第九章《多元函数微分学》

1．D　A　B　B

2．（1）$\left\{(x,y)\middle|4\le x^2+y^2\le 9\right\}$；

（2）$\left\{(x,y)\middle|,x>0,y>x+1\right\}\cup\left\{(x,y)\middle|x<0,x<y<x+1\right\}$；

（3）　全平面　；

（4）$\dfrac{2}{5}$；

（5）$\dfrac{1}{2}$；

（6）$\dfrac{\pi}{4}$.

（7）-1；

（8）$\dfrac{(x-y)dx+(x+y)dy}{x^2+y^2}$；

（9）$\{1,-1,0\}$；

（10）z；

（11）$\dfrac{x-1}{-2}=\dfrac{y+2}{1}=\dfrac{z+2}{-1}$

（12）　最大　；

（13）　模　.

3．（1）；解：$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{3-\sqrt{9+xy}}{xy}=\lim\limits_{t\to 0}\dfrac{3-\sqrt{9+t}}{t}=\lim\limits_{t\to 0}\dfrac{-t}{3+\sqrt{9+t}}=-\dfrac{1}{6}$

（2）解：$\lim\limits_{\substack{x\to+\infty\\ y\to+\infty}}(x^2+y^2)\ \mathrm{e}^{-(x+y)}=\lim\limits_{\substack{x\to+\infty\\ y\to+\infty}}\left[(x+y)^2\mathrm{e}^{-(x+y)}-2xe^{-x}ye^{-y}\right]$

由于 $\lim\limits_{t\to+\infty}te^{-t}=\lim\limits_{t\to+\infty}\dfrac{t}{e^t}=\lim\limits_{t\to+\infty}\dfrac{1}{e^t}=0$，$\lim\limits_{t\to+\infty}t^2\mathrm{e}^{-t}=\lim\limits_{t\to+\infty}\dfrac{t^2}{e^t}=\lim\limits_{t\to+\infty}\dfrac{2t}{e^t}=\lim\limits_{t\to+\infty}\dfrac{2}{e^t}=0$，

故 $\lim\limits_{\substack{x\to+\infty\\ y\to+\infty}}(x^2+y^2)\ \mathrm{e}^{-(x+y)}=\lim\limits_{\substack{x\to+\infty\\ y\to+\infty}}\left[(x+y)^2\mathrm{e}^{-(x+y)}-2xe^{-x}ye^{-y}\right]=0$

4．解：沿着曲线 $y=kx^3,(x,y)\to(0,0)$,有 $\lim\limits_{\substack{x\to 0\\ y=kx^3\to 0}}\dfrac{x^3y}{x^6+y^2}=\lim\limits_{x\to 0}\dfrac{kx^6}{x^6+k^2x^6}=\dfrac{k}{1+k^2}$ 因 k 而异，从而极限

$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^3y}{x^6+y^2}$ 不存在

5．解：由于 $f(x,0)\equiv 0, f(0,y)\equiv 0,$

从而可知在点 (0 , 0) 分别对于每个自变量 x 或 y 都连续，但沿着曲线 $y=kx,(x,y)\to(0,0)$，有

$$\lim_{\substack{x\to 0\\ y=kx\to 0}}\frac{2xy}{x^2+y^2}=\lim_{x\to 0}\frac{2kx^2}{x^2+k^2x^2}=\frac{2k}{1+k^2}$$ 因 k 而异，

从而极限 $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ 不存在，故作为二元函数在点 (0 , 0) 却不连续.

6． 解：$\dfrac{\partial u}{\partial y}=z\cdot\dfrac{1}{1+\left(\dfrac{x}{y}\right)^2}\cdot\dfrac{-x}{y^2}=\dfrac{-xz}{x^2+y^2},\dfrac{\partial^2 u}{\partial y^2}=-\dfrac{0-xz\cdot 2y}{\left(x^2+y^2\right)^2}=\dfrac{2xyz}{\left(x^2+y^2\right)^2}$

7．解： 当 $(x,y)=(0,0), f(x,y)$ 在此为分段点，用定义求偏导数

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{\Delta x^2\sin\dfrac{1}{\Delta x^2}-0}{\Delta x}=0,\ f_y(0,0)=\lim_{\Delta y\to 0}\frac{\Delta y^2\sin\dfrac{1}{\Delta y^2}-0}{\Delta y}=0$$

8．解：（1） 当 $x\neq 0,\ f_x(x,y)=\left(4x^3+2xy^2\right)\sin\dfrac{1}{x}+x^2\left(x^2+y^2\right)\cos\dfrac{1}{x}\cdot\dfrac{-1}{x^2}$

$$=\left(4x^3+2xy^2\right)\sin\frac{1}{x}-\left(x^2+y^2\right)\cos\frac{1}{x}$$

当 $x=0,\ f_x(0,y)=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,y)-f(0,y)}{\Delta x-0}=\lim\limits_{\Delta x\to 0}\dfrac{\Delta x^2\left(\Delta x^2+y^2\right)\sin\dfrac{1}{\Delta x}-0}{\Delta x}=0$

综上 $f_x(x,y)=\begin{cases}\left(4x^3+2xy^2\right)\sin\dfrac{1}{x}-\left(x^2+y^2\right)\cos\dfrac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$

当 $x\neq 0,\ f_y(x,y)=2x^2y\sin\dfrac{1}{x}$，

当 $x=0$，$f_y(0,y)=\lim\limits_{\Delta y\to 0}\dfrac{f(0,y+\Delta y)-f(0,y)}{\Delta y-0}=\lim\limits_{y\to 0}\dfrac{0-0}{\Delta y}=0$，

综上 $f_y(x,y)=\begin{cases}2x^2y\sin\dfrac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$

（2）考察偏导函数在 $(0,3)$ 点处是否连续.

$\lim\limits_{\substack{x\to 0\\ y\to 3}} f_y(x,y)=\lim\limits_{\substack{x\to 0\\ y\to 3}} 2x^2y\sin\dfrac{1}{x}=0=f_y(0,3)$，故 $f_y(x,y)$ 在 $(0,3)$ 点处连续，

$\lim\limits_{\substack{x\to0\\y\to3}} f_x(x,y)=\lim\limits_{\substack{x\to0\\y\to3}}\left[\left(4x^3+2xy^2\right)\sin\frac{1}{x}-\left(x^2+y^2\right)\cos\frac{1}{x}\right]$不存在，从而 $f_x(x,y)$ 在$(0,3)$点处不连续

9．解：因为$\frac{\partial z}{\partial x}=f+xf'\cdot\frac{-y}{x^2}+yg_1'\cdot1+yg_2'\cdot\frac{1}{y}=f-\frac{y}{x}f'+yg_1'+g_2'$

所以$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(f-\frac{y}{x}f'+yg_1'+g_2'\right)$

$=f'\cdot\frac{1}{x}-\frac{1}{x}f'-\frac{y}{x}f''\cdot\frac{1}{x}+g_1'+yg_{12}''\cdot\frac{-x}{y^2}+g_{22}''\cdot\frac{-x}{y^2}=-\frac{y}{x^2}f''+g_1'-\frac{x}{y}g_{12}''-\frac{x}{y^2}g_{22}''$

10．解：$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot y+\frac{\partial z}{\partial v}\cdot\frac{1}{y},\frac{\partial^2 z}{\partial x^2}=y\left(\frac{\partial^2 z}{\partial u^2}\cdot y+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{1}{y}\right)+\frac{1}{y}\left(\frac{\partial^2 z}{\partial v\partial u}\cdot y+\frac{\partial^2 z}{\partial v^2}\cdot\frac{1}{y}\right)$

$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot x+\frac{\partial z}{\partial v}\cdot\frac{-x}{y^2},\frac{\partial^2 z}{\partial y^2}=x\left(\frac{\partial^2 z}{\partial u^2}\cdot x+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{-x}{y^2}\right)+\frac{2x}{y^3}\cdot\frac{\partial z}{\partial v}-\frac{x}{y^2}\left(\frac{\partial^2 z}{\partial v\partial u}\cdot x+\frac{\partial^2 z}{\partial v^2}\cdot\frac{-x}{y^2}\right)$

从而由$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=0$，得 $x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=4x^2\frac{\partial^2 z}{\partial u\partial v}-\frac{2x}{y}\frac{\partial z}{\partial v}=0$， 即$2x^2\frac{\partial^2 z}{\partial u\partial v}-\frac{x}{y}\frac{\partial z}{\partial v}=0$

因为$u=xy,v=\frac{x}{y}$，所以$x^2=uv$，代入上式$2uv\frac{\partial^2 z}{\partial u\partial v}-v\frac{\partial z}{\partial v}=0$化简得$2u\frac{\partial^2 z}{\partial u\partial v}-\frac{\partial z}{\partial v}=0$

11．解 1：（求导法）$\begin{cases}\frac{dz}{dx}=f_1'+f_2'\frac{dy}{dx}\\1=\varphi_1'\frac{dy}{dx}+\varphi_2'\frac{dz}{dx}\end{cases}$，整理得$\begin{cases}f_2'\frac{dy}{dx}-\frac{dz}{dx}=-f_1'\\\varphi_1'\frac{dy}{dx}+\varphi_2'\frac{dz}{dx}=1\end{cases}$

所以$\frac{dz}{dx}=\frac{\begin{vmatrix}f_2' & -f_1'\\\varphi_1' & 1\end{vmatrix}}{\begin{vmatrix}f_2' & -1\\\varphi_1' & \varphi_2'\end{vmatrix}}=\frac{\varphi_1'f_1'+f_2'}{\varphi_1'+f_2'\varphi_2'}$

解 2：（微分法）对函数取全微分得，$dz=f_1'dx+f_2'dy,dx=\varphi_1'dy+\varphi_2'dz,$

从而$dy=\frac{-\varphi_2'dz+dx}{\varphi_1'},dz=f_1'dx+f_2'\cdot\frac{-\varphi_2'dz+dx}{\varphi_1'},\varphi_1'dz=\varphi_1'f_1'dx-f_2'\varphi_2'dz+f_2'dx$

$(\varphi_1'+f_2'\varphi_2')dz=(\varphi_1'f_1'+f_2')dx,\frac{dz}{dx}=\frac{\varphi_1'f_1'+f_2'}{\varphi_1'+f_2'\varphi_2'}$

12．解 1（求导法）：指向外侧在此即抛物面的下侧，$\vec{n}=\{2x,y,-1\}\big|_P=\{2,2,-1\},\vec{n}^\circ=\left\{\frac{2}{3},\frac{2}{3},\frac{-1}{3}\right\}$

$\ln u = \frac{1}{2}\ln(3x^2+3y^2+z^2) - \frac{1}{2}\ln x$，

$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{2}\frac{6x}{3x^2+3y^2+z^2} - \frac{1}{2x}$， $\frac{\partial u}{\partial x} = u \cdot \left(\frac{3x}{3x^2+3y^2+z^2} - \frac{1}{2x} \right)$， $\left.\frac{\partial u}{\partial x}\right|_{(1,2,3)} = -\frac{3\sqrt{6}}{4}$

$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{1}{2}\frac{6y}{3x^2+3y^2+z^2}$， $\frac{\partial u}{\partial y} = u \cdot \frac{3y}{3x^2+3y^2+z^2}$， $\left.\frac{\partial u}{\partial y}\right|_{(1,2,3)} = \frac{\sqrt{6}}{2}$

$\frac{1}{u}\frac{\partial u}{\partial z} = \frac{1}{2}\frac{2z}{3x^2+3y^2+z^2}$， $\frac{\partial u}{\partial z} = u \cdot \frac{z}{3x^2+3y^2+z^2}$， $\left.\frac{\partial u}{\partial z}\right|_{(1,2,3)} = \frac{\sqrt{6}}{4}$

从而$\frac{\partial u}{\partial n} = \left.gradu\right|_P \cdot \vec{n}^\circ = \left\{ \frac{-3\sqrt{6}}{4}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{4} \right\} \cdot \left\{ \frac{2}{3}, \frac{2}{3}, \frac{-1}{3} \right\} = -\frac{\sqrt{6}}{4}$

解 2（微分法）：指向外侧在此即抛物面的下侧，$\vec{n} = \left.\{2x, y, -1\}\right|_P = \{2, 2, -1\}, \vec{n}^\circ = \left\{ \frac{2}{3}, \frac{2}{3}, \frac{-1}{3} \right\}$

记$v = 3x^2+3y^2+z^2$，则由微分运算法则

$$du = \frac{1}{2\sqrt{v}}dv = \frac{1}{2\sqrt{v}}\frac{(6xdx+6ydy+2zdz)x - (3x^2+3y^2+z^2)dx}{x^2}$$

$$\left.du\right|_P = \left.\frac{1}{2\sqrt{v}}\frac{(6xdx+6ydy+2zdz)x - (3x^2+3y^2+z^2)dx}{x^2}\right|_P = \left\{ \frac{-3\sqrt{6}}{4}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{4} \right\}$$

从而$\frac{\partial u}{\partial n} = \left.gradu\right|_P \cdot \vec{n}^\circ = \left\{ \frac{-3\sqrt{6}}{4}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{4} \right\} \cdot \left\{ \frac{2}{3}, \frac{2}{3}, \frac{-1}{3} \right\} = -\frac{\sqrt{6}}{4}$

13．解：设切点为(x_0, y_0, z_0)，则切平面为$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$，

切平面在三个坐标轴的截距分别为$\frac{a^2}{x_0}, \frac{b^2}{y_0}, \frac{c^2}{z_0}$，从而所求体积为 $V = \frac{a^2b^2c^2}{6x_0y_0z_0}$，

因此所求为 $V = \frac{a^2b^2c^2}{6x_0y_0z_0}$ 在 $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1$ 的最值问题。

此问题与$f(x, y, z) = xyz$在$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$下的最值问题等价，只是最大值与最小值问题换位而已。

构造拉格朗日辅助函数 $L(x, y, z) = xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right)$

则$L_x = yz + \lambda\frac{2x}{a^2} = 0, L_y = xz + \lambda\frac{2y}{a^2} = 0, L_z = xy + \lambda\frac{2z}{a^2} = 0,$

与约束条件结合推得$x^2=\frac{a^2}{3},y^2=\frac{b^2}{3},z^2=\frac{c^2}{3}$

由于在第一卦限，从而切点为$\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$，由于实际问题且驻点唯一，此点即所求。

14．解：（1）当$x^2+y^2\neq 0,\quad \frac{\partial f}{\partial x}=2(x+y)\sin\frac{1}{x^2+y^2}-\frac{2x(x+y)^2}{\left(x^2+y^2\right)^2}\cos\frac{1}{x^2+y^2}$，

当$x^2+y^2=0, f(x,y)$在此为分段点，用定义求偏导数$f_x\left(0,0\right)=\lim\limits_{x\to 0}\frac{x^2\sin\frac{1}{x^2}-0}{x}=0,$

综上，$\frac{\partial f}{\partial x}=\begin{cases}2(x+y)\sin\frac{1}{x^2+y^2}-\frac{2x(x+y)^2}{\left(x^2+y^2\right)^2}\cos\frac{1}{x^2+y^2} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$，

由对称性，$\frac{\partial f}{\partial y}=\begin{cases}2(x+y)\sin\frac{1}{x^2+y^2}-\frac{2y(x+y)^2}{\left(x^2+y^2\right)^2}\cos\frac{1}{x^2+y^2} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$。

（2）因为$\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{\partial f}{\partial x}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\left[2\left(x+y\right)\sin\frac{1}{x^2+y^2}-\frac{2x\left(x+y\right)^2}{(x^2+y^2)^2}\cos\frac{1}{x^2+y^2}\right]$极限不存在，所以$\frac{\partial f}{\partial x}$在原点不连续，同理$\frac{\partial f}{\partial y}$在原点不连续。

现证明$f(x,y)$在原点可微。

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f\left(\Delta x,\Delta y\right)-f\left(0,0\right)-f_x\left(0,0\right)\Delta x-f_y\left(0,0\right)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\left(\Delta x+\Delta y\right)^2\sin\frac{1}{\Delta x^2+\Delta y^2}-0-0\cdot\Delta x-0\cdot\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\left(\Delta x+\Delta y\right)^2\sin\frac{1}{\Delta x^2+\Delta y^2}}{\sqrt{\Delta x^2+\Delta y^2}}$$

因为$0\leq\left(\Delta x+\Delta y\right)^2\leq 2(\Delta x^2+\Delta y^2)$，所以 $0\leq\frac{\left(\Delta x+\Delta y\right)^2}{\sqrt{\Delta x^2+\Delta y^2}}\leq\frac{2(\Delta x^2+\Delta y^2)}{\sqrt{\Delta x^2+\Delta y^2}}=2\sqrt{\Delta x^2+\Delta y^2}$

由夹逼准则知$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\left(\Delta x+\Delta y\right)^2}{\sqrt{\Delta x^2+\Delta y^2}}=0$，进一步知 $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\left(\Delta x+\Delta y\right)^2\sin\frac{1}{\Delta x^2+\Delta y^2}}{\sqrt{\Delta x^2+\Delta y^2}}$=0（无穷小乘有界）

由可微的定义，知 $f(x,y)$ 在原点可微。

注：9-3 节学习过可微的充分条件——函数“一阶偏导数连续”可以推出可微，此例说明一阶偏导数连续是可微的充分但不必要条件。

15．解：令 $e^x = u, y^2 = v, |z| = t$，则问题化为在约束条件 $u+v+t=3, u\ge 0, v\ge 0, t\ge 0$ 下求 $f(u,v,t)=uvt$ 的最大值问题。

构造拉格朗日辅助函数 $L = uvt + \lambda(u+v+t-3)$，

则 $L_u = vt+\lambda = 0$，$L_v = ut+\lambda = 0$，$L_t = uv+\lambda$

$\Rightarrow 3uvt + \lambda(u+v+t) = 0$，

结合约束条件 $\Rightarrow uvt = -\lambda = uv = vt = tu \Rightarrow u = v = t = 1$

由于该实际问题的最大值一定存在，又可能点唯一，因此最大值为 $f(1,1,1)=1$

从而 $e^x y^2 |z| \le 1$

16．解 1（方程组两边求导法）：方程组两边对 x 求导，得 $\begin{cases} \dfrac{dz}{dx} = 2x + 2y\dfrac{dy}{dx} \\ 2x + 4y\dfrac{dy}{dx} + 6z\dfrac{dz}{dx} = 0 \end{cases}$

整理得 $\begin{cases} 2y\dfrac{dy}{dx} - \dfrac{dz}{dx} = -2x \\ 2y\dfrac{dy}{dx} + 3z\dfrac{dz}{dx} = -x \end{cases}$，

所以 $\dfrac{dy}{dx} = \dfrac{\begin{vmatrix} -2x & -1 \\ -x & 3z \end{vmatrix}}{\begin{vmatrix} 2y & -1 \\ 2y & 3z \end{vmatrix}} = \dfrac{-6xz - x}{2y + 6yz}$，$\dfrac{dz}{dx} = \dfrac{\begin{vmatrix} 2y & -2x \\ 2y & -x \end{vmatrix}}{\begin{vmatrix} 2y & -1 \\ 2y & 3z \end{vmatrix}} = \dfrac{2xy}{2y+6yz} = \dfrac{x}{1+3z}$

解 2（方程组两边求微分法）：由已知 $\begin{cases} dz = 2xdx + 2ydy \\ 2xdx + 4ydy + 6zdz = 0 \end{cases} \Rightarrow \begin{cases} dz - 2xdx = 2ydy \\ 2xdx + 2(dz - 2xdx) + 6zdz = 0 \end{cases}$

$\Rightarrow \begin{cases} -2xdx + (2+6z)dz = 0 \\ xdx + 2ydy + 3z(2xdx + 2ydy) = 0 \end{cases} \Rightarrow \dfrac{dz}{dx} = \dfrac{x}{1+3z}, \dfrac{dy}{dx} = -\dfrac{x+6xz}{2y+6yz}$

17．证：因为 $\dfrac{\partial z}{\partial x} = f'(u)\cdot\dfrac{\partial u}{\partial x}, \dfrac{\partial z}{\partial y} = f'(u)\cdot\dfrac{\partial u}{\partial y}$，

$\dfrac{\partial u}{\partial x} = \varphi'(u)\cdot\dfrac{\partial u}{\partial x} + P(x), \dfrac{\partial u}{\partial x} = \dfrac{P(x)}{1-\varphi'(u)}$

$$\frac{\partial u}{\partial y}=\varphi'(u)\cdot\frac{\partial u}{\partial y}-P(y),\frac{\partial u}{\partial y}=\frac{-P(y)}{1-\varphi'(u)}$$

$$P(y)\frac{\partial z}{\partial x}+P(x)\frac{\partial z}{\partial y}=P(y)f'(u)\frac{P(x)}{1-\varphi'(u)}+P(x)f'(u)\frac{-P(y)}{1-\varphi'(u)}=0$$

18．解：因为$\frac{\partial z}{\partial x}=f'(u)e^x\sin y,\frac{\partial^2 z}{\partial x^2}=f''(u)\left(e^x\sin y\right)^2+f'(u)e^x\sin y$

$$\frac{\partial z}{\partial y}=f'(u)e^x\cos y,\frac{\partial^2 z}{\partial y^2}=f''(u)\left(e^x\cos y\right)^2+f'(u)e^x(-\sin y)$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=f''(u)e^{2x}=f(u)\mathrm{e}^{2x},\Rightarrow f''(u)-f(u)=0$$

特征方程为$r^2-1=0,r_1=1,r_2=-1,f(u)=c_1e^u+c_2e^{-u}$

19． 解：$gradz\big|_{(-1,1)}=\{2x-y,2y-x\}\big|_{(-1,1)}=\{-3,3\}$

$\vec{l}^{\circ}=\left\{\frac{2}{\sqrt5},\frac{1}{\sqrt5}\right\}$， $\frac{\partial z}{\partial l}=\{-3,3\}\cdot\left\{\frac{2}{\sqrt5},\frac{1}{\sqrt5}\right\}=-\frac{3\sqrt5}{5}$

z在该点沿梯度相反方向，即$\left\{\frac{1}{\sqrt2},\frac{-1}{\sqrt2}\right\}$方向减少得最快；

沿与梯度垂直的那个方向，即$\pm\left\{\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right\}$方向$z$的值不变