THREE-PERIOD ORBITS IN BILLIARDS ON THE SURFACES OF CONSTANT CURVATURE

VICTORIA BLUMEN, KI YEUN KIM, JOE NANCE, VADIM ZHARNITSKY

ABSTRACT. An approach due to Wojtkovski [13], based on the Jacobi fields, is applied to study sets of 3-period orbits in billiards on hyperbolic plane and on two-dimensional sphere. It is found that the set of 3-period orbits in billiards on hyperbolic plane, as in the planar case, has zero measure. For the sphere, a new proof of Baryshnikov's theorem is obtained which states that 3-period orbits can form a set of positive measure if and only if a certain a natural condition on the orbit length is satisfied.

1. INTRODUCTION

This article provides a unified approach, based on the Jacobi fields, to study open sets of 3-period orbits in billiards on manifolds with constant curvature. Specifically, we consider spherical and hyperbolic cases. While the spherical as well as Euclidean case has been treated previously, our result for the billiards on hyperbolic plane is a new one. Billiards on manifolds of constant curvature have been studied earlier in [7], [9]. In [9], billiard domains on the sphere, containing open sets of periodic orbits, have been constructed. In [2] and in this article, the main goal is to assure that except for those special cases, 3-period orbits have zero measure.

The billiard system on a two dimensional Riemannian manifold (M, g) consists of the domain $\mathcal{Q}$ with a piecewise smooth boundary $\partial\mathcal{Q}$ and a mass point moving along the geodesics inside the domain. Whenever the mass hits the boundary, it reflects according to Fermat's principle so as to extremize the path length. That leads to the familiar law: the angle of incidence is equal to the angle of reflection.

Periodic orbits are a natural object of study in dynamical systems. One important question concerns the presence of large sets, in particular sets of positive measure, of periodic orbits in the billiard ball problem. Informally speaking, measure corresponds to the probability that a given orbit is periodic. This question has been originally motivated by spectral geometry problems. The second term of the Weyl asymptotics for the Dirichlet problem in a bounded domain has a certain special form if periodic orbits of the associated billiard problem have zero measure [14], see also a survey article by Gutkin [8]. There is a natural invariant measure for the billiard map which can be defined as follows: let s be an arclength parameter coordinatizing the boundary and let $\phi \in [0, \pi]$ be the angle of the outcoming ray from the boundary measured in the counterclockwise direction. The billiard ball map

$T : [\partial \mathcal{Q} \times [0, \pi]] \to [\partial \mathcal{Q} \times [0, \pi]]$ which takes an outcoming ray to another one obtained after reflection from the boundary, preserves the measure $\mu = \sin \phi \, d\phi \, ds$, see *e.g.* [3].

Our motivation to study the structure of the set of periodic orbits in non-Euclidean geometries is that this understanding may help one with the planar case for higher period orbits. It is also expected that eigenvalue asymptotics in non-Euclidean geometries would also require understanding the structure of the sets of periodic orbits.

For the planar billiard problem, it is easy to see that two period orbits have zero measure, since these orbits must be normal to the boundary at both ends. Similarly, this is the case for a billiard on $\mathbb{H}^2$. On the other hand, a billiard on $\mathbb{S}^2$ with boundary given by equator has two-period orbits of positive measure. This has to do with the presence of conjugated points on $\mathbb{S}^2$.

For the period 3, the problem on existence of positive measure sets is already non-trivial. The first result on zero measure of 3-period orbits in planar billiards was obtained by Rychlik, see [11], relying on symbolic calculations, which were later removed in [12]. Using Jacobi fields, Wojtkovski gave an elegant simple proof of Rychlik's theorem. Subsequently, there have been extensions to other types of billiard systems: higher dimensional ([15]), outer billiards ([5, 10]), and spherical ([2]). Recently, a proof for the period 4 case has been announced by Glutsyuk and Kudryashov [6].

Our main result is

Theorem 1. *The set of 3-period orbits in any billiard on $\mathbb{H}^2$ has zero measure.*

In order to prove this theorem, we extend the Jacobi fields approach from [13] and present the unified proof which treats all three billiard systems on the constant curvature manifolds in the same manner. Our argument proceeds independently of the underlying geometry until we get the compatibility condition. Then, using the relevant cosine formula, which depends on the geometry, we obtain the relation that must be satisfied on a neighborhood filled with 3-period orbits

$$k_g(s_0) = \sin^3(\phi_0)F(L),$$

where L is the length of 3-period orbits, $k_g(s_0)$ is geodesic curvature at one of the vertices, ϕ_0 is the angle of the billiard orbit with the tangent to the boundary at this vertex and s_0 is the value of the arclength parameter s at the vertex. The function $F(L)$ depends on the underlying Riemannian manifold

$$F(L) = \begin{cases} \frac{2}{L} & \text{on } \mathbb{E}^2 \\ \coth\left(\frac{L}{2}\right) & \text{on } \mathbb{H}^2 \\ \cot\left(\frac{L}{2}\right) & \text{on } \mathbb{S}^2. \end{cases}$$

From this formula it is possible to classify sets of 3-period orbits. In particular, we obtain a new proof of a theorem by Baryshnikov on the spherical case [2] where sub-Riemannian geometry methods were used.

Theorem 2. *Let P_3 be the set of 3-period orbits in the billiard domain ∂Q on $\mathbb{S}^2$. Assume that some orbit $(x_0, x_1, x_2) \in P_3$ has perimeter $L = \pi, 3\pi$ or 5π and that some arcs of ∂Q containing x_0, x_1, x_2 belong to great circles. Then $(x_0, x_1, x_2) \in int(P_3)$ and P_3 has positive measure. Otherwise, $(x_0, x_1, x_2) \notin int(P_3)$. In particular, if none of 3-period orbits are of the above special type, then P_3 has an empty interior and is the set of zero measure.*

Remark 1. *We only prove that the set of 3-period orbits on $\mathbb{H}^2$ (and on $\mathbb{S}^2$ when the special condition is not satisfied) has an empty interior. The stronger statement about zero measure follows verbatim the argument in [13], page 163.*

2. Billiard system on the surface of constant curvature

2.1. Jacobi fields.
Let Q be a smooth domain on a surface of constant curvature $\varkappa$. The billiard ball inside Q travels along the geodesics and reflects at the boundary. Let $\gamma(\epsilon, \tau)$ be a one-parameter family of geodesics where $|\epsilon| < \epsilon_0, -\infty < \tau < \infty$.

For the reader's convenience, we briefly recall the derivation of the Jacobi fields, see *e.g.* [4] or [1]. The Jacobi field is defined by

$$\mathbf{J}(\tau) = \frac{\partial \gamma(0, \tau)}{\partial \epsilon}$$

and it satisfies the Jacobi equation

$$\frac{\nabla}{d\tau} \frac{\nabla}{d\tau} \mathbf{J}(\tau) + R(\mathbf{J}(\tau), \dot{\gamma})\dot{\gamma} = 0,$$

where ∇ denotes the covariant derivative and R is the curvature tensor. As usual, we are interested in the component of the Jacobi field, that is perpendicular to $\dot{\gamma}$. Therefore, it can be expressed as

$$\mathbf{J}(\tau) = J(\tau)\mathbf{n}(\tau),$$

where $J(\tau)$ is a scalar function and $\mathbf{n}(\tau)$ is a unit vector field perpendicular to $\dot{\gamma}$. If the surface has constant curvature $\varkappa$, then one obtains a scalar equation with constant coefficients

(1) $$J''(\tau) + \varkappa J(\tau) = 0.$$

According to the standard result in the theory of differential equations, the solution of the Jacobi equation is uniquely defined if two initial conditions $J(0)$ and $J'(0)$ are given.

2.2. Evolution and reflection matrices.
Consider billiards on the hyperbolic plane $\mathbb{H}^2$ and the 2-sphere $\mathbb{S}^2$ which have the curvature $\varkappa = -1$ and $\varkappa = 1$ respectively. Solving the Jacobi equation (1), we obtain

$$J(\tau) = \begin{cases} J(0)\cosh(\tau) + J'(0)\sinh(\tau) & \text{on } \mathbb{H}^2 \\ J(0)\cos(\tau) + J'(0)\sin(\tau) & \text{on } \mathbb{S}^2. \end{cases}$$

In each case, we obtain the evolution matrix $P(\tau)$

$$\begin{pmatrix} J(\tau) \\ J'(\tau) \end{pmatrix} = P(\tau) \begin{pmatrix} J(0) \\ J'(0) \end{pmatrix},$$

where $P(\tau) = \begin{cases} \begin{pmatrix} \cosh(\tau) & \sinh(\tau) \\ \sinh(\tau) & \cosh(\tau) \end{pmatrix} & \text{on } \mathbb{H}^2 \\ \\ \begin{pmatrix} \cos(\tau) & \sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{pmatrix} & \text{on } \mathbb{S}^2 \end{cases}$

which describes the changes of the Jacobi field over time.

Note that the corresponding evolution matrix in the Euclidean case is given by

$$P(\tau) = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}.$$

When the billiard ball hits the boundary at $x = (s, \phi)$, the Jacobi field is transformed by the linear map $R(x)$ which is essentially the same as the reflection map in the Euclidean case

$$\begin{pmatrix} J_{out} \\ J'_{out} \end{pmatrix} = R(x) \begin{pmatrix} J_{in} \\ J'_{in} \end{pmatrix},$$

where $R(x) = \begin{pmatrix} -1 & 0 \\ \dfrac{2k_g(s)}{\sin(\phi)} & -1 \end{pmatrix}.$

Note that the reflection matrix is directly related to the classical mirror formula, see [13].

One should expect that the reflection matrix $R(x)$ for the billiard on a two dimensional Riemannian manifold should have the same form as in the Euclidean case [13]. Nevertheless, we provide some justification. Consider a one-parameter family of geodesics $\boldsymbol{\gamma}(\epsilon, \tau)$ reflecting from the billiard boundary ∂Q on a two dimensional Riemannian manifold. In an ϵ-neighborhood of the reflection point x_0 of $\boldsymbol{\gamma}(0, \tau_0)$, the manifold can be represented as a smooth two dimensional surface in $\mathbb{R}^3$. Projecting the geodesics and the boundary onto the tangent plane at x_0, we obtain the corresponding structure on the plane: a family of orbits reflecting from the boundary. It is easy to estimate that the angles as well as distances before and after the projection, differ by $O(\epsilon^2)$. This is mainly due to the expansion $\cos \epsilon = 1 - \epsilon^2/2 + O(\epsilon^4)$. Also, straightforward estimates show that the projected boundary curve will have the curvature equal to the geodesic curvature of ∂Q with the accuracy $O(\epsilon^2)$. As quadratic terms do not affect linear transformations, the reflection map will have the same form as in the Euclidean case with k replaced by k_g.
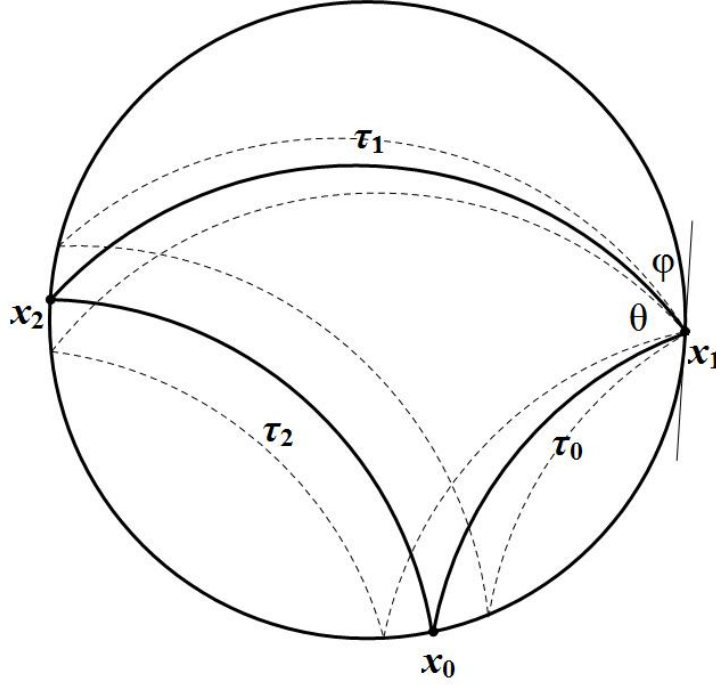
FIGURE 1. Periodic orbits with period 3 in a billiard on $\mathbb{H}^2$.

3. BILLIARD ON THE HYPERBOLIC PLANE

In this section we prove Theorem 1. Assume that there is an open set of 3-period orbits. Then we must have T^3 and DT^3 equal to the identity, which implies

$$(2) \qquad P(\tau_1)R(x_1)P(\tau_0)R(x_0)P(\tau_2)R(x_2) = I,$$

where I is the identity map, x_0, x_1 and x_2 are the collision points, and τ_0, τ_2, τ_1, are the distances between collision points, see Figure 1. This relation can be also rewritten as

$$(3) \qquad P(\tau_1)R(x_1)P(\tau_0) = R^{-1}(x_2)P^{-1}(\tau_2)R^{-1}(x_0)$$

which takes the form

$$\begin{bmatrix} \cosh(\tau_1) & \sinh(\tau_1) \\ \sinh(\tau_1) & \cosh(\tau_1) \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ \frac{2k_g(x_1)}{\sin(\phi_1)} & -1 \end{bmatrix} \cdot \begin{bmatrix} \cosh(\tau_0) & \sinh(\tau_0) \\ \sinh(\tau_0) & \cosh(\tau_0) \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & 0 \\ -\frac{2k_g(x_2)}{\sin(\phi_2)} & -1 \end{bmatrix} \cdot \begin{bmatrix} \cosh(\tau_2) & -\sinh(\tau_2) \\ -\sinh(\tau_2) & \cosh(\tau_2) \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ -\frac{2k_g(x_0)}{\sin(\phi_0)} & -1 \end{bmatrix}.$$

After simplification, we equate the top right components to get[1],

$$(4) \qquad \sinh(\tau_0 + \tau_1) - \sinh(\tau_2) = \frac{2k_g(x_1)\sinh(\tau_0)\sinh(\tau_1)}{\sin(\phi_1)}.$$

We define θ to be the interior angle between two adjacent segments of an orbit, that is, $\theta = \pi - 2\phi$, see Figure 1.

Then we alter the hyperbolic cosine formula into

$$\cosh(\tau_2) = \cosh(\tau_0 + \tau_1) - \sinh(\tau_0)\sinh(\tau_1) - \sinh(\tau_0)\sinh(\tau_1)\cos(\theta_1).$$

We use the half angle formula to get

$$(5) \qquad \cosh(\tau_0 + \tau_1) - \cosh(\tau_2) = 2\cos^2\left(\frac{\theta_1}{2}\right)\sinh(\tau_0)\sinh(\tau_1).$$

Combining (4) and (5), we arrive at

$$\frac{\cosh(\tau_0 + \tau_1) - \cosh(\tau_2)}{\cos^2\left(\frac{\theta_1}{2}\right)} = \frac{\sinh(\tau_0 + \tau_1) - \sinh(\tau_2)\cos\left(\frac{\theta_1}{2}\right)}{k_g(x_1)}.$$

Note that the length of an orbit $L = \tau_0 + \tau_1 + \tau_2$ is invariant. Therefore,

$$\begin{aligned}
k_g(x_1) &= \frac{\sinh(\tau_0 + \tau_1) - \sinh(\tau_2)}{\cosh(\tau_0 + \tau_1) - \cosh(\tau_2)}\cos^3\left(\frac{\theta_1}{2}\right) \\
&= \frac{\sinh(L - \tau_2) - \sinh(\tau_2)}{\cosh(L - \tau_2) - \cosh(\tau_2)}\cos^3\left(\frac{\theta_1}{2}\right) \\
&= \cos^3\left(\frac{\theta_1}{2}\right)\coth\left(\frac{L}{2}\right) \\
&= \sin^3(\phi_1)\coth\left(\frac{L}{2}\right).
\end{aligned}$$

This relation must hold for all nearby orbits. In particular, for all orbits starting at the same point on the boundary with different angles of reflection. Thus, we obtain a contradiction because the right-hand side of the equation is not constant in any interval. Therefore, the set of 3-period orbits has an empty interior. Next, following an argument in [13] we obtain that the set has zero measure, which ends the proof of the Theorem 1.

---

[1]Compare with the corresponding formula in the Euclidean case: $\tau_0 + \tau_1 - \tau_2 = \frac{2\,k(x_1)\,\tau_0\,\tau_1}{\sin(\phi_1)}$, which was derived in [13].

4. Billiard on the 2-sphere

Now we prove Theorem 2 using the same method. Assuming there is an open set of 3-period orbits on $\mathbb{S}^2$, we again obtain that T^3 and DT^3 are equal to the identity. Therefore, using (3) again

$$P(\tau_1)R(x_1)P(\tau_0) = R^{-1}(x_2)P^{-1}(\tau_2)R^{-1}(x_0)$$

we get

(6) $$\sin(\tau_0)\cos(\tau_1) + \cos(\tau_0)\sin(\tau_1) - \sin(\tau_2) = \frac{2k_g(x_1)\sin(\tau_0)\sin(\tau_1)}{\sin(\phi_1)}.$$

Note that this relation is the same as (4) if trigonometric functions are replaced with their hyperbolic counterparts.

Combining (6) and the modified version of spherical cosine formula, we arrive at

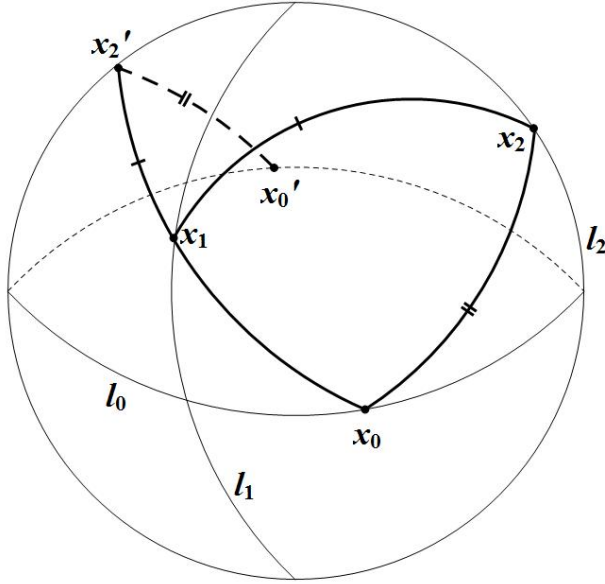(7) $$k_g(x_1) = \sin^3(\phi_1)\cot\left(\frac{L}{2}\right).$$

FIGURE 2. The special case $L = \pi$. The billiard boundary, which is an octant, is formed by three mutually orthogonal great circles l_0, l_1, l_2. The billiard orbit (x_0, x_1, x_2) is obtained from the half of the great circle (x_0, x_0') by reflecting from the boundaries.

If $\cot\left(\frac{L}{2}\right) \neq 0$, then we have the same contradiction as in the hyperbolic case. When $L = (2n + 1)\pi$, we have $\cot\left(\frac{L}{2}\right) = 0$ and $k_g = 0$. In this case, there could exist open sets of 3-period orbits.

Now, we discuss the characteristics of the billiards on which open sets of 3-period orbits exist. Note that only orbits without repetition are considered, and this assumption limits our cases to $L = \pi, 3\pi$, or 5π, see also [9].

Proposition 1. *Consider a spherical triangle x_0, x_1, x_2 on the unit sphere with perimeter $L = \pi, 3\pi$ or 5π. Let l_0, l_1, l_2 be the great circles passing through the vertices orthogonal to the corresponding bisectors. Then, these great circles intersect at the right angles and any billiard boundary containing segments of l_0, l_1, l_2 passing through x_0, x_1, x_2 will have an open set of 3-period orbits.*

Proof. Let l_0 be a geodesic on $\mathbb{S}^2$ and x_0 be any point on l_0. Create two geodesics, l_1 and l_2, that are perpendicular to l_0 and to each other, but do not pass through x_0. Consider any geodesic segment, τ, of length π, whose endpoint is x_0. Denote the angle between τ and l_0 as σ. Through two reflections over l_1 and l_2, this line τ forms a triangle of length π within the boundary created by l_0, l_1, and l_2. Since x_0 and σ were arbitrary, any 3-period orbit of length π must be contained in one octant, which is formed by l_0, l_1, and l_2, whose intersections are orthogonal. In particular, this implies that all orbits in the octant are 3-periodic except those which hit the corners.

Consider three great circles that intersect at x_0, x_1, and x_2. The total length of the lines is 6π. This implies that an orbit of length 5π is the complement of $\triangle x_0 x_1 x_2$. It follows that an orbit of length 5π must have vertices on l_0, l_1, l_2 as in the π case.

Now we consider a 3-period orbit of length 3π. Since it is impossible to create an orbit where $\tau_0 = \tau_1 = \tau_2 = \pi$, we look at the two other possible cases; when $0 < \tau_0 < \pi$, $\pi < \tau_1, \tau_2 < 2\pi$, and $0 < \tau_0, \tau_1 < \pi$, $\pi < \tau_2 < 2\pi$. In case 1, as shown in Figure 3, we know that $\triangle x_0 x_1 x_2'$ has perimeter π and that x_2 is antipodal to x_2'. This implies that x_2 lies on l_2. Note that case 2 is simply the complement of case 1. Therefore, we conclude that a 3-period orbit of length 3π also has vertices on l_0, l_1, and l_2. $\qquad\square$

The last proposition completely classifies the special cases when open sets of 3-period orbits occur. If a given 3-period orbit has perimeter $L \neq \pi, 3\pi, 5\pi$ then the relation (7) implies that this orbit has an empty interior in P_3. If $L = \pi, 3\pi$ or 5π but for some vertex x_i the geodesic curvature $k_g(s)$ does not vanish identically on any open boundary arc containing x_i, then (7) again leads to the same contradiction.

Finally applying the argument in [13], we obtain that if the special cases do not occur the set of 3-period orbits has zero measure. This ends the proof of Theorem 2.

5. Acknowledgements
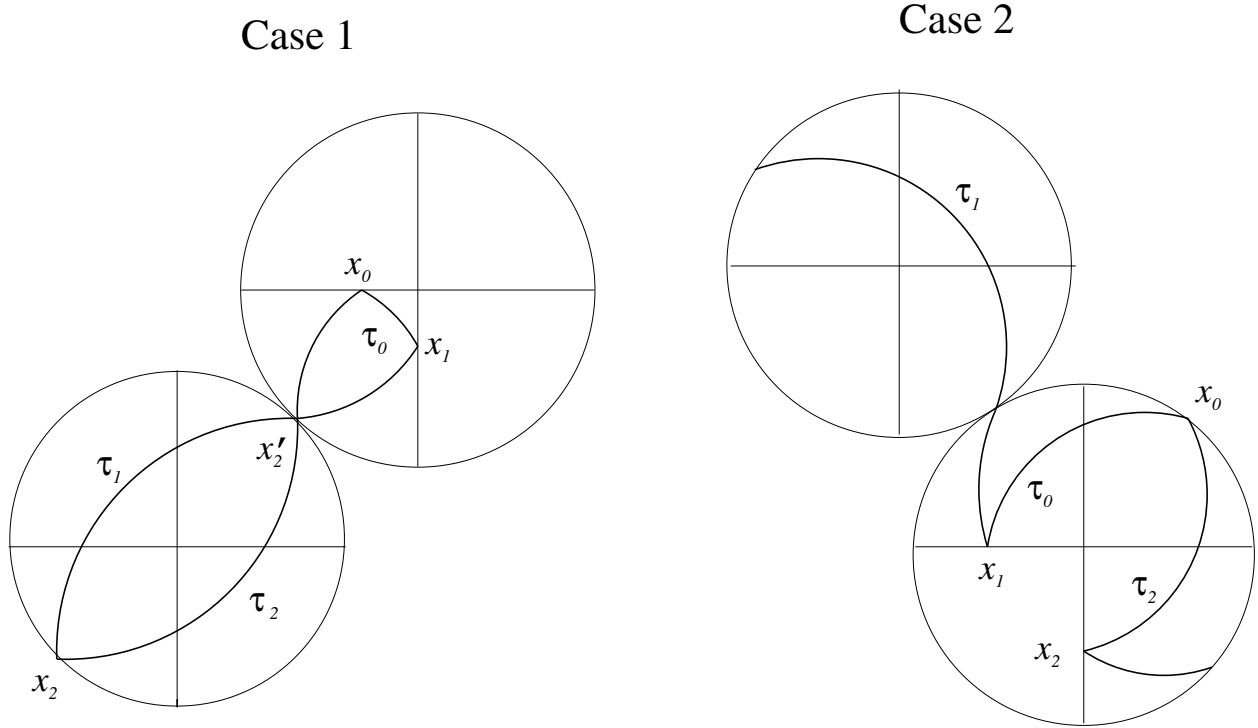
FIGURE 3. The special cases $L = 3\pi$ and $L = 5\pi$.

REFERENCES

[1] C. Bar, Elementary Differential Geometry, New York: Cambridge University Press, 2010.

[2] Y.M. Baryshnikov, Spherical billiards with periodic orbits, preprint.

[3] G.D. Birkhoff, Dynamical Systems, revised edition, Amer. Math. Soc. Colloq. Publ., vol. IX, Amer. Math. Soc., Providence, RI, 1966.

[4] M. do Carmo, Differential Geometry of Curves and Surfaces, PrenticeHall Inc., Englewood Cliffs, New Jersey (1976).

[5] D. Genin, S. Tabachnikov. On configuration space of plane polygons, sub-Riemannian geometry and periodic orbits of outer billiards. J. Modern Dynamics, 1 (2007), 155-173.

[6] A.A. Glutsyuk, Yu.G. Kudryashov, On Quadrilateral Orbits in Planar Billiards, Doklady Mathematics, 2011, Vol. 83, No. 3, pp. 371373, 2011.

[7] B. Gutkin, U. Smilansky, E. Gutkin, Hyperbolic billiards on surfaces of constant curvature, Comm. Math. Phys. 208 (1999), no. 1, 65-90.

[8] E. Gutkin, Billiard dynamics: a survey with the emphasis on open problems, Regul. Chaotic Dyn. 8 (2003), no. 1, 1-13.

[9] E. Gutkin, S. Tabachnikov, Complexity of piecewise convex transformations in two dimensions, with applications to polygonal billiards on surfaces of constant curvature, Mosc. Math. J. 6 (2006), no. 4, 673701, 772.

[10] A. Tumanov and V. Zharnitsky, Periodic orbits in outer billiard, International Mathematics Research Notices, vol. 2006, 1-17.

[11] M. R. Rychlik, Periodic points of the billiard ball map in a convex domain, Journal of Differential Geometry 30 (1989), no. 1, 191-205.

[12] L. Stojanov, Note on the periodic points of the billiard, Journal of Differential Geometry 34 (1991), no. 3, 835-837.

[13] M. P. Wojtkowski, Two applications of Jacobi fields to the billiard ball problem, Journal of Differential Geometry 40 (1994), no. 1, 155-164.

[14] V. Ya. Ivrii, The second term of the spectral asymptotics for a laplace-beltrami operator on manifolds with boundary, Func. Anal. Appl. 14 (2) (1980), 98106.

[15] Ya. B. Vorobets, On the measure of the set of periodic points of a billiard, Mathematical Notes 55(1994), no. 5, 455-460.

Department of Mathematics, University of Illinois, Urbana, IL 61801