One Page Summary:

Background:
Decentralised Exchanges (DEXes) have transformed the trading landscape by enabling users to trade directly through smart contracts, thus removing the need for a central counterparty. Automated Market Makers (AMMs) are crucial in this ecosystem, setting the price for asset swaps. Liquidity providers (LPs), deposit assets into liquidity pools that traders access. However, LPs encounter a challenge termed "Impermanent Loss" - a relative loss compared to a buy-and-hold strategy when asset prices deviate from their initial rate. In compensation, LPs earn trading fees.

Risk-neutral Valuation of Liquidity Provision:
The paper utilises risk-neutral valuation methods to determine the equitable price for liquidity provision in a complete market. It begins with a basic binomial model, a foundational tool in finance for modelling asset price fluctuations over discrete time intervals. The discussion then shifts to a continuous model, leveraging the renowned Black-Scholes framework, a continuous-time model for option pricing. This quantitative approach pinpoints the fair price of liquidity provision, essentially quantifying the returns LPs should accrue from fees to match the buy-and-hold strategy's outcomes. By expanding upon the paper's original simulations, I discerned that elements like implied volatility and the risk-free rate also influence the fair price of liquidity provision.

Weighted Variance Swaps:
The paper highlights a close relationship between the hedging of Impermanent Loss and financial instruments like variance swaps and gamma swaps. The paper shows that gamma swaps tend to over-hedge impermanent loss, while variance swaps tend to underhedge impermanent loss. The paper then introduces the concept of weighted variance swaps. Essentially, weighted variance swaps are an optimal blend of gamma and variance swaps that can almost perfectly hedge impermanent loss. The authors provide a detailed mathematical framework for pricing these swaps and understanding their payoff structure.

Numerical Analysis:
The paper offers numerical examples and simulations to illustrate the concepts introduced, making the theoretical constructs more tangible. I replicated all the paper simulations using python and further extended some of the models to better understand the dynamics of the concepts introduced.

Conclusion:
The paper suggests that by using weighted variance swaps, liquidity providers can potentially hedge against the risks of impermanent loss. This is a significant contribution as it provides a bridge between traditional finance tools and the emerging challenges in the DeFi space.

2. Liquidity provision

In the world of decentralised finance (DeFi), Automated Market Makers (AMMs) have become a popular mechanism for trading. These AMMs rely on liquidity providers (LPs) who deposit assets into a pool, allowing users to trade against this pool rather than against other users. The paper focuses on a specific type of AMM known as the Constant Product Market.

Key Concepts:

Constant Product Markets

Exchange Rate: The paper defines the exchange rate in a Constant Function Market as the no-fee infinitesimal price. In simpler terms, it's the rate at which one asset can be swapped for another in the pool.

- **Constant L:** For Constant Product Markets, there's a constant, denoted as (L), which is the product of the initial amounts of the two tokens in the liquidity pool. This constant remains unchanged regardless of trades. So, if someone buys one token using another, the product of the two token amounts in the pool will always equal (L).
- **Exchange Rate Dynamics:** The paper notes that the initial exchange rate between two assets, say asset X and asset Y, is $(S_0 = y_0/x_0)$. This means that for the initial amounts of tokens in the pool, (x_0) and (y_0), the rate at which X can be swapped for Y is (S_0). To further illustrate the dynamics, imagine asset X is being swapped for asset Y by traders, the ratio of those two assets in the pool will change. To ensure the product of those two assets remains constant, their prices will adapt. The larger the trade the more the price X will drop. In essence AMM is an alternative way to approximate order book dynamics in DeFi.
- **Liquidity Provider's Portfolio Value:** The value of the LP's portfolio, in the absence of fees and other LPs, is determined by the formula $(V_{LP}(0) = y_0 + x_0 S_0)$. This represents the total value of assets the LP has in the pool.

Multiple Liquidity Providers:

- Pool Share Tokens: In practice, when an LP deposits assets into a pool, they receive "pool share tokens." These tokens represent their share of the pool. When they want to withdraw their assets, they redeem these tokens. They will receive trading fees proportional to their share in the pool.
- Value Independence: The paper makes an important observation: the value of one LP's claim to the pool is independent of other LPs' actions. This means that if one LP adds or removes assets from the pool, it doesn't affect the value of another LP's share.

Intuitive Explanation of Value Independence for Multiple Lps:

Imagine a big pot of soup (the liquidity pool) in a community kitchen. People can come and take soup (trade) or add ingredients (provide liquidity). The soup's taste (the exchange rate) changes based on the ingredients added. Now, if you're one of the people who added ingredients, you'd want to ensure that no matter what others add or take, your contribution's value remains the same. This paper mathematically proves that in the world of AMMs, this is indeed the case. Your share remains unaffected by others' actions. In essence, the "Liquidity provision" section breaks down the mechanics of how assets are valued in a liquidity pool and assures that each liquidity provider's share remains fair and independent of others.

Intuitive Explanation for Impermanent Loss:

Imagine you lend two types of toys to a playground's toy-sharing pool, say dolls and trucks, with an agreement that you can take them back whenever you want. Over time, dolls become super popular and trucks less so. When you decide to get your toys back, you might get fewer dolls (because they're in high demand) and more trucks (because they're not as popular). The change in the number of dolls and trucks you get back compared to what you lent is similar to impermanent loss in the world of cryptocurrencies. If dolls and trucks become equally popular again, and you haven't taken your toys out, there's no "loss" — hence the term "impermanent."

3.1. Binomial model

To explain quantitatively the mechanics behind liquidity provision the paper starts with a simple Binomial Model. This model is a simplified representation of how the price of an asset can evolve over time. It's like a coin flip: the price can either go up or down in the next period. This model is foundational in understanding more complex financial models.

The paper contrasts two strategies:

Liquidity Provider (LP): This is someone who provides liquidity to a decentralised exchange. In return, they earn fees from the trades that use their liquidity. However, being an LP comes with risks, especially the risk of "impermanent loss." This loss occurs when the relative prices of the assets in the pool change, causing the LP's holdings to be worth less than if they had just held onto their assets outside the pool.

Buy-and-Hold: This strategy involves buying an asset and holding onto it, regardless of price fluctuations.

What if a liquidity provider could combine their liquidity provision with another financial instrument in such a way that their portfolio's value behaves just like the buy-and-hold strategy? The paper's goal is to find a way to combine the liquidity provision with another financial instrument (a straddle). A straddle is a financial strategy that involves buying both a call and a put option on the same asset with the same strike price and expiration date. In simpler terms, it's a bet on the asset's price volatility, not its direction. If the asset's price moves significantly in either direction (up or down), the straddle will be profitable.

Given this simple model the paper derives a value, which represents the amount of the straddle needed to achieve this replication. The crucial takeaway is that this value doesn't depend on whether the asset's price goes up or down. This is powerful because it means that, with the right combination of liquidity provision and this straddle, a liquidity provider can effectively hedge against price changes and achieve the same outcome as if they just held onto the asset It's like having the best of both worlds: you earn fees as an LP, but your portfolio's value remains stable, just like the buy-and-hold strategy.

However, by extending the simulations of the authors I demonstrated that in the context of the model the price of the straddle depends 1) on the risk free interest rate, which can also be thought as the opportunity cost of the investment 2) implied volatility of the assets, which will in turn determine the cost of hedging.

For the LP to be profitable:

The fees they collect must exceed the sum of the straddle premium and any other associated costs represented as the risk free rate (ie e transaction fees, the opportunity cost of locking up funds in a liquidity pool instead of other investments, or any other costs associated with participating in the DeFi platform.)

Numerical Example

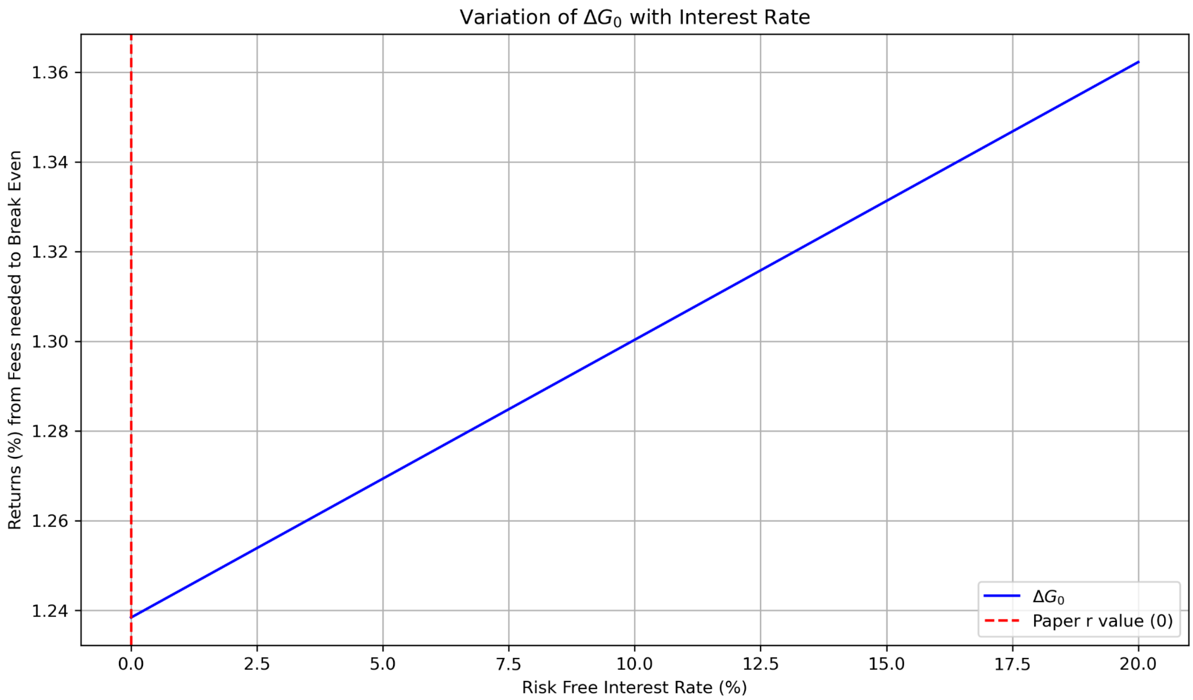
To make things concrete, the paper provides a numerical example. Let's say both tokens start with a value of 100. The authors set the parameters in a way to imply that the price can either increase by 25% or decrease by 20%. In the context of the model this is about 22% volatility. Given these parameters, the paper calculates various values, including the straddle premium and the expected returns of providing liquidity and the simple buy and hold strategy.

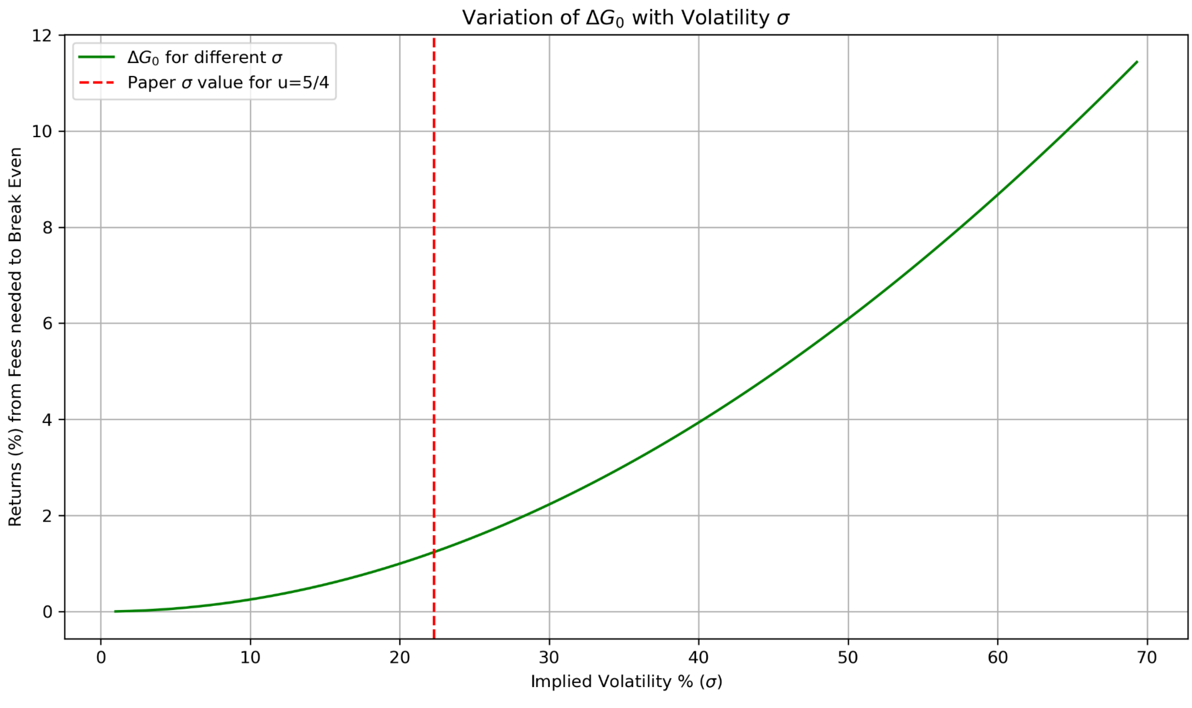
Suppose you buy a straddle with a strike price of $100 (the initial asset price).If the asset's price goes up to $125 (25% increase), the call option will be in the money by $25, while the put option will expire worthless. Your profit from the call option will be $25 minus the premium you paid for the straddle. If the asset's price drops to $80 (20% decrease), the put option will be in the money by $20, while the call option will expire worthless. Your profit from the put option will be $20 minus the straddle's premium.

Returns: In the context of the paper, the wealth of a liquidity provider (LP) is represented by the total value of the LP's assets in the liquidity pool. The paper contrasts this with the wealth of someone following a buy-and-hold strategy

Premium: This is the cost of the straddle. It's the amount you pay upfront to buy both the call and put options. The paper calculates the premium using the binomial model's parameters and the risk-neutral probability (a probability measure used in finance to price derivatives).

To explore the model mechanics I replicated the numerical simulation in python and further experimented by changing the interest rate and implied volatility. As it is clear from the graphs below, higher interest rates as well as higher implied volatility will raise the hedging cost, therefore will in turn require higher fees to make liquidity providers break even.


Variation of ΔG_0 with Interest Rate
Returns (%) from Fees needed to Break Even
1.36
1.34
1.32
1.30
1.28
1.26
1.24
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Risk Free Interest Rate (%)
ΔG_0
Paper r value (0)


Variation of ΔG_0 with Volatility σ
Returns (%) from Fees needed to Break Even
12
10
8
6
4
2
0
0
10
20
30
40
50
60
70
Implied Volatility % (σ)
ΔG_0 for different σ
Paper σ value for u=5/4

3.2. Continuous model

After understanding the basics with the simple Binomial Model, the paper dives deeper into a more realistic scenario using the Continuous Model. Think of the Binomial Model as a digital watch that only shows hours and minutes, while the Continuous Model is like an analog watch with a sweeping second hand, capturing every moment in between.

In the real world, asset prices don't just jump up or down at fixed intervals; they continuously change, influenced by countless factors. This continuous model captures these subtle changes, providing a more detailed and accurate representation of how asset prices evolve.

Understanding the Exchange Rate:
The paper introduces the concept of an exchange rate, which is the ratio of the values of the two assets in the liquidity pool. This rate continuously changes, and its behaviour is likened to a random walk - imagine a drunk person trying to walk in a straight line but keeps veering off randomly.

Two Strategies Revisited:
Just as in the Binomial Model, we again look at the Liquidity Provider (LP) and the Buy-and-Hold strategies. But this time, we're observing them in a world that's constantly in flux, not just waiting for the next coin flip.

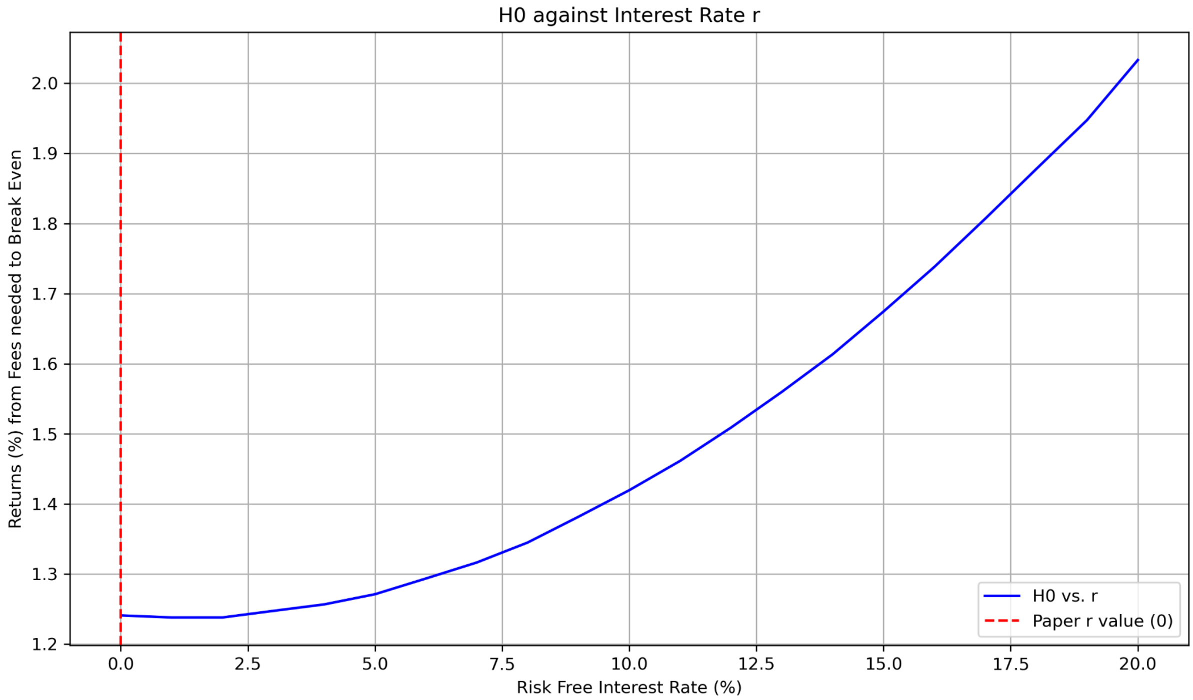
Hedge Strategy:
The paper's goal remains the same: How can an LP protect themselves from the risks of price changes? In this continuous setting, the paper introduces a mathematical tool that acts like a protective shield against these risks. This tool is a combination of two financial instruments that pay off based on how much the exchange rate changes, not necessarily in which direction.The aim is to ensure that our hedge (the straddle) works effectively, even when the weather (asset prices) is unpredictable.

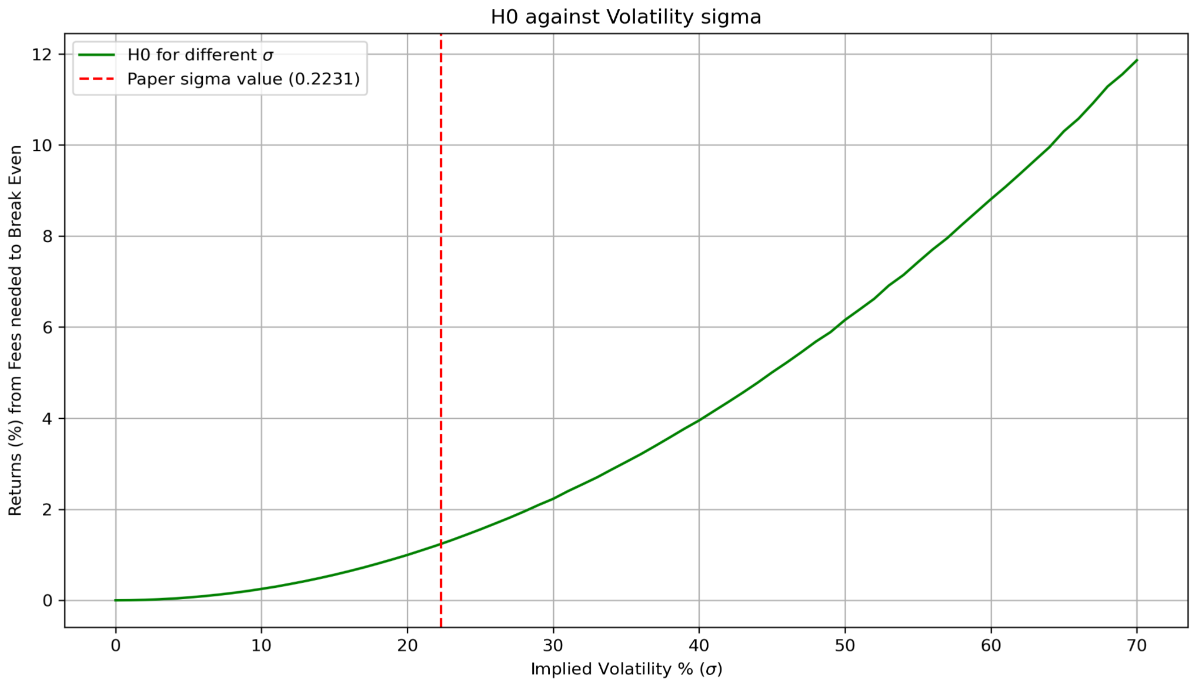
The Cost of Hedging:
Just as in the simpler model, the cost of this hedge (the straddle) depends on factors like the risk-free interest rate and the asset's volatility. The more unpredictable the asset's price, the costlier it is to hedge against its movements. For an LP to benefit, the fees they earn should outweigh this cost.

The Continuous Model paints a more detailed picture of the world of decentralized finance. It shows that with the right tools and strategies, a liquidity provider can navigate this complex landscape, earning fees while also protecting themselves from the unpredictable nature of asset prices. It's like sailing in choppy waters with a sturdy ship and a reliable compass.

Numerical Example

Through a Monte Carlo simulation, I validated the paper's results. Additionally I replicated the prior experiments. Higher interest rates as well as higher implied volatility will raise the hedging cost, requiring higher fees for liquidity providers to break even.





3.3 Dynamic hedging in a Black-Scholes market

The Black-Scholes model is a renowned tool in finance, often used to price options. Think of it as a compass that helps traders navigate the financial markets. In this section, the paper introduces how this model can be used for dynamic hedging in the context of liquidity provision. Dynamic hedging is about adjusting your protective measures based on market movements, ensuring you're always shielded against risks like Impermanent Loss.

The section also draws a fascinating connection between Impermanent Loss and two financial instruments called variance swaps and gamma swaps. These instruments are tools that can help manage volatility in investments. The section suggests that Impermanent Loss can be hedged using a combination of these swaps, providing a more comprehensive protective shield.

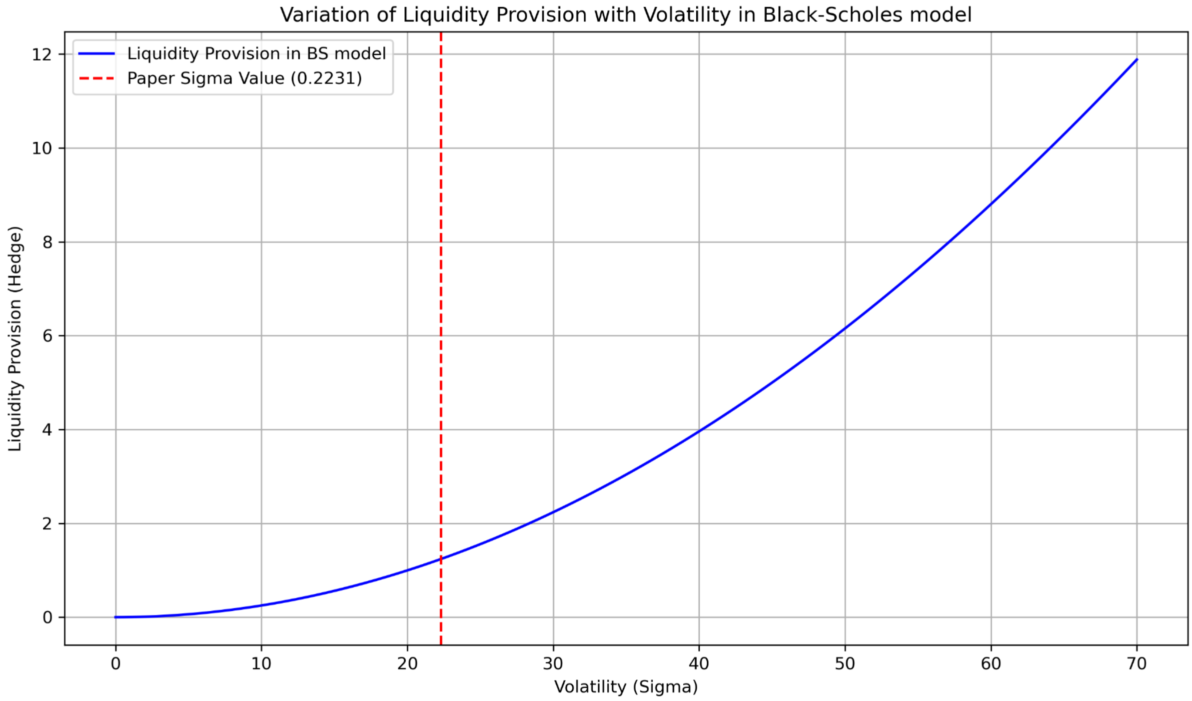
A variance swap is a financial instrument that allows parties to trade future realised (or actual) volatility against current implied (or expected) volatility over a specific time frame. They can be used by liquidity providers to hedge against this volatility. By entering into a variance swap, an LP can receive a payout if the actual volatility exceeds the expected volatility, potentially offsetting some of the impermanent loss. Additionally variance swaps allow LPs to isolate and trade volatility without having to deal with the directional risk of holding an asset. This can be particularly useful in DeFi environments where asset prices can be highly volatile.

A gamma swap allows one to gain exposure to the gamma, or the rate of change of an option's delta. It's a way to trade or hedge against the convexity of an option's payoff. Impermanent loss is a non-linear risk, meaning it doesn't change uniformly with price movements. Gamma, being a measure of the rate of change of delta, is also about non-linearity. A gamma swap can be used by LPs to manage this kind of non-linear exposure in their portfolios. If the assets in a liquidity pool experience rapid price changes, the impermanent loss can escalate quickly. Since gamma measures the sensitivity to rapid price changes, a gamma swap can be a tool for LPs to hedge against these swift movements.

The essence of this section is adaptability. By dynamically adjusting the hedge using principles from the Black-Scholes model, a liquidity provider can navigate the ever-changing financial markets, ensuring their investments remain protected against unforeseen price changes.

Numerical Example

As before I successfully replicated the results of the paper and additionally experimented by adjusting the implied volatility as well as the interest rate. The results are similar: higher implied volatility will raise the hedging cost. Due to time constraints I didn't replicate the interest rate experiment. However in the Black Scholes world higher Interest rates will affect the discount factor of the present value of the options, which should also increase the hedging cost in a similar manner as before.

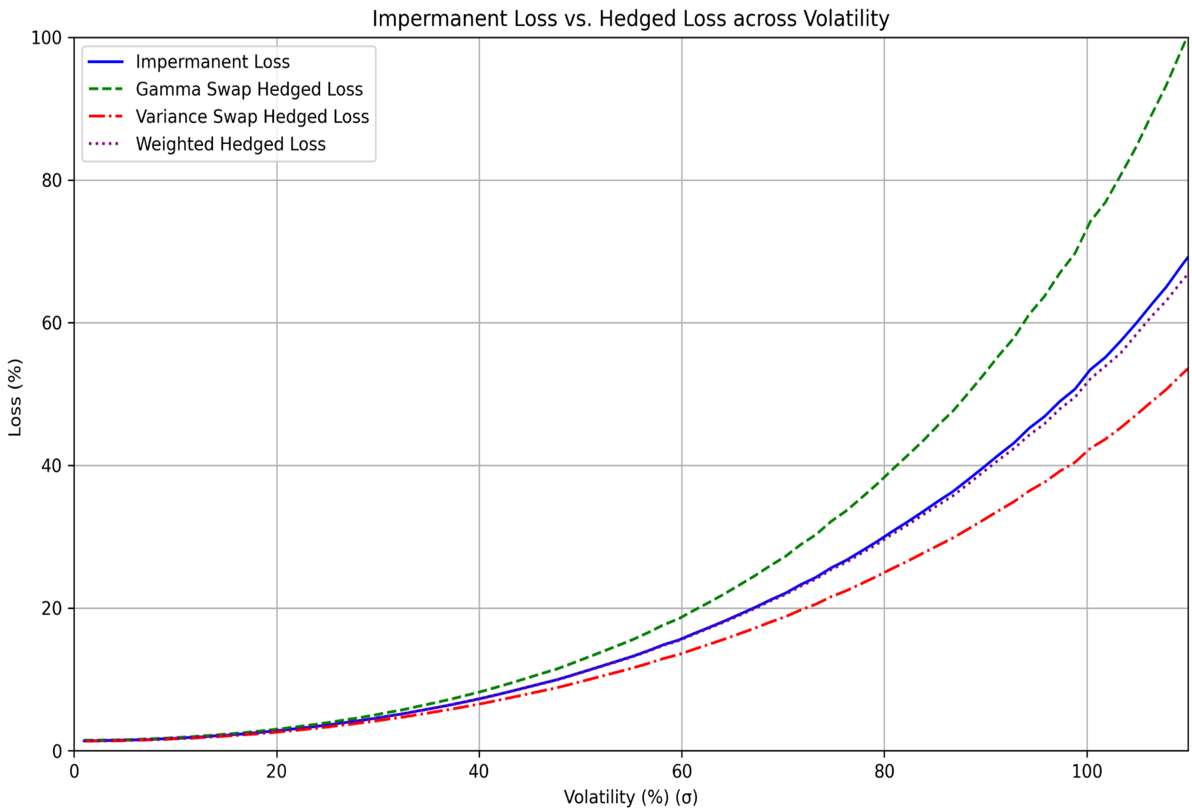


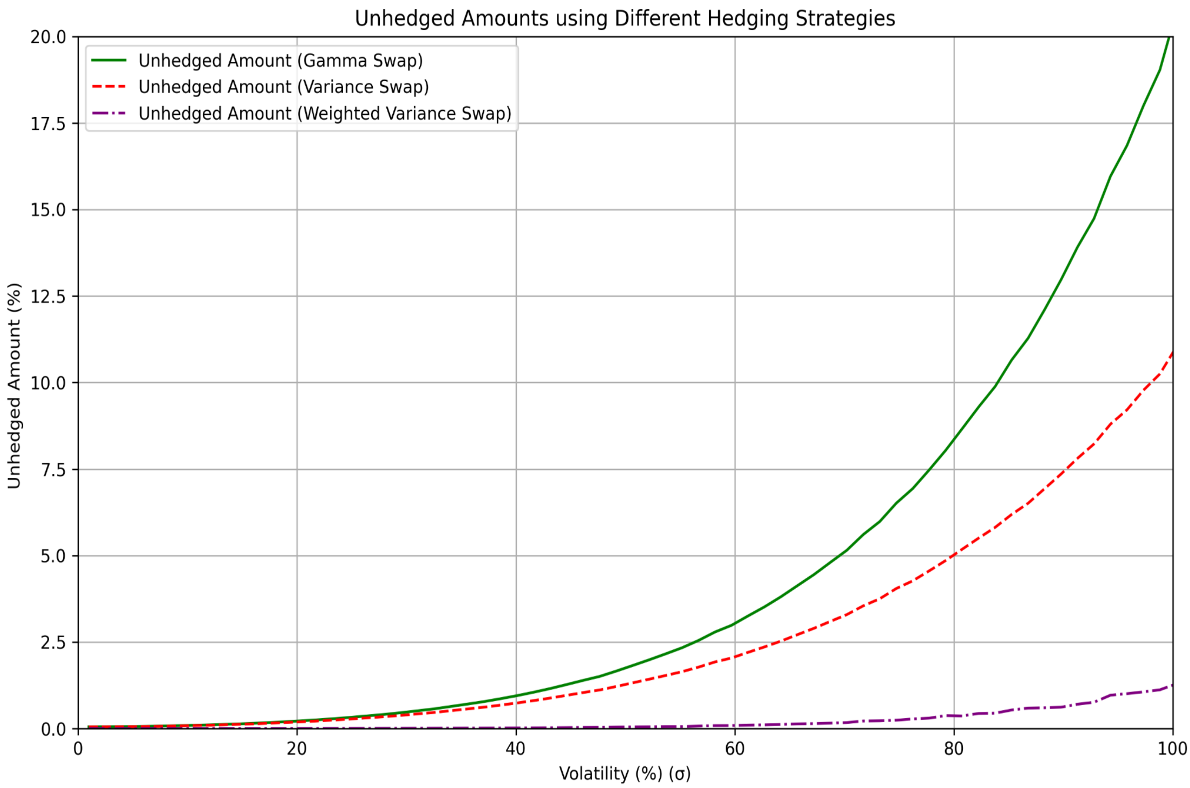
4 Impermanent Loss Hedge

The paper introduces the concept of weighted variance swaps as a means to hedge against impermanent loss in decentralised finance (DeFi) settings, specifically in the context of Automated Market Makers (AMMs). The Impermanent Loss hedge can be viewed as lying between a variance swap and a gamma swap.

The paper shows that gamma swaps tend to over-hedge impermanent loss, while variance swaps tend to underhedge impermanent loss. Essentially, weighted variance swaps are an optimal blend of gamma and variance swaps that can almost perfectly hedge impermanent loss.

I followed the paper's methodology and conducted simulations which clearly validated the paper's main results. Weighted variance swaps can almost perfectly hedge impermanent loss and clearly outperform the other strategies.





Unhedged Amounts using Different Hedging Strategies
Unhedged Amount (Gamma Swap)
Unhedged Amount (Variance Swap)
Unhedged Amount (Weighted Variance Swap)
Unhedged Amount (%)
Volatility (%) (σ)
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
0
20
40
60
80
100