1. (a) $f(x)$ is increasing on $\mathbb{R}$, so $x^* = 0$, $f(x^*) = \log 2$

(b) $\phi(\mu) = \inf_x \left[\log(1+e^x) - \mu x\right]$

$$= \begin{cases} (\mu-1)\log(1-\mu) - \mu\log\mu, & 0<\mu<1 \\ 0, & \mu=0 \text{ or } \mu=1 \\ -\infty, & \mu<0 \text{ or } \mu>1 \end{cases}$$

The dual problem is

$$\max_\mu \ \phi(\mu)$$
$$\text{s.t. } \mu \geq 0$$

(c) Let $g(\mu) = (\mu-1)\log(1-\mu) - \mu\log\mu$, $0<\mu<1$

$g'(\mu) = \log(1-\mu) - \log\mu$

$g''(\mu) = -\left(\frac{1}{1-\mu} + \frac{1}{\mu}\right) < 0$, so $g'(\mu)$ is decreasing.

$g'(\frac{1}{2}) = 0$, $g(\frac{1}{2}) = \log 2 > 0$

so $\mu^* = \frac{1}{2}$, $\phi(\mu^*) = \log 2$

$f(x^*) = \phi(\mu^*)$, so the strong equality holds.

2. (a) $\phi(\mu) = \inf\limits_{x} \left\{ x_1^2 + x_2^2 + \mu_1\left[(x_1-1)^2+(x_2-1)^2-1\right] + \mu_2\left[(x_1-1)^2+(x_2+1)^2-1\right] \right\}$

$$= \begin{cases} \mu_1+\mu_2 - \dfrac{2(\mu_1^2+\mu_2^2)}{1+\mu_1+\mu_2}, & \mu_1+\mu_2 > -1 \\ -\infty, & \mu_1+\mu_2 \le -1 \end{cases}$$

The dual problem is

$$\max_{\mu} \ \phi(\mu)$$
$$\text{s.t. } \mu \geq 0$$

(b) When $\mu_1+\mu_2 > -1$

$$\phi(\mu) \le \mu_1+\mu_2 - \frac{(\mu_1+\mu_2)^2}{1+\mu_1+\mu_2} = \frac{\mu_1+\mu_2}{1+\mu_1+\mu_2} < 1$$

The dual optimal value $\phi^* = 1 = f^*$,
so the strong duality holds.

(c) $\begin{cases} (x_1-1)^2+(x_2-1)^2 < 1 \\ (x_1-1)^2+(x_2+1)^2 < 1 \end{cases}$ has no solutions,

so Slater's condition doesn't hold.

Thus, Slater's condition is not necessary.

(d) ϕ^* cannot be attained by any dual feasible point.
This is expected because the KKT conditions cannot be satisfied.

3. (a) Let $g(x) = x^3 - \mu x$ $(x \geq 0)$. Then $g'(x) = 3x^2 - \mu$

If $\mu > 0$, $\min g(x) = g\left(\sqrt{\frac{\mu}{3}}\right) = -\frac{2\sqrt{3}}{9}\mu^{\frac{3}{2}}$

If $\mu \leq 0$, $\min g(x) = g(0) = 0$

$$\phi(\mu) = \inf_{x \in D} \{ g(x_1) + g(x_2) + \mu \}$$
$$= \begin{cases} -\frac{4\sqrt{3}}{9}\mu^{\frac{3}{2}} + \mu, & \mu \geq 0 \\ \mu, & \mu < 0 \end{cases}$$

(b) When $\mu \geq 0$, $\phi'(\mu) = -\frac{2\sqrt{3}}{3}\sqrt{\mu} + 1$. $\phi'\left(\frac{3}{4}\right) = 0$.

Thus, $\mu^* = \frac{3}{4}$, $\phi^* = \frac{1}{4}$

(c) $$x_1^3 + x_2^3 = (x_1 + x_2)(x_1^2 + x_2^2 - x_1 x_2)$$
$$\geq (x_1 + x_2)\left[\frac{1}{2}(x_1 + x_2)^2 - \frac{1}{4}(x_1 + x_2)^2\right]$$
$$\geq \frac{1}{4}$$

The equality holds iff $x_1 = x_2 = \frac{1}{2}$.

Thus, the primal optimal value $f^* = \frac{1}{4}$.

(d) $\phi(\mu) = \inf_{x} \{ x_1^3 + x_2^3 + \mu_1(1 - x_1 - x_2) - \mu_2 x_1 - \mu_3 x_2 \} = -\infty$

$\phi^* = -\infty$. Obviously, strong duality doesn't hold for (P2).

4. (a) The KKT conditions hold, so

$$\mu_i^*\left[y_i(x_i^T w^* + b^*) - 1\right] = 0$$

$\mu_i^* > 0$, so $y_i(x_i^T w^* + b^*) = 1$

$y_i = \pm 1 \Rightarrow \frac{1}{y_i} = y_i$

so $b^* = y_i - x_i^T w^*$

(b) The output is

```
primal optimal:
        w = [-1.09090895  1.45454542]
        b = [-0.0909093]

dual optimal:
        mu = [1.65289246e+00 0.00000000e+00 0.00000000e+00 0.00000000e+00
 0.00000000e+00 0.00000000e+00 0.00000000e+00 1.65289239e+00
 0.00000000e+00 0.00000000e+00 7.15316507e-08 0.00000000e+00
 0.00000000e+00]
```