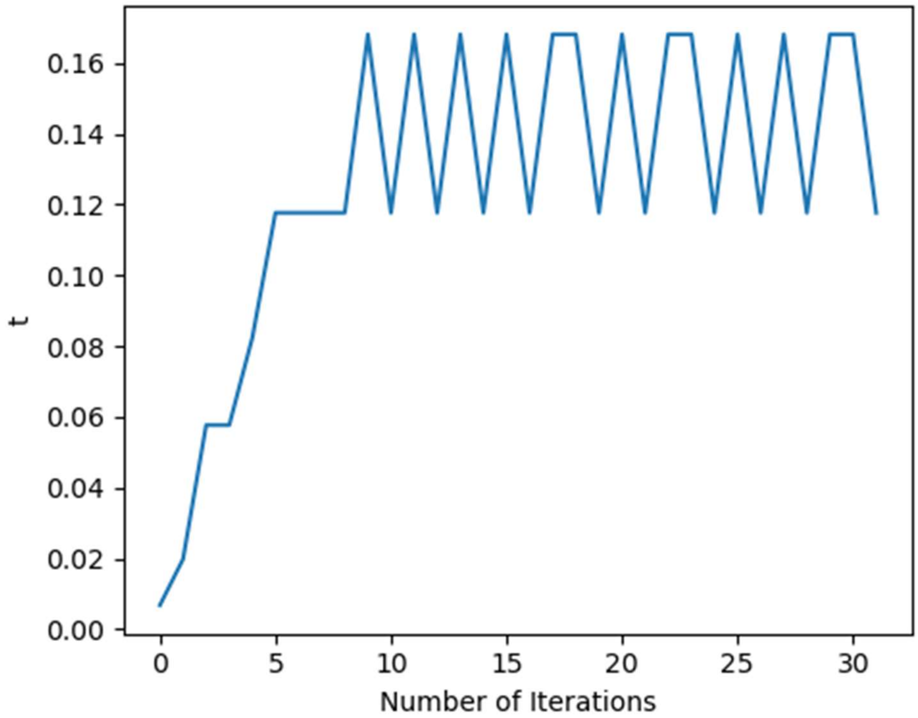
(d) Use $t_0 = 1$. The output is

```
solution: [-3.4657238e-01  6.5447655e-07]
number of iterations in the outer loop: 33
number of iterations in the inner loop: 197
```

The figures are







(e) The output is

```
solution: [-3.46569713e-01 -7.62280416e-18]
number of iterations: 985
```

If we use the step sizes in part (c), there will be a floating number overflow error.

2.

(a) $$\|x_{k+1}-x^*\| = \|(\tilde{x}_{k+1}-x^*)+(x_{k+1}-\tilde{x}_{k+1})\|$$
$$\le \|\tilde{x}_{k+1}-x^*\| + \|x_{k+1}-\tilde{x}_{k+1}\|$$
$$= \|\tilde{x}_{k+1}-x^*\| + t\|\varepsilon_k\|$$
$$\le \|\tilde{x}_{k+1}-x^*\| + tE$$

(b) We already know that
$$\|\tilde{x}_{k+1}-x^*\|^2 \le (1-mt)\|x_k-x^*\|^2$$
So $$\|x_{k+1}-x^*\| \le \|\tilde{x}_{k+1}-x^*\| + tE$$
$$\le q\|x_k-x^*\| + tE$$

(c) Problem (b) is the case when $k=0$.

Assume that $\|x_k-x^*\| \le q^k\|x_0-x^*\| + \frac{1-q^k}{1-q}tE$

Then $$\|x_{k+1}-x^*\| \le q\|x_k-x^*\| + tE$$
$$\le q\left(q^k\|x_0-x^*\| + \frac{1-q^k}{1-q}tE\right) + tE$$
$$= q^{k+1}\|x_0-x^*\| + \frac{1-q^{k+1}}{1-q}tE$$

Therefore, $\|x_k-x^*\| \le q^k\|x_0-x^*\| + \frac{1-q^k}{1-q}tE$ holds for any non-negative integer k.

(d) $\sup \|x_k - x^*\| \le q^k \|x_0 - x^*\| + \frac{1-q^k}{1-q} tE$

$q = \sqrt{1-mt} < 1$, so letting $k \to \infty$ on both sides will get

$$\lim_{k\to\infty} \sup \|x_k - x^*\| \le \lim_{k\to\infty} \left(q^k \|x_0 - x^*\| + \frac{1-q^k}{1-q} tE \right)$$

$$= \frac{tE}{1-q}$$

$$(q-1)^2 \ge 0$$

$$\Rightarrow 2(1-q) \ge 1-q^2 = mt$$

$$\Rightarrow \frac{2}{m} \ge \frac{t}{1-q}$$

Thus, $\lim_{k\to\infty} \sup \|x_k - x^*\| \le \frac{tE}{1-q} \le \frac{2E}{m}$