Point-Pushing Homeomorphisms on a Genus-*g* Surface

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Point-Pushing Homeomorphisms

- [Definition]: Each closed curve γ with a base point *pt* on a surface S determines a natural homeomorphism $\mathcal{P}(\gamma)$ of the surface $S \setminus \{pt\}$ via **point-pushing**:

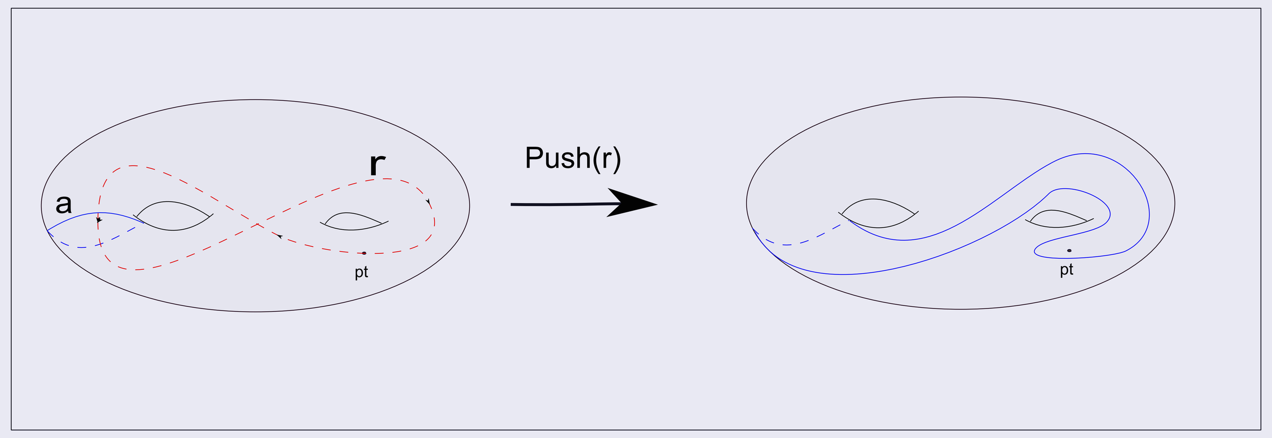

Figure: **Point-push** around γ by dragging basepoint *pt* once around γ.

- [Theorem]: Every point-pushing homeomorphism on a genus-g surface with a base point $p \in S$ can be written as a composition of *generating* point-pushing homeomorphisms and their inverses.

Self-Intersection Numbers

- [Definition] Let γ be a closed curve on a surface S and $[\gamma]$ be the homotopy class of γ. The self-intersection number of γ is the minimum of the set of all self-intersection numbers of curves in $[\gamma]$. i.e., the minimum of the set of all self-intersection numbers of the curves that are homotopic to γ. (For example, the self-intersection number of γ in the figure above will be at most 1.)

Brief Description of Our Project

Goal

The goal of this project is to study point-pushing homeomorphisms on the genus-g surface, and to relate their topological entropy (mesured by Dilatation) to their combinatorial complexity (mesured by Self-Intersection Number).

Procedure (Algorithm for our computer program)

1. Randomly pick a closed curve γ with a base point *pt* on a Genus-g surface. (We choosed g to be 2.) And let $\mathcal{P}(\gamma)$ be the associated point-pushing homeomorphism around γ.
2. Express $\mathcal{P}(\gamma)$ as a composition of generating dehn-twists in order to calculate its topological entropy.
3. Calculate the Self-Intersection Number of γ and the Dilatation of $\mathcal{P}(\gamma)$.
4. Repeat this process, enough times, so that we can get enough data.
5. Analyze the distribution of the Self-Intersection Numbers of randomly selected closed curves.
6. Analyze the distribution of the Dilataion of randomly selected point-pushing homeomorphisms.
8. Analyze the correlation between the Self-Intersection Numbers and Dilatation.

Distribution of Dilatations and Self-Intersection Numbers in the Genus-2 surface

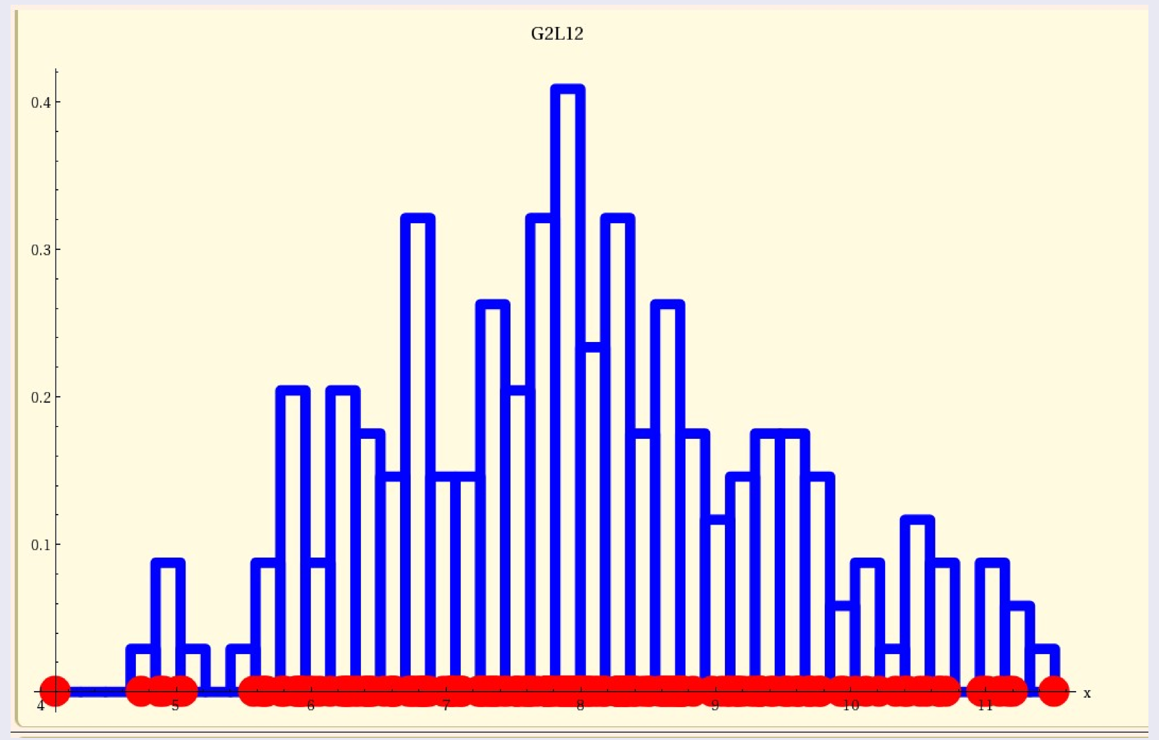

Figure: Distribution of Topological Entropies of randomly selected point-pushing homeomorphism on genus-2 surface.

Correlation between Dilatations and Self-Intersection Numbers in the Genus-2 surface, and Analysis

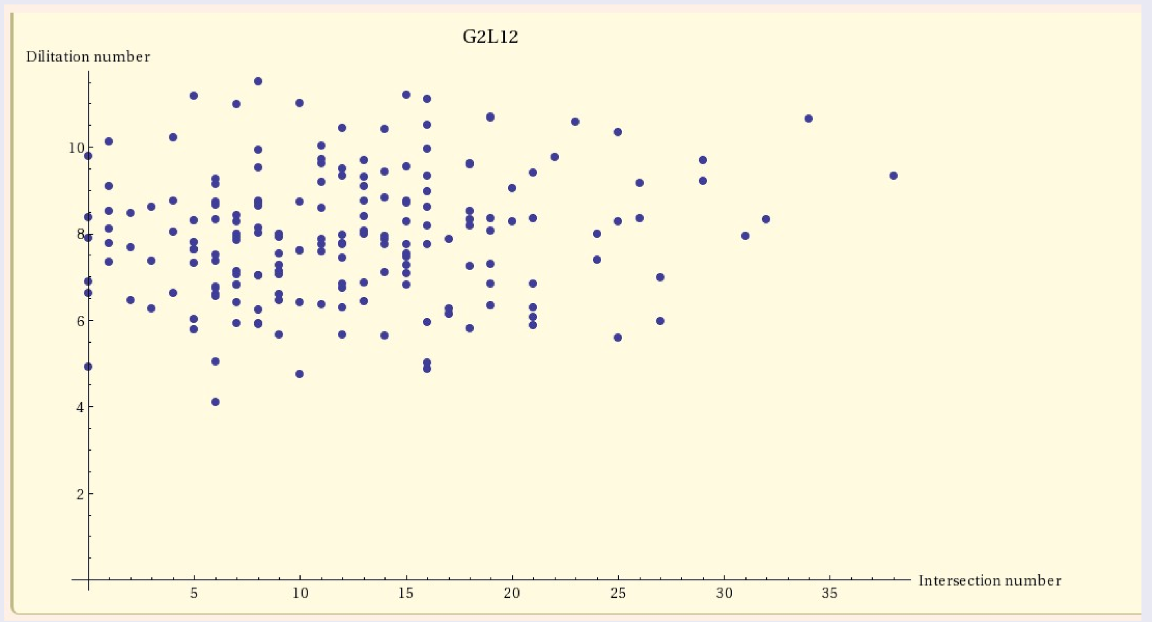

Figure: Scatter plot of Self-Intersection Number vs Topological Entropies of randomly selected point-pushing homeomorphism on genus-2 surface.

Dehn-Twists

- [Definition]: A dehn-twsit about a closed curve γ on a surface S is a self-homeomorphism on S that can be obtained by twsiting 2π about the given curve γ.
- [Theorem]: Every homeomorphism on a genus-g surface can be written as a composition of *generating* dehn-twists (about the following generating curves in the picture below) and their inverses.

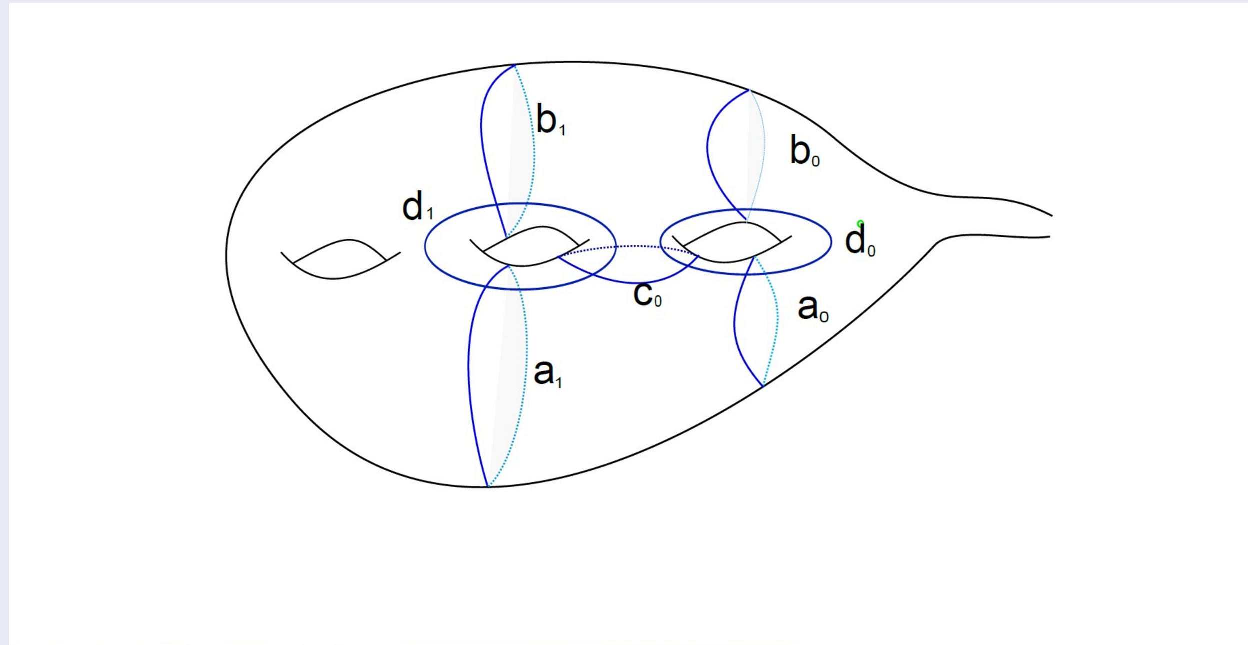

Figure: Example of Generating Cuves on a genus-g surface.

- [Consequence] Every point-pushing homeomorphism on a surface can be written as a product of some "generating" point-pushing homeomorphisms, which then, can be written as a product of some "generating" Dehn-Twists.

Topological Entropy

- [Definition] Topological entropy is a nonnegative number which measures the complexity of a topological dynamical system. In this project, the topological dynamical system will be a surface with a point-pushing homeomorphism on that surface. Moreover, we measured the topological entropy of a point-pushing homeomorphism on a surface with the dilatation number of the given homeomorphism, so that the topological entropy of the given homeomorphism would be the logorithm of the dilatation.

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