The Dynamics of Propagation Fronts on Sets with a Negative Fractal Dimension

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Abstract. In sets with a fractal dimension greater than 1, the average number of neighbors increases with distance. For that reason spherical pulses propagate outward in systems with nearest neighbor interactions. In sets with a negative fractal dimension, such as the set of individual coordinates of a population of a small city, the average number of neighbors decreases with distance in a precise way relating the number of neighbor to the fractal dimension of the set. We study the propagation of diffusive pulses and waves on such sets. We find that on sets with negative fractal dimension, the velocity of pulse peak is negative (*i.e.* the median radius of circular pulses decreases as a function of time). Eventually the pulse broadens and disappears. We discuss applications in physical systems, such as the spreading of heat and sound, as well as applications in social systems, such as the spread of infectious diseases and the spread of rumors.

Keywords: Negative fractal dimension, diffusion, propagation front.

1 Introduction

Many interesting objects can be thought of as finite sets with a large number of elements. Social networks consist of a finite set of people. Solid objects, such as a copper wire, a snow crystal, or the branches of a tree are made of a finite number of molecules. The relative position of the elements with respect to their neighbors (or *distribution*) determines if the object is called one dimensional, two dimensional, or three dimensional. For instance, in a sheet of paper, the molecules are mostly located near a two dimensional plane. Therefore the sheet is called two dimensional. When the same molecules are arranged along a line, like in a thread, the resulting object is called one dimensional. And if the molecules agglomerate into a spherical lump, the object is called three dimensional.

There are several methods to quantify the dimension of a set. The Hausdorff dimension [1] and fractal dimension [2] use estimates of the volume of the set at various levels of course graining to determine the dimension of the set. The correlation dimension [3] uses a count of the number of neighboring elements in close proximity to determine the dimension of the set.

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The algorithm for computing the correlation dimension is much simpler than the algorithm for computing the fractal dimension, but the correlation dimension sometimes produces unintuitive results. For instance, the correlation dimension of a sheet of paper is 3, because at distances less than the thickness of the sheet, the number of neighbors with radius R scales like in a three dimensional object.

There is another issue with the correlation dimension: the surface problem. That is, for elements which are close to the surface of a D dimensional object with a "lump" geometry and with a dimension D > 1, such as a sphere or a circle, the object appears D-1 dimensional near the surface. If the elements are sorted in terms of their distance from the center of mass, the largest group is the group near the surface of the object. Therefore for the largest group of elements the object appears to be D-1 dimensional.

In this paper we introduce a definition of dimension which overcomes these problems.

2 Objects with a Negative Fractal Dimension

We consider objects which are made of N elements. The positions of the elements are x_i, where i=1,..,N. We compute the distance of the elements from the center of mass $x_c = (1/N)\, \Sigma_{i=1,\ldots,N}\, x_i$, and count the number of elements C(r), which are within a shell of radius r = k Δr and width Δr from x_c, where k=1,2,3,…, that is,

$$C(r)= \Sigma_{i=1,\ldots,N}\, H(|x_i-x_c|- k\, \Delta r) - H(|x_i-x_c|- k\, \Delta r - \Delta r), \tag{1}$$

where $r=|x_i-x_c|$ is the distance of the element from the center. H is the Heaviside step function. For an object with a common shape, such as a sphere or a square, C(r) is a power law within a certain range of r-values, where the exponent is one less than the intuitive dimension of the object. Therefore the fractal dimension is defined as

$$D = 1 + d\,(\ln C) / d(\ln r). \tag{2}$$

Figure 1 and Fig. 2 show a circular object and a star-shaped object along with their corresponding functions C(r) and D(r). Both objects consist of about 3200 elements. In the star shaped object, the elements are on a square grid with side length 1. In the star-shaped object, the position of the elements (x,y) satisfies the condition: $|x|<200$ and $|y|<200$ and ($|y|<=|x|$ and $|y| <= 250/|x|^{\alpha}$ or $|x|<=|y|$ and $|x| <= 250/|y|^{\alpha}$), where $\alpha > 0$.

The fitting function for the number of elements at distance r from the center for the circular object in Fig. 1a is: ln C = 1.0467 ln r + 1.6791, for 0 < r < 32. The continuous line in Fig. 1b is a graph of the fitting function. With Eq. 1 we obtain a numerical value for the dimension D = 2.05, for 0 < r < 32. Fig 1c is a graph of the numerical value of the fractal dimension D versus the radius and the theoretical value. Fig. 1d shows that the numerical value for the fractal dimension is in good agreement with the intuitive value.

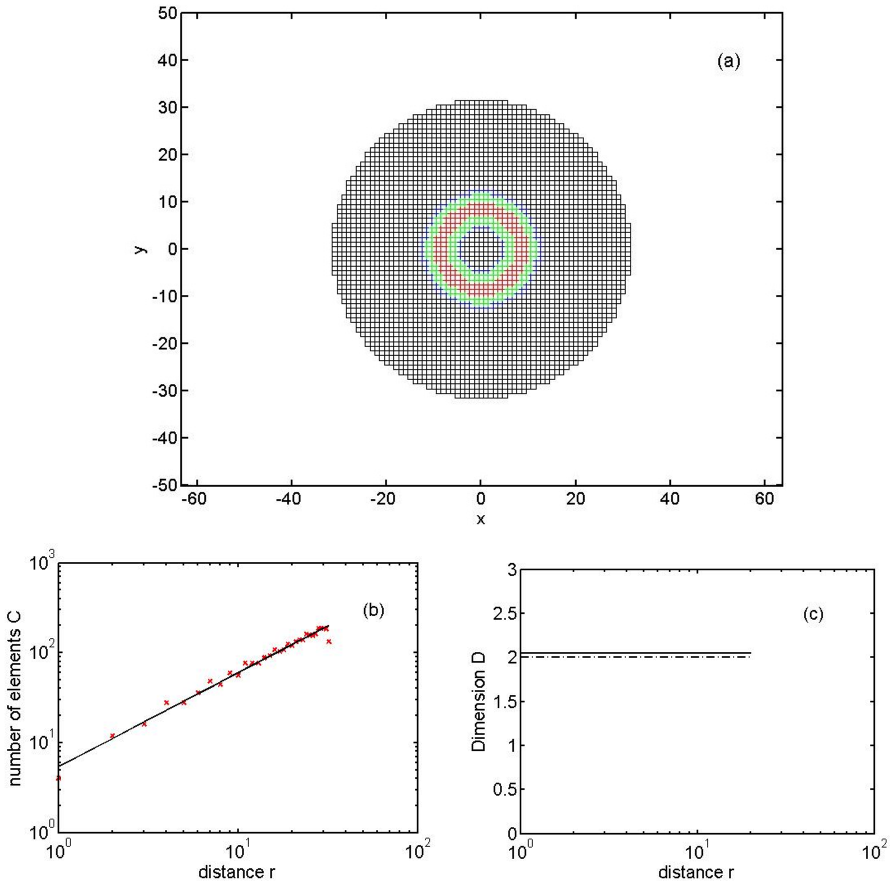

Fig. 1. Plot (a) shows a circular object. The colors indicate the amplitude of a spreading pulse (*red =very high, green=high, blue =low, black = very low*). The elements are located at the intersections and corners. The lines indicate connections between elements. Plot (b) shows a histogram of the numbers of elements versus their distance from the center of mass for bin size $\Delta r=1$. The lines are least-square fits. Plot (c) shows the dimension D versus the distance from the center. The dashed line indicates the theoretical value, D=2.

The fitting function for the number of elements at distance r from the center for the star-shaped object (Fig. 2a, Fig. 2b) is: $\ln C = 1.0643 \ln r + 1.6403$, for $0 < r < 19$ and $\ln C = -1.3339 \ln r + 8.1032$, for $18 < r < 100$. The continuous line in Fig. 2c is a graph of the fitting functions. With Eq. 1, we obtain a numerical value for the dimension $D = 2.06$, for $0 < r < 32$ and $D = -0.33$, for $18 < r < 100$. Fig 2d is a graph of the numerical value of the dimension D versus the radius and the theoretical value.

We find that the numerical values for the dimension are in good agreement with the intuitive value. The star is expected to be 2-dimensional near the center, whereas for distances that exceed the solid area in the center, the dimension of the object is expected to be $D = \alpha+1$, because the number of points inside the rays is approximately equal to the width of the rays $W= 500/|y|^{\alpha}$.

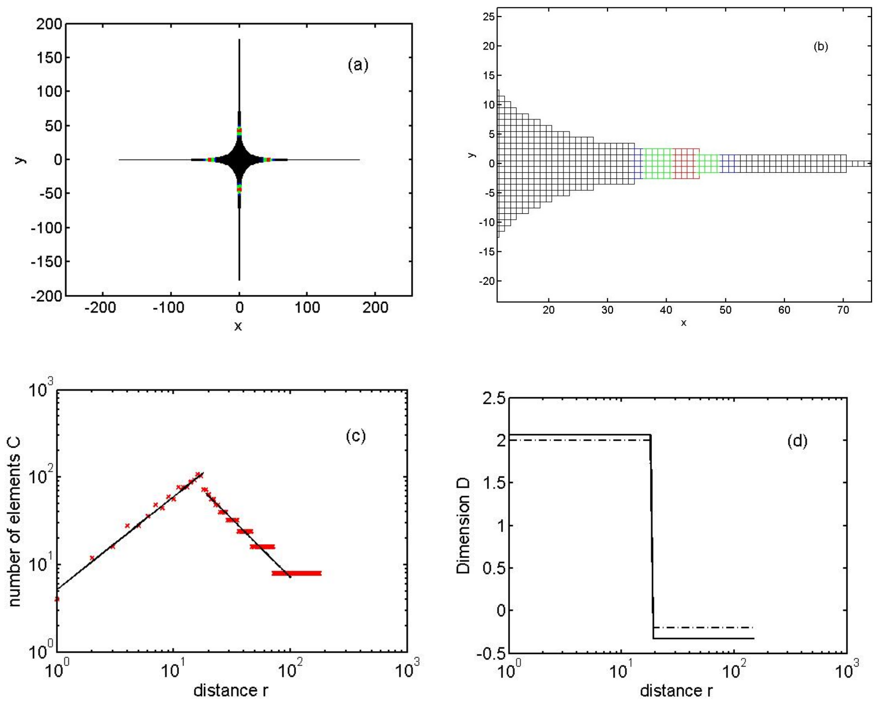

Fig. 2. Plot (a) shows a star-shaped object with α = 1.2. The colors indicate the amplitude of a spreading pulse (*red =very high, green=high, blue =low, black = very low*). Plot (b) shows an enlargement of Plot (a). The elements are located at the intersections and corners. The lines indicate connections between elements. Plot (c) shows is a histogram of the numbers of elements versus their distance from the center of mass for bin size $\Delta r = 1$. The lines are least-square fits. Plot (d) shows the dimension D versus the distance. The dashed line indicates the theoretical value, D=2 for small distances and $D= -\alpha +1 = -0.2$ for large distances from the center of mass.

3 Diffusion on Objects with a Negative Fractal Dimension

We assume that each element stores $q_i(n)$ particles at time step n, where n=1,2,3,...,T. The initial particle distribution is a square wave between R_1 and R_2, *i.e.* if $R_0 < r < R_1$ then $q_i(0) = 1$, where $r^2=x_i^2+y_i^2$, otherwise $q_i(0) = 0$. The elements and the connections form a grid.

We assume that the particles do a random walk on this grid. The dynamics of the density of random walkers is asymptotically equivalent to a diffusion process. We model the diffusion process with a relaxation dynamics

$$q_i(n+1) = q_i(n) + \lambda \Sigma_j (q_j(n) - q_i(n)), \tag{3}$$

where λ=0.1 and where j represents indexes of all neighbors, *i.e.* all elements which are connected to element i. The colors in Figure 1a and Figure 1b represent the density of particles at time step n=10, where R_0=7 and R_1=10. Figure 2a and Figure 2b represent the density of particles at time step n=50, where R_0=40 and R_1=45.

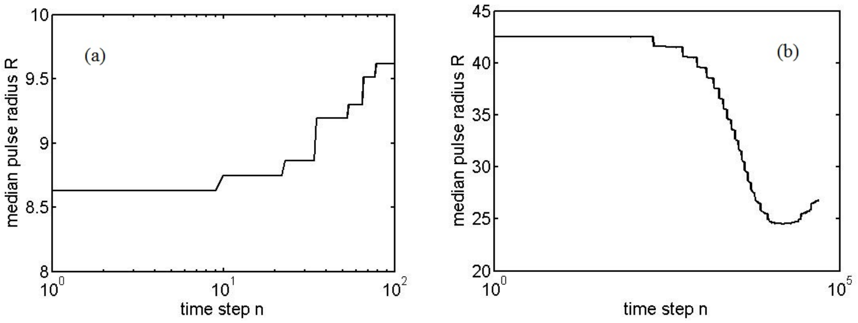

Fig. 3. The median distance from the center of diffusion particles versus time for an object with dimension D=2 (a), and an object with dimension D= -0.33 (b)

We find that the particles initially have a tendency to migrate towards the center in negative dimensional objects because there are more connected elements towards the center. On objects with a dimension greater than D=1, the particles tend to migrate away from the center because there are more connections to neighbors which are further away from the center. The median distance of the particles from the center is defined by the following inequality

$$\Sigma_{i=1,2,3,..,N}\ q_i\ H(R^2-x_i^2-y_i^2) < Q/2 \tag{4}$$

where Q= $\Sigma_{i=1,2,3,..,N}\ q_i$. Fig. 3 shows the dynamics of R.

For circular objects, R increases (Fig 3a), whereas for the negative dimensional object R initially decreases (Fig. 3b).

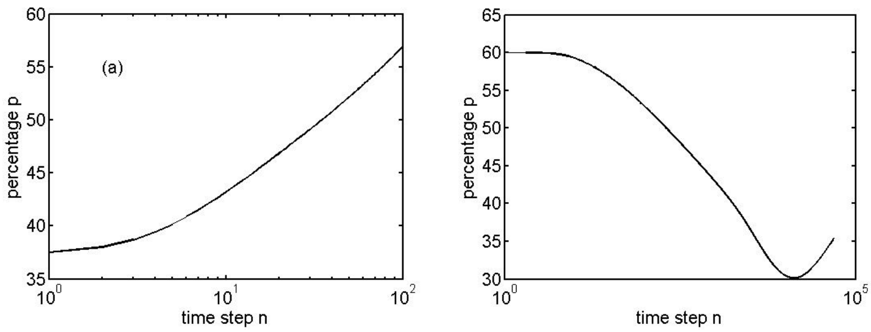

Fig. 4. The percentage of diffusion particles which are further away than the center of the initial square wave distribution versus time for an object with dimension D=2 (left) and an object with dimension D= -0.33 (right)

The quantity p tracks the percentage of particles which is further away than the center of the initial square wave

$$p = (1/Q)\ \Sigma_{i=1,2,3,...,N}\ H(x_i^2+y_i^2 - R_w^2), \tag{5}$$

where $R_w^2 = (R_1 - R_2)^2/4$.. R_w is the center of the initial square wave distribution. Figure 4 shows the dynamics of p for an object with dimension 2 and object with dimension D=-0.33. On the D=2 object, the number of p increases whereas on the negative dimensional object p initially decreases.

Another quantity which illustrates these tendencies to migrate towards or away from the center is the mean square deviation of the particles from the center

$$\langle r^2\rangle = (1/N)\ \Sigma_{i=1,2,3,...,N}\ q_i\ (x_i^2+y_i^2). \tag{6}$$

Figure 5 shows the dynamics of $\langle r^2\rangle$ for an object with dimension 2 and object with dimension D=-0.33. On the D=2 object the number of $\langle r^2\rangle$ increases whereas on the negative dimensional object $\langle r^2\rangle$ initially decreases.

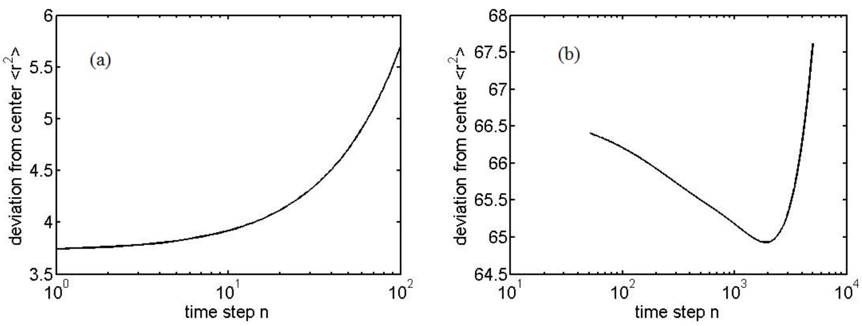

Fig. 5. The mean square deviation $\langle r^2\rangle$ of diffusing particles from the center of the object versus time for a circular object (a) and for a star-shaped object with dimension D= -.33

4 Discussion

Diffusion on a two dimensional grid differs from diffusion on a three dimensional grid, because each node on a two dimensional grid has 4 neighbors and each node on a three dimensional grid has 6 neighbors. Similarly, diffusion particles appear to migrate away from the center of two and three dimensional objects because most neighboring elements are further away from the center than the element itself. This is different for objects with a negative dimension, since most neighboring elements are closer to the center than the element itself. Fig. 3 and Fig. 4 illustrate that in contrast to objects with dimension greater than 1, on negative dimensional objects, diffusing particles appear to migrate towards the center because the median distance from the center decreases due to an increasing number of particles is closer to the center than initially and because the mean square deviation of the particles from the center decreases.

Diffusing particles are equally likely to move to one of the neighboring elements. Many other systems behave in the same way. A sound wave spreads equally to all neighboring locations inside an object if the object is homogeneous. Similarly, heat spreads to all neighboring locations equally in solids, liquids and gases and charges in a resistor/capacitor network spread to all neighboring capacitors equally. Therefore we expect that acoustic pulses, heat pulses and charges spread like the pulse in Fig. 1 and Fig 2. We expect that in objects with a dimension greater than D=1, sound pulses, heat pulses, and charges distributions with radial symmetry tend to move outward, whereas in negative dimensional objects, these pulses move toward the center of the object.

It is conceivable that social systems can have a negative dimension. For example, in most cities, the density of people is highest in the center and decreases gradually as a function of the distance from the city center. If the density ρ decreases like a power law with an exponent less than -1 as a function of the distance from the city, *i.e.* $\rho = a\ r^{\alpha}$, where $\alpha < -1$, then the set of people in the city is an object with a negative dimension. If we assume that a rumor or infection starts at a certain distance from the city center and travels quickly from person to person, then the rumor or infection travels mostly towards the city center, *i.e.* more people close to the city center know about the rumor or are infected than those people further away.

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