Mathematics

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Linear algebra

- *Linear:* having to do with lines, planes, etc.
- *Algebra:* solving equations involving unknowns.

What is system of linear equations ?

$$\begin{cases} x + 3y - z = 4 \\ 2x - y + 3z = 17 \\ y - 4z = -3. \end{cases}$$

Definition. An equation in the unknowns $x, y, z, \ldots$ is called **linear** if both sides of the equation are a sum of (constant) multiples of $x, y, z, \ldots$, plus an optional constant.

For instance,

$$\begin{aligned} 3x + 4y &= 2z \\ -x - z &= 100 \end{aligned}$$

are linear equations, but

$$\begin{aligned} 3x + yz &= 3 \\ \sin(x) - \cos(y) &= 2 \end{aligned}$$

are not.

Write few linear as well as non linear equations ?

- **Linear Equation**:

  A linear equation is an equation where the variables are either not present or are only raised to the first power. Linear equations form a straight line when graphed on a coordinate plane.

  Standard form:

  $$ax + by + c = 0$$

  Here, a, b, and c are constants, and x and y are variables. No variable is raised to a power greater than one.

  Examples:

  - $2x + 3y = 6$
  - $y = 4x + 5$
  - $x - y/2 = 3$

- **Nonlinear Equation**:

  A nonlinear equation is any equation that is not linear. This means that the variables can be raised to powers other than one, and the equation can involve products of variables or other more complex operations.

  Examples:

  - $y = x^2 + 2x + 1$ (quadratic)
  - $y = \sin(x)$ (trigonometric)
  - $y = e^x$ (exponential)
  - $x^2 + y^2 = 16$ (circle equation)
  - $y = \log(x)$ (logarithmic)

2. **Graphical Representation**

- **Linear Equation**:
  - The graph of a linear equation is always a straight line. The slope and y-intercept can be easily determined from the equation, especially when written in the slope-intercept form $y = mx + b$.
- **Nonlinear Equation**:
  - The graph of a nonlinear equation can take various forms, such as curves, circles, parabolas, hyperbolas, or more complex shapes depending on the equation's form.

- It will be very important to us to understand systems of linear equations both *algebraically* and *geometrically.*

- *Algebraically* (writing equations for their solutions).
- *Geometrically* (drawing pictures and visualizing).

- Engineers need to solve many, many equations in many, many variables. Here is a tiny example:

$$\begin{cases} 3x_1 + 4x_2 + 10x_3 + 19x_4 - 2x_5 - 3x_6 = 141 \\ 7x_1 + 2x_2 - 13x_3 - 7x_4 + 21x_5 + 8x_6 = 2567 \\ -x_1 + 9x_2 + \frac{3}{2}x_3 + x_4 + 14x_5 + 27x_6 = 26 \\ \frac{1}{2}x_1 + 4x_2 + 10x_3 + 11x_4 + 2x_5 + x_6 = -15. \end{cases}$$

A **system** of linear equations is a collection of several linear equations, like

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2. \end{cases} \tag{1.1.1}$$

Definition (Solution sets).

- A **solution** of a system of equations is a list of numbers $x, y, z, \ldots$ that make all of the equations true simultaneously.
- The **solution set** of a system of equations is the collection of all solutions.
- **Solving** the system means finding all solutions with formulas involving some number of parameters.

- Solve this system of equations:

$$\begin{cases} x + 2y = 3 \\ x + 2y = -3. \end{cases}$$

A system of linear equations need not have a solution. For example, there do not exist numbers *x* and *y* making the following two equations true simultaneously:

Definition. A system of equations is called **inconsistent** if it has no solutions. It is called **consistent** otherwise.

1.1.1 Line, Plane, Space, Etc.

We use $\mathbf{R}$ to denote the set of all real numbers, i.e., the number line. This contains numbers like $0, \frac{3}{2}, -\pi, 104, \ldots$

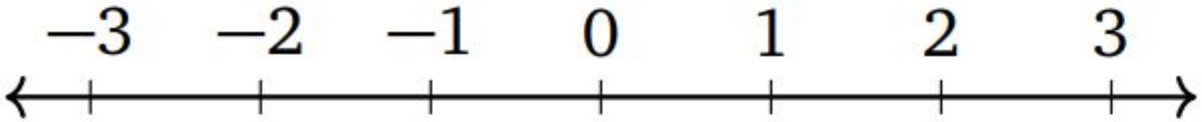
Definition. Let n be a positive whole number. We define

$$\mathbf{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

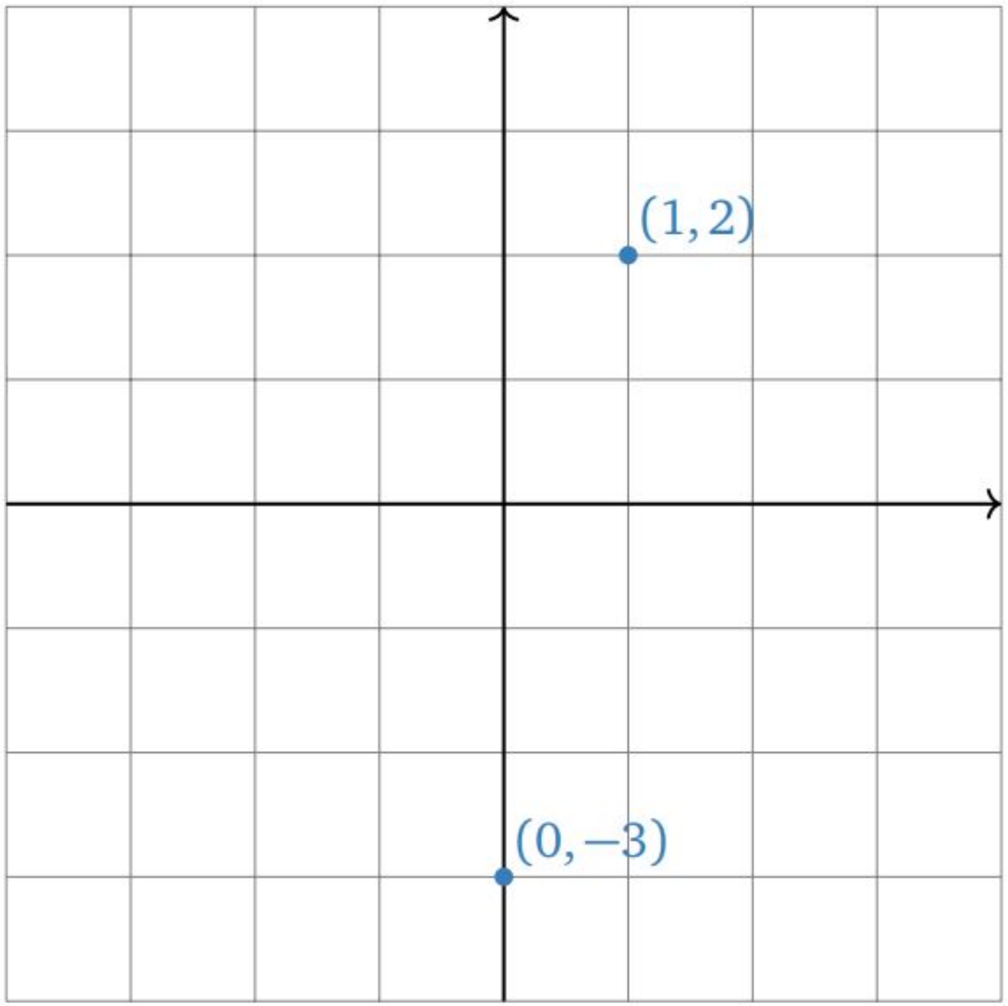
An n-tuple of real numbers is called a **point** of $\mathbf{R}^n$.

In other words, $\mathbf{R}^n$ is just the set of all (ordered) lists of n real numbers. We will draw pictures of $\mathbf{R}^n$ in a moment, but keep in mind that *this is the definition*. For example, $(0, \frac{3}{2}, -\pi)$ and $(1, -2, 3)$ are points of $\mathbf{R}^3$.

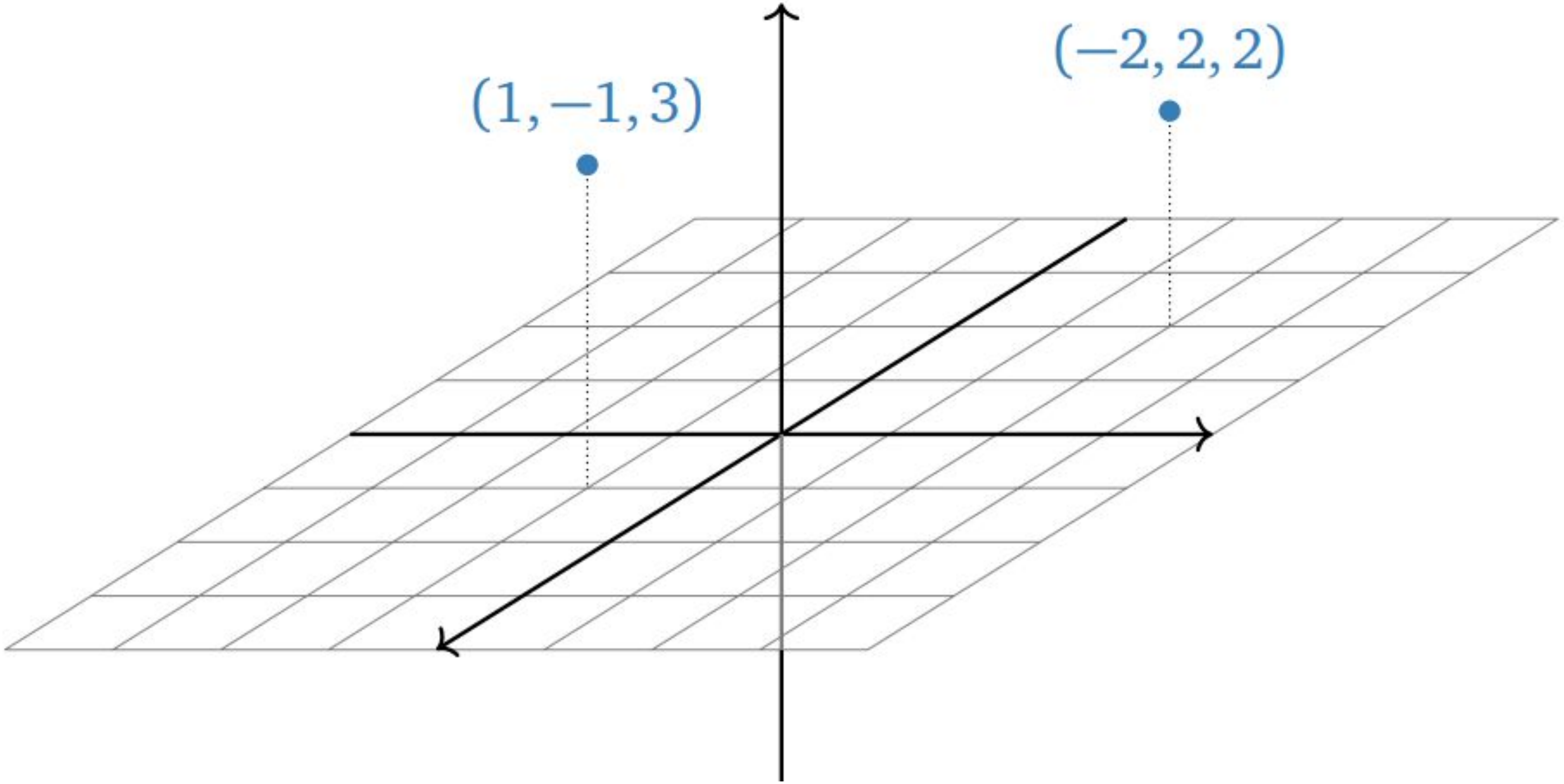
Example (The number line)**.** When $n = 1$, we just get $\mathbf{R}$ back: $\mathbf{R}^1 = \mathbf{R}$. Geometrically, this is the number line.



Example (The Euclidean plane)**.** When $n = 2$, we can think of $\mathbf{R}^2$ as the xy-plane. We can do so because every point on the plane can be represented by an ordered pair of real numbers, namely, its x- and y-coordinates.



Example (3-Space)**.** When $n = 3$, we can think of $\mathbf{R}^3$ as the *space* we (appear to) live in. We can do so because every point in space can be represented by an ordered triple of real numebrs, namely, its x-, y-, and z-coordinates.



So what is $\mathbf{R}^4$? or $\mathbf{R}^5$? or $\mathbf{R}^n$? These are harder to visualize, so you have to go back to the definition: $\mathbf{R}^n$ is the set of all ordered n-tuples of real numbers $(x_1, x_2, x_3, \ldots, x_n)$.

They are still "geometric" spaces, in the sense that our intuition for $\mathbf{R}^2$ and $\mathbf{R}^3$ often extends to $\mathbf{R}^n$.

We will make definitions and state theorems that apply to any $\mathbf{R}^n$, but we will only draw pictures for $\mathbf{R}^2$ and $\mathbf{R}^3$.

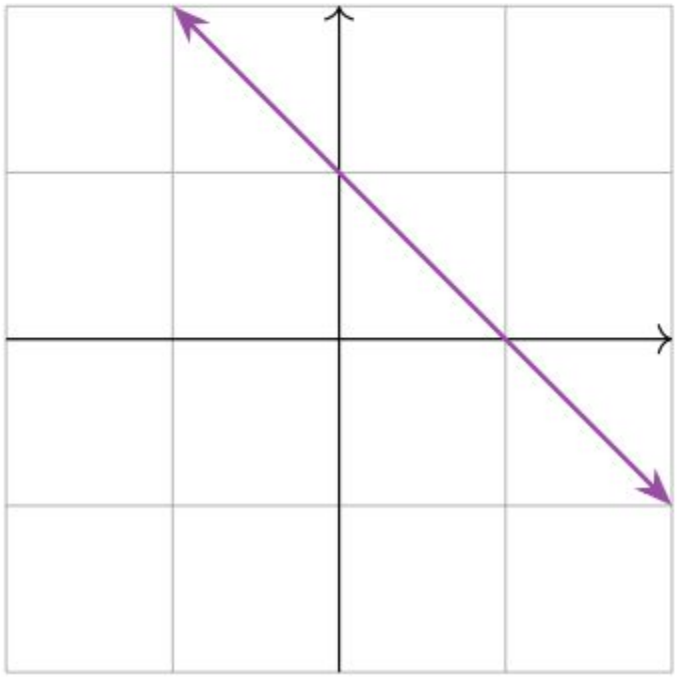
The power of using these spaces is the ability to *label* various objects of interest, such as geometric objects and solutions of systems of equations, by the points of $\mathbf{R}^n$.

Pictures of Solution Sets

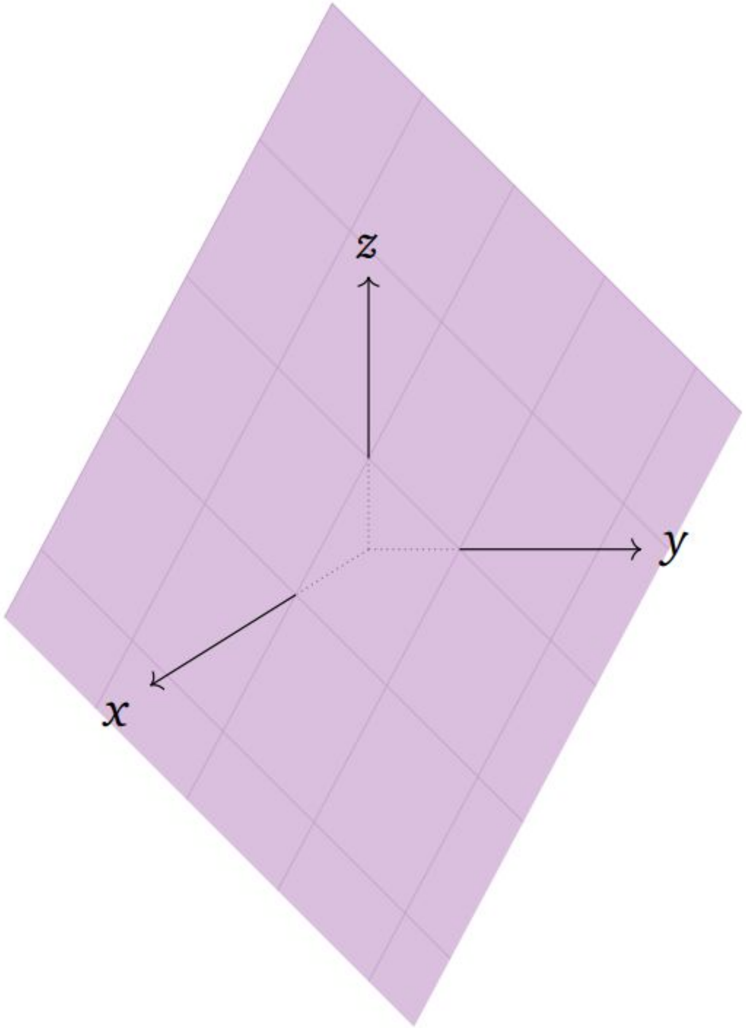
Before discussing how to solve a system of linear equations below, it is helpful to see some pictures of what these solution sets look like geometrically

One Equation in Two Variables.

Consider the linear equation $x + y = 1$. We can rewrite this as $y = 1 - x$, which defines a line in the plane: the slope is -1, and the x-intercept is 1.



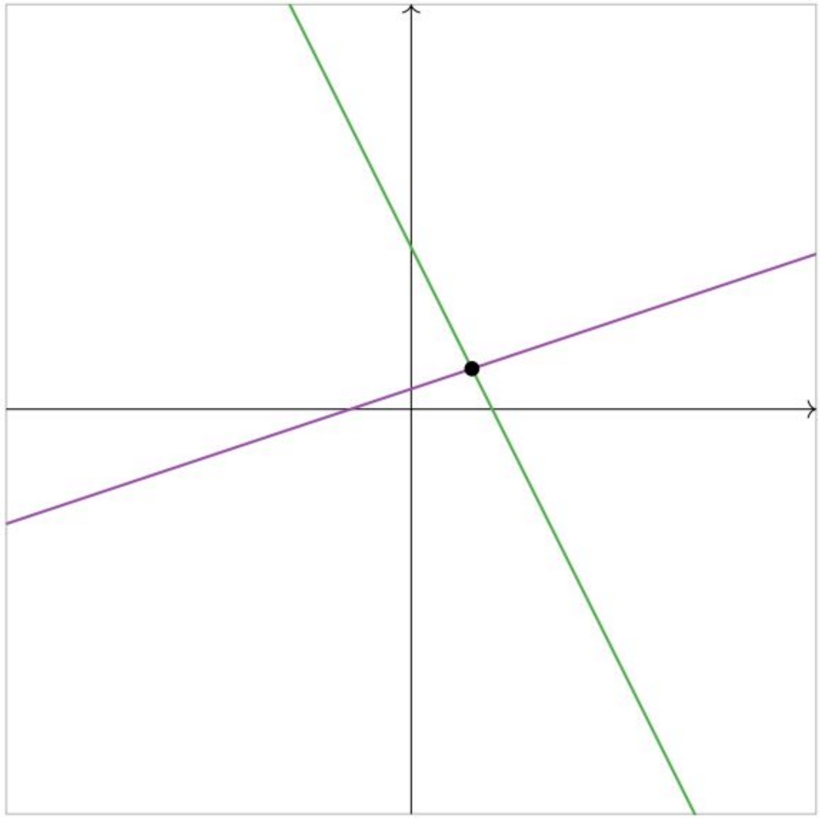
One Equation in Three Variables. Consider the linear equation $x + y + z = 1$. This is the **implicit equation** for a plane in space.



Two Equations in Two Variables. Now consider the system of two linear equations

$$\begin{cases} x - 3y = -3 \\ 2x + y = 8. \end{cases}$$

Each equation individually defines a line in the plane, pictured below.

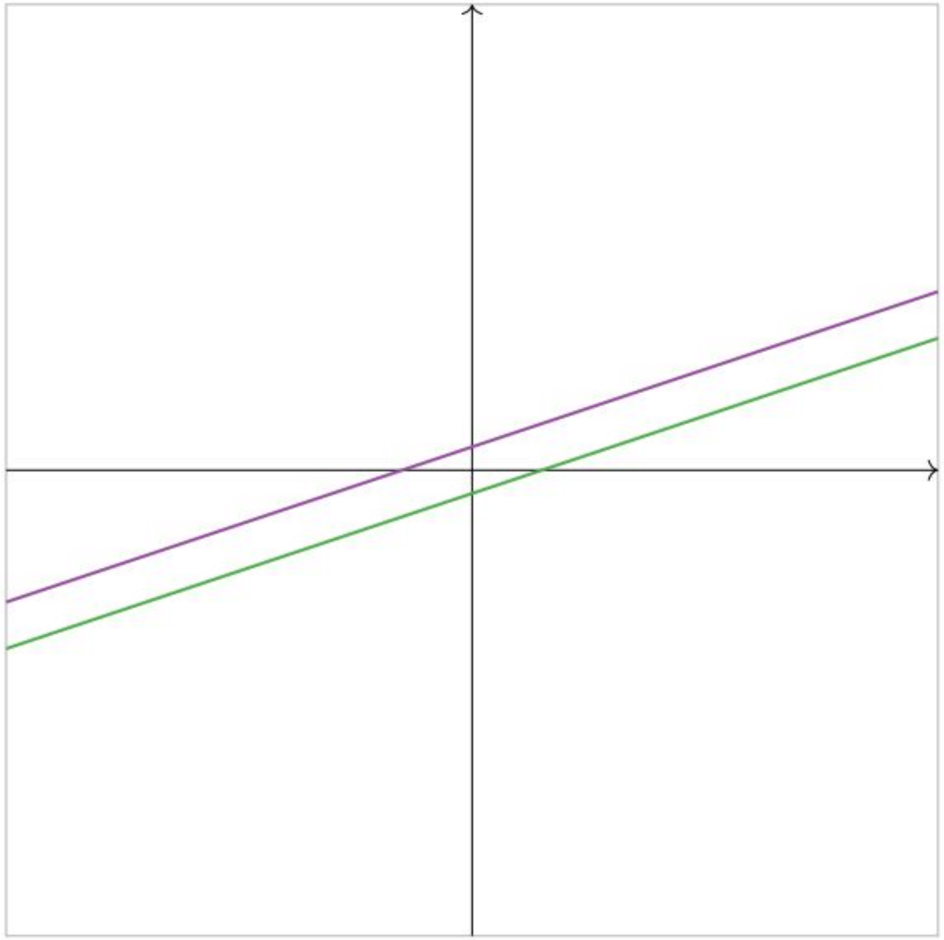


A solution to the *system* of both equations is a pair of numbers (x, y) that makes both equations true at once. In other words, it as a point that lies on both lines simultaneously. We can see in the picture above that there is only one point where the lines intersect: therefore, this system has exactly one solution. (This solution is (3,2))

Usually, two lines in the plane will intersect in one point, but of course this is not always the case.

Consider now the system of equations $\begin{cases} x - 3y = -3 \\ x - 3y = 3. \end{cases}$

These define *parallel* lines in the plane.

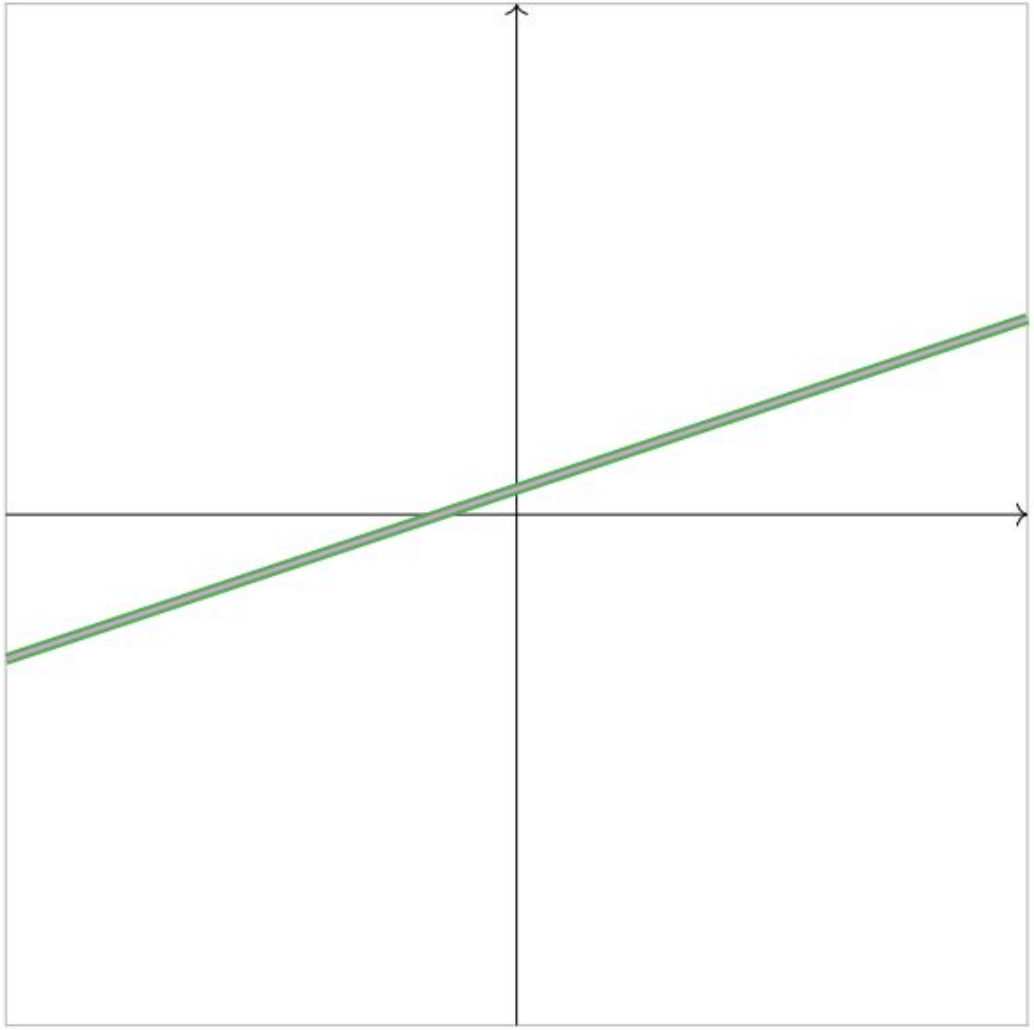


The fact that that the lines do not intersect means that the system of equations has no solution. Of course, this is easy to see algebraically: if $x - 3y = -3$, then it is cannot also be the case that $x - 3y = 3$.

There is one more possibility. Consider the system of equations

$$\begin{cases} x - 3y = -3 \\ 2x - 6y = -6. \end{cases}$$

The second equation is a multiple of the first, so these equations define the *same* line in the plane.



In this case, there are infinitely many solutions of the system of equations.

So the system of linear equations can have:

- Only one solution (unique solution)
- No solution
- Infinite many solutions

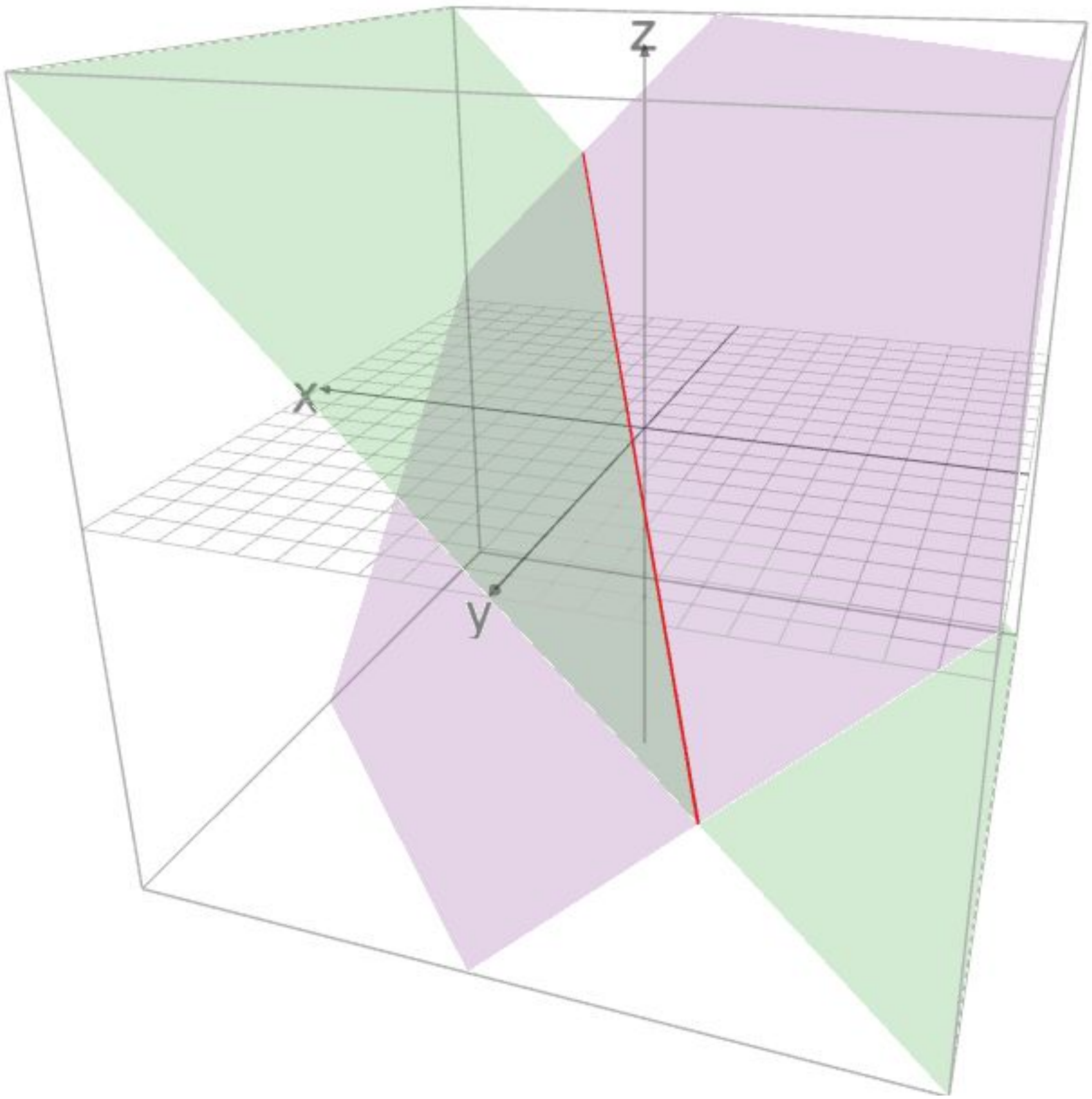
Two Equations in Three Variables. Consider the system of two linear equations

$$\begin{cases} x + y + z = 1 \\ x \qquad\; - z = 0. \end{cases}$$

Each equation individually defines a plane in space. The solutions of the system of both equations are the points that lie on both planes. We can see in the picture below that the planes intersect in a line. In particular, this system has infinitely many solutions.

The planes defined by the equations $x + y + z = 1$ and $x - z = 0$ intersect in the red line, which is the solution set of the system of both equations.

Remark. In general, the solutions of a system of equations in n variables is the intersection of "$(n-1)$-planes" in n-space. This is always some kind of linear space, as we will discuss in Section 2.4.



The planes defined by the equations $x + y + z = 1$ *and* $x - z = 0$ *intersect in the red line, which is the solution set of the system of both equations.*

Row Reduction

Objectives

1. Learn to replace a system of linear equations by an augmented matrix.
2. Learn how the elimination method corresponds to performing row operations on an augmented matrix.
3. Understand when a matrix is in (reduced) row echelon form.
4. Learn which row reduced matrices come from inconsistent linear systems.
5. *Recipe:* the row reduction algorithm.
6. *Vocabulary words:* **row operation**, **row equivalence**, **matrix**, **augmented matrix**, **pivot**, **(reduced) row echelon form**.

The Elimination Method

We will solve systems of linear equations algebraically using the **elimination** method. In other words, we will combine the equations in various ways to try to eliminate as many variables as possible from each equation. There are three valid operations we can perform on our system of equations:

- ***Scaling:*** we can multiply both sides of an equation by a nonzero number.

$$\left\{\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}\right. \xrightarrow{\text{multiply 1st by } -3} \left\{\begin{aligned} -3x - 6y - 9z &= -18 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}\right.$$

- ***Replacement:*** we can add a multiple of one equation to another, replacing the second equation with the result.

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \xrightarrow{\text{2nd} \;=\; \text{2nd}-2\times\text{1st}} \begin{cases} x + 2y + 3z = 6 \\ -7y - 4z = 2 \\ 3x + y - z = -2 \end{cases}$$

- ***Swap:*** we can swap two equations.

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \xrightarrow{\text{3rd} \longleftrightarrow \text{1st}} \begin{cases} 3x + y - z = -2 \\ 2x - 3y + 2z = 14 \\ x + 2y + 3z = 6 \end{cases}$$

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2. \end{cases} \tag{1.1.1}$$

Solve (1.1.1) using the elimination method.

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \xrightarrow{\text{2nd } = \text{ 2nd}-2\times\text{1st}} \begin{cases} x + 2y + 3z = 6 \\ -7y - 4z = 2 \\ 3x + y - z = -2 \end{cases}$$

$$\xrightarrow{\text{3rd } = \text{ 3rd}-3\times\text{1st}} \begin{cases} x + 2y + 3z = 6 \\ -7y - 4z = 2 \\ -5y - 10z = -20 \end{cases}$$

$$\xrightarrow{\text{2nd} \longleftrightarrow \text{3rd}} \left\{\begin{array}{rrrr} x + & 2y + & 3z = & 6 \\ & -5y - & 10z = & -20 \\ & -7y - & 4z = & 2 \end{array}\right.$$

$$\xrightarrow{\text{divide 2nd by } -5} \left\{\begin{array}{rrrr} x + & 2y + & 3z = & 6 \\ & y + & 2z = & 4 \\ & -7y - & 4z = & 2 \end{array}\right.$$

$$\xrightarrow{\text{3rd } = \text{ 3rd} + 7 \times \text{2nd}} \left\{\begin{array}{rrrr} x + & 2y + & 3z = & 6 \\ & y + & 2z = & 4 \\ & & 10z = & 30 \end{array}\right.$$

At this point we've eliminated both x and y from the third equation, and we can solve $10z = 30$ to get $z = 3$. Substituting for z in the second equation gives $y + 2 \cdot 3 = 4$, or $y = -2$. Substituting for y and z in the first equation gives $x + 2 \cdot (-2) + 3 \cdot 3 = 6$, or $x = 3$. Thus the only solution is $(x, y, z) = (1, -2, 3)$.

Augmented Matrices and Row Operations Solving equations by elimination requires writing the variables x, y, z and the equals sign $=$ over and over again, merely as placeholders: all that is changing in the equations is the coefficient *numbers*. We can make our life easier by extracting only the numbers, and putting them in a box:

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \xrightarrow{\text{becomes}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right).$$

This is called an **augmented matrix**. The word "augmented" refers to the vertical line, which we draw to remind ourselves where the equals sign belongs; a **matrix** is a grid of numbers without the vertical line. In this notation, our three valid ways of manipulating our equations become **row operations**:

All three operation will still valid

- ***Scaling:*** multiply all entries in a row by a nonzero number.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_1=R_1\times -3} \left(\begin{array}{ccc|c} -3 & -6 & -9 & -18 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

  Here the notation R_1 simply means “the first row”, and likewise for R_2, R_3, etc.

- ***Replacement:*** add a multiple of one row to another, replacing the second row with the result.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2=R_2-2\times R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

- ***Swap:*** interchange two rows.

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_3} \left(\begin{array}{rrr|r} 3 & 1 & -1 & -2 \\ 2 & -3 & 2 & 14 \\ 1 & 2 & 3 & 6 \end{array}\right)$$

Example. Solve (1.1.1) using row operations.

Solution. We start by forming an augmented matrix:

$$\left\{\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}\right. \quad \xrightarrow{\text{becomes}} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right).$$

Eliminating a variable from an equation means producing a zero to the left of the line in an augmented matrix. First we produce zeros in the first column (i.e. we eliminate x) by subtracting multiples of the first row.

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2=R_2-2R_1} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ \color{red}{0} & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_3=R_3-3R_1} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ \color{red}{0} & -5 & -10 & -20 \end{array}\right)$$

This was made much easier by the fact that the top-left entry is equal to 1, so we can simply multiply the first row by the number below and subtract. In order to eliminate y in the same way, we would like to produce a 1 in the second column. We could divide the second row by -7, but this would produce fractions; instead, let's divide the third by -5.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right) \xrightarrow{R_3=R_3\div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & {\color{red}1} & 2 & 4 \end{array}\right)$$

$$\xrightarrow{R_2\longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3=R_3+7R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & {\color{red}0} & 10 & 30 \end{array}\right)$$

$$\xrightarrow{R_3=R_3\div 10} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & {\color{red}1} & 3 \end{array}\right)$$

We swapped the second and third row just to keep things orderly. Now we translate this augmented matrix back into a system of equations:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{\text{becomes}} \begin{cases} x + 2y + 3z = 6 \\ \quad\;\; y + 2z = 4 \\ \qquad\quad\;\; z = 3 \end{cases}$$

Hence $z = 3$; back-substituting as in this example gives $(x, y, z) = (1, -2, 3)$.

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Indeed, the whole point of doing these operations is to solve the equations using the elimination method.

Definition. Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.

Example (An Inconsistent System). Solve the following system of equations using row operations:

$$\begin{cases} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{cases}$$

Solution. First we put our system of equations into an augmented matrix.

$$\begin{cases} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{cases} \xrightarrow{\text{augmented matrix}} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right)$$

We clear the entries below the top-left using row replacement.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right) \xrightarrow{R_2 = R_2 - 3R_1} \left(\begin{array}{cc|r} 1 & 1 & 2 \\ \color{red}{0} & 1 & -1 \\ 4 & 5 & 9 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 4R_1} \left(\begin{array}{cc|r} 1 & 1 & 2 \\ 0 & 1 & -1 \\ \color{red}{0} & 1 & 1 \end{array}\right)$$

Now we clear the second entry from the last row.

$$\left(\begin{array}{cc|r} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{R_3=R_3-R_2} \left(\begin{array}{cc|r} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right)$$

This translates back into the system of equations

$$\begin{cases} x + y = 2 \\ \phantom{x + {}} y = -1 \\ \phantom{x + {}} 0 = 2. \end{cases}$$

Our original system has the same solution set as this system. But this system has no solutions: there are no values of x, y making the third equation true! We conclude that our original equation was inconsistent.

Row Echelon Form

We want to reduce the augmented matrix to Echelon form, to get it solved.

Definition. A matrix is in **row echelon form** if:

1. All zero rows are at the bottom.
2. The first nonzero entry of a row is to the *right* of the first nonzero entry of the row above.
3. Below the first nonzero entry of a row, all entries are zero.

Here is a picture of a matrix in row echelon form:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \star = \text{any number} \\ \boxed{\star} = \text{any nonzero number} \end{array}$$

Definition. A **pivot** is the first nonzero entry of a row of a matrix in row echelon form.

A matrix in row-echelon form is generally easy to solve using back-substitution. For example,

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{\text{becomes}} \begin{cases} x + 2y + 3z = 6 \\ y + 2z = 4 \\ 10z = 30. \end{cases}$$

We immediately see that $z = 3$, which implies $y = 4 - 2 \cdot 3 = -2$ and $x = 6 - 2(-2) - 3 \cdot 3 = 1$.

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition:

5. Each pivot is equal to 1.

6. Each pivot is the only nonzero entry in its column.

Here is a picture of a matrix in reduced row echelon form:

$$\begin{pmatrix} \color{red}{1} & 0 & \star & 0 & \star \\ 0 & \color{red}{1} & \star & 0 & \star \\ 0 & 0 & 0 & \color{red}{1} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \star = \text{any number} \\ \color{red}{1} = \text{pivot} \end{array}$$

If an augmented matrix is in reduced row echelon form, the corresponding linear system is viewed as *solved*.

Definition. A **pivot position** of a matrix is an entry that is a pivot of a row echelon form of that matrix.

A **pivot column** of a matrix is a column that contains a pivot position.

Example (Pivot Positions)**.** Find the pivot positions and pivot columns of this matrix

$$A = \left(\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right).$$

row echelon form of the matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right).$$

The pivot positions of A are the entries that become pivots in a row echelon form they are marked in red below:

$$\left(\begin{array}{ccc|c} \color{red}{0} & -7 & -4 & 2 \\ 2 & \color{red}{4} & 6 & 12 \\ 3 & 1 & \color{red}{-1} & -2 \end{array}\right).$$

The first, second, and third columns are pivot columns.

We have discussed two classes of matrices so far:

1. When the reduced row echelon form of a matrix has a pivot in every non-augmented column, then it corresponds to a system with a unique solution:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{\text{translates to}} \begin{cases} x \qquad\qquad = 1 \\ \qquad y \qquad = -2 \\ \qquad\qquad z = 3. \end{cases}$$

2. When the reduced row echelon form of a matrix has a pivot in the last (augmented) column, then it corresponds to a system with a no solutions:

$$\left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{translates to}} \begin{cases} x + 5y = 0 \\ \qquad 0 = 1. \end{cases}$$

What happens when one of the non-augmented columns lacks a pivot?   : Multiple solutions

Example (A System with Many Solutions). Solve the linear system

$$\begin{cases} 2x + y + 12z = 1 \\ x + 2y + 9z = -1 \end{cases}$$

Solution.

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 2 & 1 & 12 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & -3 & -6 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div -3} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

This row reduced matrix corresponds to the linear system

$$\begin{cases} x + 5z = 1 \\ y + 2z = -1. \end{cases}$$

Summary

1. ***The last column is a pivot column.*** In this case, the system is *inconsistent*. There are zero solutions, i.e., the solution set is empty. For example, the matrix

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

comes from a linear system with no solutions.

2. ***Every column except the last column is a pivot column.*** In this case, the system has a *unique* solution. For example, the matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$$

tells us that the unique solution is $(x, y, z) = (a, b, c)$.

3. ***The last column is not a pivot column, and some other column is not a pivot column either.*** In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). For example, in the system corresponding to the matrix

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right),$$

any values for x_2 and x_4 yield a solution to the system of equations.

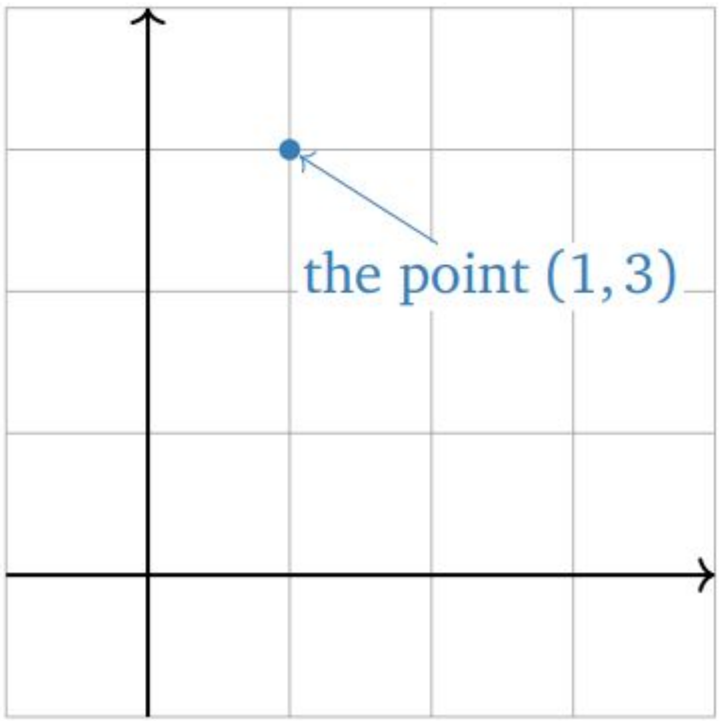
Systems of Linear Equations: Geometry

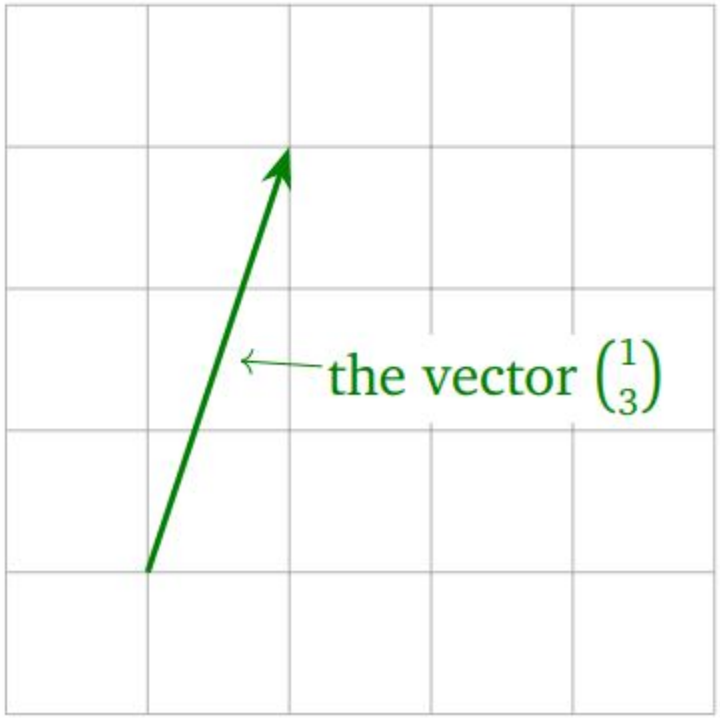
2.1 Vectors

Objectives

1. Learn how to add and scale vectors in $\mathbf{R}^n$, both algebraically and geometrically.
2. Understand linear combinations geometrically.
3. *Pictures:* vector addition, vector subtraction, linear combinations.
4. *Vocabulary words:* **vector**, **linear combination**.

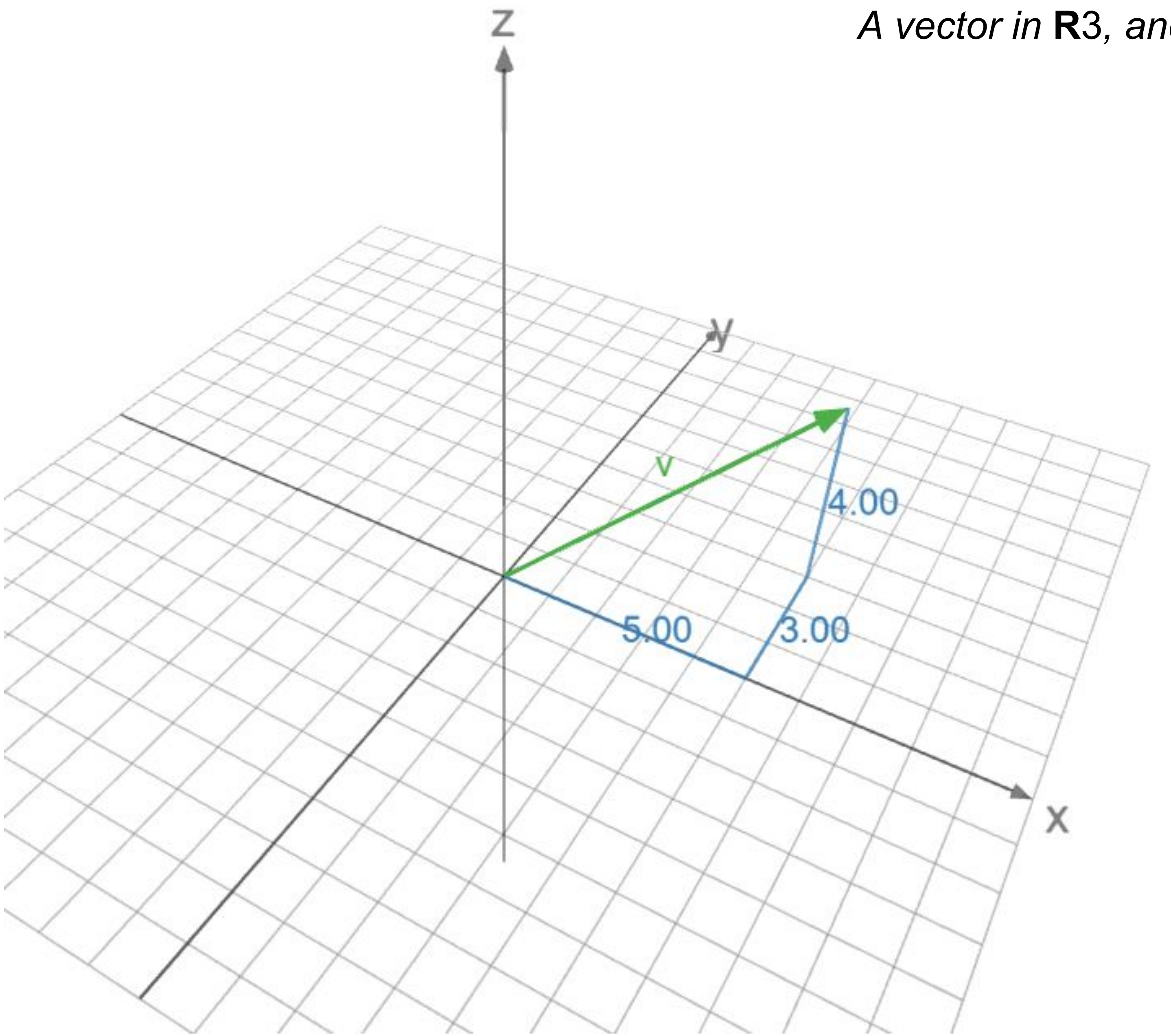
Points and Vectors. Again, a **point** in $\mathbf{R}^n$ is drawn as a dot.



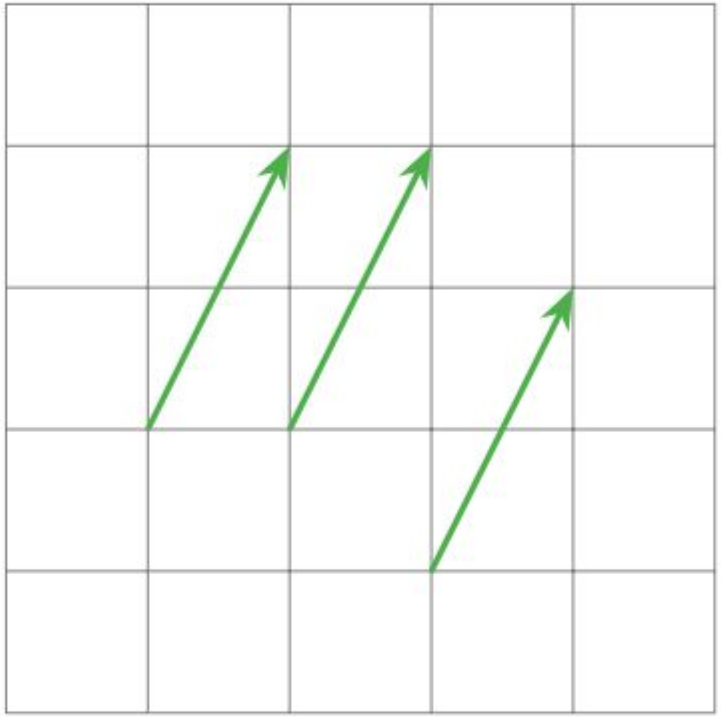


A **vector** is a point in $\mathbf{R}^n$, drawn as an arrow.

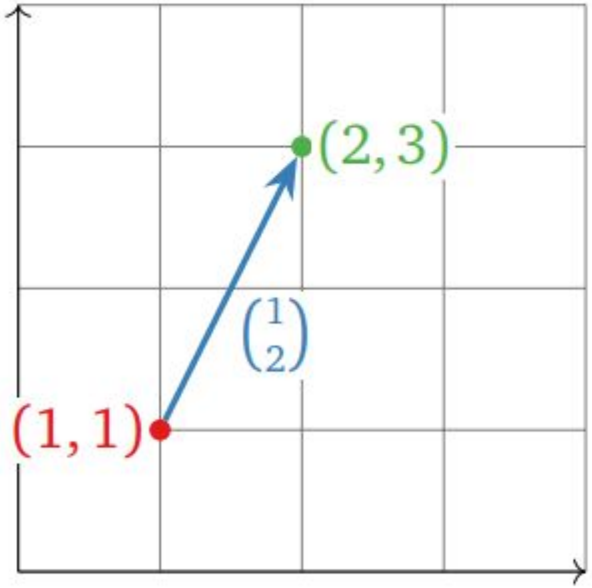
A vector in **R**3*, and its coordinates.*



Why make the distinction between points and vectors? A vector need not start at the origin: *it can be located anywhere*! In other words, an arrow is determined by its length and its direction, not by its location. For instance, these arrows all represent the vector $\begin{pmatrix}1\\2\end{pmatrix}$.



Note. Another way to think about a vector is as a *difference* between two points, or the arrow from one point to another. For instance, $\binom{1}{2}$ is the arrow from $(1,1)$ to $(2,3)$.



2.1.2 Vector Algebra and Geometry

Here we learn how to add vectors together and how to multiply vectors by numbers, both algebraically and geometrically.

Vector addition and scalar multiplication.

- We can add two vectors together:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}.$$

- We can multiply, or **scale**, a vector by a real number c:

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

  We call c a **scalar** to distinguish it from a vector. If v is a vector and c is a scalar, then cv is called a **scalar multiple** of v.

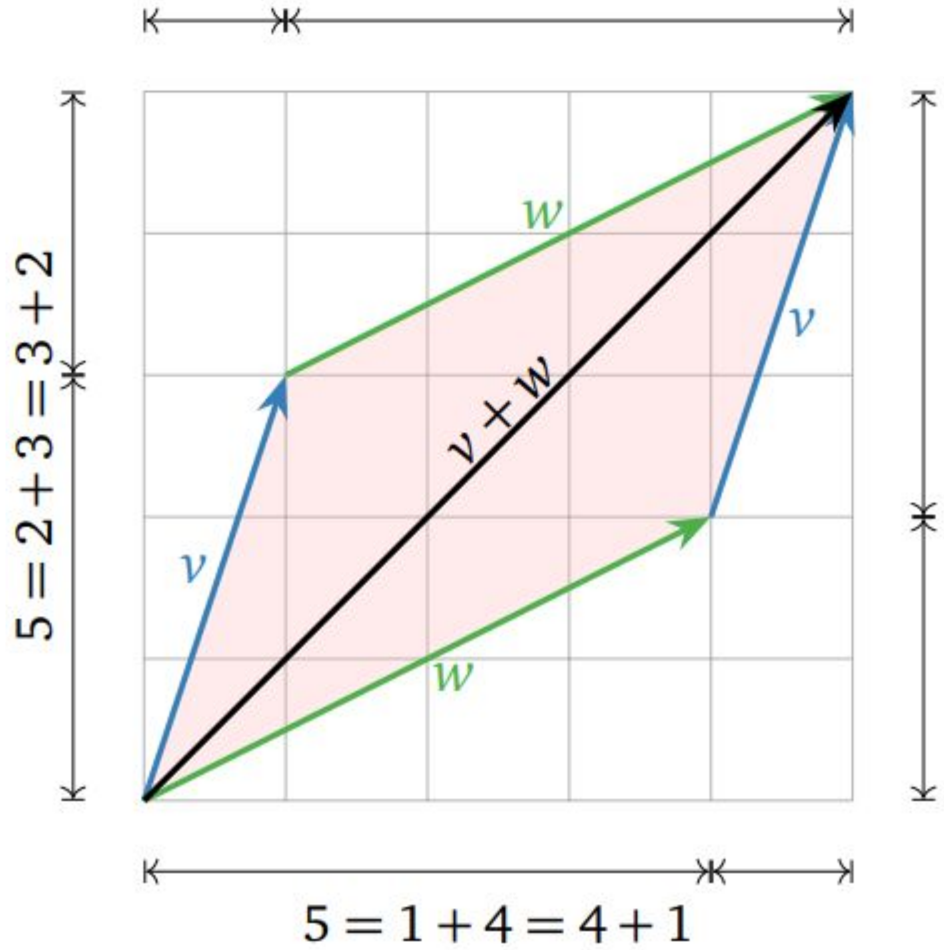
Example.

$$\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}4\\5\\6\end{pmatrix}=\begin{pmatrix}5\\7\\9\end{pmatrix} \quad \text{and} \quad -2\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-2\\-4\\-6\end{pmatrix}.$$

The Parallelogram Law for Vector Addition Geometrically, the sum of two vectors v, w is obtained as follows: place the tail of w at the head of v. Then $v+w$ is the vector whose tail is the tail of v and whose head is the head of w. Doing this both ways creates a parallelogram. For example,

$$\begin{pmatrix}1\\3\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}=\begin{pmatrix}5\\5\end{pmatrix}.$$

Add these vectors geometrically



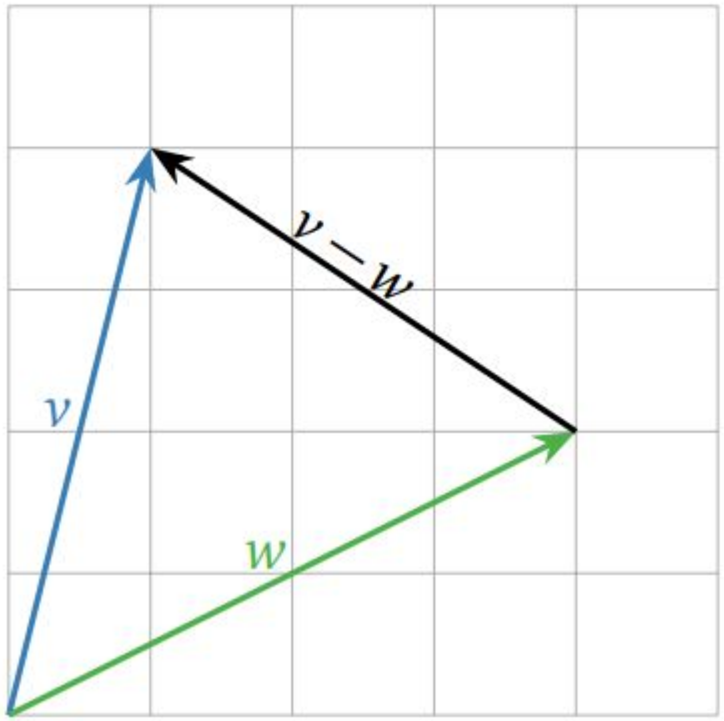
$5 = 2 + 3 = 3 + 2$
w
v
$v + w$
v
w
$5 = 1 + 4 = 4 + 1$

Vector Subtraction Geometrically, the difference of two vectors v, w is obtained as follows: place the tail of v and w at the same point. Then $v - w$ is the vector from the head of w to the head of v. For example,

$$\begin{pmatrix}1\\4\end{pmatrix} - \begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix}.$$

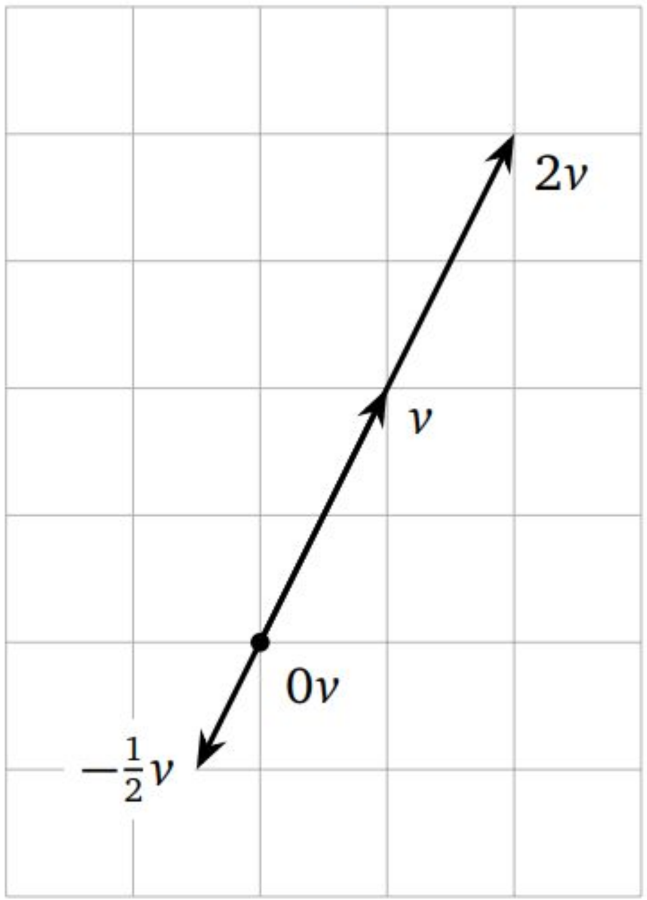
Subtract these vectors geometrically

Why? If you add $v - w$ to w, you get v.

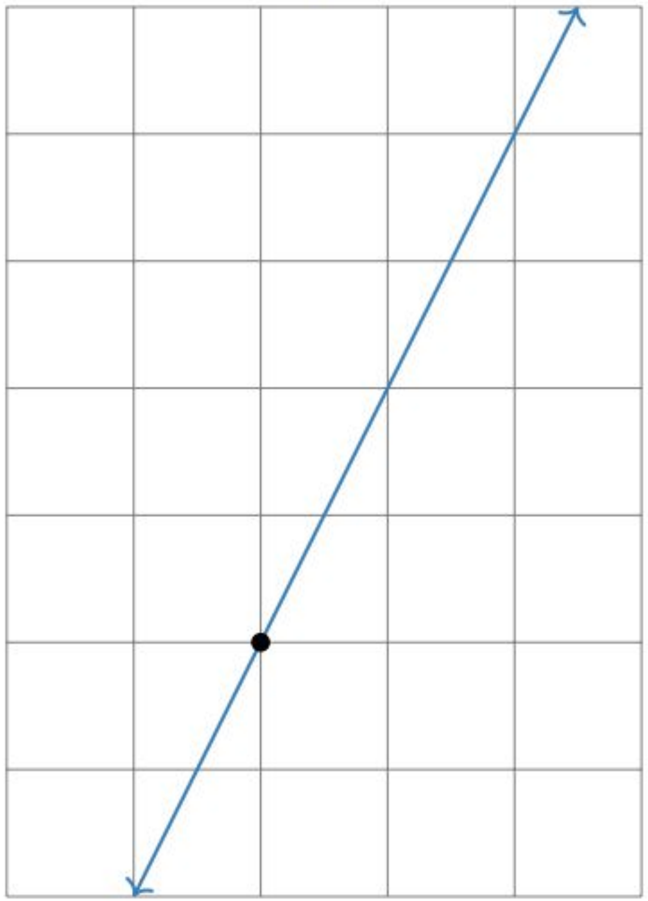


Scalar Multiplication A scalar multiple of a vector v has the same (or opposite) direction, but a different length. For instance, $2v$ is the vector in the direction of v but twice as long, and $-\frac{1}{2}v$ is the vector in the opposite direction of v, but half as long. Note that the set of all scalar multiples of a (nonzero) vector v is a *line*.

Some multiples of v.



All multiples of v.



2.1.3 Linear Combinations

We can add and scale vectors in the same equation.

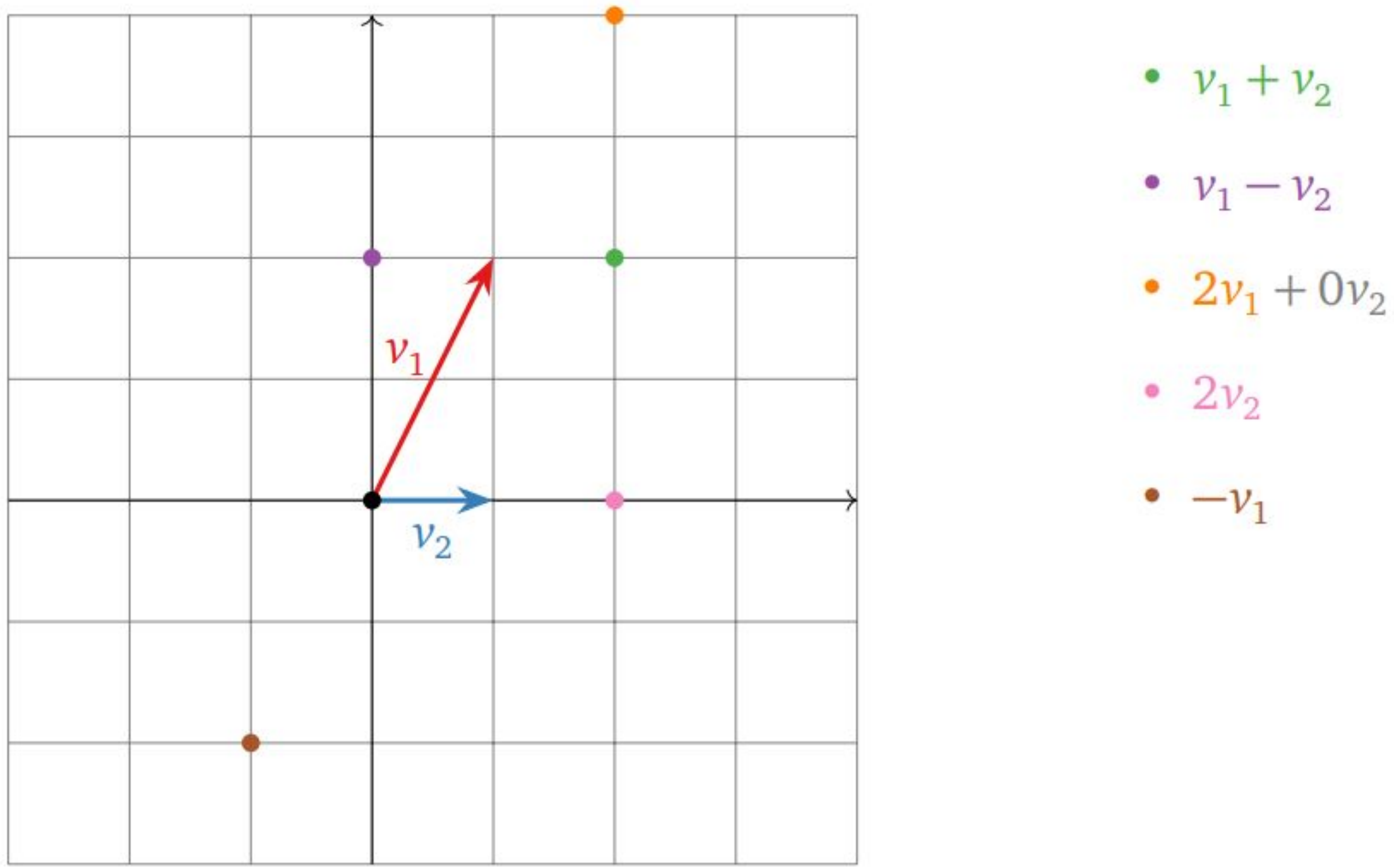
Definition. Let $c_1, c_2, \dots, c_k$ be scalars, and let $v_1, v_2, \dots, v_k$ be vectors in $\mathbf{R}^n$. The vector in $\mathbf{R}^n$

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

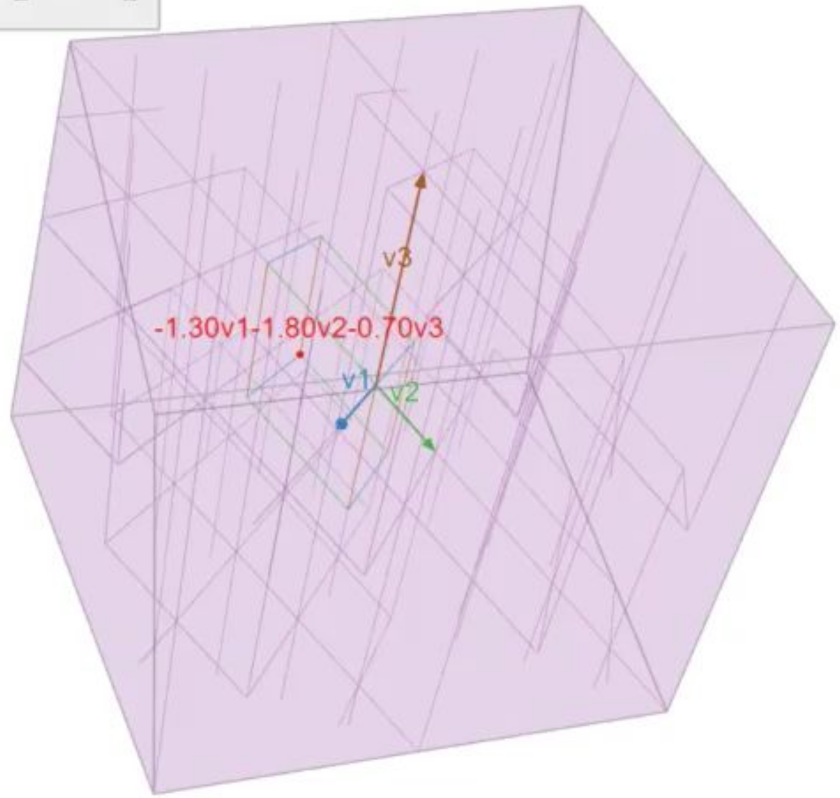
is called a **linear combination** of the vectors $v_1, v_2, \dots, v_k$, with **weights** or **coefficients** $c_1, c_2, \dots, c_k$.

Geometrically, a linear combination is obtained by stretching / shrinking the vectors $v_1, v_2, \dots, v_k$ according to the coefficients, then adding them together using the parallelogram law.

Example. Let $v_1 = \binom{1}{2}$ and $v_2 = \binom{1}{0}$. Here are some linear combinations of v_1 and v_2, drawn as points.



$$-1.30\begin{bmatrix}2.00\\-1.00\\1.00\end{bmatrix} - 1.80\begin{bmatrix}1.00\\1.00\\-1.00\end{bmatrix} - 0.70\begin{bmatrix}-1.00\\1.00\\4.00\end{bmatrix} = \begin{bmatrix}-3.70\\-1.20\\-2.30\end{bmatrix}$$



Linear Combinations of the vector

2.2 Vector Equations and Spans

Objectives

1. Understand the equivalence between a system of linear equations and a vector equation.
2. Learn the definition of Span$\{x_1, x_2, \ldots, x_k\}$, and how to draw pictures of spans.
3. *Recipe:* solve a vector equation using augmented matrices / decide if a vector is in a span.
4. *Pictures:* an inconsistent system of equations, a consistent system of equations, spans in $\mathbf{R}^2$ and $\mathbf{R}^3$.
5. *Vocabulary word:* **vector equation**.
6. *Essential vocabulary word:* **span**.