Mathematical Appendix

Binomial Model Appendix

Model Parameters

Given the parameters from the paper:

$$S_0 = 1, \quad u = \frac{5}{4}, \quad y_0 = 100, \quad x_0 = 100$$

we compute (L) as:

$$L = \sqrt{y_0 \times x_0}$$

Risk-Neutral Probabilities

The risk-neutral probabilities are used to price derivatives in a binomial model. For our one-period model, we define the upward and downward movement risk-neutral probabilities, (π_u) and (π_d) respectively, as:

$$\pi_u = \frac{(1+r) - \frac{1}{u}}{u - \frac{1}{u}}, \quad \pi_d = 1 - \pi_u$$

where (r) is the risk-free rate.

Option Premiums

In the binomial model, the value of an option at any node depends on the possible values it can take in the subsequent nodes. For a European option, which can only be exercised at expiration, the value at any node is the present value of the expected payoff.

For a Call Option with strike price (K):

- Up-move payoff:
$\max(uS_0 - K, 0)$
In our case, $(K = S_0)$, so the up-move payoff becomes:
$\max(u - 1, 0)$

- Down-move payoff:
$\max(dS_0 - K, 0)$

Given $(d = \frac{1}{u})$, the down-move payoff becomes:

$\max(\frac{1}{u} - 1, 0)$

The present value of the call is given by:

$$\text{Call Premium} = \pi_u \times \text{Up-move payoff} + \pi_d \times \text{Down-move payoff}$$

For a Put Option:

- Up-move payoff:

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$$max(K - uS_0, 0) = \max(1 - u, 0)$$

- Down-move payoff:

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$$max(K - dS_0, 0) = \max(1 - \frac{1}{u}, 0)$$

The present value of the put is:

$$\text{Put Premium} = \pi_u \times \text{Up-move payoff} + \pi_d \times \text{Down-move payoff}$$

The European straddle premium is the sum of the call and put premiums.

Liquidity Provider Portfolio Value

The value of the liquidity provider portfolio at $(t = 0)$ is:

$$VLP(0) = 2L\sqrt{S_0}$$

Delta Calculation

Delta $((\Delta))$ is a measure of how much an option's price will move for a given change in the underlying asset. In the context of this paper, (Δ) is derived as:

$$\Delta = \frac{|1 - \sqrt{u}|}{1 + \sqrt{u}}$$

Interest Rate Experiment

To investigate the role of the interest rate in the value of $(\Delta \times G_0)$, we compute $(\Delta \times G_0)$ across a range of interest rates (from 0 to 20%). The goal is to understand how much in fees (expressed as a percentage of the initial liquidity pool value) a liquidity provider would need to break even, given the impermanent loss and the prevailing risk-free rate.

Volatility Experiment

The relationship between the upward movement factor (u) and volatility (σ) in the binomial model can be expressed through:

$$\sigma = \frac{\ln(u)}{\sqrt{\Delta t}}$$

Where:
- (σ) is the volatility.
- (u) is the upward movement factor.
- (Δt) is the time increment in the binomial model, which we've assumed to be 1 for simplicity in this analysis.

In our experiments, we varied (u) and computed the corresponding (σ) values using the above relationship. The purpose of this experiment is to understand how the returns (in terms of fees) needed to break even (i.e., (ΔG_0)) vary with different market volatilities. The upward movement factor (u) represents the potential increase in the asset price, and its relationship with (σ) gives us a quantifiable measure of market volatility.

As (u) increases (indicating a more significant potential price increase), the associated volatility (σ) also rises. A higher volatility implies more uncertainty in the market, leading to potentially larger impermanent losses for liquidity providers. This relationship is crucial for liquidity providers to understand, as it directly impacts their expected returns and the associated risks.

Continuous Model Appendix

The terminal payoff of the contingent claim, which aims to cover Impermanent Loss, is given by:

$$H_T = VBH(T) - VLP(T) = y_0 \left(\sqrt{K_T} - 1\right)^2$$

Pricing the Contingent Claim

In a complete market, the fair price of this claim at any time $(t < T)$ is the present value of the expected payoff under the risk-neutral measure:

$$H(t) = y_t \mathbb{E}\left[\left(\sqrt{K_T} - 1\right)^2 |\mathcal{F}_t\right]$$

Where $(\mathcal{F}_t)$ is the filtration up to time (t) and the expectation is taken under the risk-neutral measure.

Geometric Brownian Motion

To model the exchange rate (S_t) in continuous time, we use the Geometric Brownian Motion (GBM):

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Where:
- (r) is the risk-free rate
- (σ) is the volatility of the exchange rate
- (W_t) is a standard Brownian motion

The solution to this stochastic differential equation, given (S_0), is:

$$S_t = S_0 e^{(r - 0.5\sigma^2)t + \sigma W_t}$$

Estimating Volatility

From the paper, we have:

$$u = e^{\frac{\sigma}{\sqrt{\Delta t}}}$$

Where (u) is the upward movement factor in the binomial model and (Δt) is the length of a time step. From this, we can derive (σ), the continuous-time volatility.

Monte Carlo Simulation for Pricing

To estimate the price (H_0) of the contingent claim at time $(t = 0)$, we use Monte Carlo simulation. By simulating many paths for (S_T) using GBM, we can compute the average payoff of the contingent claim and then discount it back to the present to get its price.

Experimentation

In the given code, we then vary both (r) and (σ) to observe their effects on the price (H_0) of the contingent claim, thus understanding the sensitivity of Impermanent Loss hedging cost to these parameters.

Black Scholes Appendix

Geometric Brownian Motion (GBM)

The Black-Scholes model assumes that the price of the underlying asset follows a geometric Brownian motion. This stochastic differential equation is given by:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

Where:
- (S_t) is the asset price at time (t).
- (μ) is the drift rate.
- (σ) is the volatility.
- (dW_t) is a Wiener process or Brownian motion.

Black-Scholes Solution for Impermanent Loss

Considering the specific nature of impermanent loss, the paper presents a solution to the Black-Scholes PDE for a particular function:

$$g_\alpha(z) = z^\alpha, \quad \alpha \in R$$

The impermanent loss can be hedged with a weighted variance swap with exponent $(\alpha = \frac{1}{2})$. This means the hedge for impermanent loss lies between a variance swap $((\alpha = 0))$ and a gamma swap $((\alpha = 1))$.

Given the function (g), the paper provides:

$$S_t^{\frac{1}{2}} = p(S_T, T) = p(S_0, 0) + \int_0^T \frac{\partial p}{\partial s}(S_t, t) dS_t$$

This equation relates the square root of the asset price to its initial value and a term incorporating the partial derivative of (p) with respect to (s).

The impermanent loss hedge in the Black-Scholes framework is given by:

$$x_0 \left(\sqrt{S_T} - \sqrt{S_0}\right)^2 = 2y_0 \left(1 - e^{-\frac{\sigma^2 T}{8}}\right)$$

Where:
- (x_0) and (y_0) are the initial amounts of assets.

- (S_0) is the initial asset price.
- (S_T) is the asset price at maturity(T).
- (σ) is the volatility.

The right-hand side of the equation provides a benchmark for the fair fee for providing liquidity until maturity (T).

Liquidity Provision Calculation

The liquidity provision (H) in the Black-Scholes model for a given (T) and (σ) is defined by:

$$H(T, \sigma) = 2 \times y_0 \times \left(1 - \exp\left(-\frac{\sigma^2 T}{8}\right)\right)$$

Here:
- (y_0) is the initial parameter.
-(σ) is the volatility.
- (T) is the time to maturity.
- The exponential term represents the discounting effect in the Black-Scholes model.

Experimentation

In the given code, we then vary (σ) to observe their effects on the price (H_0) of the contingent claim, thus understanding the sensitivity of Impermanent Loss hedging cost to these parameters.

The term $\exp\left(-\frac{\sigma^2 T}{8}\right)$ acts as a discount factor. To incorporate interest rates in the simulations I would have to modify to replace the term $(\exp\left(-\frac{\sigma^2 T}{8}\right))$ with $(\exp\left(-rT - \frac{\sigma^2 T}{8}\right))$ in the liquidity provision formula. However due to time constraints I skipped that simulation

Hedging Impermanent Loss Appendix

Let (S_T) be the terminal stock price, (S_0) be the initial stock price, and (T) be the time to maturity. We model the stock price using a geometric Brownian motion:

$$S_T = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma W_T\right)$$

where (μ) is the drift, (σ) is the volatility, and (W_T) is a standard Brownian motion.

Impermanent Loss

The impermanent loss for a given terminal stock price \(S_T \) is defined as:

$$\text{Impermanent Loss}(S_T, S_0) = x_0(\sqrt{S_T} - \sqrt{S_0})^2$$

Hedging Strategies

We examine three hedging strategies:

Gamma Swap Hedging

The hedged loss using a gamma swap for a given \(S_T \) is:

$$\text{Gamma Swap Hedged Loss}(S_T, S_0) = x_0 \left(\frac{1}{2} S_T \log\left(\frac{S_T}{S_0}\right) - (S_T - S_0)\right)$$

Variance Swap Hedging

The hedged loss using a variance swap for a given (S_T) is:

$$\text{Variance Swap Hedged Loss}(S_T, S_0) = x_0 \left(-\frac{1}{2} S_0 \log\left(\frac{S_T}{S_0}\right) + \frac{1}{2}(S_T - S_0)\right)$$

Weighted Variance Swap Hedging

The hedged loss using a weighted average of gamma and variance swaps for a given (S_T) is:

$$\text{Weighted Hedged Loss}(S_T, S_0, w) = w \times \text{Variance Swap Hedged Loss}(S_T, S_0) + (1 - w) \times \text{Gamma Swap Hedged Loss}(S_T, S_0)$$

where (w) is the weighting factor.

Optimization

The objective is to find an optimal (w) such that the mean squared error between the weighted hedged loss and the impermanent loss is minimized. Mathematically, this is:

$$w^* = \arg\min_w \sum_{S_T} (\text{Weighted Hedged Loss}(S_T, S_0, w) - \text{Impermanent Loss}(S_T, S_0))^2$$