Introduction to Quantum Field Theory

Notes re-written from lessons' attendance, 2022
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Corso di Laurea Magistrale in Fisica - Sapienza

Chapter 1

Settembre e Ottobre

Appunti delle lezioni del Prof. Polosa relative al mese di settembre e ottobre 2022.

1.1 22 settembre

1.2 23 settembre

1.3 27 settembre

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The rod Given a 1-dimensional rod composed by N-particles, linked each others with a "spring", the hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\sum_{n=1}^{N}\Big[P_n^2 + \Omega^2(q_n - q_{n+1})^2 + \Omega_0^2 q_n^2\Big]$$

where the last term $\Omega_0^2 q_n^2$ is relative to the equilibrium position of the n-particle. The *periodic boundaries conditions* to $N \to \infty$ and $a \to 0$.

On the other side we can write the Newtonian equation as

$$\begin{aligned} H &= \frac{1}{2}\int_0^L dx\Big[p^2(x) + v^2\Big(\frac{\partial q(x)}{\partial x}\Big)\Big] \\ p(x) &= \dot{q}(x) \\ \ddot{q}(x) &= v^2\frac{\partial^2 q(x)}{\partial x^2} \end{aligned}$$

the solution inside the boundaries is

$$\ddot{q}_n = \Omega^2\Big(q_{n+1} + q_{n-1} - 2q_n\Big)$$

Normal modes or normal frequencies

$$\begin{aligned} q_n &= \sum_j e^{ijn}\frac{Q_j}{\sqrt{N}} \\ q(x) &= \frac{1}{\sqrt{a}}\sum_n e^{\frac{2\pi l}{Na}(na)}\frac{Q_j}{\sqrt{N}} = \frac{1}{\sqrt{a}}\sum_k e^{ikx}\frac{Q_k}{\sqrt{N}} \\ k &= \frac{2\pi l}{L} \\ \Rightarrow \quad q(x) &= \sum_k e^{ikx}\frac{Q_k}{\sqrt{Na}} = \sum_k e^{ikx}\frac{Q_k}{\sqrt{L}} \end{aligned}$$

Considering now the Newtonian equation, $p^2(x) = \dot{q}^2(x)$, $\sum_{n=1}^{N} e^{in(j-j')} = \delta_{j,j'}$ where $j = \frac{2\pi l}{N}$, we can move from the sum to the integral using the following relation $\sum_{n=1}^{N} \to \frac{1}{a}\int_0^L dx$ and this leads to $\int_0^L dx e^{i(k-k')x} = L\delta_{k,k'}$.

Somehow we may land on this following expression:

$$\frac{1}{L}\sum_{k,k'} L\,\delta_{k,k'}Q_k\dot{Q}_{k'} = \sum_k Q_k\dot{Q}_k = \sum_k |\dot{Q}_k|^2$$

To finally get a *total classical description*: a discrete sum on a numerable set, as follow

$$H = \frac{1}{2}\sum_k |\dot{Q}_k|^2 + k^2v^2|Q_k|^2$$

As before, notice that the sum $\sum_{n=1}^{N}$ for $L \to \infty$ became $\frac{L}{2\pi}\int dk$ and it admits waves. Extending this to 3-dimensional space, it became

$$\sum_{\vec{k}}(\dots) \quad (\text{when } L \to \infty) \quad \frac{V}{(2\pi)^3}\int d^3k$$

Quantum system: let's consider now a quantum system, a quantum description.
Postulate the followings:

$$\begin{aligned} \Big[q_l, p_n\Big] &= i\,\delta_{l\,n} & \Big[Q_l, P_n\Big] &= i\,\delta_{ln} \\ \Big[q_l, q_n\Big] &= 0 & \Big[Q_l, Q_n\Big] &= 0 \\ \Big[p_l, p_n\Big] &= 0 & \Big[P_l, P_n\Big] &= 0 \end{aligned}$$

Where natural units are applied:

$h = 1$

$c = 1$

$$\Rightarrow \qquad q_n^\dagger = q_n \quad , \quad Q_{-j} = Q_j^\dagger \quad , \quad P_{-j} = P_j^\dagger$$

$$\text{e.g.} \quad q_n^\dagger = \Big(\sum_n e^{inj}\frac{Q_j}{\sqrt{N}}\Big)^\dagger = \sum_j e^{-inj}\frac{Q_j^\dagger}{\sqrt{N}} = q_n$$

From the hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_j \Big[P_j\, P_j^\dagger + \omega_j^2\, Q_j\, Q_j^\dagger\Big]$$

and given the following operators, we find Q_j and P_j:

$$a_j = \frac{1}{\sqrt{2\omega_j}}\Big(\omega_j Q_j + i\, P_j^\dagger\Big) \qquad\qquad Q_j = \frac{1}{\sqrt{2\omega_j}}\Big(a_j + a_{-j}^\dagger\Big)$$

$$\Rightarrow \qquad\qquad\qquad\qquad \text{and} \qquad \text{keep in mind}$$

$$a_j = \frac{1}{\sqrt{2\omega_j}}\Big(\omega_j Q_j^\dagger - i\, P_j\Big) \qquad\qquad P_j = -i\Big(\frac{\omega_j}{2}\Big)^{\frac{1}{2}}\Big(a_{-j} - a_j^\dagger\Big) \qquad\qquad \Big[a_j, a_{j'}\Big] = \delta_{j\,j'}$$

$$Q_j Q_j^\dagger = \frac{1}{2\omega_j}\Big(a_j a_j^\dagger + a_j a_{-j} + a_{-j}^\dagger a_j^\dagger + a_{-j}^\dagger a_{-j}\Big)$$

$$P_j P_j^\dagger = \Big(\frac{\omega_j}{2}\Big)^{\frac{1}{4}}\Big(a_{-j} a_{-j}^\dagger - a_{-j} a_j - a_j^\dagger a_{-j}^\dagger + a_j^\dagger a_j\Big)$$

With these lasts results we may write the $\mathcal{H}$ as

$$\mathcal{H} = \frac{1}{2}\sum_j \Big[P_j\, P_j^\dagger + \omega_j^2\, Q_j\, Q_j^\dagger\Big] = \frac{1}{2}\sum_j \omega_j\Big(a_j a_j^\dagger + a_j^\dagger a_j\Big)$$

$$= \frac{1}{2}\sum_j \omega_j\Big(2\, a_j^\dagger a_j + 1\Big) = \sum_j \omega_j\Big(a_j^\dagger a_j + \frac{1}{2}\Big)$$

Phonons description Phonons are bosons, they're used to describe the quantum problem of the rod. Phonons are like photons but in the world of sound instead of light. A n-particles system is defined with

$$|n_1, n_2, n_3, \ldots\rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3}\ldots|0\rangle$$

and for the 1-d oscillator, with energy E_n, is as follows

$$|n\rangle = (a^\dagger)^n\,|0\rangle$$

$$E_n = \hbar\omega(n + \frac{1}{2}) \overset{nu}{=} \omega(n + \frac{1}{2})$$

For the phonons is easy to *understand* which is the medium that make the transmission possible, but what about light? For the light, photons, the medium may also be the *vacuum*.

Filosofeggiamo un po' ora:

Particles are the excitation of the field

If you don't touch the piano it stays quiet, but if you play it it makes music ... song's particles.

The field is permanent.

Particles are not fixed, they live and die.

You cannot touch or see the field that you're studying, but you can see/detect the particle that pop out from the field.

Fields are NOT real but mathematical description of the world.

When you measure an energy it's always relative to an offset, a ground-state. Because you want the *vacuum* to be Lorentz invariant.

$$\Rightarrow \qquad \Big(\mathcal{H} - E_0\Big)\,|0\rangle = 0$$

Given a general operator $\Theta(t)$ and its derivate $\dot{\Theta} = i\Big[\mathcal{H}, \Theta(t)\Big]$ so that:

$$\dot{a}(t) = i\Big[\mathcal{H}, a(t)\Big] = -i\,\omega a(t) \qquad \text{where} \quad \begin{aligned}\Big[a, \mathcal{H}\Big] &= \omega a \\ \Big[\mathcal{H}, a^\dagger\Big] &= \omega a^\dagger\end{aligned} \qquad \Rightarrow \quad a(t) = e^{-i\omega t}a(0)$$

(Finire lezione del 4 ottobre manca mezza pagina di esercizio - chiedere appunti)

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$$\begin{aligned}\phi(x) &= \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k V}}\Big(a_{\vec{k}} e^{ikx} + a^\dagger_{\vec{k}} e^{-ikx}\Big) \\ &= \sum_{\vec{k}} \frac{i}{2V}\Big[e^{i\,\vec{k}(\vec{x}\cdot\vec{y})} + e^{-i\,\vec{k}(\vec{x}\cdot\vec{y})}\Big]\end{aligned} \qquad \text{considering} \quad \begin{aligned}&\Big[a_k, a^\dagger_{\vec{k}}\Big] = \delta_{\vec{k},\vec{k}'} \\ &\Big[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)\Big] = i\,V\delta^3(\vec{x}-\vec{y})\end{aligned}$$

Da capire che senso ha
e contestualizzarlo $\Rightarrow$

$$\begin{aligned}&k_\mu = (\vec{k}, i\omega_{\vec{k}}) \\ &x_\mu = (\vec{x}, it) \\ &k_\mu x_\mu = \vec{k}\cdot\vec{x} = k_\mu k_\nu \delta_{\mu\,\nu} \\ &k_\mu x_\mu = \vec{k}\cdot\vec{x} - \omega_{\vec{k}} t\end{aligned}$$

When things go to infinity $\sum_{\vec{k}} \to \frac{V}{(2\pi)^3}\int d^3\vec{k}$ and remember that "*if things doesn't work there will be some volume* V *somewhere*". Creation and destruction operators are contained into the description of the field. The energy levels' order are given from the term $n\omega$ and you can forgot about the $\frac{1}{2}$.

Classical problem Given the coordinates $q_i(t)$ time-dependents, where $i = 1, 2, 3, \dots, 3N$, we can write the system of the 2^o order derivate as follow

$$\begin{aligned}&F_i = m\ddot{q}_i \\ &F_i = -\frac{dV}{dq_i}\end{aligned} \quad \text{given the initial conditions} \quad \begin{aligned}&q_i(t_0) \\ &\dot{q}_i(t_0)\end{aligned} \quad \text{or given the boundary conditions} \quad \begin{aligned}&q_i(t_1)\,,\, q_i(t_2) \\ &\dot{q}_i(t_1)\,,\, \dot{q}_i(t_2)\end{aligned}$$

Action Functional The *Action Functional* S

$$S = \int_{t_1}^{t_2} dt\, L\Big(q_i(t), \dot{q}_i(t)\Big)$$

is defined such that a variation on the trajectory leads to a variation on S. So we can make a variables' transformation such that the new coordinates are the same as before plus a variational term

$$\begin{aligned}&q_i(t) \to q_i(t) + \delta q_i(t) \\ &\dot{q}_i(t) \to \dot{q}_i(t) + \delta\dot{q}_i(t) = \dot{q}_i(t) + \frac{d}{dt}\delta q_i(t)\end{aligned}$$

hence the action became

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} L\Big(q_i(t) + \delta q_i(t), \dot{q}_i(t) + \frac{d}{dt}\delta q_i(t)\Big)dt - \int_{t_1}^{t_2} dt\, L\Big(q_i(t), \dot{q}_i(t)\Big)dt = \\ &= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_i}\delta q_i(t), \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i(t)\Big)dt = \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\Big)\delta q_i\, dt\end{aligned}$$

in the last step we used the boundary condition at t_1 and t_2, so that $\delta q_i(t_1) = \delta q_i(t_2) = 0$. The last step leads directly to the *lagrangian equation*, that referred to the following generic L in 3-D is:

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) = \frac{\partial L}{\partial q_i} \\ &L = \frac{1}{2}m\dot{\vec{q}}^2 - V(\vec{q})\end{aligned}$$

this leads to the Newton equation for the *free motion*:

$$\begin{aligned}&F = 0 \\ &m\ddot{q}_i = -\frac{\partial V}{\partial q_i} = F\end{aligned} \quad \Rightarrow \quad \begin{aligned}&\ddot{\vec{q}} = 0 \\ &\dot{\vec{q}} = \vec{w} \\ &\vec{q} = \vec{w}t + \vec{r}\end{aligned} \quad \Rightarrow \quad \begin{aligned}&\vec{q}_1 = \vec{w}t_1 + \vec{r} \\ &\vec{q}_2 = \vec{w}t_2 + \vec{r}\end{aligned}$$

so now we can find the Lagrangian depending on q_1, q_2 and t_1, t_2

$$\begin{aligned}&\vec{w} = \frac{\vec{q}_1\cdot\vec{q}_2}{t_1 - t_2} \\ &\vec{q} = \frac{\vec{q}_1\cdot\vec{q}_2}{t_1 - t_2}t + \vec{r}\end{aligned} \quad \Rightarrow \quad \begin{aligned}&(t = t_1):\ (t_1 - t_2)\vec{q}_1 = (\vec{q}_1\cdot\vec{q}_2)t_1 + \vec{r}(t_1 - t_2) \\ &(t = t_2):\ (t_1 - t_2)\vec{q}_2 = (\vec{q}_1\cdot\vec{q}_2)t_2 + \vec{r}(t_1 - t_2)\end{aligned} \quad \Rightarrow \quad \begin{aligned}&\vec{q} = \Big(\frac{\vec{q}_1\cdot\vec{q}_2}{t_1 - t_2}\Big)t + \frac{\vec{q}_2 t_1 - \vec{q}_1 t_2}{t_1 - t_2} \\ &\dot{\vec{q}} = \frac{\vec{q}_1\cdot\vec{q}_2}{t_1 - t_2}\end{aligned}$$

where the last two equations explicit the boundary conditions. The Lagrangian of a free motion became

$$L = \frac{1}{2}m\dot{\vec{q}}^2 = \frac{1}{2}m\Big(\frac{\vec{q}_1 \cdot \vec{q}_2}{t_1 - t_2}\Big)^2$$

And the minimal Action is written as follow and represents "the true trajectory"

$$\int_{t_1}^{t_2} dtL = \frac{1}{2}m\Big(\frac{\vec{q}_1 \cdot \vec{q}_2}{t_1 - t_2}\Big)^2 (t_1 - t_2) = S_{min}$$

"*The real motion is given by the minimum action.*"

Hamiltonian equation What happen with the generic lagrangian instead substituting the free motion path? Starting from the lagrangian equation:

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} &= 0 \qquad \Big[\text{where} \quad \dot{q}_i = p_i\Big] \\
\dot{q}_i\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \dot{q}_i\frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt}\Big(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\Big) \underbrace{- \dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - \dot{q}_i\frac{\partial L}{\partial q_i}} &= 0 \qquad\qquad \Rightarrow \quad \frac{d}{dt}\Big(\dot{q}_i p_i - L(q_i, \dot{q}_i)\Big) = 0 \\
\frac{d}{dt}\Big(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{d}{dt}\Big(L(q_i, \dot{q}_i)\Big) &= 0
\end{aligned}$$

Given the lagrangian as $L(q_i, \dot{q}_i)$ let's check its *invariance* when the coordinates change: the lagrangian must not change. For small ε

$$\begin{aligned}
q_i &\to q_i + \varepsilon_i f_i(q_1, \ldots, q_i, \ldots, q_N) \\
\dot{q}_i &\to \dot{q}_i + \varepsilon_i \frac{d}{dt} f_i(q_1, \ldots, q_i, \ldots, q_N)
\end{aligned}$$

We can write the variation of the lagrangian as

$$\begin{aligned}
&\sum \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = 0 \\
&\sum \underbrace{\frac{\partial L}{\partial q_i}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} f_i + \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{\tau_i} \frac{d}{dt} f_i = 0 \qquad \longrightarrow \qquad \frac{d}{dt}\sum_i p_i f_i = 0
\end{aligned}$$

Since the lagrangian is *invariant* for transformation, as we just saw, this leads to a *conservation* $\leftrightarrow$ *symmetry.* This concept is also known as *Noether theorem.*

Now ...something here not so clear...
about changing formalism from Classical Mechanic to Quantum Mechanic.

Hamiltonian and lagrangian density Since the beginning of the course we introduce, without saying, and use the hamiltonian density $\mathcal{H}$ instead of the hamiltonian H, that is defined as:

$$H = \int_0^L dx\,\mathcal{H}$$

as well as the lagrangian density $\mathcal{L}$ instead of the lagrangian L:

$$L = \int_0^L dx\,\mathcal{L} \qquad\qquad (\text{ ! controllare se è corretto ! })$$

Back to the *rod* using now in hamiltonian formalism

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}v^2(\partial_x\phi) = \frac{1}{2}p^2 + \frac{1}{2}v^2(\partial_x\phi) \\
\mathcal{L} &= p\,\dot{q} - \mathcal{H} = \dot{\phi}\dot{\phi} - \Big(\frac{1}{2}\phi^2 + \frac{1}{2}v^2(\partial_x\phi)^2\Big) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}v^2(\partial_x\phi)^2
\end{aligned} \qquad \text{where} \quad p = \dot{\phi}$$

when the velocity is set as $v = \Omega a$ it depends strictly on the material of the medium. So when moving in the vacuum, and not along a rod, what happens? The velocity will be set to $v = c = 1$, as natural units are used, and "forget the mass" such that $m = 1$. The lagrangian density became consistent with this

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}v^2(\partial_x\phi)^2 \tag{1.1}$$

The definition of *field* goes with the definition of a proper *energy density.* The action became:

$$S = \int_{t_1}^{t_2} dt\, L = \int_{t_1}^{t_2} dt \int_0^L dx\, \mathcal{L}$$

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Continuing from the lagrangian **??**

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}v^2(\partial_x\phi)^2 = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}v^2(\partial_x\phi)^2$$

we see the lagrangian as function of these variables $\mathcal{L} = \mathcal{L}(\phi,\, \partial_t\phi,\, \partial_x\phi)$ and we write the variation and the variational principle as follow

$$S = \int \mathrm{d}t \int \mathrm{d}x\, \mathcal{L} = \int \mathrm{d}t L \qquad\qquad \delta S = 0$$

$$\begin{aligned}
\delta S &= \int \mathrm{d}t \int \mathrm{d}x\, \mathcal{L}(\phi + \delta\phi\,, \dot{\phi} + \frac{\partial}{\partial t}\,\delta\phi\,, \partial_x\phi + \partial_x\,\delta\phi) - \int \mathrm{d}t \int \mathrm{d}x\, \mathcal{L}(\phi,\, \dot{\phi},\, \partial_x\phi) \\
&= \int \mathrm{d}t \int \mathrm{d}x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\,\delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial_t\,\phi)}\partial_t\,\delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial_x\,\phi)}\partial_x\,\delta\phi \right\} \\
&= \int \mathrm{d}t \int \mathrm{d}x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t\,\phi)} - \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x\,\phi)} \right\} \delta\phi
\end{aligned}$$

Hence we may set some boundary conditions at times t_1 and t_2, for the rod problem

$$\begin{aligned}
\delta\phi\,(\vec{x}, t_1) &= 0 \\
\delta\phi\,(\vec{x}, t_2) &= 0
\end{aligned}$$

Notice that the partial derivative of $\mathcal{L}$ with respect to ϕ is the following, and trough which we find the *Euler-Lagrange equation* in *Quantum Field Theory*

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t\phi)} + \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x\phi)} + \partial_y \frac{\partial \mathcal{L}}{\partial(\partial_y\phi)} + \partial_z \frac{\partial \mathcal{L}}{\partial(\partial_z\phi)} \qquad \Rightarrow \qquad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \\
\text{with} \quad \partial_\mu &= (\partial_1, \partial_2, \partial_3, \partial_4) = (\partial_t, \partial_x, \partial_y, \partial_z)
\end{aligned}$$

The lagrangian density may be generalized in 3-D:

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - \frac{1}{2}(\partial_y\phi)^2 - \frac{1}{2}(\partial_z\phi)^2 \\
=& \frac{1}{2}(\partial_\mu\phi)^2
\end{aligned} \qquad \text{where} \quad (\partial_\mu\phi)^2 = \partial_\mu\phi\,\partial_\mu\phi$$

This shorter notation implicitly implement the relativity, with $v = 1$ and natural units. If you want to describe a physical system, like the rod, you must instead keep it explicit with $v \neq 0$ and the Lagrangian will need one term more, such that

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$$

Solve the motion equation starting from the lagrangian above, we derive it w.r.t. ϕ and w.r.t. $\partial_\mu\phi$

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial\phi} & = -m^2\phi \\ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} & = -(\partial_\mu\phi) \end{cases}$$

Using the *laplacian* we find

$$\begin{aligned}
\partial_\mu\,\partial_\mu &= \partial_1\,\partial_1 + \partial_2\,\partial_2 + \partial_3\,\partial_3 - \partial_t\,\partial_t \\
&= \nabla_1\,\nabla_1 + \nabla_2\,\nabla_2 + \nabla_3\,\nabla_3 - \nabla_t\,\nabla_t \\
&= \nabla^2 - \partial_t^2 \; = \Box
\end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \partial_\mu(-\partial_\mu\phi) &= -m^2\phi \\ -\Box\,\phi &= -m^2\phi \end{aligned} \qquad \Rightarrow \qquad (\Box - m^2)\,\phi = 0$$

obtaining the *Klein-Gordon equation.*

$$\phi = e^{i\,(\vec{p}\cdot\vec{x} - \phi_0\,t)} = e^{i\,P_\mu x_\mu} \quad \Rightarrow \quad \Big(-\vec{P}^2 + P_0^2 - m^2 \Big) = 0 \quad \Rightarrow \quad P_0^2 = \vec{P}^2 + m^2$$

that describes the relativistic motion of a particle of mass m. Then the *free particle* may have a positive or negative energy value P_0

$$P_0 = \pm\sqrt{\vec{P}^2 + m^2}$$

Time evolution If we also want to consider the time dependency as QM time evolution we may write

$$e^{-i\,E\,t} = e^{-i\,(m+K_E)} \quad \Rightarrow \quad \phi = e^{-i\,m\,t}\psi \quad \Rightarrow \quad \partial_t\phi = \dot{\phi} = \Big((-i\,m)e^{-i\,m\,t}\psi + e^{-i\,m\,t}\dot{\psi}\Big)$$

and also the second time derivative

$$\begin{aligned}\partial_t^2\phi = \ddot{\phi} =&\Big((-i\,m)(-i\,m)e^{-i\,m\,t}\psi + (-i\,m)e^{-i\,m\,t}\dot{\psi} + (-i\,m)e^{-i\,m\,t}\dot{\psi} + \underbrace{e^{-i\,m\,t}\ddot{\psi}}\Big)\\ =&\Big(-m^2e^{-i\,m\,t}\psi - 2i\,me^{-i\,m\,t}\dot{\psi}\Big)\end{aligned}$$

With the approximation to the first order derivative $e^{-i\,m\,t}\ddot{\psi} = 0$. Finally we find the solution that's something familiar to us

$$\begin{aligned}&e^{-i\,m\,t}\Big(\nabla^2\psi + 2i\,m\dot{\psi} + m^2\psi - m^2\psi\Big) = 0\\ &\Rightarrow \quad i\dot{\psi} = -\frac{i}{2m}\nabla^2\psi\end{aligned} \qquad \text{where} \quad \nabla^2\phi = e^{-i\,m\,t}\nabla^2\psi$$

that's the *Schrodinger equation* with $\hbar = 1$ that is the non-relativistic limit of the *Klein-Gordon equation* that is a relativistic equation.

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The electromagnetic field theory approach starts with a postulate: *it is possible to derive the Maxwell's equations starting from a lagrangian.* From this postulate, we now ask to ourself which is that lagrangian that solves the Euler-Lagrangian equation (**??**)?

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \qquad \text{where} \quad \phi = \begin{cases} \phi & \text{scalar potential} \\ \vec{A} & \text{vector potential} \end{cases}$$

To do so, we start postulating that it may be

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) - \rho\,\phi + \vec{J}\cdot\vec{A} \qquad \text{where} \quad \begin{cases} \phi & \mapsto \quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{A} & \mapsto \quad \vec{B} = \vec{\nabla}\times\vec{A} \end{cases}$$

Where the magnetic field's components may be also written as $B_i = \epsilon_{ijk}\partial_j\, A_k$, using the Levi-Civita symbol. We can define a 4-vector that transforms following the Lorentz transformations as $A_\mu = (\vec{A}, i\,\phi)$.

Maxwell's equations regression

$$\left.\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 & \\ \vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} & \text{corrent density - } \textit{source} \\ \vec{\nabla}\cdot\vec{E} = \rho & \text{charge density - } \textit{source} \\ \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 & \end{cases}\right\} \Rightarrow \quad \text{continuity equation} \quad \frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$$

Now, similarly as before, we define a 4-vector that transforms following the Lorentz transformations as $J_\mu = (\vec{J}, i\,\rho)$. Which leads to the following lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) - \rho\,\phi + J_\mu\, A_\mu$$

What's left is to check if our new postulates and definitions are compatible with what we already know.

$$\left(\ldots\ \textit{calculations}\ \ldots\right)$$

Electromagnetic tensor One we've our new lagrangian we may now introduce the *electromagnetic tensor* $F_{\mu\nu}$, as follow:

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -i\,E_1 \\ -B_3 & 0 & B_1 & -i\,E_2 \\ B_2 & -B_1 & 0 & -i\,E_3 \\ i\,E_1 & i\,E_2 & i\,E_3 & 0 \end{bmatrix} \quad \Rightarrow \quad \textit{antisimetrical tensor}$$

and also

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \Longleftrightarrow \quad \begin{cases} B_i = & \epsilon_{ijk}\,\partial_j\,A_k = \left(\vec{\nabla}\times\vec{A}\right)_i \\ B_i = & \frac{1}{2}\epsilon_{ijk}\,F_{jk} \quad \text{(inverse relation)} \\ E_i = & F_{i\,4} = -F_{4\,i} \end{cases}$$

$$\textit{examples}: \quad \begin{cases} F_{1\,4} = \partial_1 A_4 - \partial_4 A_1 = i\,\partial_1\phi - i\,\frac{\partial A_1}{\partial t} = i\,(-E_1) = -i\,E_1 \\ F_{1\,2} = \partial_1 A_2 - \partial_2 A_1 = \left(\vec{\nabla}\times\vec{A}\right)_3 = \begin{vmatrix} \partial_1 & \partial_2 \\ A_1 & A_2 \end{vmatrix}_{\hat{k}} = (\partial_1 A_2 - \partial_2 A_1)_{\hat{k}} = B_3 \end{cases}$$

This leads to a compact way to write the Maxwell's equations, in only one tensorial equation:

$$\begin{cases} \partial_\mu F_{i\,\mu} = J_i \\ \partial_i F_{4\,i} = J_4 \end{cases} \quad \Rightarrow \quad \partial_\mu F_{\nu\mu} = J_\nu\ .$$

The electromagnetic tensor and its dual Starting from the definition ?? of the electromagnetic tensor, we may see

$$\begin{aligned} F_{\mu\nu}F_{\mu\nu} &= -F_{\mu\nu}F_{\nu\mu} = -T_{\mu}(F\cdot F) = 2(\vec{B}^2-\vec{E}^2) \\ \text{from which} \quad \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + J_{\mu}A_{\mu} \end{aligned}$$

Trough the *antisymmetric tensor* $T_{\mu\nu\lambda}$ it's possible to write the following identity:

$$\partial_{\mu}F_{\nu\lambda} + \partial_{\nu}F_{\lambda\mu} + \partial_{\lambda}F_{\mu\nu} = 0$$

The *dual tensor* of F is as well invariant for *cage transformations*

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho} = \begin{bmatrix} 0 & -i\,E_3 & i\,E_2 & B_1 \\ i\,E_3 & 0 & -i\,E_1 & B_2 \\ -i\,E_2 & i\,E_1 & 0 & B_3 \\ -B_1 & -B_2 & -B_3 & 0 \end{bmatrix}$$

Levi-Civita symbols:

3-D $\epsilon_{\mu\nu\lambda} \quad \rightarrow \quad \epsilon_{1\,2\,3} = 1$

4-D $\epsilon_{\mu\nu\lambda\rho} \quad \rightarrow \quad \epsilon_{1\,2\,3\,4} = -1$

$\rightsquigarrow$ An interesting exercise, calculate: $F_{\mu\nu}\tilde{F}_{\mu\nu} = -4i\,\vec{E}\cdot\vec{B}$ and verify the equivalence.

> Confusione con gli indici? Il Prof. considera, al fine del corso, equivalente scrivere:
>
> $$A_{\mu}B^{\mu} \longleftrightarrow A_{\mu}B_{\mu}$$
>
> quindi attenzione, soprattutto rispetto alla notazione usata nel corso di *General Relativity.*

We know want to demonstrate that the tensor F is *cage invariant*, so let's introduce the scalar quantity Λ and write the transformation:

$$A_{\mu} \longrightarrow A_{\mu} + \partial_{\mu}\Lambda$$

$$\begin{aligned} F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \longrightarrow F_{\mu\nu} &= \partial_{\mu}A_{\nu} + \partial_{\mu}\partial_{\nu}\Lambda - \partial_{\nu}A_{\mu} - \partial_{\nu}\partial_{\mu}\Lambda \\ &= \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \end{aligned}$$

We may say that $F_{\mu\nu}$ "has the right combination of the magnetic and electron field", to make two terms cancelled each other.
A generic vector changes under Lorentz transformations as

$$C_{\mu} \longrightarrow L_{\mu\nu}C_{\nu}$$

How does the $F_{\mu\nu}$ tensor changes under Lorentz transformations?

$$F_{\mu\nu} \longrightarrow L_{\mu\mu'}L_{\nu\nu'}F_{\mu'\nu'}$$

1.8 13 ottobre

Since the lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + J_\mu A_\mu \qquad \text{where:} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

has to be considered in the action S:

$$S = \int \mathrm{d}^4 x \Big(-\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + J_\mu A_\mu \Big)$$

where the integrand is *cage invariant* for the transformation

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda \qquad \Rightarrow \qquad \int \mathrm{d}^4 x \Big(\partial_\mu J_\mu \Big) \Lambda = 0 \qquad (\leftarrow \textit{controllare})$$

Rotations Rotation in 3 dimension are given by the matrix $R(\phi)$ and where $\tilde{R}$ is its transposition.

$$R(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$R \cdot \tilde{R} = 1$$

Similar condition are true for Lorentz transformations L, such that

$$L \cdot \tilde{L} = 1$$
$$L_{\mu\nu} \cdot (\tilde{L})_{\nu\rho} = \delta_{\mu\rho} \quad \longrightarrow \quad L_{\mu\nu} \delta_{\nu\beta} (\tilde{L})_{\beta\alpha} = \delta_{\mu\alpha}$$

Meanwhile in terms of g, as metric tensor, instead of δ we have:

$$L^\mu{}_\nu (-g^{\nu\beta})\, \tilde{L}_\beta{}^\alpha = -g^{\mu\alpha}$$
$$L^\mu{}_\nu\, g^{\nu\beta}\, \tilde{L}_\beta{}^\alpha = g^{\mu\alpha}\ .$$

Boost along x-direction The Lorentz transformation of a boost along the x-direction

$$L = \begin{bmatrix} \cos\theta & 0 & 0 & \sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta & 0 & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cosh\psi & 0 & 0 & i\,\sinh\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\,\sinh\psi & 0 & 0 & \cosh\psi \end{bmatrix}$$

where in the second row we used:

$$\sin\theta = \frac{i\,v}{\sqrt{1-v^2}}$$
$$\theta = i\,\psi$$
$$\sin i\psi = i\,\sinh\psi$$
$$\cos\theta = \sqrt{1-v^2}$$

When we consider the application of the Lorentz transformations L to a vector $v = (x, y, z, it)$, it gives

$$v' = L\,v = \begin{bmatrix} \cosh\psi & 0 & 0 & i\,\sinh\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\,\sinh\psi & 0 & 0 & \cosh\psi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ i\,t \end{bmatrix} = \begin{bmatrix} \cosh\psi\, x - i\,\sinh\psi\, t \\ y \\ z \\ -i\,\sinh\psi\, x + i\,\cosh\psi\, t \end{bmatrix}$$

$$\begin{cases} x' = & \cosh\psi\, x - \sinh\psi\, t \\ y' = & y \\ z' = & z \\ t' = & -i\,\sinh\psi\, x + i\,\cosh\psi\, t \end{cases} \qquad \text{in the origin} \qquad \begin{cases} x' = & \cancel{\cosh\psi\, x} - \sinh\psi\, t \\ y' = & y \\ z' = & z \\ t' = & \cancel{-i\,\sinh\psi\, x} + i\,\cosh\psi\, t \end{cases}$$

The velocity of the boost is given by

$$v = \frac{x}{t} = \frac{\sinh\psi}{\cosh\psi} = \tanh\psi$$

Infinitesimal rotation The first thing to do is to consider the first order approximation. Consider n a large number, such that $\frac{\phi}{n}$ is a *small* angle, then

$$R\Big(\frac{\phi}{n}\Big) = \begin{bmatrix} 1 & \frac{\phi}{n} & 0 & 0 \\ -\frac{\phi}{n} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 + \frac{\phi}{n} L_3$$

since the generator of the rotation ($\hat{z}$-ax), and the classical rotation are

$$L_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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1.10 18 ottobre

1.11 20 ottobre

1.12 21 ottobre

1.13 25 ottobre

1.14 27 ottobre

1.15 28 ottobre

Chapter 2

Novembre

Appunti delle lezioni del Prof. Polosa relative al mese di novembre 2022.

2.1 November 3rd

We have learned a very general method to look for conserved quantities, once we are aware of certain symmetries. Let us consider the symmetry under rotations of space coordinates. The variation of space coordinates under the most general rotation can be expressed as:

$$\delta x_\mu = \theta_{\mu\nu} x_\nu \tag{2.1}$$

where θ is the antisymmetric matrix containing the infinitesimal angle of rotation. A more refined way to write it is by using the matrices $K_{\alpha\beta}$ introduced in update here. Just remember that the matrix $K_{\alpha\beta}$ has a 1 in row α, column β, a -1 in row β, column α, and zeroes everywhere else. The more refined way is:

$$\begin{aligned}
\delta x_\mu = &\frac{1}{2}\theta_{\alpha\beta}\left(K_{\alpha\beta}\right)_{\mu\nu} x_\nu = \frac{1}{2}\theta_{\alpha\beta}\left(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}\right)_{\mu\nu} x_\nu = \\
= &\frac{1}{2}\theta_{\alpha\beta}\left(\delta_{\alpha\mu}x_\beta - \delta_{\beta\mu}x_\alpha\right) = \frac{1}{2}\left(\theta_{\mu\beta}x_\beta - \theta_{\alpha\mu}x_\alpha\right) = \\
= &\theta_{\mu\beta}x_\beta
\end{aligned}$$

where the last step is justified by the antisymmetry of θ and by the fact that α, being a dumb index, can be renamed as β. While expanding the expression, we have found a specific way of writing the transformation (highlighted in blue) which is perfectly consistent with the way we wrote the most general coordinate transformation: $\delta x_\mu = M_{\mu_{(s)}}\xi_{(s)}$. It is easy to realize that, in our case, $M_{\mu_{\alpha\beta}} = \frac{1}{2}\left(\delta_{\alpha\mu}x_\beta - \delta_{\beta\mu}x_\alpha\right)$ and $\xi_{\alpha\beta} = \theta_{\alpha\beta}$. Recovering the expression for the conserved current we found somewhere:

$$J_{\mu_{(s)}} = \frac{\partial\mathcal{L}}{\partial(\partial\phi_a)}\left(\psi_{\alpha_{(s)}} - M_{\nu_{(s)}}\partial_\nu\phi_a\right) + M_{\nu_{(s)}}\mathcal{L}$$

we can apply it to the specific case of the scalar field lagrangian $\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2$. A scalar lagrangian has obviously no components to be acted upon by $\psi_{\alpha_{(s)}}$, therefore this last quantity is simply absent in the expression of the conserved current. Let's evaluate $J_{\mu_{\alpha\beta}}$:

$$\begin{aligned}
J_{\mu_{(\alpha\beta)}} &= -\partial_\mu\phi\left(0 - \frac{1}{2}\left(\delta_{\alpha\nu}x_\beta - \delta_{\beta\nu}x_\alpha\right)\partial_\nu\phi\right) + \frac{1}{2}\left(\delta_{\alpha\mu}x_\beta - \delta_{\beta\mu}x_\alpha\right)\mathcal{L} = \\
&= \frac{1}{2}(\partial_\mu\phi)x_\beta(\partial_\alpha\phi) + \frac{1}{2}\delta_{\alpha\mu}x_\beta\mathcal{L} - \frac{1}{2}(\partial_\mu\phi)x_\alpha(\partial_\beta\phi) - \frac{1}{2}\delta_{\beta\mu}x_\alpha\mathcal{L}) = \\
&= \frac{1}{2}\left(x_\beta(\partial_\mu\partial_\alpha\phi + \delta_{\alpha\mu}\mathcal{L} - x_\alpha(\partial_\mu\phi\partial_\beta\phi + \delta_{\beta\mu}\mathcal{L})\right) = \frac{1}{2}(x_\beta T_{\mu\alpha} - x_\alpha T_{\mu\beta})
\end{aligned}$$

The expression we arrived to closely resembles that of an orbital angular momentum, if we recall that the tensor $T_{\alpha\beta}$ is associated to the conservedl inear momentum.

Things change when we consider a vector field, like the A_μ of electromagnetism. This one will be changed when we act with Lorentz transformations upon the space. The variation δA_μ will be expressible as:

$$\delta A_\mu = \psi_{\mu_{(s)}} \xi_{(s)} \tag{2.2}$$

Since vectors will change under Lorentz trasformations exactly like coordinates, we can observe that:

$$\psi_{\mu_{(\alpha\beta)}} \xi_{(\alpha\beta)} = \frac{1}{2}\theta_{\alpha\beta}(\delta_{\alpha\mu}A_\beta - \delta_{\beta\mu}A_\alpha)$$

Now this variation is also equal to $A'_\mu(x') - A_\mu(x)$. This quantity would be zero for a scalar field: *the transformed field evaluated in the transformed point must always be equal to the non-transformed field evaluated in the non-transformed point.* This is the definition of a scalar field. Think of a temperature field: the temperature of a point should not depend on the velocity or on the orientation of the observer. On the contrary, the values of a vector field depend on the observer: the velocity of a point that lies somewhere in space is different for observers travelling at different velocities. This difference is precisely the one expressed by (**??**). We can expand on these findings to obtain some more interesting results:

$$\begin{aligned}\delta A_\mu &= A'_\mu(x') - A_\mu(x) = A'_\mu(x') - A'_\mu(x) + A'_\mu(x) - A_\mu(x) = \\ &= \delta x_\mu \partial_\mu A'_\mu(x) + \bar{\delta} A_\mu = \delta x_\mu \partial_\mu A_\mu(x) - \delta x_\nu \partial_\nu A_\mu(x)\end{aligned}$$

where:

- we have used the equality: $x'_\mu = x_\mu + \delta x_\mu$ to go from the first line to the second;
- we have recalled that $A'_\mu(x) = A_\mu(x) + \delta A_\mu(x) = A_\mu(x) + \psi_{\mu_{(\alpha\beta)}}\xi_{(\alpha\beta)}$ so that $\delta x_\mu \partial_\mu A'_\mu(x) = \delta x_\mu \partial_\mu A_\mu(x) + o(\xi^2)$ (we neglect all terms $o(\xi^2)$);
- we have observed that $\bar{\delta} A_\mu = A'_\mu(x) - A_\mu(x) = A_\mu(x_\mu - \delta x_\mu) - A_\mu(x) = -\delta x_\nu \partial_\nu A_\mu(x)$

Exploiting these results, we can write the *variation in form* (namely, the variation of the functional dependence on spacetime coordinates) of the electromagnetic lagrangian as:

$$\begin{aligned}\bar{\delta}\mathcal{L} &= \bar{\delta}\left(-\frac{1}{4}F_{\alpha\beta}F_{\alpha\beta}\right) = -F_{\alpha\beta}\frac{1}{2}\bar{\delta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = -F_{\alpha\beta}\bar{\delta}(\partial_\alpha A_\beta) \\ &= -F_{\alpha\beta}\partial_\alpha(\bar{\delta}A_\beta) = F_{\alpha\beta}\partial_\alpha(\delta x_\nu \partial_\nu A_\beta)\end{aligned}$$

Now expand the derivative to obtain:

$$\begin{aligned}F_{\alpha\beta}\partial_\alpha(\delta x_\nu \partial_\nu A_\beta) &= F_{\alpha\beta}(\partial_\alpha \delta x_\nu)\partial_\nu A_\beta + F_{\alpha\beta}\delta x_\nu \partial_\nu \partial_\alpha A_\beta = \\ =&\partial_\alpha(F_{\alpha\beta}\delta x_\nu \partial_\nu A_\beta) - \delta x_\nu \partial_\alpha(F_{\alpha\beta}\partial_\nu A_\beta) + \delta x_\nu \partial_\nu\left(\frac{1}{4}F_{\alpha\beta}F_{\alpha\beta}\right)\end{aligned}$$

where we have applied the inverse rule of the derivative of a product to go from the first to the second line and we have played with the derivatives of A_μ to make another $F_{\alpha\beta}$ appear. If we now want to evaluate the variation of the action due to the variation in form of $\mathcal{L}$, as usual we need to take the integral: $\bar{\delta}S = \int d^4x \bar{\delta}\mathcal{L}$. This integral will kill the first term of $\bar{\delta}\mathcal{L}$, namely $\partial_\alpha(F_{\alpha\beta}\delta x_\nu \partial_\nu A_\beta)$, since the argument of this 4-divergence vanishes on the boundaries. Using the remaining terms, we are now able to say that the relevant part of $\bar{\delta}\mathcal{L}$ is:

$$\bar{\delta}\mathcal{L} = -\delta x_\nu \partial_\alpha(F_{\alpha\beta}\partial_\nu A_\beta + \delta_{\alpha\nu}\mathcal{L}) \tag{2.3}$$

If we find that our lagrangian is invariant for rotations, we are sure that $\partial_\alpha T_{\nu\alpha} = 0$, with $T_{\nu\alpha} = F_{\alpha\beta}\partial_\nu A_\beta + \delta_{\alpha\nu}\mathcal{L}$. It is possible to define a gauge invariant (and also symmetric) version of T, namely $\bar{T}_{\nu\alpha} = F_{\alpha\beta}F_{\nu\beta} + \delta_{\nu\alpha}\mathcal{L}$, which is again divergenceless.

Chapter 3

Dicembre

Appunti delle lezioni del Prof. Polosa relative al mese di dicembre 2022.