1. For any x_1, x_2 that satisfy $f(x_1) \in C$, $f(x_2) \in C$, because C is convex, we have that

$$\theta f(x_1) + \bar{\theta} f(x_2) = A(\theta x_1 + \bar{\theta} x_2) + b = f(\theta x_1 + \bar{\theta} x_2) \in C$$

Thus $\theta x_1 + \bar{\theta} x_2 \in f^{-1}(C)$, $\forall \theta \in [0,1]$. $f^{-1}(C)$ is convex.

2. C_1 and C_2 are nonempty, so C is nonempty.

Let $\eta_1 = x_1 - x_2$ ($x_1 \in C_1$, $x_2 \in C_2$), $\eta_2 = x_3 - x_4$ ($x_3 \in C_1$, $x_4 \in C_2$). Then $\theta\eta_1 + \bar{\theta}\eta_2 = (\theta x_1 + \bar{\theta} x_3) - (\theta x_2 + \bar{\theta} x_4)$.

C_1 and C_2 are convex, so $\theta x_1 + \bar{\theta} x_3 \in C_1$, $\theta x_2 + \bar{\theta} x_4 \in C_2$.

Thus, $\theta\eta_1 + \bar{\theta}\eta_2 \in C$, $\forall \theta \in [0,1]$. C is convex.

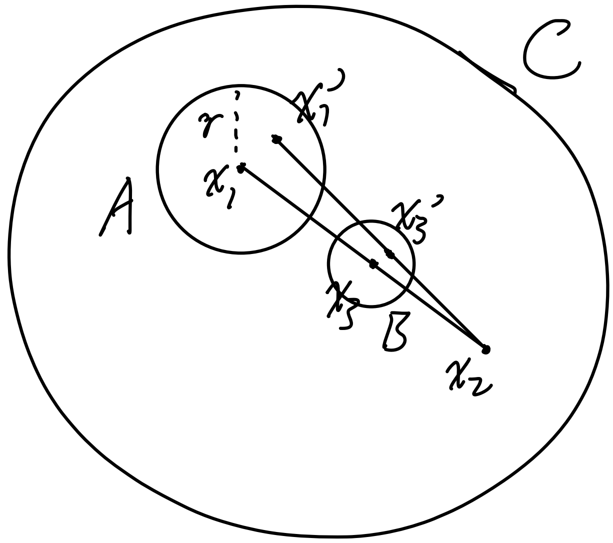
If $0 \in C$, then $\exists x_1$, $x_1 \in C_1$, $x_1 \in C_2$, from which we conclude that $C_1 \cap C_2 \neq \emptyset$. It is a contradiction, so $0 \notin C$.

3. (a) For any $x_1, x_2 \in \operatorname{int} C$, there exists an open ball $A = U(x_1, r)$, s.t. $A \subset C$. Let $x_3 = \theta x_1 + \bar{\theta} x_2$, $\theta \in (0,1)$, and open ball $B = U(x_3, \theta r)$. $\forall x_3' \in B$, we can find x_1' that satisfies $x_3' = \theta x_1' + \bar{\theta} x_2$.

$\|x_1 - x_1'\| = \frac{1}{\theta} \|x_3 - x_3'\| < r$, so $x_1' \in A$, $x_1' \in C$.

C is convex, so $x_3' \in C$. Thus, $B \subset C$, $x_3 \in \operatorname{int} C$.

$\operatorname{int} C$ is convex.



(b) $\forall x, y \in \bar{C}$, $\exists \{x_n\}, \{y_n\}$, s.t. $x_i, y_i \in C$, $x_i \neq x_j$, $y_i \neq y_j$ $(i \neq j)$, $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$.

$\forall \theta \in (0,1)$, $\theta x_i + \bar{\theta} y_i \in C$, $\lim\limits_{n\to\infty} (\theta x_n + \bar{\theta} y_n)$

$$= \theta x + \bar{\theta} y \in \bar{C}$$

Thus, $\bar{C}$ is convex.

4. (a) Let $x \in C$ s.t. $x_i \in S$, $\alpha_i \geq 0 \, (i=1,2,\cdots,m_1)$, $\sum_{i=1}^{m_1} \alpha_i = 1$, $x = \sum_{i=1}^{m_1} \alpha_i x_i$

$y \in C$ s.t. $y_i \in S$, $\beta_i \geq 0 \, (i=1,2,\cdots,m_2)$, $\sum_{i=1}^{m_2} \beta_i = 1$, $y = \sum_{i=1}^{m_2} \beta_i y_i$

Then $\forall \theta \in (0,1)$, $\theta x + \bar{\theta} y = \sum_{i=1}^{m_1+m_2} \lambda_i z_i$, where

$$\lambda_i = \begin{cases} \theta \alpha_i, & 1 \leq i \leq m_1 \\ \bar{\theta} \beta_{i-m_1}, & m_1 < i \leq m_1 + m_2 \end{cases}, \quad \sum_{i=1}^{m_1+m_2} \lambda_i = 1, \; \lambda_i \geq 0$$

$$z_i = \begin{cases} x_i, & 1 \leq i \leq m_1 \\ y_{i-m_1}, & m_1 < i \leq m_1 + m_2 \end{cases}, \quad z_i \in S,$$

so $\theta x + \bar{\theta} y \in C$. C is convex.

(b) Let $\mathcal{A} = \{X \mid X$ is convex and $S \subset X\}$.

$\forall X \in \mathcal{A}$, $\forall x = \sum_{i=1}^{m} \theta_i x_i \in C$, because $x_i \subset X$ and X is convex, we know that $x \in X$ and $C \subset X$.

$\operatorname{conv} S \in \mathcal{A}$, so $C \subset \operatorname{conv} S$.

By definition, $\forall X \in \mathcal{A}$, $\operatorname{conv} S \subset X$. Fix $m=1$. Then we know that $C \in \mathcal{A}$, so $\operatorname{conv} S \subset C$.

$C \subset \operatorname{conv} S \wedge \operatorname{conv} S \subset C \Rightarrow C = \operatorname{conv} S$.

5. $\|x-x_0\| \le \|x-x_i\| \iff \|x-x_0\|^2 \le \|x-x_i\|^2$

$$(x-x_0)^T(x-x_0) \le (x-x_i)^T(x-x_i)$$

$$(x^T-x_0^T)(x-x_0) \le (x^T-x_i^T)(x-x_i)$$

$$\|x\|^2+\|x_0\|^2-x^Tx_0-x_0^Tx \le \|x\|^2+\|x_i\|^2-x^Tx_i-x_i^Tx$$

$$\|x_0\|^2-\|x_i\|^2+(x_i^T-x_0^T)x \le x^T(x_0-x_i) = (x_0^T-x_i^T)x$$

$$(x_i^T-x_0^T)x \le \frac{\|x_i\|^2-\|x_0\|^2}{2}$$

Thus, $A=\begin{pmatrix} x_1^T-x_0^T \\ x_2^T-x_0^T \\ \cdots \\ x_K^T-x_0^T \end{pmatrix}$, $b=\frac{1}{2}\begin{pmatrix} \|x_1\|^2-\|x_0\|^2 \\ \|x_2\|^2-\|x_0\|^2 \\ \cdots \\ \|x_K\|^2-\|x_0\|^2 \end{pmatrix}$

An example of V in $\mathbb{R}^2$ is: