Formulation of the variational principle

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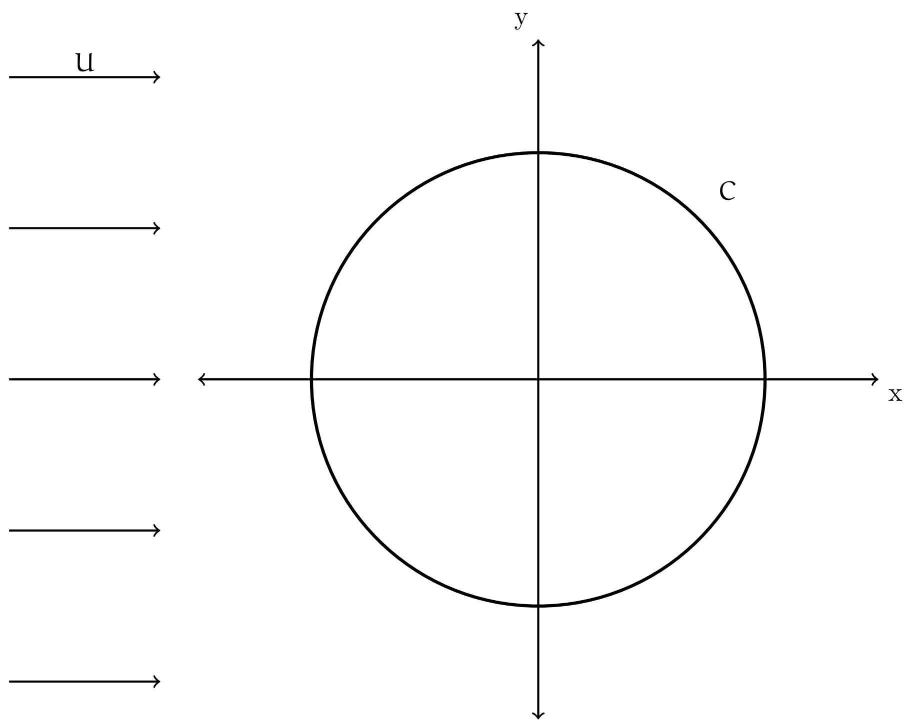

Figure 1: Sketch of the flow field

1 Introduction

The classical problem of steady inviscid subsonic flow past an aerofoil can be formulated in two ways. The usual formulation is as a boundary value problem consisting of a set non linear partial differential equations and a set of boundary conditions. However, it can also be formulated in terms of complementary variational principles. This report describes a variational method for obtaining numerical solutions. The method consists of replacing the infinitely dimensional variational problem by a finitely dimensional problem by means of finite differences, and an approximate maximizing function is then found by standard methods.

2 Formulation

The boundary value problem for plane subsonic flow past an aerofoil can be formulated as follows.

Let $\vec{u} = (u_1, u_2)$ be the velocity vector in the Cartesian coordinate system. Far from the aerofoil, C $\vec{u} = (U, 0)$ where U is a constant.

For an irrotational flow, a velocity potential can be defined as

$$\vec{u} = \nabla\phi$$

The pressure and density are defined by p and ρ respectively. The speed of sound is defined by

$$c^2 = \frac{dp}{d\rho}$$

$$p = p_0\left(1 - \frac{q^2}{2\beta c_0^2}\right)^\alpha$$

$$\rho = \rho_0\left(1 - \frac{q^2}{2\beta c_0^2}\right)^\beta$$

where

$$q^2 = \vec{u}.\vec{u}, \qquad \alpha = \frac{\gamma}{\gamma - 1}, \qquad \beta = \frac{1}{\gamma - 1}$$

the suffix 0 indicates stagnation values.

The boundary value problem for ϕ is equivalent to a variational problem of maximizing the integral

$$J[\phi] = \int_{R_1} p dV + \int_B \phi h dA \tag{1}$$

(dV = area element, dA = arc length element of $B = C$). Then $J[\phi]$ is a maximum if $\nabla.(\rho\vec{u} = 0$ and $\rho\vec{u}.\hat{n} = h$ on B. Here the normal mass flu h is prescribes on B such that

$$\text{outflow} = \int_B h dA = 0$$

If the flow region R_1 becomes infinite, the variational integral (1) becomes unbounded. To remove this difficulty, the integral can be formulated as follows

$$J[\phi] = \int\int_\infty [p - p_\infty + p_\infty.\nabla(\phi - \phi_\infty)] dx dy \tag{2}$$

where

p_∞ = pressure at infinity,
ρ_∞ = density at infinity,
ϕ_∞ = potential for a uniform stream
ϕ_0 = potential for incompressible flow past C.

To extend the method to a general class of aerofoils, a conformal mapping from the aerofoil C onto a unit circle can be used. Let $z = x + iy$ and $\sigma = r(\cos\theta + i\sin\theta)$, then,, the transform modulus is given by

$$T = \left|\frac{dz}{d\sigma}\right| = (x_r^2 + y_r^2)^{\frac{1}{2}}$$

The Jacobian of the transformation is

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = x_r y_\theta - x_\theta y_r$$

Since the transformation is conformal

$$y_\theta = r x_r \qquad \text{and} \qquad y_r = -\frac{1}{r} x_\theta$$

so

$$T^2 = x_r^2 + \frac{1}{r^2} x_t heta^2$$

and

$$J = r\left(x_r^2 + \frac{1}{r^2} x_\theta^2\right)$$

Hence

$$J = rT^2 \tag{3}$$

The coordinates r, θ are orthogonal so the element of length $ds = |dz|$ is given by

$$ds^2 = h_1^2 dr^2 + h_2^2 d\theta^2$$

Also

$$ds^2 = |dz|^2 = \left|\frac{dz}{d\sigma}\right|^2 |d\sigma|^2$$
$$= T^2(dr^2 + r^2 d\theta^2)$$

Therefore

$$h1 = T \qquad \text{and} \qquad h_2 = rT$$

so

$$\nabla\phi = \frac{1}{T}\left(\hat{r}\phi_r + \frac{\hat{\theta}}{r}\phi_\theta\right)$$

Since

$$q^2 = (\nabla\phi)^2$$

and

$$\phi = U(r\cos\theta + \chi)$$

we have

$$q^2 = \frac{U^2}{T^2}\left[1 + 2\cos\theta.\chi_r - \frac{2}{r}\sin\theta.\chi_\theta + \chi_r^2 + \frac{1}{r^2}\chi_\theta^2\right] \tag{4}$$

Also

$$p = p_0\left(1 - \frac{q^2}{2\beta c_0^2}\right)^\alpha$$

and by using the relation

$$p_0 = \left(\frac{\rho_0}{\rho_\infty}\right)^\gamma p_\infty$$

we get after some manipulation

$$p = p_\infty\left[1 + \frac{(\gamma-1)M_\infty^2}{2T^2}\left(T^2 - 1 - 2\cos\theta.\chi_r + \frac{2}{r}\sin\theta.\chi_\theta - \chi_r^2 - \frac{1}{r^2}\chi_\theta^2\right)\right]^\alpha \tag{5}$$

where the free stream Mach number M_∞ is defined by

$$M_\infty^2 = \frac{2\beta U^2}{2\beta c_0^2 - U^2}$$

Now

$$\frac{\gamma p_\infty}{\rho_\infty} = c_0^2 - \frac{U^2}{2\beta}$$

Thus since

$$\phi_o = U\left(r + \frac{1}{r}\right)\cos\theta,$$

$$\rho_\infty \nabla\phi_0.\nabla(\phi - \phi_\infty) = p_\infty \frac{\gamma M_\infty^2}{T^2}\left[\frac{r^2-1}{r^2}\cos_t heta.\chi_r - \frac{r^2-1}{r^3}\sin\theta.\chi_\theta\right] \tag{6}$$

When the expressions (3), (5) and (6) are used in (2), the variational integral becomes

$$J[\chi] = p_\infty \int_0^{2\pi}\int_1^\infty \left\{\left[1 + \frac{(\gamma-1)M_\infty^2}{2T^2}\left(T^2 - 1 - 2\cos\theta.\chi_r + \frac{2}{r}\sin\theta.\chi_\theta - \chi_r^2 - \frac{1}{r^2}\chi_\theta^2\right)\right]^\alpha \right.$$
$$\left. -1 + \frac{\gamma M_\infty^2}{T^2}\left(\frac{r^2-1}{r^2}\cos\theta.\chi_r - \frac{r^2+1}{r^3}\sin\theta.\chi_\theta\right)\right\} rT^2\,dr \tag{7}$$

The boundary conditions on χ are

$$\begin{aligned} \frac{\partial\chi}{\partial r} &= -\cos\theta \qquad \text{at} \qquad r = 1, \\ \chi &= 0\left(\frac{1}{r}\right) \qquad \text{as} \qquad r \to \infty \end{aligned} \tag{8}$$

The local Mach number$M$ and the local non dimensional pressure p_L are given by

$$M = M_\infty \frac{q}{U}\left[1 + \frac{1}{2}(\gamma-1)M_\infty^2\left(1 - \left(\frac{q}{U}\right)^2\right)\right]^{\frac{1}{2}} \tag{9}$$

$$p_L = \frac{p}{p_\infty} = \left[1 + \frac{1}{2}(\gamma-1)M_\infty^2\left(1 - \left(\frac{q}{U}\right)^2\right)\right]^{\frac{\gamma}{\gamma-1}} \tag{10}$$

where q is given by equation (4)

The transform modulus for an ellipse is given by

$$T^2 = \frac{1}{r^4}\left[\left(r^2 + \lambda^2\right)^2 - 4\lambda^2 r^2 \cos^2\theta\right] \tag{11}$$

where λ^2 is the following function of τ, the thickness ratio of the ellipse,

$$\lambda^2 = \frac{1-\tau}{1+\tau} \tag{12}$$

3 Numerical Method

The objective of the calculation is to find for given M_∞ and aerofoil shape a function χ which maximizes $J[\chi]$ as given by (7) subject to boundary conditions (8).

3.1 Domain

For non lifting symmetric bodies (about $y = 0$), it is only necessary to treat the interval $0 \leq \theta \leq \pi$.

Since the derivatives in both directions are approximated by finite differences, it is necessary to have a finite computation region. This is obtained by replacing the infinite integration limit on r by a finite limit R and insisting that the reduced potential χ satisfies an appropriate condition at $r = R$. The simplest condition to impose is that χ equals the reduced potential for incompressible flow at $r = R$.

Thus the variational integral reduces to

$$J[\chi] = p_\infty \int_0^\pi \int_1^R F(r, \theta, \chi_r, \chi_\theta) dr \tag{13}$$

where

$$F = \left\{ \left[1 + \frac{(\gamma - 1)M_\infty^2}{2T^2} \left(T^2 - 1 - 2\cos\theta.\chi_r + \frac{2}{r}\sin\theta.\chi_\theta - \chi_r^2 - \frac{1}{r^2}\chi_\theta^2 \right) \right]^\alpha \right.$$
$$\left. -1 + \frac{\gamma M_\infty^2}{T^2} \left(\frac{r^2 - 1}{r^2}\cos\theta.\chi_r - \frac{r^2 + 1}{r^3}\sin\theta.\chi_\theta \right) \right\} rT^2$$

The boundary conditions are now

$$\begin{aligned}
\frac{\partial\chi}{\partial\theta} &= 0 && \text{at} \quad \theta = 0, \pi \\
\frac{\partial\chi}{\partial r} &= -\cos\theta && \text{at} \quad r = 1 \\
\chi &= \frac{1}{R}\cos\theta && \text{at} \quad r = R
\end{aligned} \tag{14}$$

3.2 Discretisation

$$r = 1, \ldots, R; \qquad k_j = r_{j+1} - r_j$$
$$\theta = 0, \ldots, \pi; \qquad h_i = \theta_{i+1} - \theta_i$$

θ varies linearly from 0 to π. r is mapped as $r = 1/\sigma$, where σ varies linearly in the range [1/R, 1].

Thus, the infinitely dimensional variational problem is now replaced by a finite-dimensional problem. Consider four neighbouring points as shown in figure (3). The derivatives of χ in the rectangle (i, j) can be approximated as

$$\frac{\partial\chi}{\partial\theta} = \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi i, j + 1}{2h_i}$$

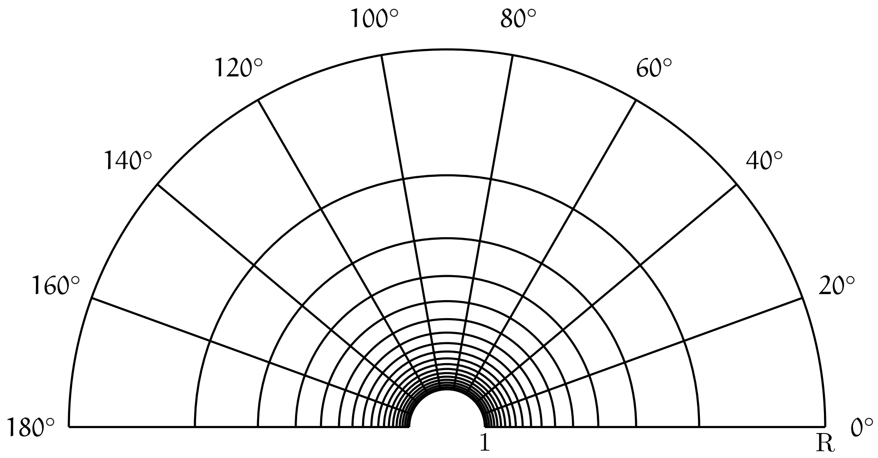

Figure 2: Mesh around the body

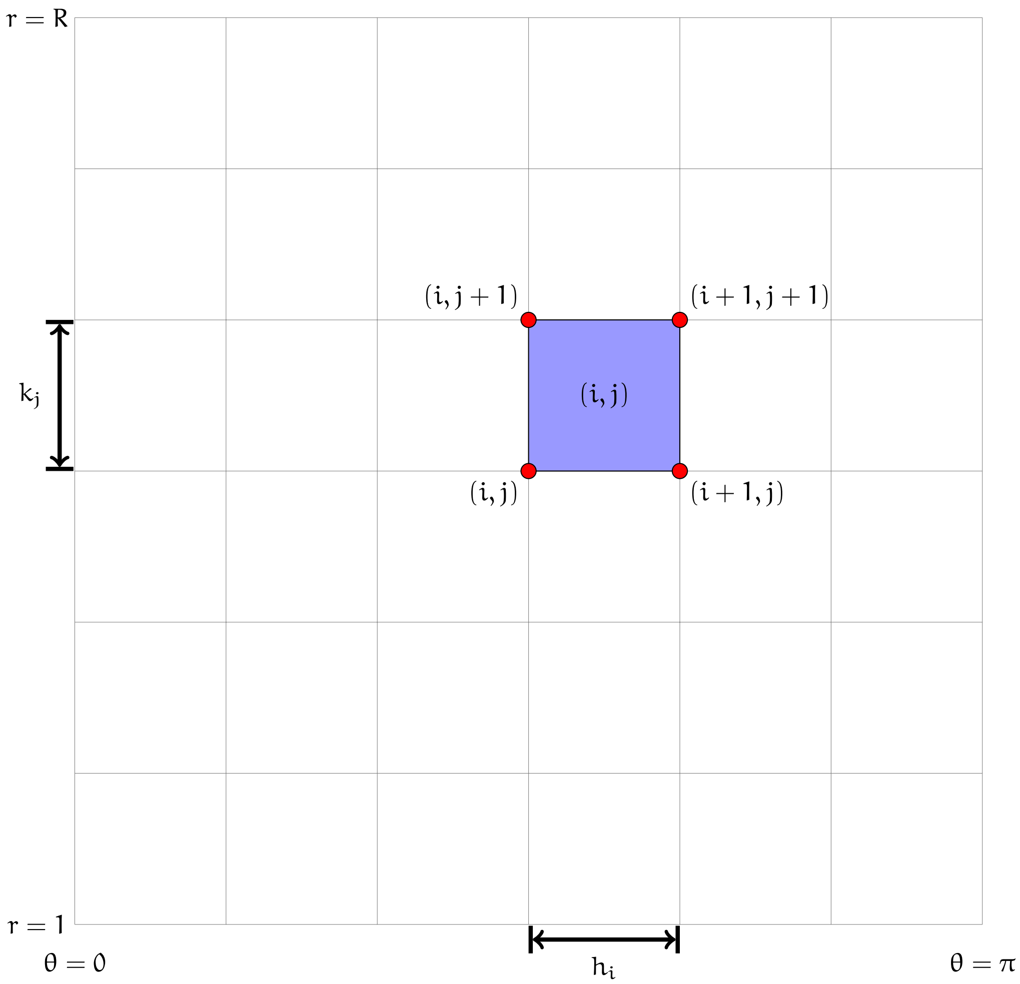

Figure 3: Computational domain

$$\frac{\partial \chi}{\partial \theta} = \frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi i + 1, j}{2k_j}$$

For a rectangular region i, j, $J[\chi]$ can be approximated as

$$\begin{aligned} J_{i,j} = p_\infty \Bigg\{ \Bigg[1 + \frac{(\gamma - 1)M_\infty^2}{2T_{i,j}^2} \Bigg(T^2 - 1 - 2\cos\theta_i' \frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2k_j} \\ + \frac{2}{r_j'} \sin\theta_i' \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i} \\ - \left(\frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2k_j} \right)^2 \\ - \frac{1}{r^2} \left(\frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i} \right)^2 \Bigg) \Bigg]^\alpha \\ - 1 + \frac{\gamma M_\infty^2}{T_{i,j}^2} \Bigg(\left(\frac{r_j'^2 - 1}{r_j'^2} \right) \cos\theta_i' \frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2k_j} \\ - \left(\frac{r_j'^2 - 1}{r_j'^3} \right) \sin\theta_i' \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i} \Bigg) \Bigg\} \end{aligned} \tag{15}$$

$$\theta_i' = \theta_i + 0.5h_i; \qquad r_j' = r_j + 0.5k_j$$

$$T_{i,j}^2 = \frac{1}{r_j'^4} \left[(r_j'^2 + \lambda^2)^2 - 4\lambda^2 r_j'^2 \cos^2\theta_i' \right]$$

$$\lambda^2 = \frac{1 - \tau}{1 + \tau}$$

Boundary conditions for χ-

1.
$$\frac{\partial \chi}{\partial r} = -\cos\theta; \text{ at } r = 1$$
$$\chi_{i,1} = \frac{1}{k_2 + 2k_1} \left[\frac{1}{k_2} ((k_1 + k_2)^2 \chi_{1,2} - k_1^2 \chi_{i,3}) + k_1(k_1 + k_2)\cos\theta \right] \tag{16}$$

2.
$$\frac{\partial \chi}{\partial \theta} = 0 \text{ at } \theta = 0, \pi$$
$$\chi_{m+1,j} = \chi_{m-1,j} \qquad \chi_{0,j} = \chi_{2,j} \tag{17}$$

3.
$$\chi = \frac{1}{R}\cos\theta \text{ as } r \to \infty$$
$$\chi_{i,n} = \frac{1}{R}\cos\theta_i \tag{18}$$

Summing the contributions for each rectangle, $J[\chi]$ can be approximated as

$$J[\chi] = \bar{J} = \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} J_{i,j} \tag{19}$$

Therefore, the values of $\chi_{i,j}$ which maximizes this expression is given by the solutions of the equations

$$\frac{\partial \bar{J}}{\partial \chi_{i,j}} = 0 \tag{20}$$
$$i = 1, \ldots, n$$
$$j = 2, \ldots, m-1$$

for a given i, j, the equation to maximize can be written in the form (by summing the contributions from the four surrounding rectangles)

$$g(\chi_{i,j}) = \frac{\partial \bar{J}}{\partial \chi_{i,j}} \sum_{s=(i,j),(i-1,j),(i,j-1),(i-1,j-1)} \Big[\alpha \left(A_s \chi_{i,j}^2 + B_s \chi i, j + C_s \right)^{\alpha - 1} (2A_s \chi_{i,j} + B_s) + D_s \Big] H_s = 0 \tag{21}$$

Using Newton Raphson method, an improved estimate is given by

$$\chi_{i,j}^{(q+1)} = \chi_{i,j}^{(q)} - \frac{g(\chi_{i,j}^{(q)})}{g'(\chi_{i,j}^{(q)})} \tag{22}$$

$$g' = \frac{\partial g_{i,j}}{\partial \chi_{i,j}} \tag{23}$$

3.3 Coefficients A, B, C, D, H

The coefficients A, B, C, D, H are given by -

$$\begin{aligned}
A_{i,j} &= -\frac{(\gamma-1)M_\infty^2}{2T_{i,j}^2} \left(\frac{1}{4k_j^2} + \frac{1}{4r'^2_j h_i} \right) \\
A_{i,j-1} &= -\frac{(\gamma-1)M_\infty^2}{2T_{1,j-1}^2} \left(\frac{1}{4k_{j-1}^2} + \frac{1}{4r'^2_{j-1} h_i} \right) \\
A_{i-1,j} &= -\frac{(\gamma-1)M_\infty^2}{2T_{i-1,j}^2} \left(\frac{1}{4k_j^2} + \frac{1}{4r'^2_j h_{i-1}} \right) \\
A_{i-1,j-1} &= -\frac{(\gamma-1)M_\infty^2}{2T_{i-1,j-1}^2} \left(\frac{1}{4k_{j-1}^2} + \frac{1}{4r'^2_{j-1} h_{i-1}} \right)
\end{aligned} \tag{24}$$

$$
\begin{aligned}
B_{i,j} = \frac{(\gamma-1)M_\infty^2}{2T_{i,j}^2}\Bigg(& 2\cos\theta_i' \frac{1}{2k_j} - \frac{2}{r_j'}\sin\theta_i' \frac{1}{2h_i} \\
&+ \frac{1}{2k_j^2}(\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i+1,j}) \\
&+ \frac{1}{2r_j'^2 h_i^2}(\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j+1})\Bigg) \\
B_{i,j-1} = \frac{(\gamma-1)M_\infty^2}{2T_{i,j-1}^2}\Bigg(& -2\cos\theta_i' \frac{1}{2k_{j-1}} - \frac{2}{r_{j-1}'}\sin\theta_i' \frac{1}{2h_i} \\
&- \frac{1}{2k_{j-1}^2}(\chi_{i+1,j} - \chi_{i,j-1} - \chi_{i+1,j-1}) \\
&+ \frac{1}{2r_{j-1}'^2 h_i^2}(\chi_{i+1,j-1} + \chi_{i+1,j} - \chi_{i,j-1})\Bigg) \\
B_{i-1,j} = \frac{(\gamma-1)M_\infty^2}{2T_{i-1,j}^2}\Bigg(& 2\cos\theta_{i-1}' \frac{1}{2k_j} + \frac{2}{r_j'}\sin\theta_{i-1}' \frac{1}{2h_{i-1}} \\
&+ \frac{1}{2k_j^2}(\chi_{i-1,j+1} + \chi_{i,j+1} - \chi_{i-1,j}) \\
&- \frac{1}{2r_j'^2 h_{i-1}^2}(\chi_{i,j+1} - \chi_{i-1,j+1} - \chi_{i-1,j+1})\Bigg) \\
B_{i-1,j-1} = \frac{(\gamma-1)M_\infty^2}{2T_{i-1,j-1}^2}\Bigg(& -2\cos\theta_{i-1}' \frac{1}{2k_{j-1}} + \frac{2}{r_{j-1}'}\sin\theta_{i-1}' \frac{1}{2h_{i-1}} \\
&- \frac{1}{2k_{j-1}^2}(\chi_{i-1,j} - \chi_{i-1,j-1} - \chi_{i,j-1}) \\
&- \frac{1}{2r_{j-1}'^2 h_{i-1}^2}(\chi_{i,j-1} - \chi_{i-1,j-1} - \chi_{i-1,j})\Bigg)
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
C_{i,j} = 1 + \frac{(\gamma-1)M_\infty^2}{2T_{i,j}^2}\Bigg(T_{i,j}^2 - 1 - \frac{\cos\theta_i'}{k_j}(\chi_{i,j+1}+\chi_{i+1,j+1}-\chi_{i+1,j}) \\
+\frac{\sin\theta_i'}{r_j'h_i}(\chi_{i+1,j}+\chi_{i+1,j+1}-\chi_{i,j+1}) \\
-\frac{1}{4k_j^2}(\chi_{i,j+1}+\chi_{i+1,j+1}-\chi_{i+1,j})^2 \\
-\frac{1}{4r_j'^2h_i^2}(\chi_{i+1,j}+\chi_{i+1,j+1}-\chi_{i,j+1})^2\Bigg) \\
C_{i,j-1} = 1 + \frac{(\gamma-1)M_\infty^2}{2T_{i,j-1}^2}\Bigg(T_{i,j-1}^2 - 1 - \frac{\cos\theta_i'}{k_{j-1}}(\chi_{i+1,j}-\chi_{i,j-1}-\chi_{i+1,j-1}) \\
+\frac{\sin\theta_i'}{r_{j-1}'h_i}(\chi_{i+1,j-1}+\chi_{i+1,j}-\chi_{i,j-1}) \\
-\frac{1}{4k_{j-1}^2}(\chi_{i+1,j}-\chi_{i,j-1}-\chi_{i+1,j-1})^2 \\
-\frac{1}{4r_{j-1}'^2h_i^2}(\chi_{i+1,j-1}+\chi_{i+1,j}-\chi_{i,j-1})^2\Bigg) \\
C_{i-1,j} = 1 + \frac{(\gamma-1)M_\infty^2}{2T_{i-1,j}^2}\Bigg(T_{i-1,j}^2 - 1 - \frac{\cos\theta_{i-1}'}{k_j}(\chi_{i-1,j+1}+\chi_{i,j+1}-\chi_{i-1,j}) \\
+\frac{\sin\theta_{i-1}'}{r_j'h_i}(\chi_{i,j+1}-\chi_{i-1,j+1}-\chi_{i-1,j+1}) \\
-\frac{1}{4k_j^2}(\chi_{i-1,j+1}+\chi_{i,j+1}-\chi_{i-1,j})^2 \\
-\frac{1}{4r_j'^2h_{i-1}^2}(\chi_{i,j+1}-\chi_{i-1,j+1}-\chi_{i-1,j+1})^2\Bigg) \\
C_{i-1,j-1} = 1 + \frac{(\gamma-1)M_\infty^2}{2T_{i-1,j-1}^2}\Bigg(T_{i-1,j-1}^2 - 1 - \frac{\cos\theta_{i-1}'}{k_{j-1}}(\chi_{i-1,j}-\chi_{i-1,j-1}-\chi_{i,j-1}) \\
+\frac{\sin\theta_{i-1}'}{r_{j-1}'h_{i-1}}(\chi_{i,j-1}-\chi_{i-1,j-1}-\chi_{i-1,j}) \\
-\frac{1}{4k_{j-1}^2}(\chi_{i-1,j}-\chi_{i-1,j-1}-\chi_{i,j-1})^2 \\
-\frac{1}{4r_{j-1}'^2h_{i-1}^2}(\chi_{i,j-1}-\chi_{i-1,j-1}-\chi_{i-1,j})^2\Bigg)
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
D_{i,j} &= -1 + \frac{\gamma M_\infty^2}{T_{i,j}^2}\left[-\left(\frac{r_j'^2 - 1}{r_j'^2}\right)\frac{\cos\theta_i'}{2k_j} + \left(\frac{r_j'^2 - 1}{r_j'^3}\right)\frac{\sin\theta_i'}{2h_i}\right] \\
D_{i,j-1} &= -1 + \frac{\gamma M_\infty^2}{T_{i,j-1}^2}\left[\left(\frac{r_{j-1}'^2 - 1}{r_{j-1}'^2}\right)\frac{\cos\theta_i'}{2k_{j-1}} + \left(\frac{r_{j-1}'^2 - 1}{r_{j-1}'^3}\right)\frac{\sin\theta_i'}{2h_i}\right] \\
D_{i-1,j} &= -1 + \frac{\gamma M_\infty^2}{T_{i-1,j}^2}\left[-\left(\frac{r_j'^2 - 1}{r_j'^2}\right)\frac{\cos\theta_{i-1}'}{2k_j} - \left(\frac{r_j'^2 - 1}{r_j'^3}\right)\frac{\sin\theta_{i-1}'}{2h_{i-1}}\right] \\
D_{i-1,j-1} &= -1 + \frac{\gamma M_\infty^2}{T_{i-1,j-1}^2}\left[\left(\frac{r_{j-1}'^2 - 1}{r_{j-1}'^2}\right)\frac{\cos\theta_{i-1}'}{2k_{j-1}} - \left(\frac{r_{j-1}'^2 - 1}{r_{j-1}'^3}\right)\frac{\sin\theta_{i-1}'}{2h_{i-1}}\right]
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
H_{i,j} &= r_j' T_{i,j}^2 h_i k_j \\
H_{i,j-1} &= r_{j-1}' T_{i,j-1}^2 h_i k_{j-1} \\
H_{i-1,j} &= r_j' T_{i-1,j}^2 h_{i-1} k_j \\
H_{i-1,j-1} &= r_{j-1}' T_{i-1,j-1}^2 h_{i-1} k_{j-1}
\end{aligned} \tag{28}
$$