1. (a) $(\log x)'' = -\frac{1}{x^2} < 0$, so $y = \log x$ is concave on $(0, +\infty)$.

Let $k = \|x\|_0$. Assume that $x_i \neq 0$, $i = 1, 2, \cdots, k$.

Then $H(x) = -\sum_{i=1}^{k} x_i \log x_i = \sum_{i=1}^{k} x_i \log \frac{1}{x_i} \leq \log k \leq \log n$.

(b) $H(\bar{x}) = \log n$, so $\bar{x}$ is a maximum point.

Let $C = \{x \in \Delta_{n-1} : x_i > 0, i = 1, 2, \cdots, n\}$

If $x \in C$, $\nabla^2 H(x) = \mathrm{diag}\left(-\frac{1}{x_1}, -\frac{1}{x_2}, \cdots, -\frac{1}{x_n}\right)$

Obviously $\nabla^2 H(x)$ is negative definite, so $H(x)$ is strictly concave on $C$. Thus $\bar{x}$ is the unique maximum point on $C$.

If $x \in \Delta_{n-1} \setminus C$, $k < n$, $H(x) \leq \log k < \log n$.

Thus, $\bar{x}$ is the unique maximum point on $\Delta_{n-1}$.

2. (a) $\dfrac{(\mu-s)f(u)+(u-\mu)f(s)}{u-s} \geq f\left(\dfrac{\mu-s}{u-s}\cdot u+\dfrac{u-\mu}{u-s}\cdot s\right)=f(\mu)$

$(u-\mu)(f(\mu)-f(s)) \leq (\mu-s)(f(u)-f(\mu))$

so $\dfrac{f(\mu)-f(s)}{\mu-s} \leq \dfrac{f(u)-f(\mu)}{u-\mu}$

(b) Let $\beta=\sup\limits_{a<s<\mu} \dfrac{f(\mu)-f(s)}{\mu-s}$. $\exists u_0\in(\mu,b)$, $\forall s\in(a,\mu)$,

$\dfrac{f(\mu)-f(s)}{\mu-s} \leq \dfrac{f(u_0)-f(\mu)}{u_0-\mu} < +\infty$, so $-\infty<\beta<+\infty$.

If $a<x<\mu$, $\dfrac{f(\mu)-f(x)}{\mu-x} \leq \beta$, so $f(x)\geq f(\mu)+\beta(x-\mu)$

If $\mu<x<b$, $\dfrac{f(x)-f(\mu)}{x-\mu} \geq \beta$ (otherwise we can find $u_0$,

s.t. $\dfrac{f(\mu)-f(s)}{\mu-s} \leq \dfrac{f(u_0)-f(\mu)}{u_0-\mu} < \beta$, so $\beta$ is not the supremum)

so $f(x)\geq f(\mu)+\beta(x-\mu)$.

If $x=\mu$, the proposition is trivial

(c) $X\in(a,b)$, so $f(X)\geq f(\mu)+\beta(X-\mu)$

$E[f(X)] \geq E[f(\mu)+\beta(X-\mu)]$

$= Ef(\mu)+\beta(\mu-E\mu)$

$= f(EX)$ (If $k$ is constant, then $Ek=k$.)

3. $\|x\|$ is convex. $x^3$ is increasing $\Rightarrow \|x\|^3$ is convex.

$\Rightarrow \|Ax+b\|^3$ is convex.

$e^x$ is convex. $f(x_1, x_2) = \log(x_1 + x_2)$ is increasing

$\Rightarrow \log(e^{x_1} + e^{x_2})$ is convex $\Rightarrow \log(1 + e^{3x_1 + 2x_2})$ is convex

(Let $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.)

$\|Ax+b\|^3$ and $\log(1 + e^{3x_1 + 2x_2})$ are convex

$\Rightarrow \max\{\|Ax+b\|^3, \log(1 + e^{3x_1 + 2x_2})\}$ is convex.

Sublevel sets of convex functions are convex sets, so $S$ is convex.

4. (a) Let $f(x) = x_1^2 - 2x_1x_2 + x_2^2 + x_1 + x_2$.

$$\nabla^2 f(x) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$|2| > 0$, $\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 0$, so $\nabla^2 f(x) \succeq 0$.

$f(x)$ is a convex function.

$x^2$ and $e^x$ are convex functions. Similar to problem 4. of homework 3, we know that $x_1^2 + e^{x_2}$ is a convex function. Let $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then

$(x_1 + x_2)^2 + e^{x_1+x_2} = (x_1 - x_2)^2 + 4x_1x_2 + e^{x_1+x_2}$

is a convex function.

$x_1 - 3x_2$ is an affine function.

Thus, (a) is a convex optimization problem.

(b) $6x_1^2 - 7x_2$ is not an affine function, so (b) is not a convex optimization problem