Mathematics

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Linear algebra

- *Linear:* having to do with lines, planes, etc.
- *Algebra:* solving equations involving unknowns.

What is system of linear equations ?

$$\begin{cases} x + 3y - z = 4 \\ 2x - y + 3z = 17 \\ y - 4z = -3. \end{cases}$$

Definition. An equation in the unknowns $x, y, z, \ldots$ is called **linear** if both sides of the equation are a sum of (constant) multiples of $x, y, z, \ldots$, plus an optional constant.

For instance,

$$3x + 4y = 2z$$
$$-x - z = 100$$

are linear equations, but

$$3x + yz = 3$$
$$\sin(x) - \cos(y) = 2$$

are not.

Write few linear as well as non linear equations ?

- **Linear Equation**:

  A linear equation is an equation where the variables are either not present or are only raised to the first power. Linear equations form a straight line when graphed on a coordinate plane.

  Standard form:

  $$ax + by + c = 0$$

  Here, a, b, and c are constants, and x and y are variables. No variable is raised to a power greater than one.

  Examples:

  - $2x + 3y = 6$
  - $y = 4x + 5$
  - $x - y/2 = 3$

- **Nonlinear Equation**:

  A nonlinear equation is any equation that is not linear. This means that the variables can be raised to powers other than one, and the equation can involve products of variables or other more complex operations.

  Examples:

  - $y = x^2 + 2x + 1$ (quadratic)
  - $y = \sin(x)$ (trigonometric)
  - $y = e^x$ (exponential)
  - $x^2 + y^2 = 16$ (circle equation)
  - $y = \log(x)$ (logarithmic)

2. **Graphical Representation**

- **Linear Equation**:
  - The graph of a linear equation is always a straight line. The slope and y-intercept can be easily determined from the equation, especially when written in the slope-intercept form $y = mx + b$.
- **Nonlinear Equation**:
  - The graph of a nonlinear equation can take various forms, such as curves, circles, parabolas, hyperbolas, or more complex shapes depending on the equation's form.

- It will be very important to us to understand systems of linear equations both *algebraically* and *geometrically.*

- *Algebraically* (writing equations for their solutions).
- *Geometrically* (drawing pictures and visualizing).

- Engineers need to solve many, many equations in many, many variables. Here is a tiny example:

$$\begin{cases} 3x_1 + 4x_2 + 10x_3 + 19x_4 - 2x_5 - 3x_6 = 141 \\ 7x_1 + 2x_2 - 13x_3 - 7x_4 + 21x_5 + 8x_6 = 2567 \\ -x_1 + 9x_2 + \frac{3}{2}x_3 + x_4 + 14x_5 + 27x_6 = 26 \\ \frac{1}{2}x_1 + 4x_2 + 10x_3 + 11x_4 + 2x_5 + x_6 = -15. \end{cases}$$

A **system** of linear equations is a collection of several linear equations, like

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2. \end{cases} \tag{1.1.1}$$

Definition (Solution sets)**.**

- A **solution** of a system of equations is a list of numbers $x, y, z, \ldots$ that make all of the equations true simultaneously.
- The **solution set** of a system of equations is the collection of all solutions.
- **Solving** the system means finding all solutions with formulas involving some number of parameters.

- Solve this system of equations:

$$\begin{cases} x + 2y = 3 \\ x + 2y = -3. \end{cases}$$

A system of linear equations need not have a solution. For example, there do not exist numbers *x* and *y* making the following two equations true simultaneously:

Definition. A system of equations is called **inconsistent** if it has no solutions. It is called **consistent** otherwise.

1.1.1 Line, Plane, Space, Etc.

We use $\mathbf{R}$ to denote the set of all real numbers, i.e., the number line. This contains numbers like $0, \frac{3}{2}, -\pi, 104, \ldots$

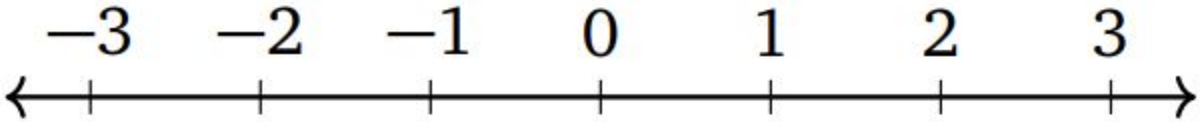
Definition. Let n be a positive whole number. We define

$$\mathbf{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

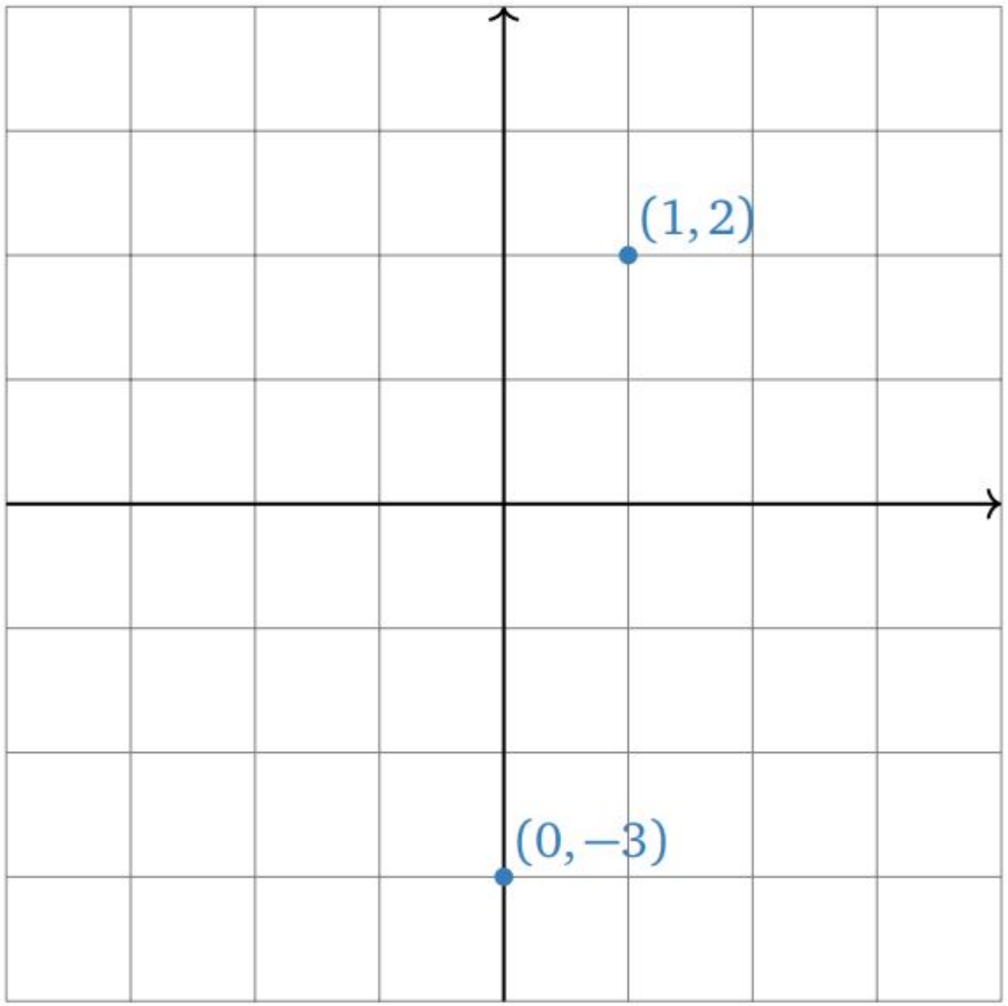
An n-tuple of real numbers is called a **point** of $\mathbf{R}^n$.

In other words, $\mathbf{R}^n$ is just the set of all (ordered) lists of n real numbers. We will draw pictures of $\mathbf{R}^n$ in a moment, but keep in mind that *this is the definition*. For example, $(0, \frac{3}{2}, -\pi)$ and $(1, -2, 3)$ are points of $\mathbf{R}^3$.

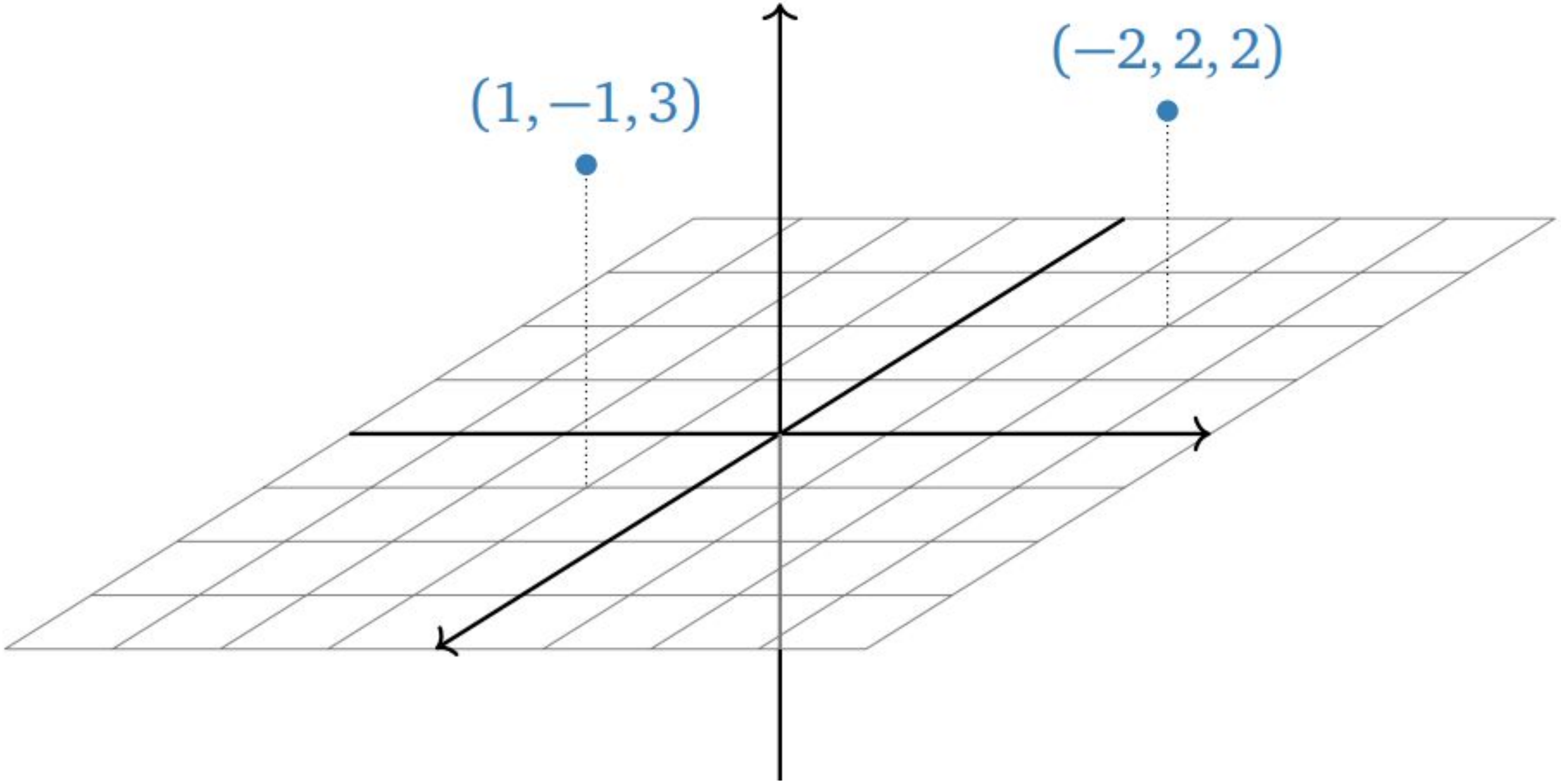
Example (The number line)**.** When $n = 1$, we just get $\mathbf{R}$ back: $\mathbf{R}^1 = \mathbf{R}$. Geometrically, this is the number line.



Example (The Euclidean plane)**.** When $n = 2$, we can think of $\mathbf{R}^2$ as the xy-plane. We can do so because every point on the plane can be represented by an ordered pair of real numbers, namely, its x- and y-coordinates.



Example (3-Space)**.** When $n = 3$, we can think of $\mathbf{R}^3$ as the *space* we (appear to) live in. We can do so because every point in space can be represented by an ordered triple of real numebrs, namely, its x-, y-, and z-coordinates.



So what is $\mathbf{R}^4$? or $\mathbf{R}^5$? or $\mathbf{R}^n$? These are harder to visualize, so you have to go back to the definition: $\mathbf{R}^n$ is the set of all ordered n-tuples of real numbers $(x_1, x_2, x_3, \ldots, x_n)$.

They are still "geometric" spaces, in the sense that our intuition for $\mathbf{R}^2$ and $\mathbf{R}^3$ often extends to $\mathbf{R}^n$.

We will make definitions and state theorems that apply to any $\mathbf{R}^n$, but we will only draw pictures for $\mathbf{R}^2$ and $\mathbf{R}^3$.

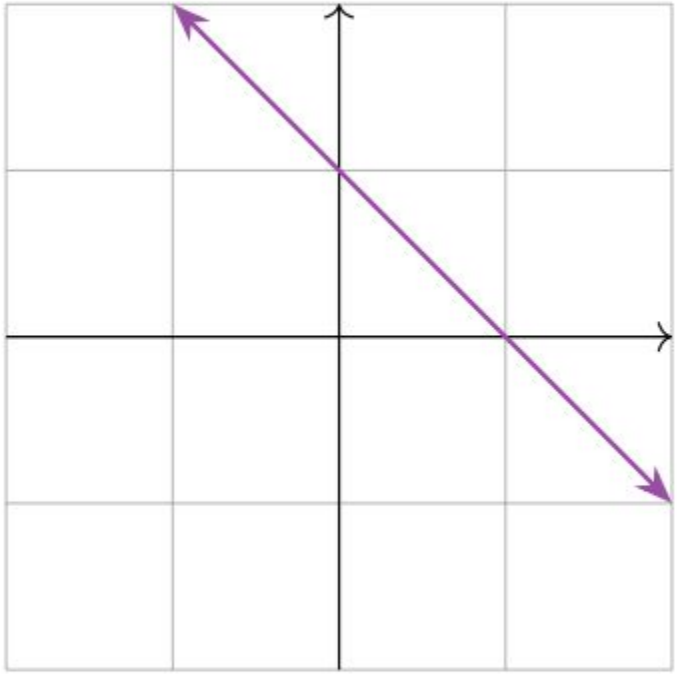
The power of using these spaces is the ability to *label* various objects of interest, such as geometric objects and solutions of systems of equations, by the points of $\mathbf{R}^n$.

Pictures of Solution Sets

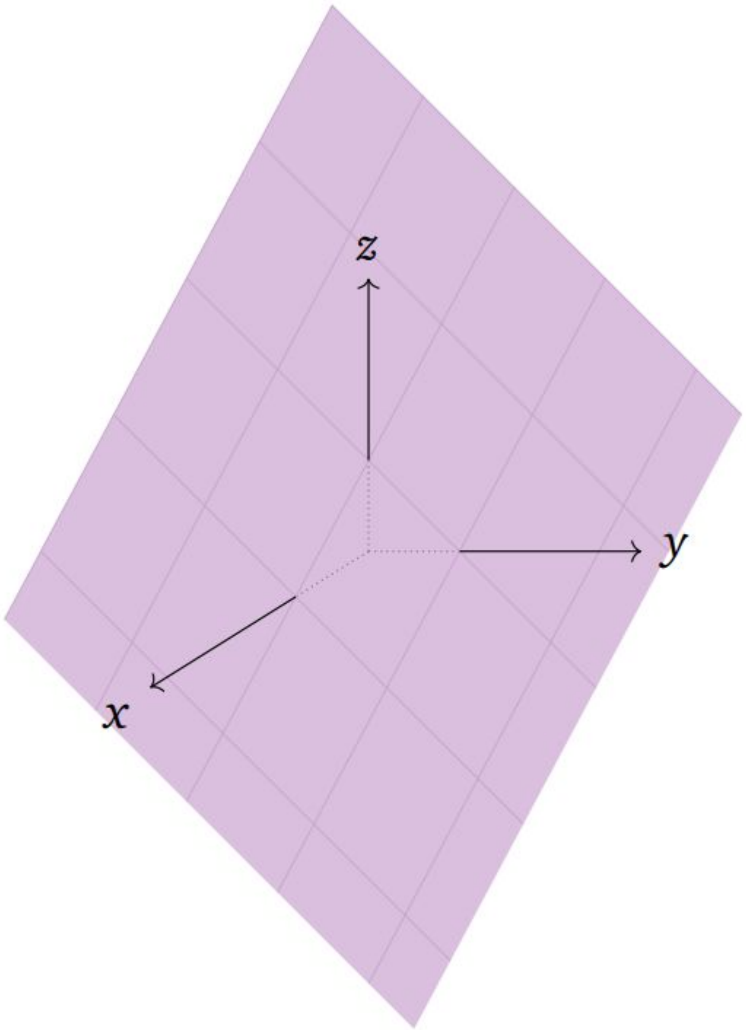
Before discussing how to solve a system of linear equations below, it is helpful to see some pictures of what these solution sets look like geometrically

One Equation in Two Variables.

Consider the linear equation $x + y = 1$. We can rewrite this as $y = 1 - x$, which defines a line in the plane: the slope is -1, and the x-intercept is 1.



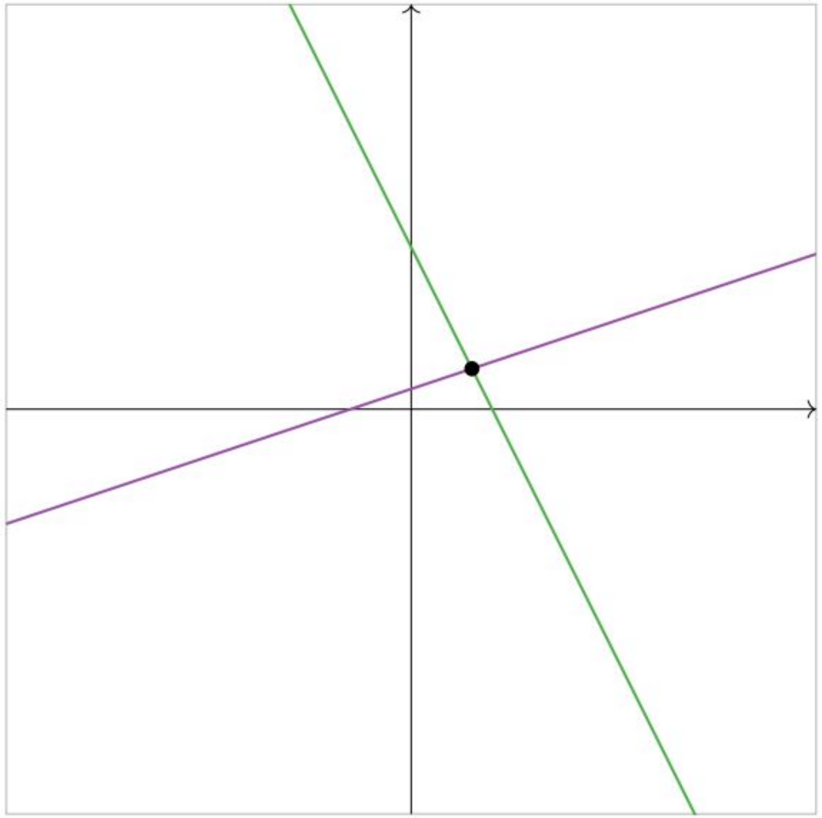
One Equation in Three Variables. Consider the linear equation $x + y + z = 1$. This is the **implicit equation** for a plane in space.



Two Equations in Two Variables. Now consider the system of two linear equations

$$\begin{cases} x - 3y = -3 \\ 2x + y = 8. \end{cases}$$

Each equation individually defines a line in the plane, pictured below.

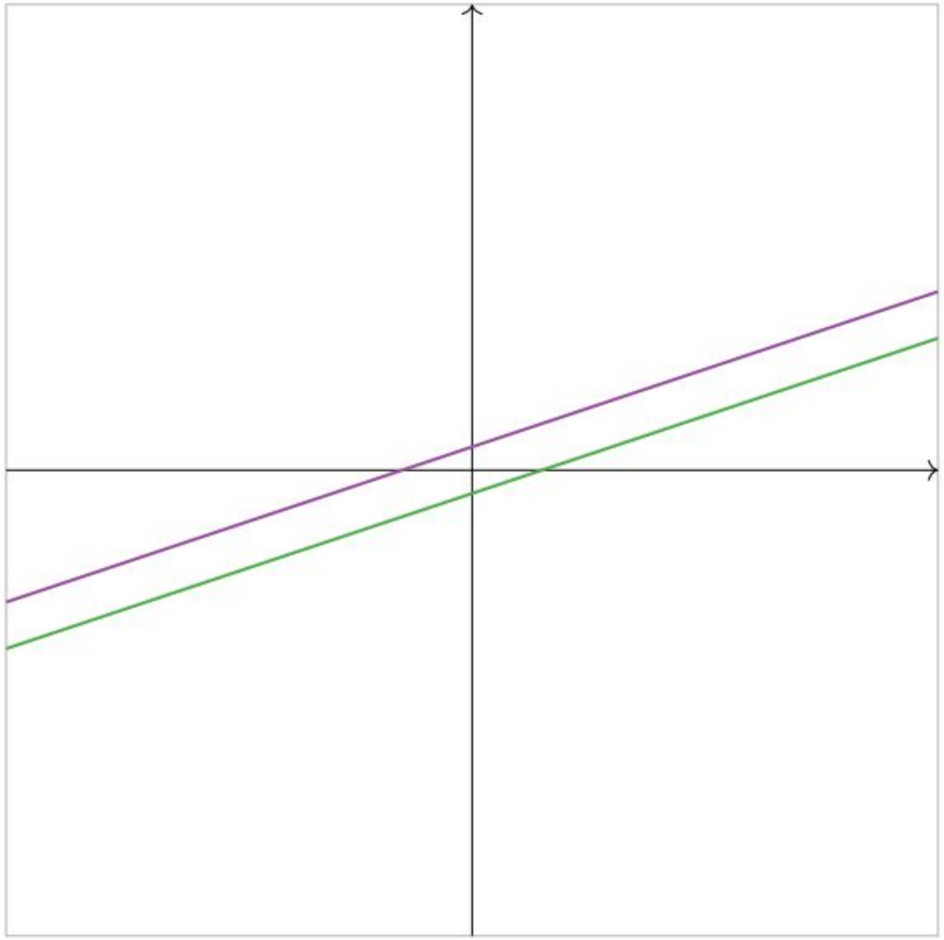


A solution to the *system* of both equations is a pair of numbers (x, y) that makes both equations true at once. In other words, it as a point that lies on both lines simultaneously. We can see in the picture above that there is only one point where the lines intersect: therefore, this system has exactly one solution. (This solution is (3,2))

Usually, two lines in the plane will intersect in one point, but of course this is not always the case.

Consider now the system of equations
$$\begin{cases} x - 3y = -3 \\ x - 3y = 3. \end{cases}$$

These define *parallel* lines in the plane.

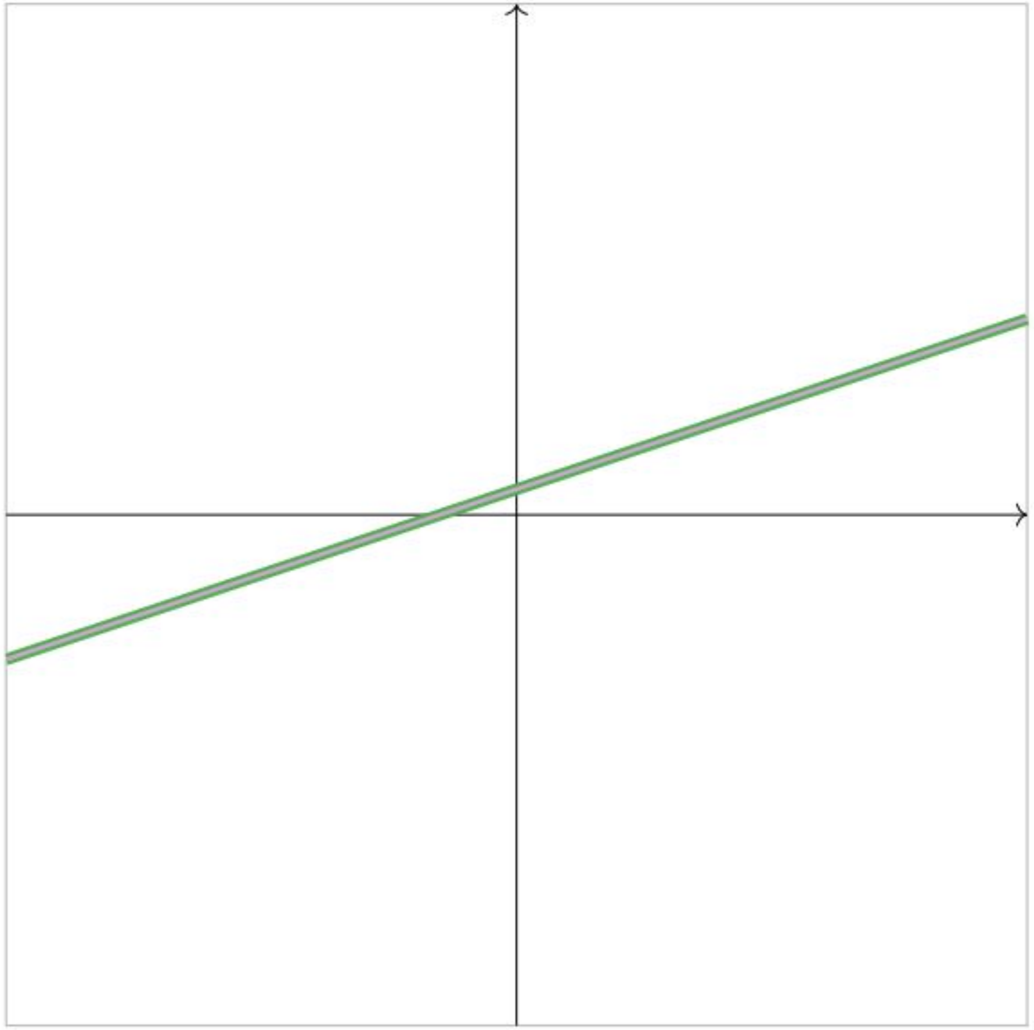


The fact that that the lines do not intersect means that the system of equations has no solution. Of course, this is easy to see algebraically: if $x - 3y = -3$, then it is cannot also be the case that $x - 3y = 3$.

There is one more possibility. Consider the system of equations

$$\begin{cases} x - 3y = -3 \\ 2x - 6y = -6. \end{cases}$$

The second equation is a multiple of the first, so these equations define the *same* line in the plane.



In this case, there are infinitely many solutions of the system of equations.

So the system of linear equations can have:

- Only one solution (unique solution)
- No solution
- Infinite many solutions

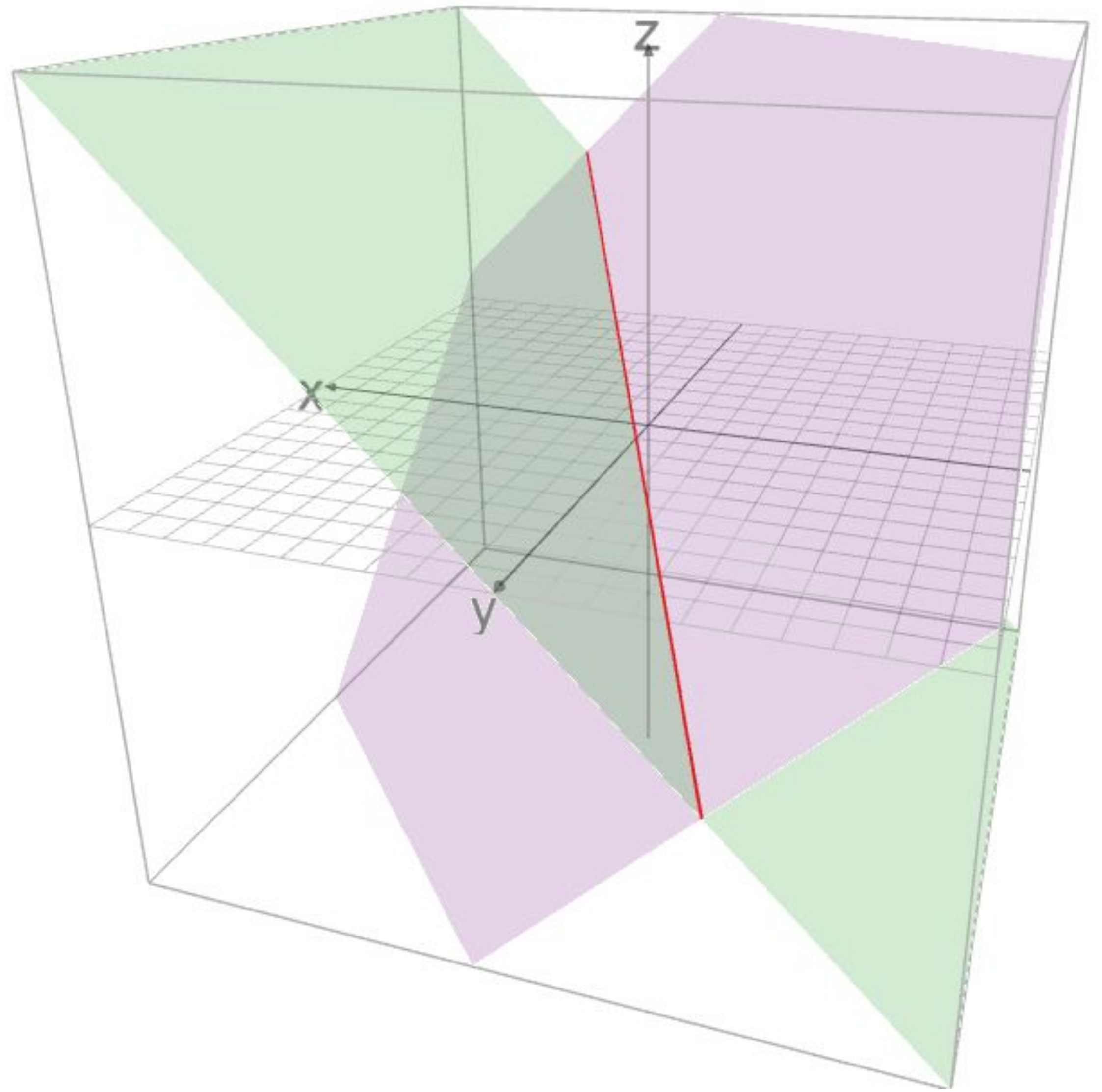
Two Equations in Three Variables. Consider the system of two linear equations

$$\begin{cases} x + y + z = 1 \\ x - z = 0. \end{cases}$$

Each equation individually defines a plane in space. The solutions of the system of both equations are the points that lie on both planes. We can see in the picture below that the planes intersect in a line. In particular, this system has infinitely many solutions.

The planes defined by the equations $x+y+z=1$ and $x-z=0$ intersect in the red line, which is the solution set of the system of both equations.

Remark. In general, the solutions of a system of equations in n variables is the intersection of "$(n-1)$-planes" in n-space. This is always some kind of linear space, as we will discuss in Section 2.4.



The planes defined by the equations $x + y + z = 1$ *and* $x - z = 0$ *intersect in the red line, which is the solution set of the system of both equations.*

Row Reduction

Objectives

1. Learn to replace a system of linear equations by an augmented matrix.
2. Learn how the elimination method corresponds to performing row operations on an augmented matrix.
3. Understand when a matrix is in (reduced) row echelon form.
4. Learn which row reduced matrices come from inconsistent linear systems.
5. *Recipe:* the row reduction algorithm.
6. *Vocabulary words:* **row operation**, **row equivalence**, **matrix**, **augmented matrix**, **pivot**, **(reduced) row echelon form**.

The Elimination Method

We will solve systems of linear equations algebraically using the **elimination** method. In other words, we will combine the equations in various ways to try to eliminate as many variables as possible from each equation. There are three valid operations we can perform on our system of equations:

- ***Scaling:*** we can multiply both sides of an equation by a nonzero number.

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \xrightarrow{\text{multiply 1st by } -3} \begin{cases} -3x - 6y - 9z = -18 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases}$$

- ***Replacement:*** we can add a multiple of one equation to another, replacing the second equation with the result.

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \xrightarrow{\text{2nd} = \text{2nd}-2\times\text{1st}} \begin{cases} x + 2y + 3z = 6 \\ -7y - 4z = 2 \\ 3x + y - z = -2 \end{cases}$$

- ***Swap:*** we can swap two equations.

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \xrightarrow{\text{3rd} \longleftrightarrow \text{1st}} \begin{cases} 3x + y - z = -2 \\ 2x - 3y + 2z = 14 \\ x + 2y + 3z = 6 \end{cases}$$

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2. \end{cases} \tag{1.1.1}$$

Solve (1.1.1) using the elimination method.

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \xrightarrow{\text{2nd} \;=\; \text{2nd}-2\times\text{1st}} \begin{cases} x + 2y + 3z = 6 \\ -7y - 4z = 2 \\ 3x + y - z = -2 \end{cases}$$

$$\xrightarrow{\text{3rd} \;=\; \text{3rd}-3\times\text{1st}} \begin{cases} x + 2y + 3z = 6 \\ -7y - 4z = 2 \\ -5y - 10z = -20 \end{cases}$$

$$\xrightarrow{\text{2nd} \longleftrightarrow \text{3rd}} \begin{cases} x + 2y + 3z = 6 \\ -5y - 10z = -20 \\ -7y - 4z = 2 \end{cases}$$

$$\xrightarrow{\text{divide 2nd by } -5} \begin{cases} x + 2y + 3z = 6 \\ y + 2z = 4 \\ -7y - 4z = 2 \end{cases}$$

$$\xrightarrow{\text{3rd} \ = \ \text{3rd}+7\times\text{2nd}} \begin{cases} x + 2y + 3z = 6 \\ y + 2z = 4 \\ 10z = 30 \end{cases}$$

At this point we've eliminated both x and y from the third equation, and we can solve $10z = 30$ to get $z = 3$. Substituting for z in the second equation gives $y + 2 \cdot 3 = 4$, or $y = -2$. Substituting for y and z in the first equation gives $x + 2 \cdot (-2) + 3 \cdot 3 = 6$, or $x = 3$. Thus the only solution is $(x, y, z) = (1, -2, 3)$.

Augmented Matrices and Row Operations Solving equations by elimination requires writing the variables x, y, z and the equals sign $=$ over and over again, merely as placeholders: all that is changing in the equations is the coefficient *numbers*. We can make our life easier by extracting only the numbers, and putting them in a box:

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \xrightarrow{\text{becomes}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right).$$

This is called an **augmented matrix**. The word "augmented" refers to the vertical line, which we draw to remind ourselves where the equals sign belongs; a **matrix** is a grid of numbers without the vertical line. In this notation, our three valid ways of manipulating our equations become **row operations**:

All three operation will still valid

- ***Scaling:*** multiply all entries in a row by a nonzero number.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_1 = R_1 \times -3} \left(\begin{array}{ccc|c} -3 & -6 & -9 & -18 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

  Here the notation R_1 simply means "the first row", and likewise for R_2, R_3, etc.

- ***Replacement:*** add a multiple of one row to another, replacing the second row with the result.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2 \times R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

- ***Swap:*** interchange two rows.

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_3} \left(\begin{array}{rrr|r} 3 & 1 & -1 & -2 \\ 2 & -3 & 2 & 14 \\ 1 & 2 & 3 & 6 \end{array}\right)$$

Example. Solve (1.1.1) using row operations.

Solution. We start by forming an augmented matrix:

$$\left\{\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}\right. \xrightarrow{\text{becomes}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right).$$

Eliminating a variable from an equation means producing a zero to the left of the line in an augmented matrix. First we produce zeros in the first column (i.e. we eliminate x) by subtracting multiples of the first row.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2=R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ \color{red}{0} & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_3=R_3-3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ \color{red}{0} & -5 & -10 & -20 \end{array}\right)$$

This was made much easier by the fact that the top-left entry is equal to 1, so we can simply multiply the first row by the number below and subtract. In order to eliminate y in the same way, we would like to produce a 1 in the second column. We could divide the second row by -7, but this would produce fractions; instead, let's divide the third by -5.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right) \xrightarrow{R_3=R_3\div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

$$\xrightarrow{R_2\longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3=R_3+7R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

$$\xrightarrow{R_3=R_3\div 10} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We swapped the second and third row just to keep things orderly. Now we translate this augmented matrix back into a system of equations:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{\text{becomes}} \begin{cases} x + 2y + 3z = 6 \\ y + 2z = 4 \\ z = 3 \end{cases}$$

Hence $z = 3$; back-substituting as in this example gives $(x, y, z) = (1, -2, 3)$.

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Indeed, the whole point of doing these operations is to solve the equations using the elimination method.

Definition. Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.

Example (An Inconsistent System). Solve the following system of equations using row operations:

$$\begin{cases} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{cases}$$

Solution. First we put our system of equations into an augmented matrix.

$$\begin{cases} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{cases} \xrightarrow{\text{augmented matrix}} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right)$$

We clear the entries below the top-left using row replacement.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right) \xrightarrow{R_2=R_2-3R_1} \left(\begin{array}{cc|r} 1 & 1 & 2 \\ {\color{red}0} & 1 & -1 \\ 4 & 5 & 9 \end{array}\right)$$

$$\xrightarrow{R_3=R_3-4R_1} \left(\begin{array}{cc|r} 1 & 1 & 2 \\ 0 & 1 & -1 \\ {\color{red}0} & 1 & 1 \end{array}\right)$$

Now we clear the second entry from the last row.

$$\left(\begin{array}{cc|r} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{R_3=R_3-R_2} \left(\begin{array}{cc|r} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & \color{red}{0} & 2 \end{array}\right)$$

This translates back into the system of equations

$$\begin{cases} x + y = 2 \\ \phantom{x + {}} y = -1 \\ \phantom{x + {}} 0 = 2. \end{cases}$$

Our original system has the same solution set as this system. But this system has no solutions: there are no values of x, y making the third equation true! We conclude that our original equation was inconsistent.

Row Echelon Form

We want to reduce the augmented matrix to Echelon form, to get it solved.

Definition. A matrix is in **row echelon form** if:

1. All zero rows are at the bottom.
2. The first nonzero entry of a row is to the *right* of the first nonzero entry of the row above.
3. Below the first nonzero entry of a row, all entries are zero.

Here is a picture of a matrix in row echelon form:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} \star &= \text{any number} \\ \boxed{\star} &= \text{any nonzero number} \end{aligned}$$

Definition. A **pivot** is the first nonzero entry of a row of a matrix in row echelon form.

A matrix in row-echelon form is generally easy to solve using back-substitution. For example,

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{\text{becomes}} \begin{cases} x + 2y + \ 3z = \ 6 \\ \qquad y + \ 2z = \ 4 \\ \qquad\quad 10z = 30. \end{cases}$$

We immediately see that $z = 3$, which implies $y = 4 - 2 \cdot 3 = -2$ and $x = 6 - 2(-2) - 3 \cdot 3 = 1$.

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition:

5. Each pivot is equal to 1.

6. Each pivot is the only nonzero entry in its column.

Here is a picture of a matrix in reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \star = \text{any number} \\ 1 = \text{pivot} \end{array}$$

If an augmented matrix is in reduced row echelon form, the corresponding linear system is viewed as *solved*.

Definition. A **pivot position** of a matrix is an entry that is a pivot of a row echelon form of that matrix.

A **pivot column** of a matrix is a column that contains a pivot position.

Example (Pivot Positions)**.** Find the pivot positions and pivot columns of this matrix

$$A = \left(\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right).$$

row echelon form of the matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right).$$

The pivot positions of A are the entries that become pivots in a row echelon form they are marked in red below:

$$\left(\begin{array}{ccc|c} \color{red}{0} & -7 & -4 & 2 \\ 2 & \color{red}{4} & 6 & 12 \\ 3 & 1 & \color{red}{-1} & -2 \end{array}\right).$$

The first, second, and third columns are pivot columns.

We have discussed two classes of matrices so far:

1. When the reduced row echelon form of a matrix has a pivot in every non-augmented column, then it corresponds to a system with a unique solution:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{\text{translates to}} \begin{cases} x \qquad\qquad = 1 \\ \qquad y \qquad = -2 \\ \qquad\qquad z = 3. \end{cases}$$

2. When the reduced row echelon form of a matrix has a pivot in the last (augmented) column, then it corresponds to a system with a no solutions:

$$\left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{translates to}} \begin{cases} x + 5y = 0 \\ \qquad\; 0 = 1. \end{cases}$$

What happens when one of the non-augmented columns lacks a pivot? : Multiple solutions

Example (A System with Many Solutions)**.** Solve the linear system

$$\begin{cases} 2x + y + 12z = 1 \\ x + 2y + 9z = -1 \end{cases}$$

Solution.

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} {\color{red}1} & 2 & 9 & -1 \\ 2 & 1 & 12 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ {\color{red}0} & -3 & -6 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div -3} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & {\color{red}1} & 2 & -1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & {\color{red}0} & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

This row reduced matrix corresponds to the linear system

$$\begin{cases} x + 5z = 1 \\ y + 2z = -1. \end{cases}$$

Summary

1. ***The last column is a pivot column.*** In this case, the system is *inconsistent*. There are zero solutions, i.e., the solution set is empty. For example, the matrix

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

comes from a linear system with no solutions.

2. ***Every column except the last column is a pivot column.*** In this case, the system has a *unique* solution. For example, the matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$$

tells us that the unique solution is $(x, y, z) = (a, b, c)$.

3. ***The last column is not a pivot column, and some other column is not a pivot column either.*** In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). For example, in the system corresponding to the matrix

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right),$$

any values for x_2 and x_4 yield a solution to the system of equations.

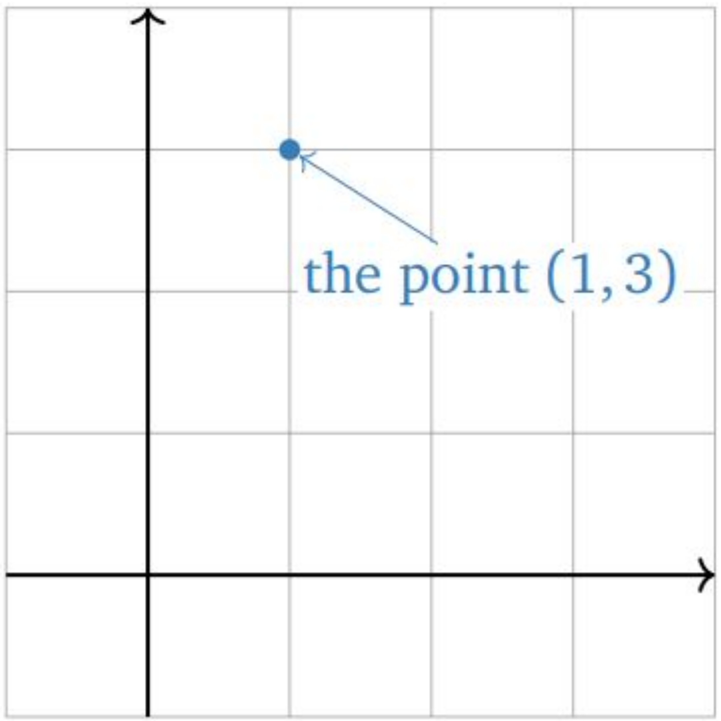
Systems of Linear Equations: Geometry

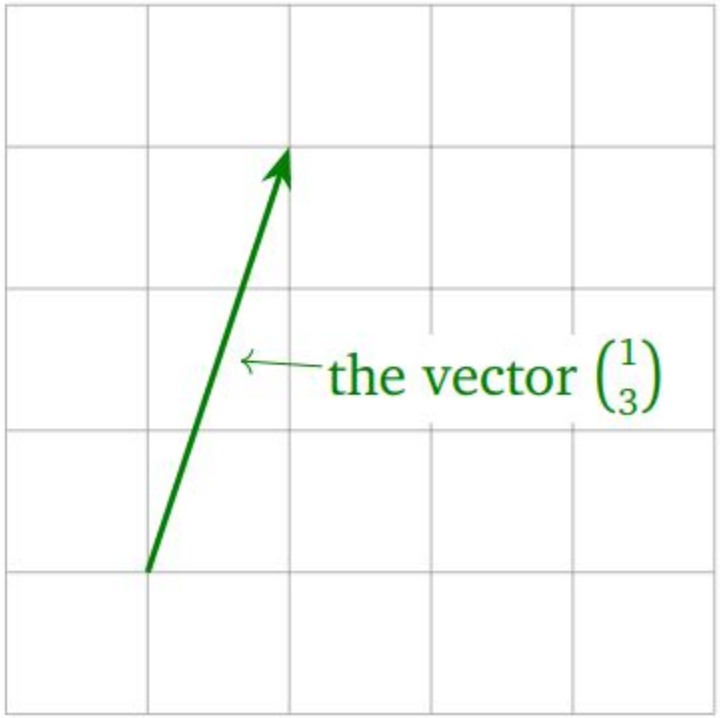
2.1 Vectors

Objectives

1. Learn how to add and scale vectors in $\mathbf{R}^n$, both algebraically and geometrically.
2. Understand linear combinations geometrically.
3. *Pictures:* vector addition, vector subtraction, linear combinations.
4. *Vocabulary words:* **vector**, **linear combination**.

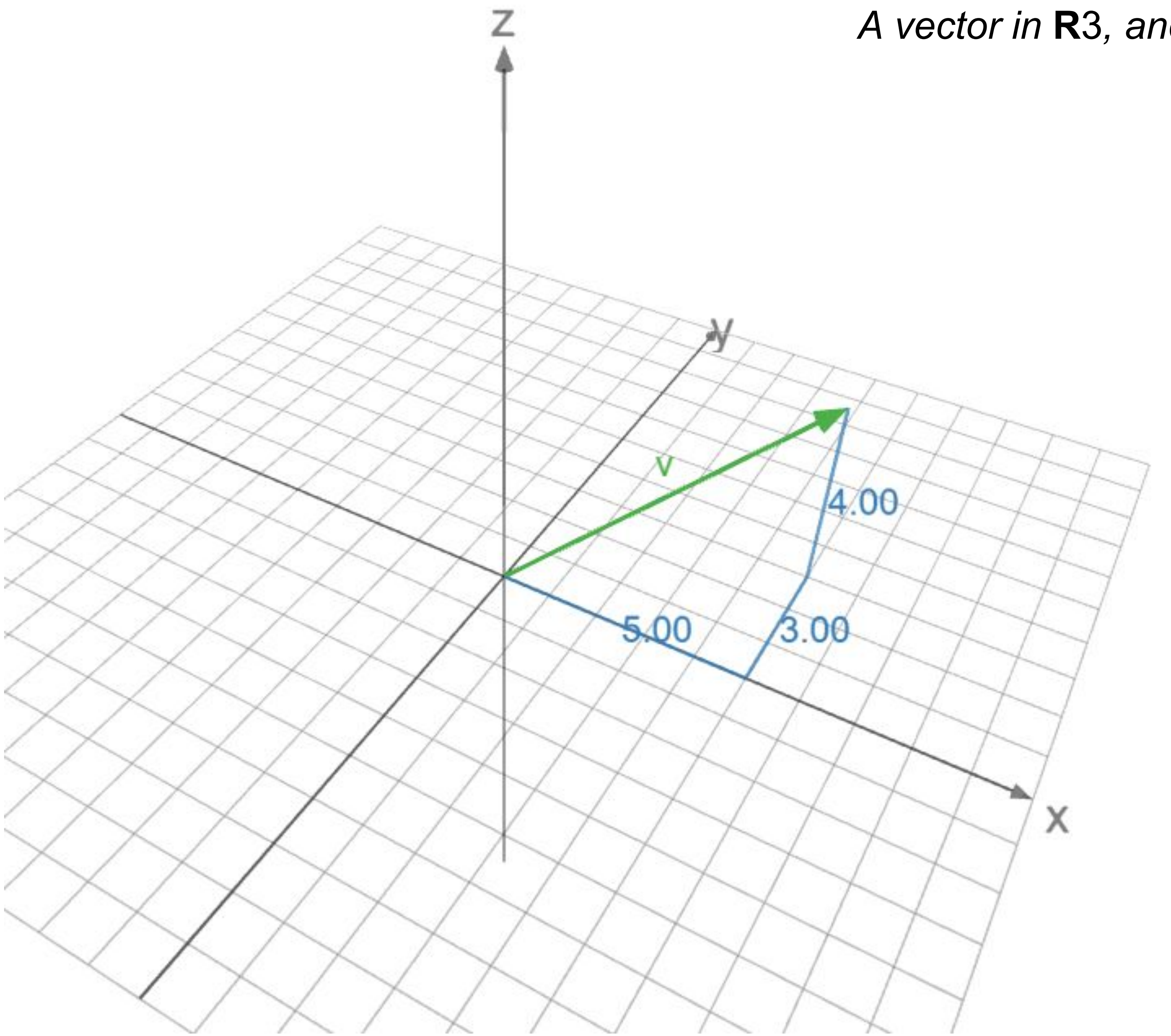
Points and Vectors. Again, a **point** in $\mathbf{R}^n$ is drawn as a dot.



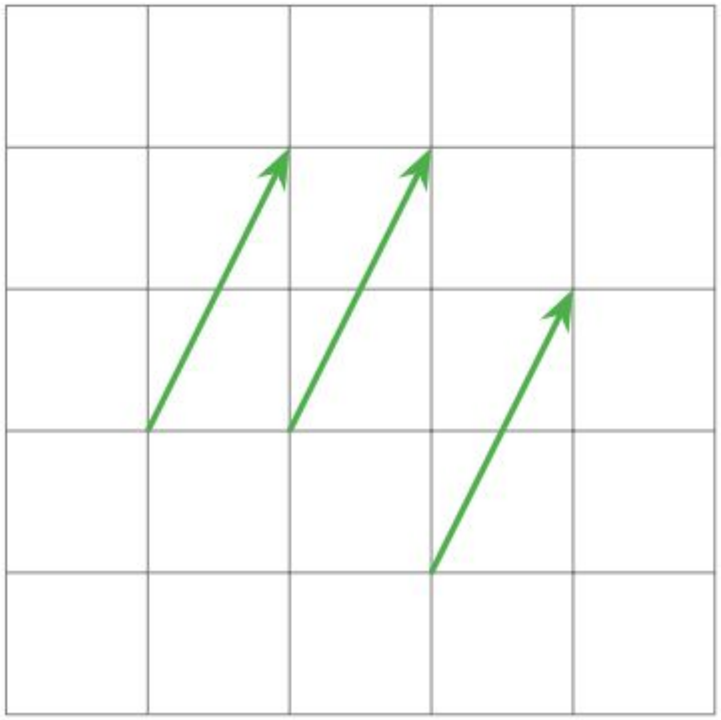


A **vector** is a point in $\mathbf{R}^n$, drawn as an arrow.

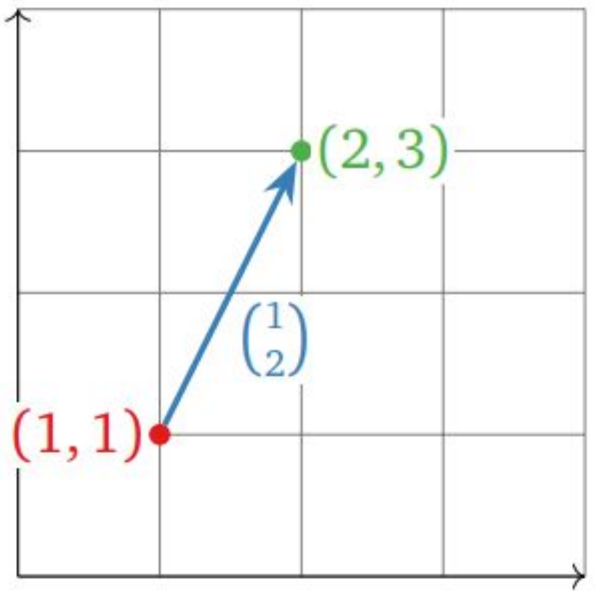
A vector in **R**3*, and its coordinates.*



Why make the distinction between points and vectors? A vector need not start at the origin: *it can be located anywhere*! In other words, an arrow is determined by its length and its direction, not by its location. For instance, these arrows all represent the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.



Note. Another way to think about a vector is as a *difference* between two points, or the arrow from one point to another. For instance, $\binom{1}{2}$ is the arrow from $(1,1)$ to $(2,3)$.



2.1.2 Vector Algebra and Geometry

Here we learn how to add vectors together and how to multiply vectors by numbers, both algebraically and geometrically.

Vector addition and scalar multiplication.

- We can add two vectors together:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}.$$

- We can multiply, or **scale**, a vector by a real number c:

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

  We call c a **scalar** to distinguish it from a vector. If v is a vector and c is a scalar, then cv is called a **scalar multiple** of v.

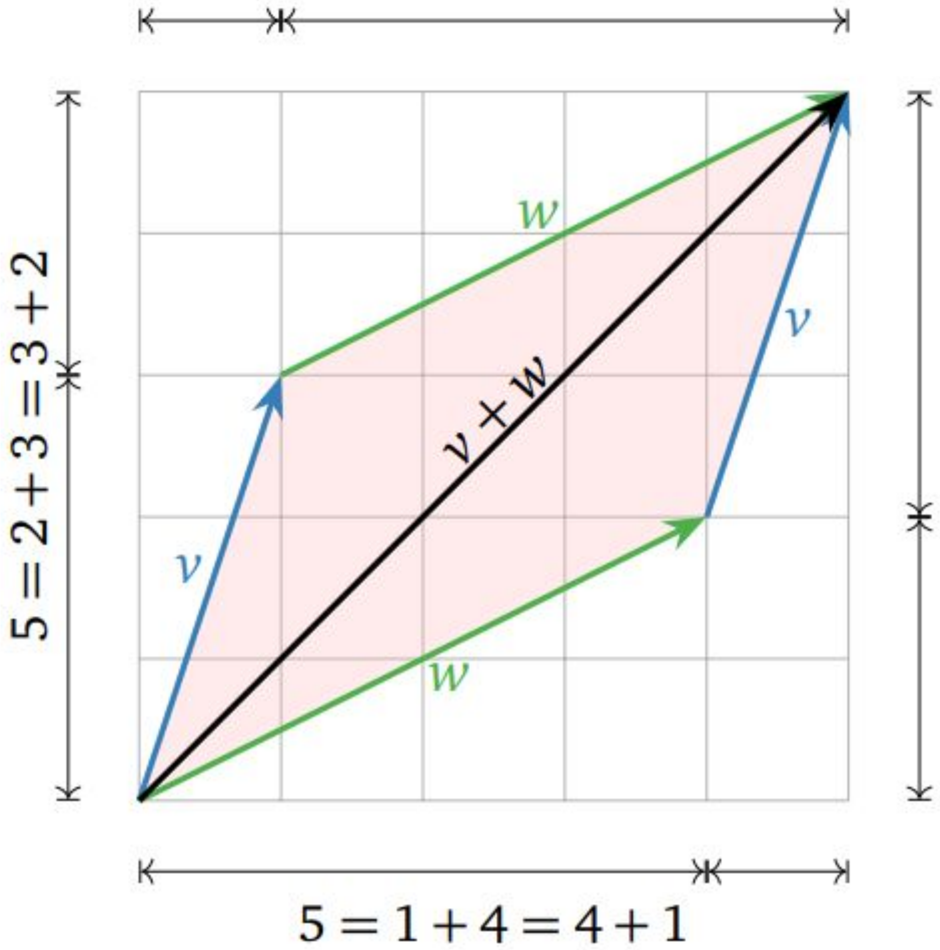
Example.

$$\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}4\\5\\6\end{pmatrix}=\begin{pmatrix}5\\7\\9\end{pmatrix} \quad \text{and} \quad -2\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-2\\-4\\-6\end{pmatrix}.$$

The Parallelogram Law for Vector Addition Geometrically, the sum of two vectors v, w is obtained as follows: place the tail of w at the head of v. Then $v+w$ is the vector whose tail is the tail of v and whose head is the head of w. Doing this both ways creates a parallelogram. For example,

$$\begin{pmatrix}1\\3\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}=\begin{pmatrix}5\\5\end{pmatrix}.$$

Add these vectors geometrically



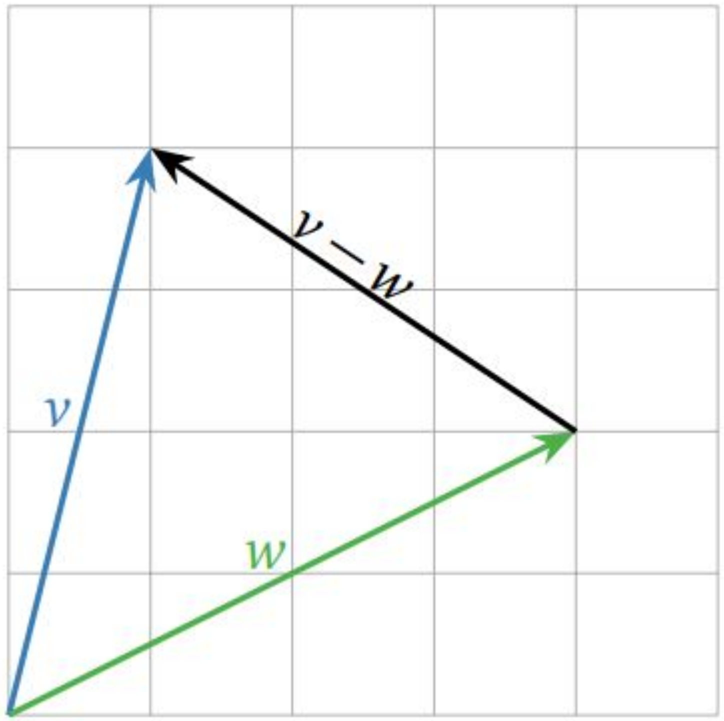
w
v
$v+w$
v
w
$5 = 2 + 3 = 3 + 2$
$5 = 1 + 4 = 4 + 1$

Vector Subtraction Geometrically, the difference of two vectors v, w is obtained as follows: place the tail of v and w at the same point. Then $v - w$ is the vector from the head of w to the head of v. For example,

$$\begin{pmatrix}1\\4\end{pmatrix} - \begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix}.$$

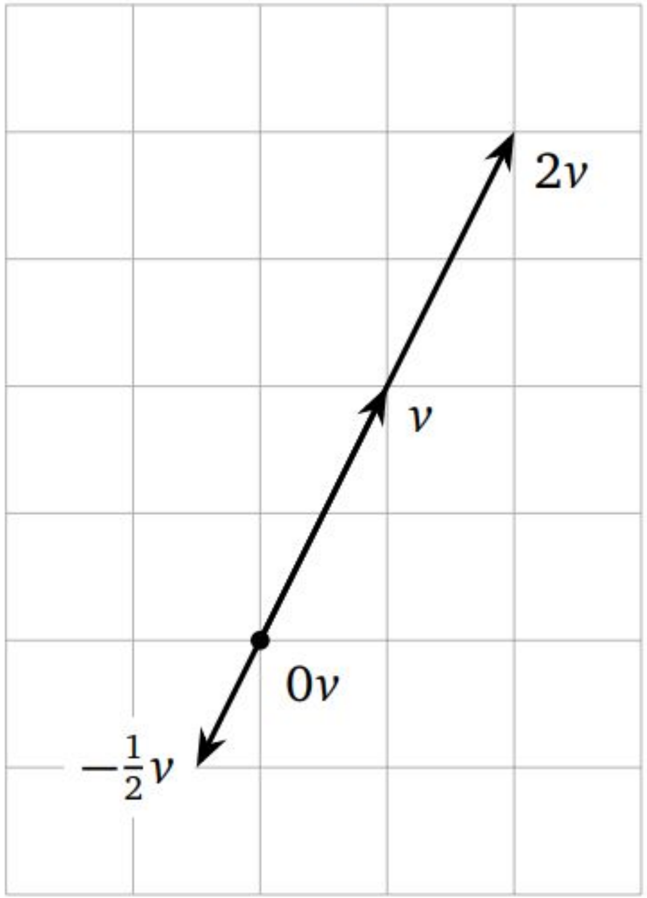
Subtract these vectors geometrically

Why? If you add $v - w$ to w, you get v.

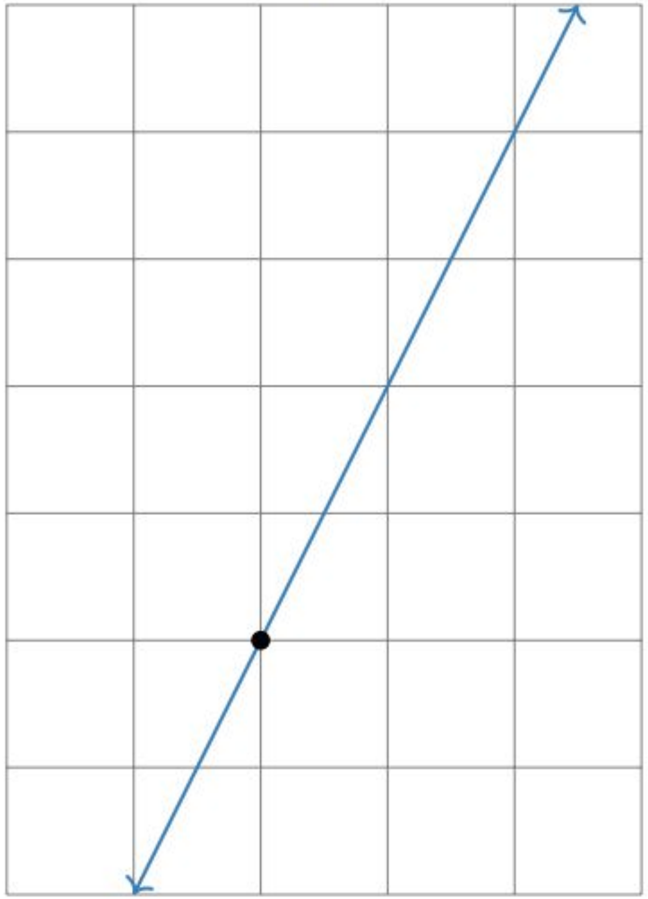


Scalar Multiplication A scalar multiple of a vector v has the same (or opposite) direction, but a different length. For instance, $2v$ is the vector in the direction of v but twice as long, and $-\frac{1}{2}v$ is the vector in the opposite direction of v, but half as long. Note that the set of all scalar multiples of a (nonzero) vector v is a *line*.

Some multiples of v.



All multiples of v.



2.1.3 Linear Combinations

We can add and scale vectors in the same equation.

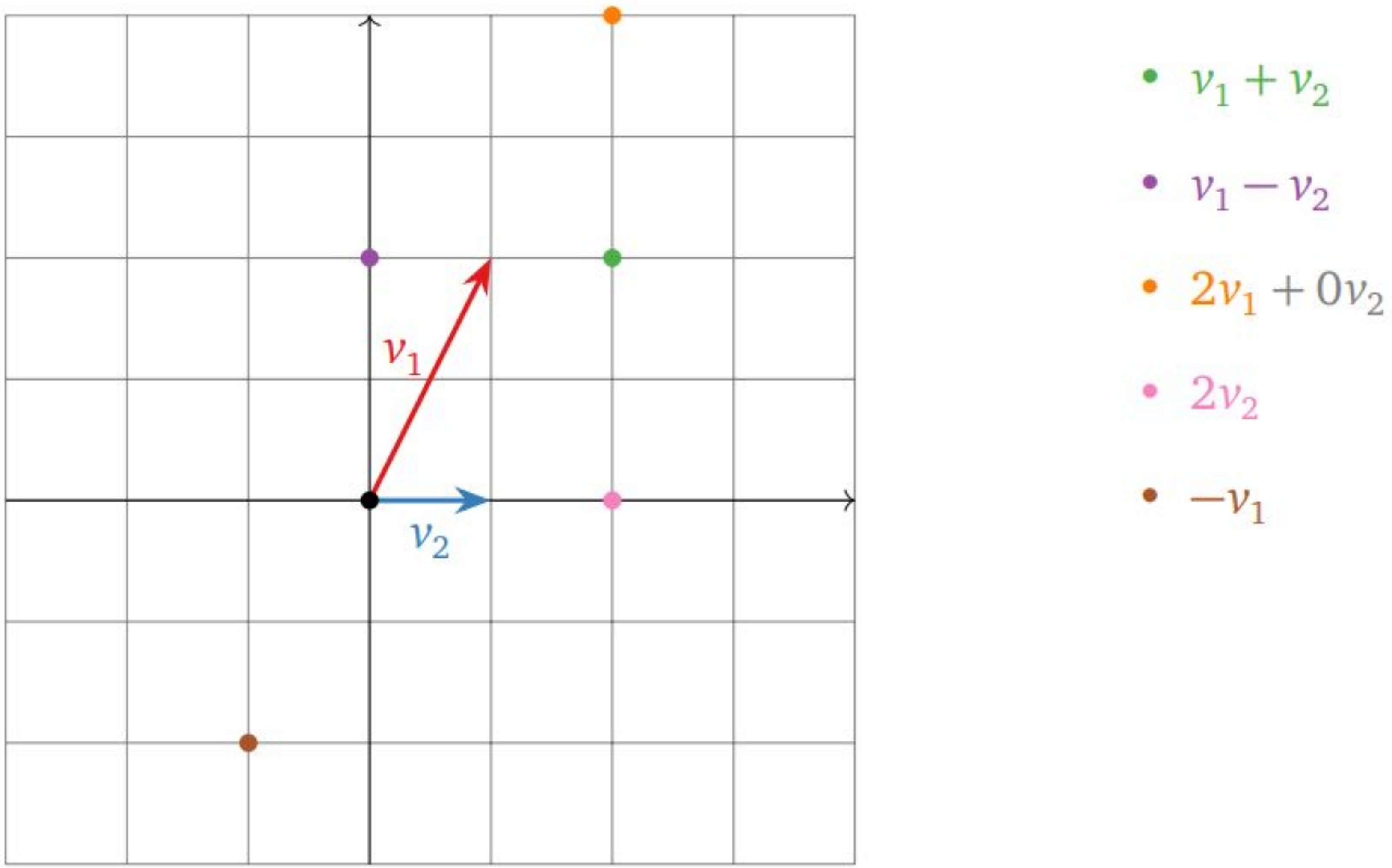
Definition. Let $c_1, c_2, \ldots, c_k$ be scalars, and let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbf{R}^n$. The vector in $\mathbf{R}^n$

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

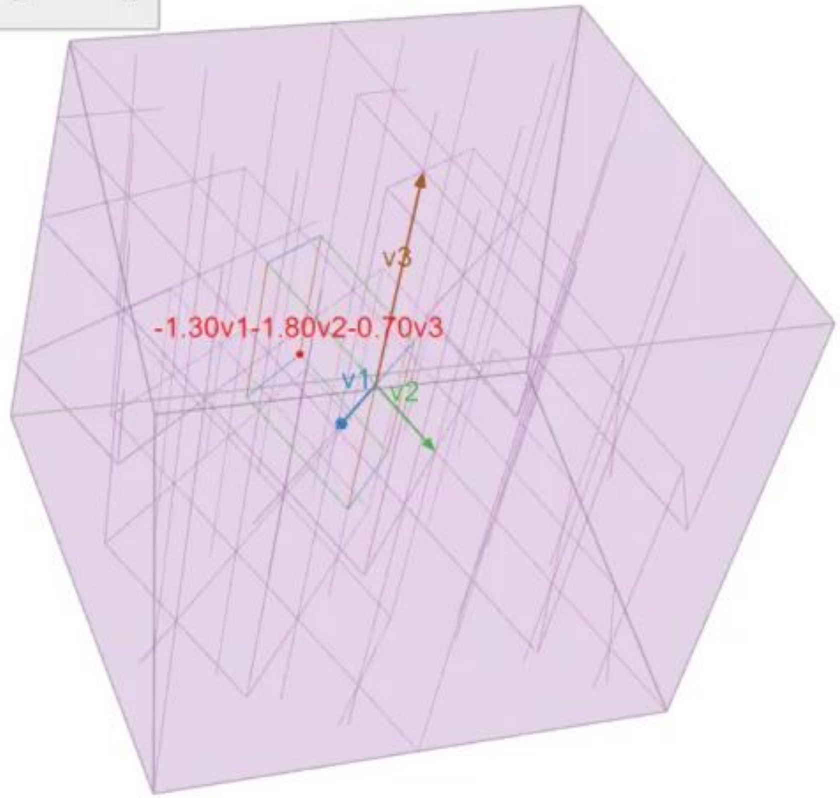
is called a **linear combination** of the vectors $v_1, v_2, \ldots, v_k$, with **weights** or **coefficients** $c_1, c_2, \ldots, c_k$.

Geometrically, a linear combination is obtained by stretching / shrinking the vectors $v_1, v_2, \ldots, v_k$ according to the coefficients, then adding them together using the parallelogram law.

Example. Let $v_1 = \binom{1}{2}$ and $v_2 = \binom{1}{0}$. Here are some linear combinations of v_1 and v_2, drawn as points.



$$-1.30\begin{bmatrix}2.00\\-1.00\\1.00\end{bmatrix}-1.80\begin{bmatrix}1.00\\1.00\\-1.00\end{bmatrix}-0.70\begin{bmatrix}-1.00\\1.00\\4.00\end{bmatrix}=\begin{bmatrix}-3.70\\-1.20\\-2.30\end{bmatrix}$$



Linear Combinations of the vector

2.2 Vector Equations and Spans

Objectives

1. Understand the equivalence between a system of linear equations and a vector equation.
2. Learn the definition of Span$\{x_1, x_2, \ldots, x_k\}$, and how to draw pictures of spans.
3. *Recipe:* solve a vector equation using augmented matrices / decide if a vector is in a span.
4. *Pictures:* an inconsistent system of equations, a consistent system of equations, spans in $\mathbf{R}^2$ and $\mathbf{R}^3$.
5. *Vocabulary word:* **vector equation**.
6. *Essential vocabulary word:* **span**.

2.2.1 Vector Equations

An equation involving vectors with n coordinates is the same as n equations involving only numbers. For example, the equation

$$x\begin{pmatrix}1\\2\\6\end{pmatrix}+y\begin{pmatrix}-1\\-2\\-1\end{pmatrix}=\begin{pmatrix}8\\16\\3\end{pmatrix}$$

simplifies to

$$\begin{pmatrix}x\\2x\\6x\end{pmatrix}+\begin{pmatrix}-y\\-2y\\-y\end{pmatrix}=\begin{pmatrix}8\\16\\3\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}x-y\\2x-2y\\6x-y\end{pmatrix}=\begin{pmatrix}8\\16\\3\end{pmatrix}.$$

For two vectors to be equal, all of their coordinates must be equal, so this is just the system of linear equations

$$\begin{cases}x-y=8\\2x-2y=16\\6x-y=3.\end{cases}$$

Definition. A **vector equation** is an equation involving a linear combination of vectors with possibly unknown coefficients.

Asking whether or not a vector equation has a solution is the same as asking if a given vector is a linear combination of some other given vectors.

For example the vector equation above is asking if the vector $(8,16,3)$ is a linear combination of the vectors $(1,2,6)$ and $(-1,2,-1)$.

The thing we really care about is solving systems of linear equations, not solving vector equations. The whole point of vector equations is that they give us a different, and more geometric, way of viewing systems of linear equations.

In order to actually solve the vector equation

$$x\begin{pmatrix}1\\2\\6\end{pmatrix}+y\begin{pmatrix}-1\\-2\\-1\end{pmatrix}=\begin{pmatrix}8\\16\\3\end{pmatrix},$$

one has to solve the system of linear equations

$$\begin{cases} x - y = 8\\ 2x-2y=16\\ 6x-y=3.\end{cases}$$

This means forming the augmented matrix

$$\left(\begin{array}{cc|c}1 & -1 & 8\\ 2 & -2 & 16\\ 6 & -1 & 3\end{array}\right)$$

Example. Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

Solution. As discussed above, this question boils down to a row reduction:

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right).$$

From this we see that the equation is consistent, and the solution is $x = -1$ and $y = -9$. We conclude that $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ is indeed a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$, with coefficients -1 and -9:

$$-\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

Recipe: Solving a vector equation. In general, the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = b$$

where $v_1, v_2, \ldots, v_k$, b are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_k$ are unknown scalars, has the same solution set as the linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right)$$

whose columns are the v_i's and the b's.

Now we have *three* equivalent ways of thinking about a linear system:

1. As a system of equations:

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 = 7 \\ x_1 - x_2 - 3x_3 = 5 \end{cases}$$

2. As an augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 7 \\ 1 & -1 & -3 & 5 \end{array}\right)$$

3. As a vector equation ($x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

The third is geometric in nature: it lends itself to drawing pictures.

2.2.2 Spans

It will be important to know what are *all* linear combinations of a set of vectors $v_1, v_2, \dots, v_k$ in $\mathbf{R}^n$. In other words, we would like to understand the set of all vectors b in $\mathbf{R}^n$ such that the vector equation (in the unknowns $x_1, x_2, \dots, x_k$)

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = b$$

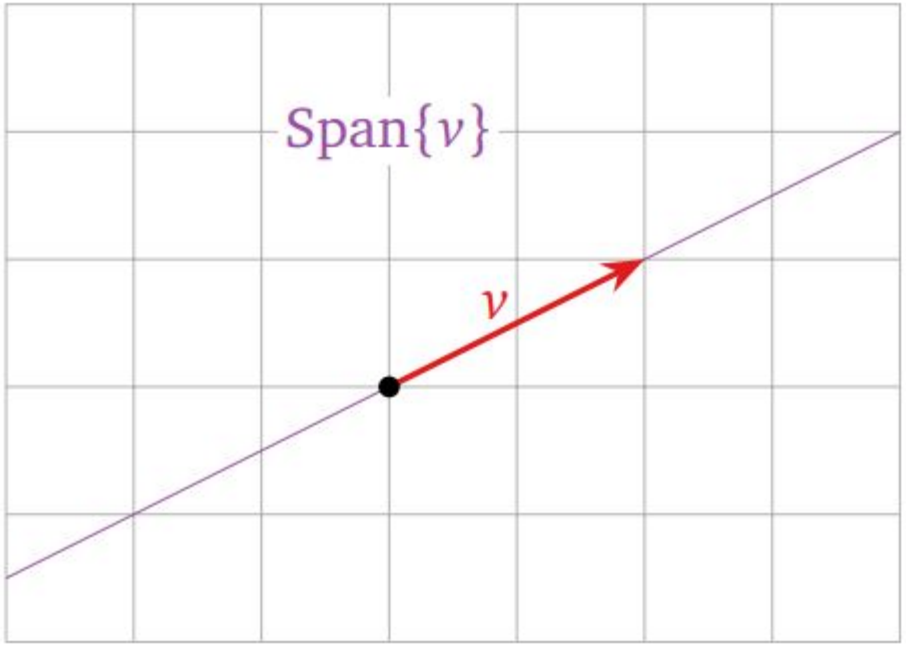
has a solution (i.e. is consistent).

Essential Definition. Let $v_1, v_2, \dots, v_k$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \dots, v_k$ is the collection of all linear combinations of $v_1, v_2, \dots, v_k$, and is denoted $\mathrm{Span}\{v_1, v_2, \dots, v_k\}$. In symbols:

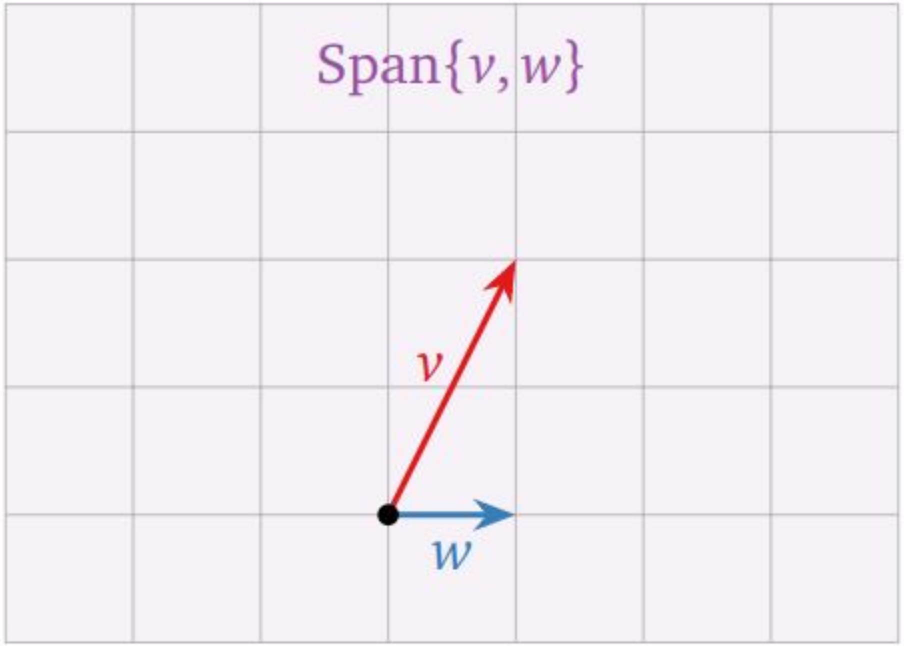
$$\mathrm{Span}\{v_1, v_2, \dots, v_k\} = \bigl\{x_1 v_1 + x_2 v_2 + \cdots + x_k v_k \mid x_1, x_2, \dots, x_k \text{ in } \mathbf{R}\bigr\}$$

We also say that $\mathrm{Span}\{v_1, v_2, \dots, v_k\}$ is the subset **spanned by** or **generated by** the vectors $v_1, v_2, \dots, v_k$.

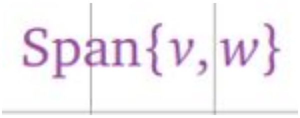
Pictures of Spans Drawing a picture of $\mathrm{Span}\{v_1, v_2, \ldots, v_k\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_k$.

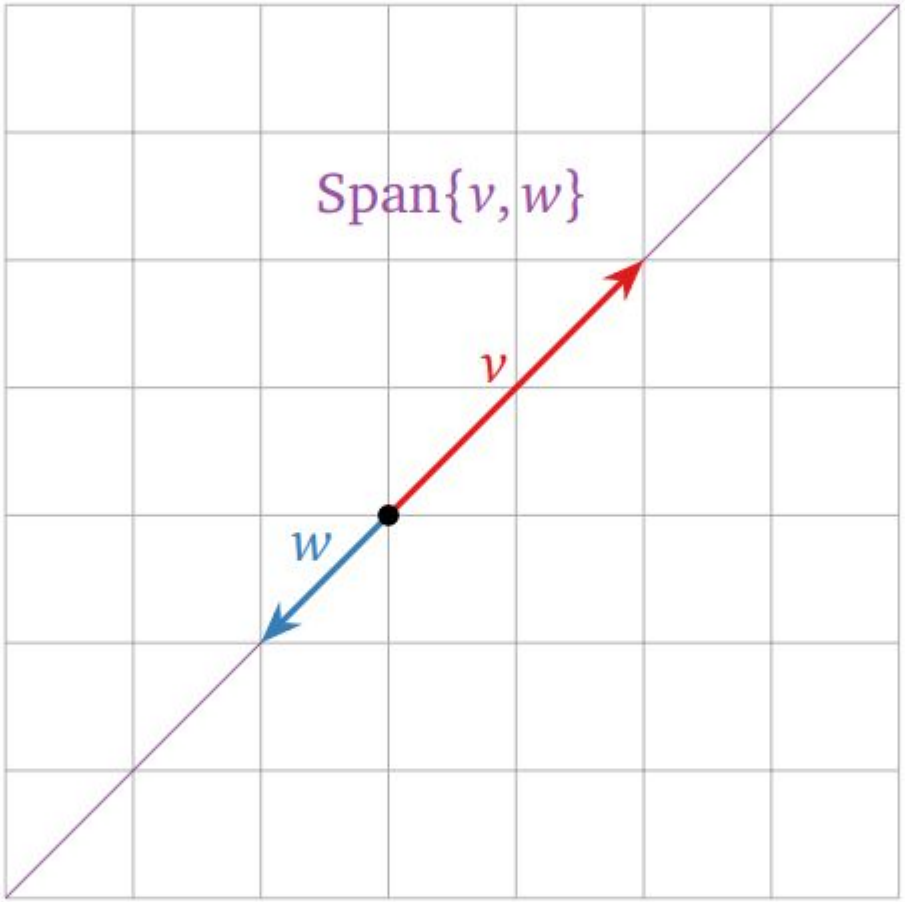


$\mathrm{Span}\{v\}$

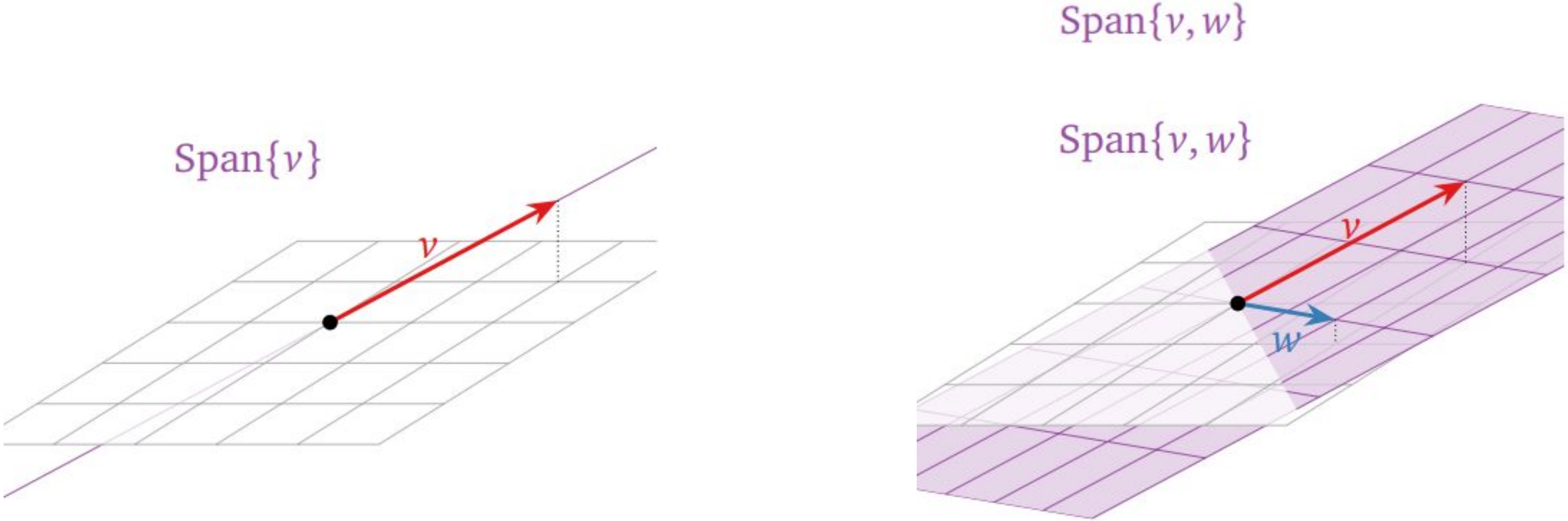


$\mathrm{Span}\{v, w\}$



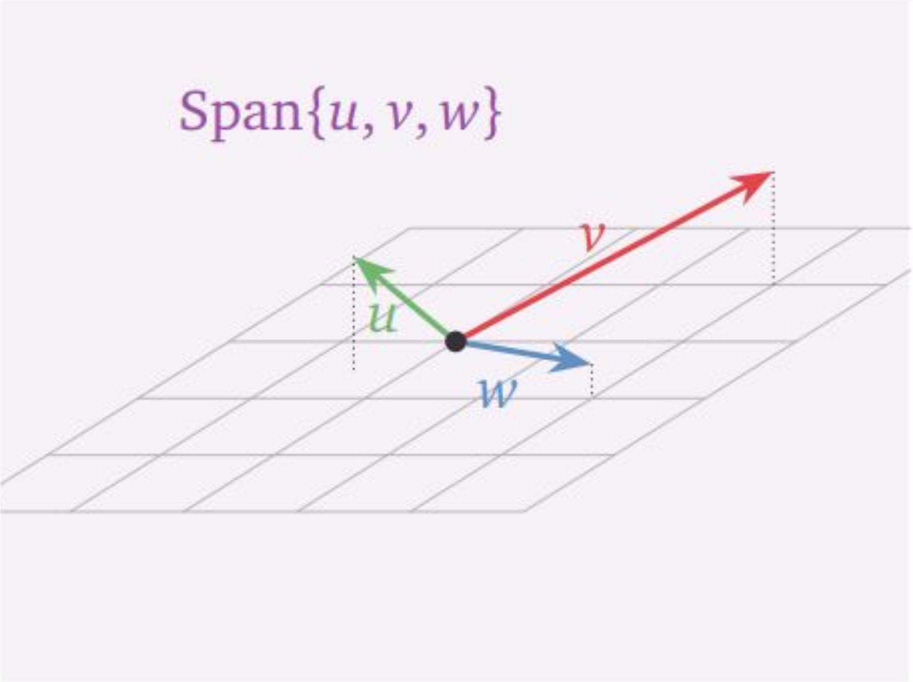


Pictures of spans in $\mathbf{R}^2$.

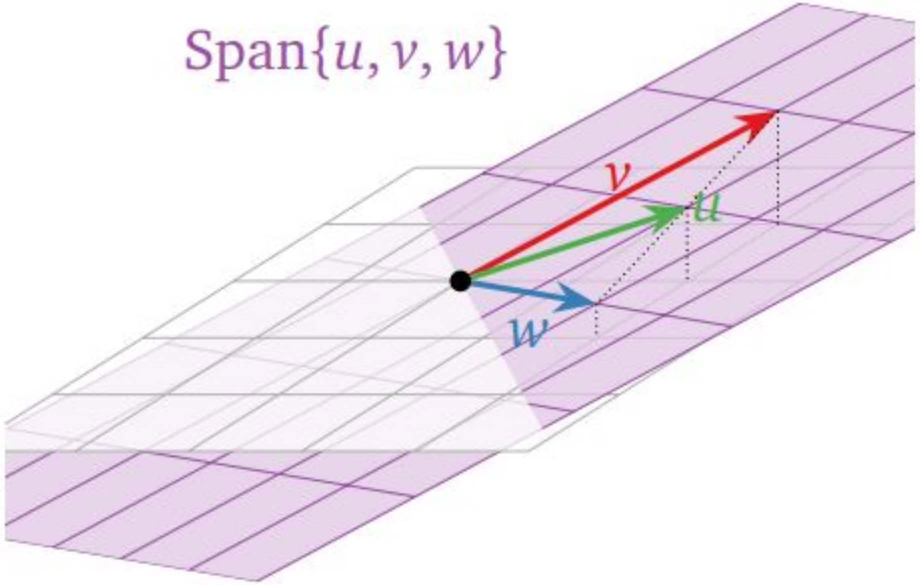


Pictures of spans in $\mathbf{R}^3$. *The span of two noncollinear vectors is the plane containing the origin and the heads of the vectors. Note that three coplanar (but not collinear) vectors span a plane and not a 3-space, just as two collinear vectors span a line and not a plane.*

$\mathrm{Span}\{u, v, w\}$



$\mathrm{Span}\{u, v, w\}$



2.3 Matrix Equations

Objectives

1. Understand the equivalence between a system of linear equations, an augmented matrix, a vector equation, and a matrix equation.
2. Characterize the vectors b such that $Ax = b$ is consistent, in terms of the span of the columns of A.
3. Characterize matrices A such that $Ax = b$ is consistent for all vectors b.
4. *Recipe:* multiply a vector by a matrix (two ways).
5. *Picture:* the set of all vectors b such that $Ax = b$ is consistent.
6. *Vocabulary word:* **matrix equation**.

2.3.1 The Matrix Equation $Ax = b$.

In this section we introduce a very concise way of writing a system of linear equations: $Ax = b$. Here A is a matrix and x, b are vectors (generally of different sizes), so first we must explain how to multiply a matrix by a vector.

When we say "A is an $m \times n$ matrix," we mean that A has m rows and n columns.

Definition. Let A be an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

The **product** of A with a vector x in $\mathbf{R}^n$ is the linear combination

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is a vector in $\mathbf{R}^m$.

Example.

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

-

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1\begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2\begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3\begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

In order for Ax to make sense, the number of entries of x has to be the same as the number of columns of A: we are using the entries of x as the coefficients of the columns of A in a linear combination. The resulting vector has the same number of entries as the number of *rows* of A, since each column of A has that number of entries.

> If A is an $m \times n$ matrix (m rows, n columns), then Ax makes sense when x has n entries. The product Ax has m entries.

Properties of the Matrix-Vector Product. *Let A be an $m \times n$ matrix, let u, v be vectors in $\mathbf{R}^n$, and let c be a scalar. Then:*

- $A(u+v) = Au + Av$
- $A(cu) = cAu$

Matrix Equations and Vector Equations. Let $v_1, v_2, \ldots, v_n$ and b be vectors in $\mathbf{R}^m$. Consider the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

This is equivalent to the matrix equation $Ax = b$, where

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Conversely, if A is any $m \times n$ matrix, then $Ax = b$ is equivalent to the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b,$$

where $v_1, v_2, \ldots, v_n$ are the columns of A, and $x_1, x_2, \ldots, x_n$ are the entries of x.

Example. Write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

as a matrix equation, where v_1, v_2, v_3 are vectors in $\mathbf{R}^3$.

Solution. Let A be the matrix with columns v_1, v_2, v_3, and let x be the vector with entries $2, 3, -4$. Then

$$Ax = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 2v_1 + 3v_2 - 4v_3,$$

so the vector equation is equivalent to the matrix equation $Ax = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$.

Four Ways of Writing a Linear System. We now have *four* equivalent ways of writing (and thinking about) a system of linear equations:

1. As a system of equations:

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 = 7 \\ x_1 - x_2 - 3x_3 = 5 \end{cases}$$

2. As an augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 7 \\ 1 & -1 & -3 & 5 \end{array}\right)$$

3. As a vector equation ($x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation ($Ax = b$):

$$\begin{pmatrix} 2 & 3 & -2 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}.$$

2.3.2 Spans and Consistency

Let A be a matrix with columns $v_1, v_2, \ldots, v_n$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Then

$Ax = b$ has a solution

2.3.2 Spans and Consistency

Let A be a matrix with columns $v_1, v_2, \ldots, v_n$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Then

$$\begin{aligned} & Ax = b \text{ has a solution} \\ & \iff \text{there exist } x_1, x_2, \ldots, x_n \text{ such that } A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b \\ & \iff \text{there exist } x_1, x_2, \ldots, x_n \text{ such that } x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b \\ & \iff b \text{ is a linear combination of } v_1, v_2, \ldots, v_n \\ & \iff b \text{ is in the span of the columns of } A. \end{aligned}$$

Spans and Consistency. The matrix equation $Ax = b$ has a solution if and only if b is in the span of the columns of A.

Example (An Inconsistent System)**.** Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

Solution. First we answer the question geometrically. The columns of A are

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

and the target vector (on the right-hand side of the equation) is $w = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$. The equation $Ax = w$ is consistent if and only if w is contained in the span of the columns of A. So we draw a picture:

Demo to show *w* is not in Span of *v1* and *v2*

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&target=0,2,2&range=5

The vector w is not contained in Span $\{v_1, v_2\}$*, so the equation Ax = b is inconsistent.*

Let us check our geometric answer by solving the matrix equation using row reduction. We put the system into an augmented matrix and row reduce:

$$\left(\begin{array}{rr|r} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$

The last equation is $0 = 1$, so the system is indeed inconsistent, and the matrix equation

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

has no solution.

Example (A Consistent System). Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

Solution. First we answer the question geometrically. The columns of A are

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

and the target vector (on the right-hand side of the equation) is $w = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. The equation $Ax = w$ is consistent if and only if w is contained in the span of the columns of A. So we draw a picture:

Demo to show *w* is in Span of *v1* and *v2*

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&target=1,-1,2&range=5

The vector w is contained in Span $\{v_1, v_2\}$, *so the equation Ax = b is consistent.*

Let us check our geometric answer by solving the matrix equation using row reduction. We put the system into an augmented matrix and row reduce:

$$\left(\begin{array}{rr|r} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right).$$

This gives us $x = 1$ and $y = -1$, which is consistent with the picture:

$$1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{or} \quad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

2.4.1 Homogeneous Systems

The equation $Ax = b$ is easier to solve when $b = 0$, so we start with this case.

Definition. A system of linear equations of the form $Ax = 0$ is called **homogeneous.**

A system of linear equations of the form $Ax = b$ for $b \neq 0$ is called **inhomogeneous.**

A homogeneous system is just a system of linear equations where all constants on the right side of the equals sign are zero.

A homogeneous system always has the solution $x = 0$. This is called the **trivial solution**. Any nonzero solution is called **nontrivial**.

Example (No nontrivial solutions)**.** What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

Solution. We form an augmented matrix and row reduce:

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

The only solution is the trivial solution $x = 0$.

How to solve AX = 0 ?

Parametric Vector Form (homogeneous case). Consider the following matrix in reduced row echelon form:

$$A = \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix equation $Ax = 0$ corresponds to the system of equations

$$\begin{cases} x_1 \qquad\quad - 8x_3 - 7x_4 = 0 \\ \qquad x_2 + 4x_3 + 3x_4 = 0. \end{cases}$$

We can write the parametric form as follows:

$$\begin{cases} x_1 = & 8x_3 + 7x_4 \\ x_2 = & -4x_3 - 3x_4 \\ x_3 = & x_3 \\ x_4 = & x_4. \end{cases}$$

We wrote the redundant equations $x_3 = x_3$ and $x_4 = x_4$ in order to turn the above system into a *vector equation*:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

This vector equation is called the **parametric vector form** of the solution set. Since x_3 and x_4 are allowed to be anything, this says that the solution set is the set of all linear combinations of $\begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$. In other words, the solution set is

$$\mathrm{Span}\left\{\begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Here is the general procedure.

Recipe: Parametric vector form (homogeneous case). Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation $Ax = 0$ are, for example, x_3, x_6, and x_8.

1. Find the reduced row echelon form of A.
2. Write the parametric form of the solution set, including the redundant equations $x_3 = x_3$, $x_6 = x_6$, $x_8 = x_8$. Put equations for all of the x_i in order.
3. Make a single vector equation from these equations by making the coefficients of x_3, x_6, and x_8 into vectors v_3, v_6, and v_8, respectively.

The solutions to $Ax = 0$ will then be expressed in the form

$$x = x_3 v_3 + x_6 v_6 + x_8 v_8$$

for some vectors v_3, v_6, v_8 in $\mathbf{R}^n$, and any scalars x_3, x_6, x_8. This is called the **parametric vector form** of the solution.
In this case, the solution set can be written as $\mathrm{Span}\{v_3,\, v_6,\, v_8\}$.

Example (The solution set is a line). Compute the parametric vector form of the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}.$$

Since RHS is 0. so we can ignore Construction of augmented matrix.

Solution. We row reduce (without augmenting, as suggested in the above observation):

$$\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}.$$

This corresponds to the single equation $x_1 - 3x_2 = 0$. We write the parametric form including the redundant equation $x_2 = x_2$:

$$\begin{cases} x_1 = 3x_2 \\ x_2 = x_2. \end{cases}$$

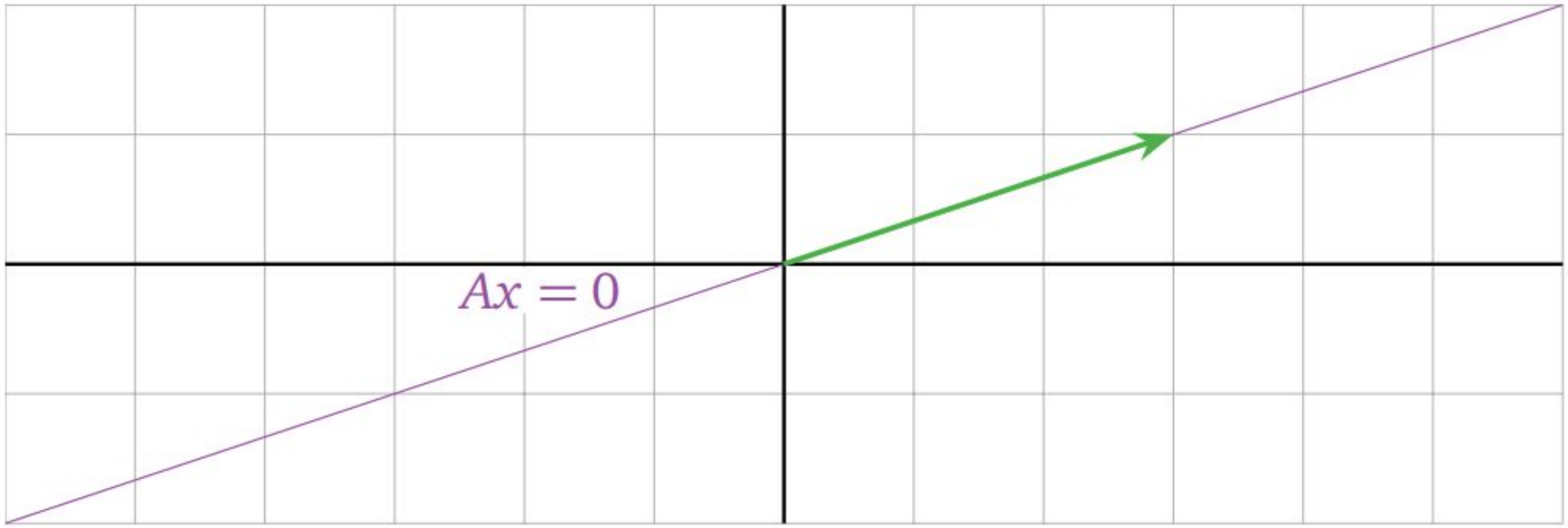
We turn these into a single vector equation:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

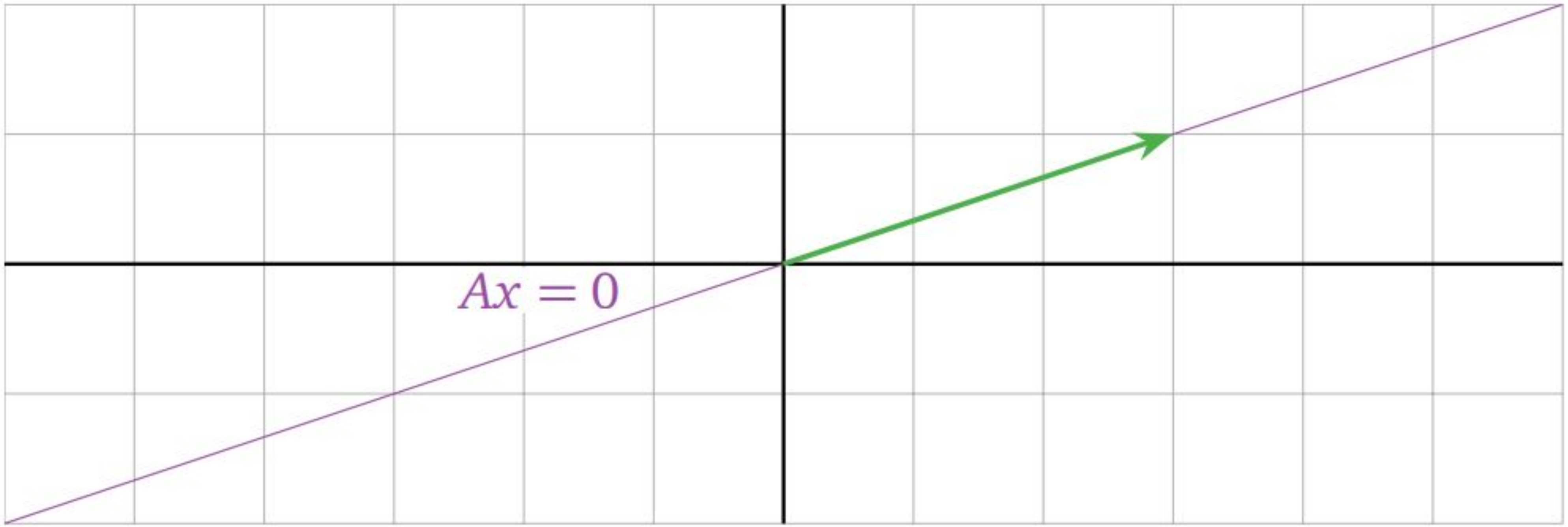
This is the parametric vector form of the solution set. Since x_2 is allowed to be anything, this says that the solution set is the set of all scalar multiples of $\binom{3}{1}$, otherwise known as

$$\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}.$$

We know how to draw the picture of a span of a vector: it is a line. Therefore, this is a picture of the the solution set:



Since there were *two* variables in the above example, the solution set is a subset of $\mathbf{R}^2$. Since *one* of the variables was free, the solution set is a *line*:



Example (The solution set is a plane). Compute the parametric vector form of the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

Solution. We row reduce (without augmenting, as suggested in the above observation):

$$\begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

This corresponds to the single equation $x_1 - x_2 + 2x_3 = 0$. We write the parametric form including the redundant equations $x_2 = x_2$ and $x_3 = x_3$:

$$\begin{cases} x_1 = x_2 - 2x_3 \\ x_2 = x_2 \\ x_3 = x_3. \end{cases}$$

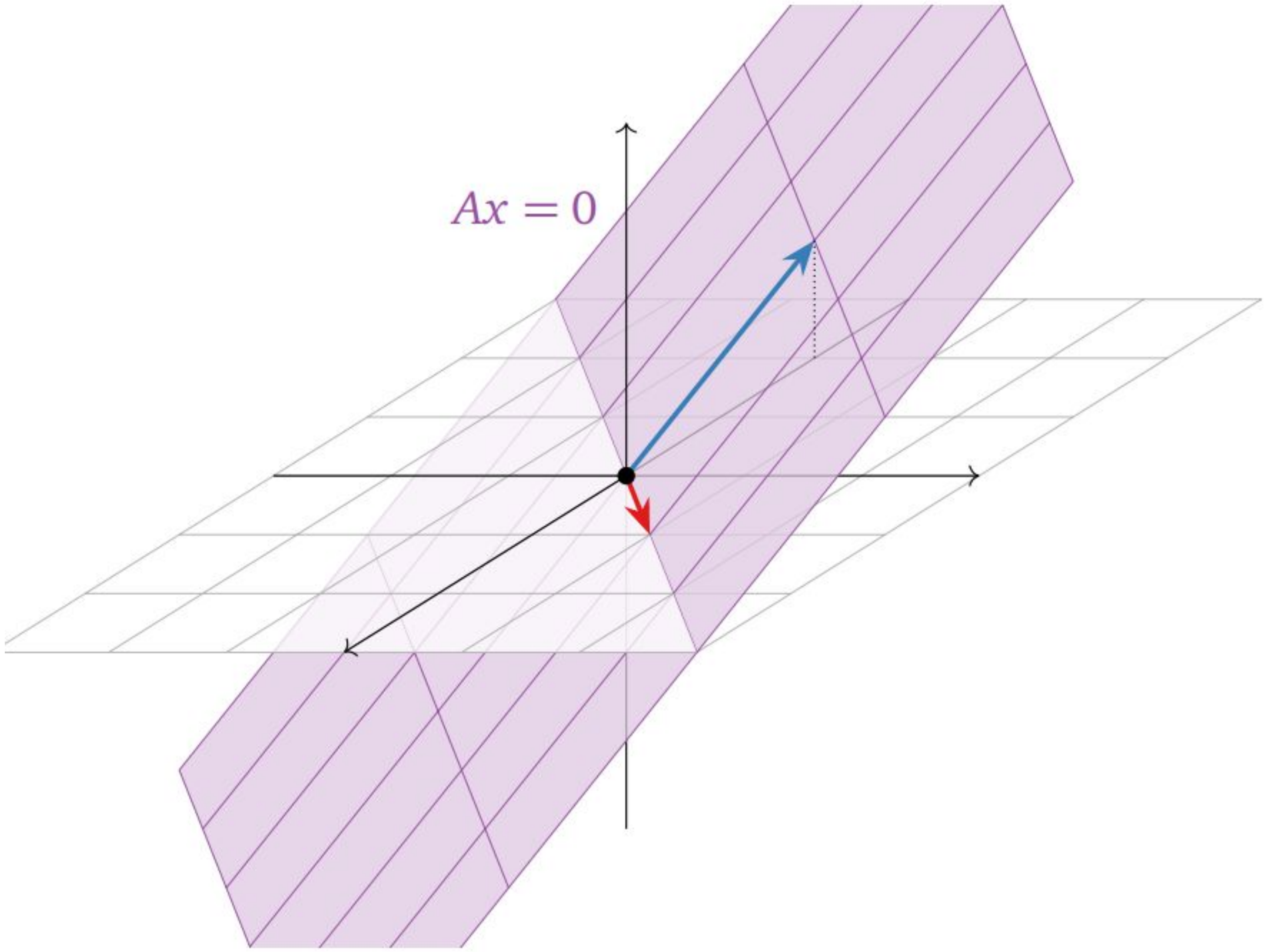
We turn these into a single vector equation:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

This is the parametric vector form of the solution set. Since x_2 and x_3 are allowed to be anything, this says that the solution set is the set of all linear combinations of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. In other words, the solution set is

$$\operatorname{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}.$$

We know how to draw the span of two noncollinear vectors in $\mathbf{R}^3$: it is a plane. Therefore, this is a picture of the solution set:



Since there were *three* variables in the above example, the solution set is a subset of $\mathbf{R}^3$. Since *two* of the variables were free, the solution set is a *plane*.

2.4.2 Inhomogeneous Systems

Recall that a matrix equation $Ax = b$ is called **inhomogeneous** when $b \neq 0$.

Example (The solution set is a line). What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

(Compare to this example, where we solved the corresponding homogeneous equation.)

Solution. We row reduce the associated augmented matrix:

$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -6 & -6 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 0 & 0 \end{array}\right).$$

This corresponds to the single equation $x_1 - 3x_2 = -3$. We can write the parametric form as follows:

$$\begin{cases} x_1 = 3x_2 - 3 \\ x_2 = x_2 + 0. \end{cases}$$

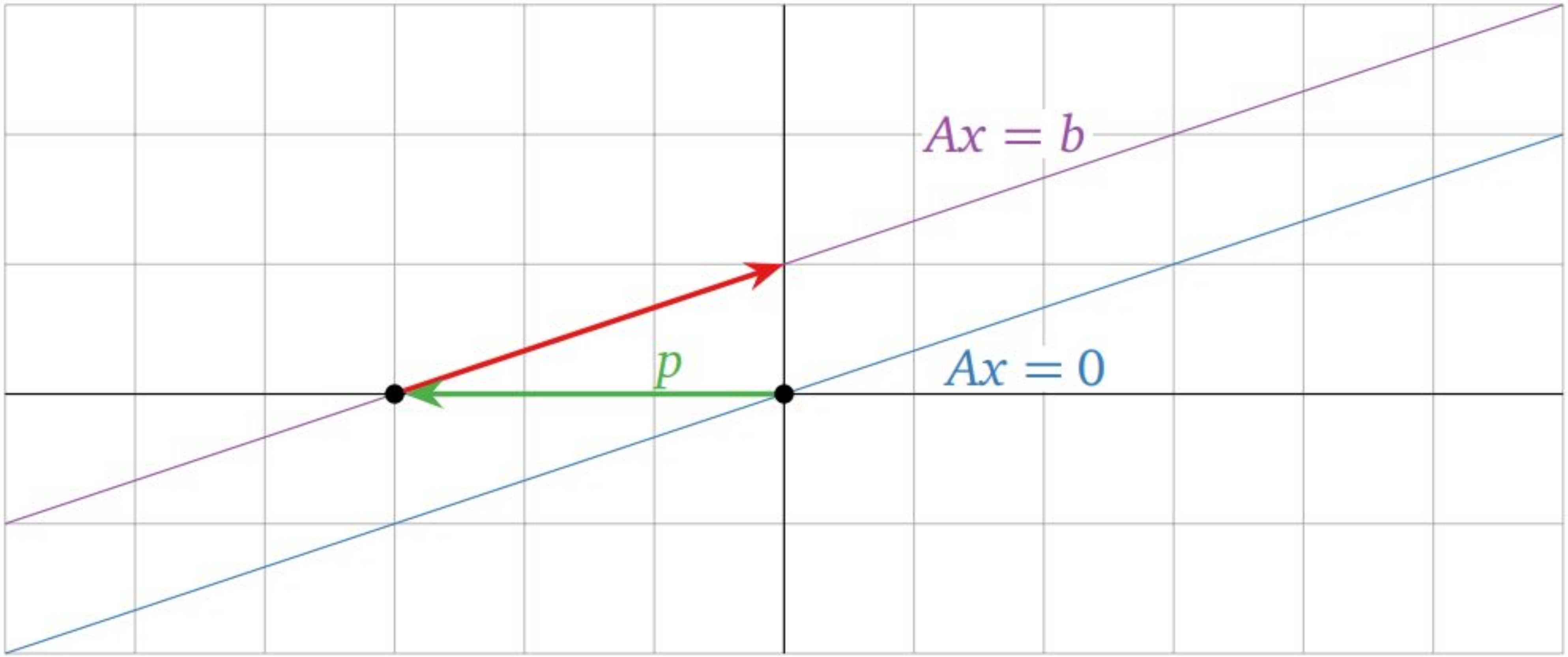
We turn the above system into a *vector equation*:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

This vector equation is called the **parametric vector form** of the solution set. We write the solution set as

$$\mathrm{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

Here is a picture of the the solution set:



In the above example, the solution set was all vectors of the form

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

where x_2 is any scalar. The vector $p = \binom{-3}{0}$ is also a solution of $Ax = b$: take $x_2 = 0$. We call p a **particular solution**.

In the solution set, x_2 is allowed to be anything, and so the solution set is obtained as follows: we take all scalar multiples of $\binom{3}{1}$ and then add the particular solution $p = \binom{-3}{0}$ to each of these scalar multiples. Geometrically, this is accomplished by first drawing the span of $\binom{3}{1}$, which is a line through the origin (and, not coincidentally, the solution to $Ax = 0$), and we *translate*, or push, this line along $p = \binom{-3}{0}$. The translated line contains p and is parallel to $\mathrm{Span}\{\binom{3}{1}\}$: it is a *translate of a line*.

Example (The solution set is a plane). What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$

(Compare this example.)

Solution. We row reduce the associated augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ -2 & 2 & -4 & -2 \end{array}\right) \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This corresponds to the single equation $x_1 - x_2 + 2x_3 = 1$. We can write the parametric form as follows:

$$\begin{cases} x_1 = x_2 - 2x_3 + 1 \\ x_2 = x_2 \qquad\quad\;\; + 0 \\ x_3 = \qquad\quad\; x_3 + 0. \end{cases}$$

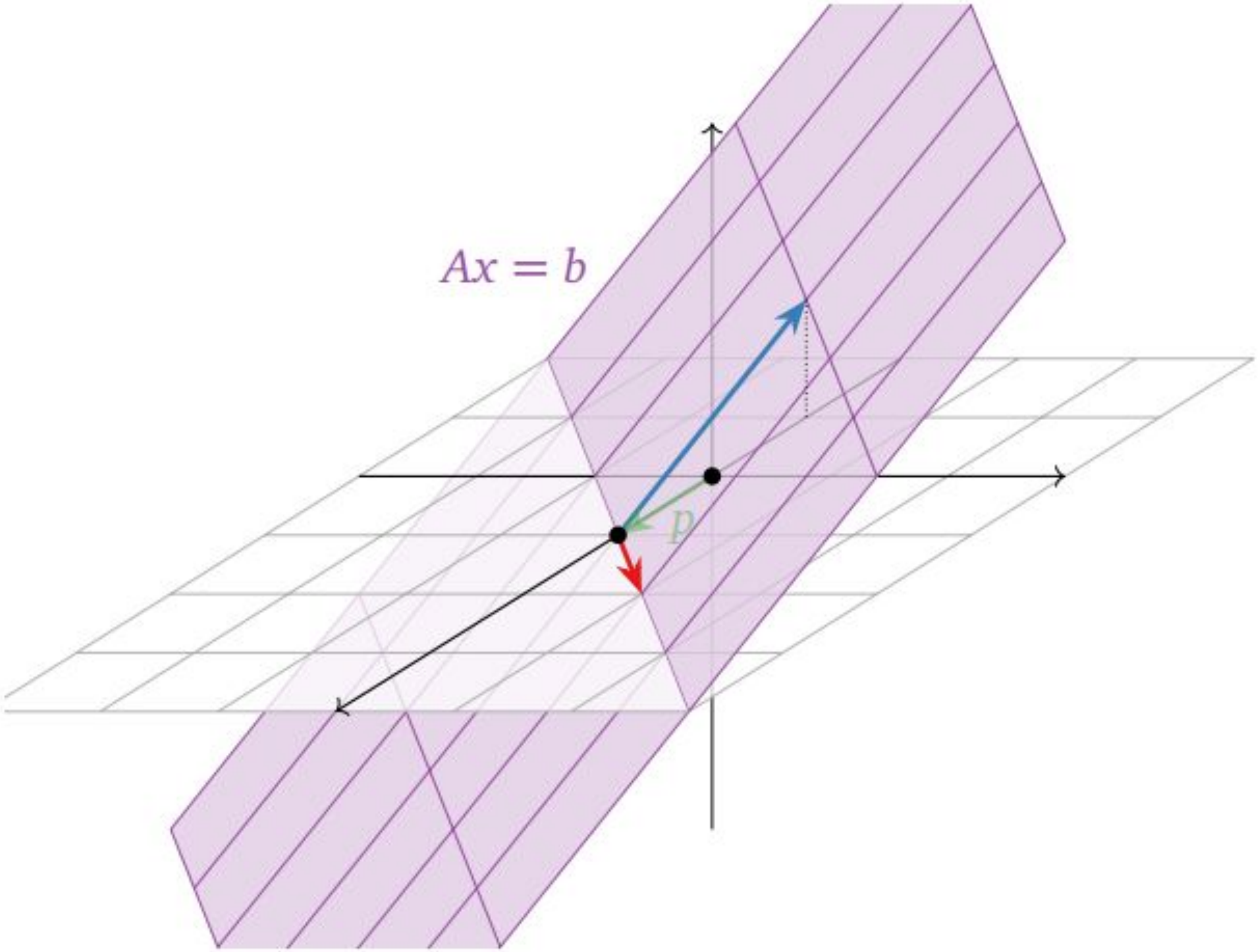
We turn the above system into a *vector equation*:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This vector equation is called the **parametric vector form** of the solution set. Since x_2 and x_3 are allowed to be anything, this says that the solution set is the set of all linear combinations of $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-2\\0\\1\end{pmatrix}$, *translated* by the vector $p = \begin{pmatrix}1\\0\\0\end{pmatrix}$. This is a plane which contains p and is parallel to Span $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}$: it is a *translate of a plane*. We write the solution set as

$$\text{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right\} + \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Here is a picture of the solution set:



In the above example, the solution set was all vectors of the form

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

where x_2 and x_3 are any scalars. In this case, a particular solution is $p = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Key Observation. If $Ax = b$ is consistent, the set of solutions to is obtained by taking one **particular solution** p of $Ax = b$, and adding all solutions of $Ax = 0$.
In particular, if $Ax = b$ is consistent, the solution set is a *translate of a span*.

The **parametric vector form** of the solutions of $Ax = b$ is just the parametric vector form of the solutions of $Ax = 0$, plus a particular solution p.

Example (The solution set is a point). What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}?$$

Solution. We form an augmented matrix and row reduce:

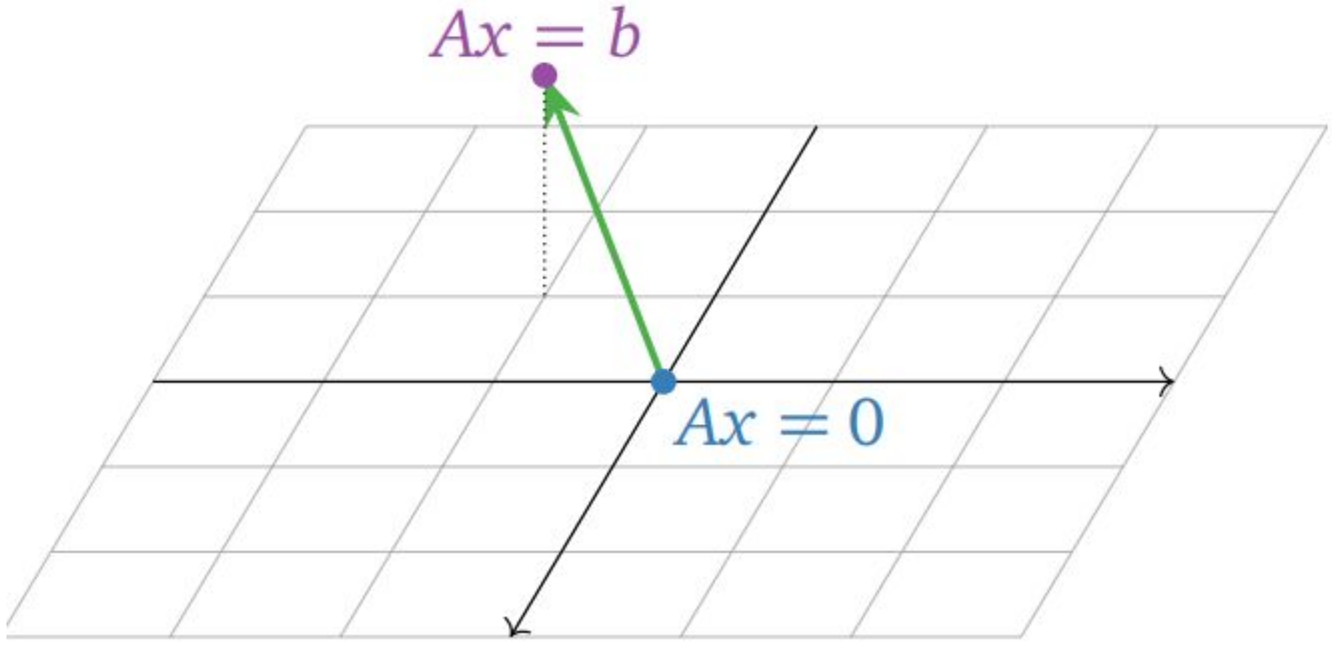
$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

The only solution is $p = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

According to the key observation, this is supposed to be a translate of a span by p. Indeed, we saw in the first example that the only solution of $Ax = 0$ is the trivial solution, i.e., that the solution set is the one-point set $\{0\}$. The solution set of the inhomogeneous equation $Ax = b$ is

$$\{0\} + \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Note that $\{0\} = \mathrm{Span}\{0\}$, so the homogeneous solution set is a span.



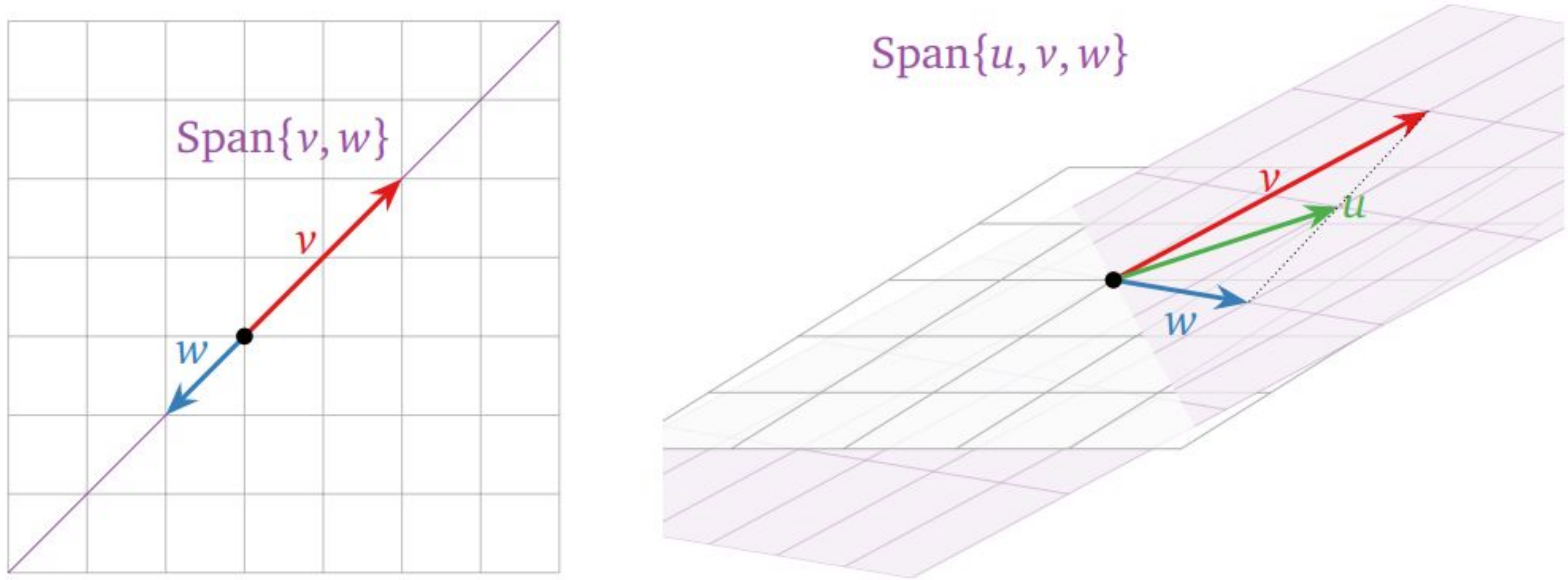
$Ax = b$
$Ax = 0$

2.5 Linear Independence

Objectives

1. Understand the concept of linear independence.
2. Learn two criteria for linear independence.
3. Understand the relationship between linear independence and pivot columns / free variables.
4. *Recipe:* test if a set of vectors is linearly independent / find an equation of linear dependence.
5. *Picture:* whether a set of vectors in $\mathbf{R}^2$ or $\mathbf{R}^3$ is linearly independent or not.
6. *Vocabulary words:* **linear dependence relation / equation of linear dependence.**

Sometimes the span of a set of vectors is “smaller” than you expect from the number of vectors, as in the picture below. This means that (at least) one of the vectors is redundant: it can be removed without affecting the span. In the present section, we formalize this idea in the notion of *linear independence*.



Pictures of sets of vectors that are linearly dependent. Note that in each case, one vector is in the span of the others—so it doesn't make the span bigger.

2.5.1 The Definition of Linear Independence

Essential Definition. A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_k = 0$. The set $\{v_1, v_2, \ldots, v_k\}$ is **linearly dependent** otherwise.

In other words, $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if there exist numbers $x_1, x_2, \ldots, x_k$, not all equal to zero, such that

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0.$$

This is called a **linear dependence relation** or **equation of linear dependence**.

Example (Checking linear dependence). Is the set linearly independent?

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$$

Solution. Equivalently, we are asking if the homogeneous vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a nontrivial solution. We solve this by forming a matrix and row reducing

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This says $x = -2z$ and $y = -z$. So there exist nontrivial solutions: for instance, taking $z = 1$ gives this equation of linear dependence:

$$-2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Example (Checking linear independence). Is the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$$

linearly independent?

Solution. Equivalently, we are asking if the homogeneous vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a nontrivial solution. We solve this by forming a matrix and row reducing

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -2 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This says $x = y = z = 0$, i.e., the only solution is the trivial solution. We conclude that the set is linearly independent.

Recipe: Checking linear independence. A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

has only the trivial solution, if and only if the matrix equation $Ax = 0$ has only the trivial solution, where A is the matrix with columns $v_1, v_2, \ldots, v_k$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{pmatrix}.$$

This is true if and only if A has a pivot position in every column.
Solving the matrix equatiion $Ax = 0$ will either verify that the columns $v_1, v_2, \ldots, v_k$ are linearly independent, or will produce a linear dependence relation by substituting any nonzero values for the free variables.

Suppose that A has more columns than rows. Then A cannot have a pivot in every column (it has at most one pivot per row), so its columns are automatically linearly dependent.

A wide matrix (a matrix with more columns than rows) has linearly dependent columns.

For example, four vectors in $\mathbf{R}^3$ are automatically linearly dependent. Note that a tall matrix may or may not have linearly independent columns.

Facts about linear independence.

1. *Two vectors are linearly dependent if and only if they are collinear, i.e., one is a scalar multiple of the other.*

2. *Any set containing the zero vector is linearly dependent.*

3. *If a subset of $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent, then $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent as well.*

2.5.2 Criteria for Linear Independence

In this subsection we give two criteria for a set of vectors to be linearly independent. Keep in mind, however, that the actual definition is above.

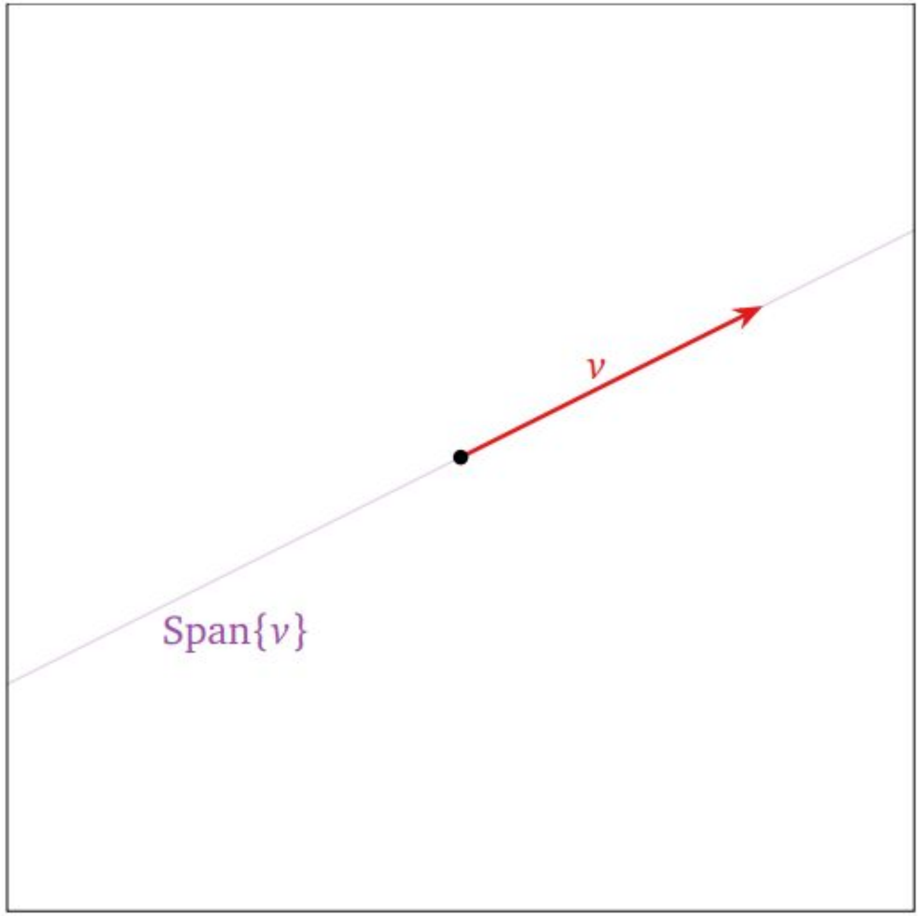
Theorem. *A set of vectors* $\{v_1, v_2, \ldots, v_k\}$ *is linearly dependent if and only if one of the vectors is in the span of the other ones.*

Any such vector may be removed without affecting the span.

Theorem (Increasing Span Criterion)**.** *A set of vectors* $\{v_1, v_2, \ldots, v_k\}$ *is linearly independent if and only if, for every* j*, the vector* v_j *is not in* $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

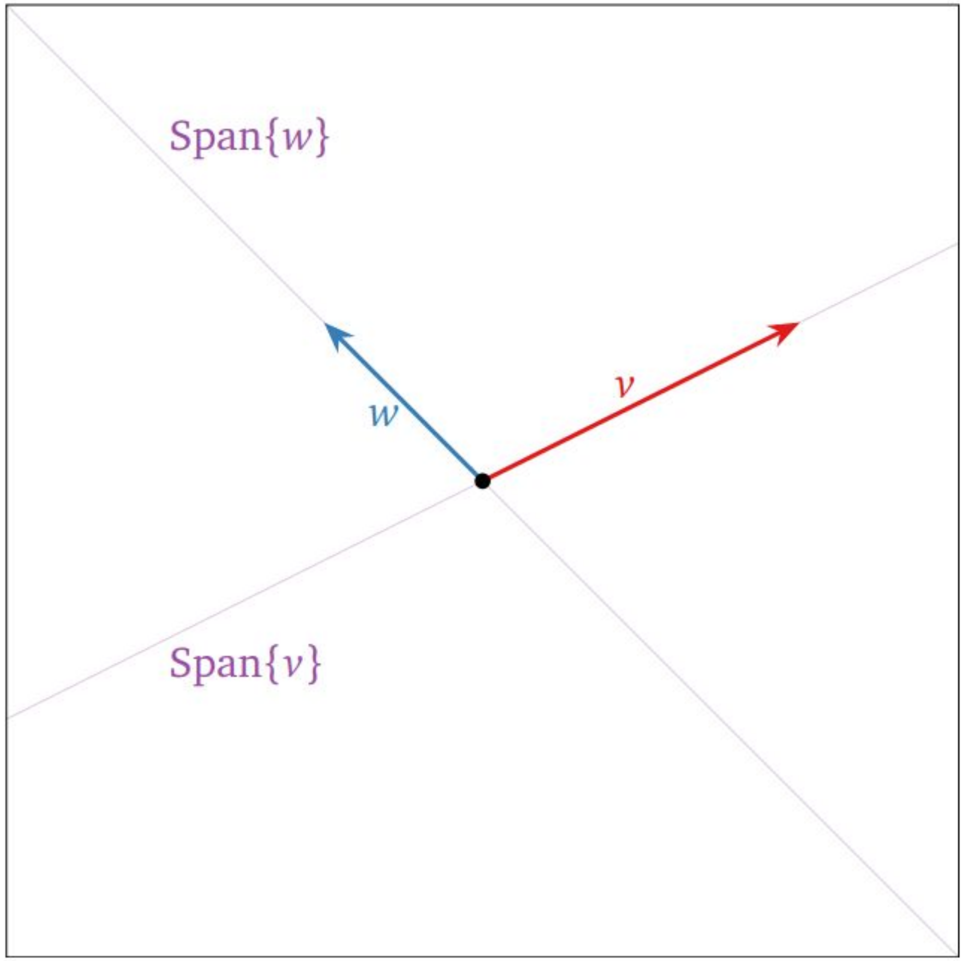
2.5.3 Pictures of Linear Independence

A set containg one vector $\{v\}$ is linearly independent when $v \neq 0$, since $xv = 0$ implies $x = 0$.



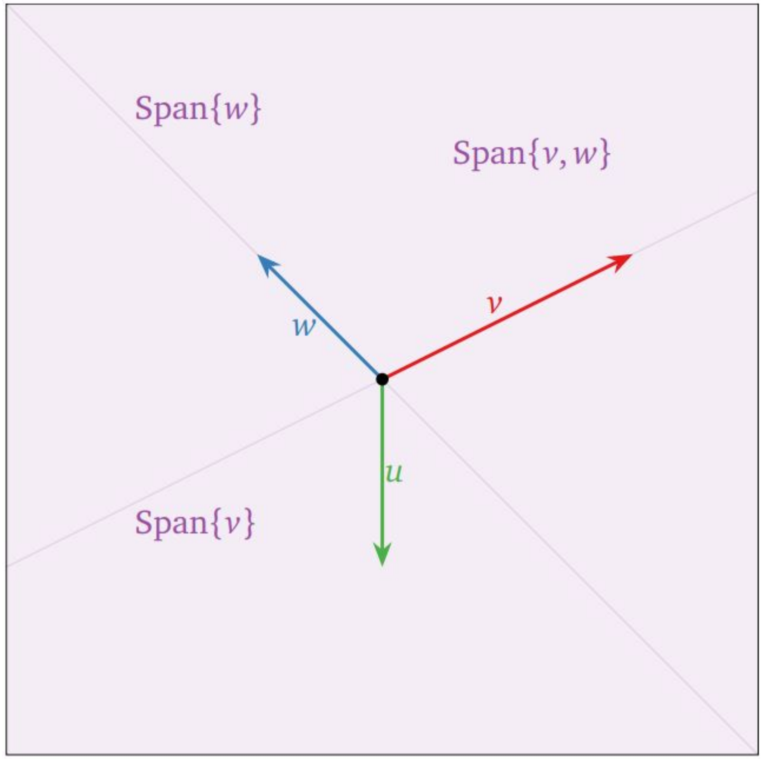
A set of two noncollinear vectors v, w is linearly independent:

- Neither is in the span of the other, so we can apply the first criterion.
- The span got bigger when we added w, so we can apply the increasing span criterion.



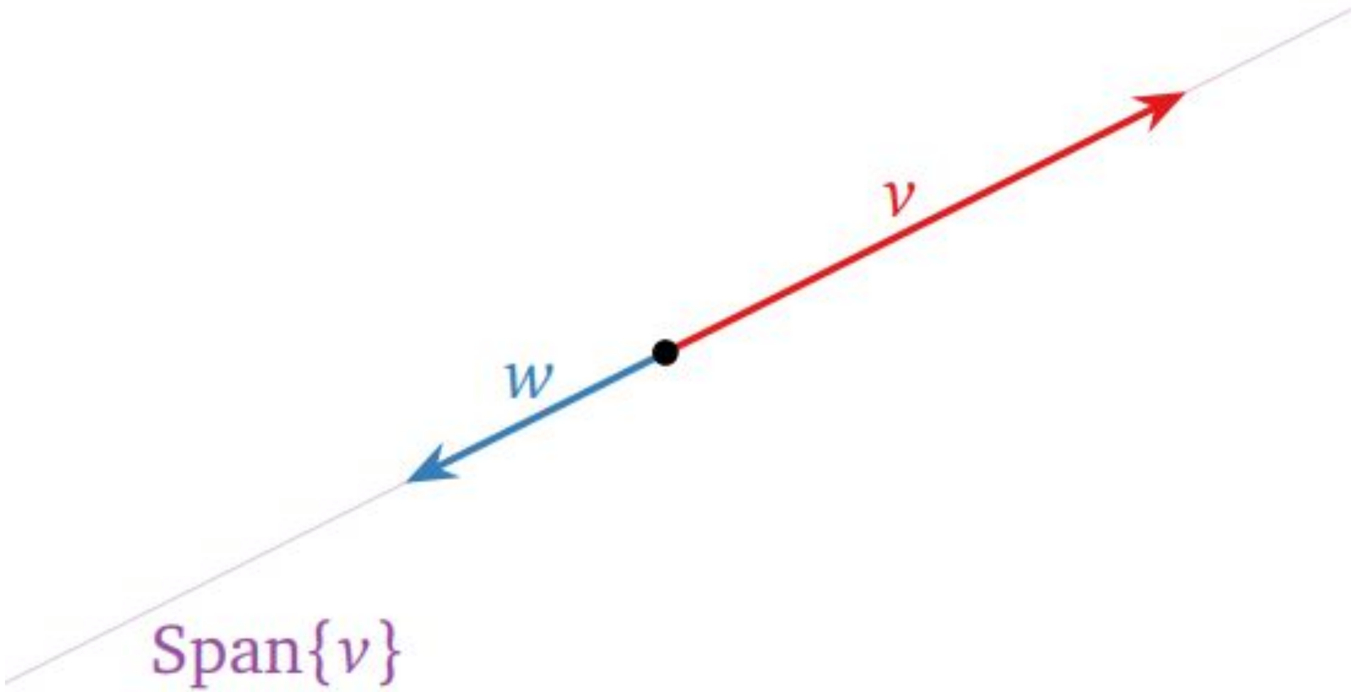
The set of three vectors $\{v, w, u\}$ below is linearly dependent:

- u is in Span $\{v, w\}$, so we can apply the first criterion.
- The span did not increase when we added u, so we can apply the increasing span criterion
- In the picture below, note that v is in Span $\{u, w\}$, and w is in Span $\{u, v\}$, so we can remove any of the three vectors without shrinking the span.

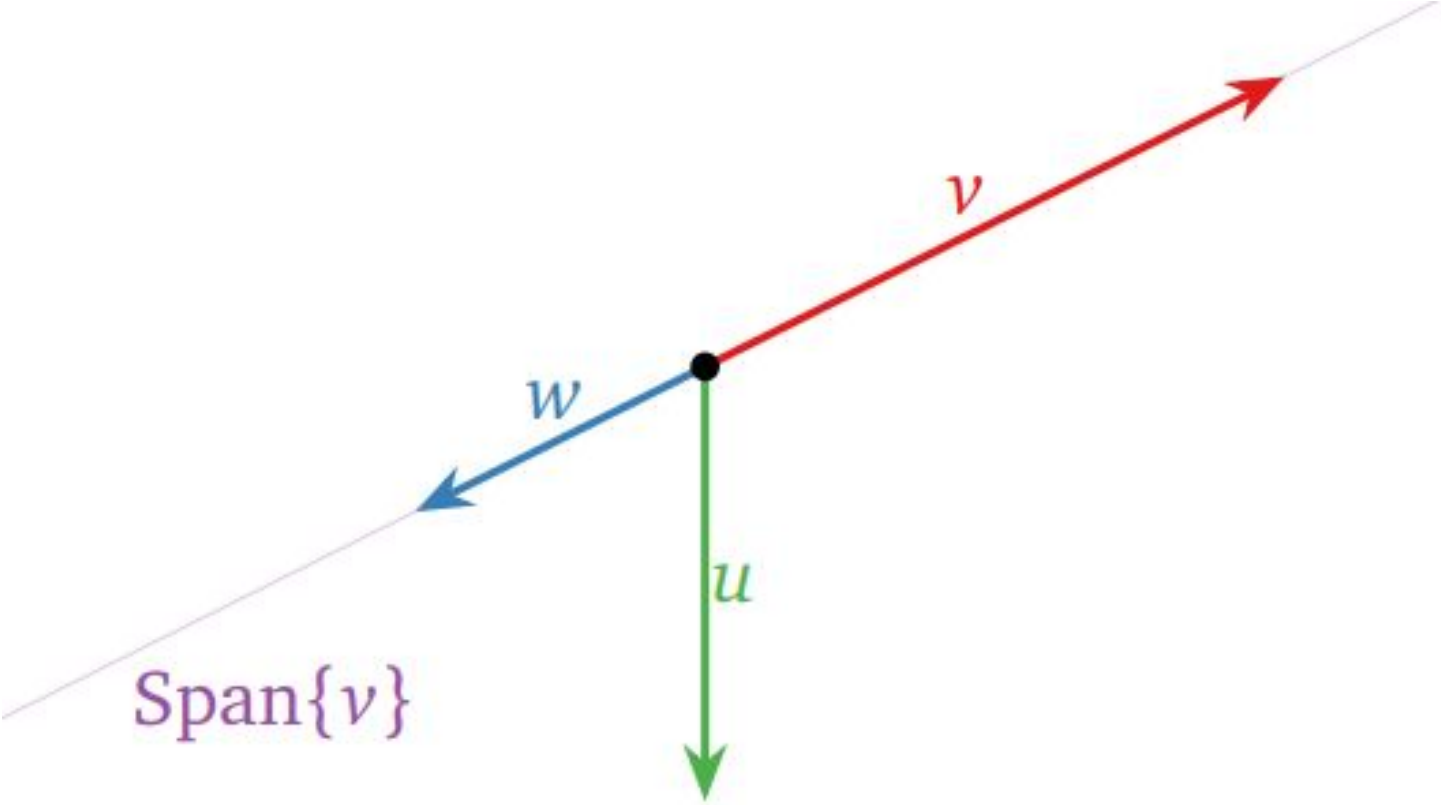


Two collinear vectors are always linearly dependent:

- w is in Span$\{v\}$, so we can apply the first criterion.
- The span did not increase when we added w, so we can apply the increasing span criterion.



These three vectors $\{v, w, u\}$ are linearly dependent: indeed, $\{v, w\}$ is already linearly dependent, so we can use the third fact.



2.5.4 Linear Dependence and Free Variables

In light of this important note and this criterion, it is natural to ask which columns of a matrix are redundant, i.e., which we can remove without affecting the column span.

Theorem. *Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbf{R}^n$, and consider the matrix*

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{pmatrix}.$$

Then we can delete the columns of A without *pivots (the columns corresponding to the free variables), without changing* $\operatorname{Span}\{v_1, v_2, \ldots, v_k\}$.

The pivot columns are linearly independent, so we cannot delete any more columns without changing the span.

Example. The matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the first two columns of A are the pivot columns, so we can delete the others without changing the span:

$$\operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} \right\}.$$

Moreover, the first two columns are linearly independent.

Pivot Columns and Dimension. Let d be the number of pivot columns in the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{pmatrix}.$$

- If $d = 1$ then $\mathrm{Span}\{v_1, v_2, \ldots, v_k\}$ is a line.
- If $d = 2$ then $\mathrm{Span}\{v_1, v_2, \ldots, v_k\}$ is a plane.
- If $d = 3$ then $\mathrm{Span}\{v_1, v_2, \ldots, v_k\}$ is a 3-space.
- Et cetera.

2.6 Subspaces

Objectives

1. Learn the definition of a subspace.
2. Learn to determine whether or not a subset is a subspace.
3. Learn the most important examples of subspaces.
4. Learn to write a given subspace as a column space or null space.
5. *Recipe:* compute a spanning set for a null space.
6. *Picture:* whether a subset of $\mathbf{R}^2$ or $\mathbf{R}^3$ is a subspace or not.
7. *Vocabulary words:* **subspace**, **column space**, **null space**.