Assignment 1

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1. Prove that for positive integer n, $169 \mid 3^{3n+3} - 26n - 27$

Sol. Given, $n \in \mathbb{N}$.

Suppose, $169 \mid 3^{3n+3} - 26n - 27$ for a given n.

For, n+1, we have

$3^{3(n+1)+3} - 26(n+1) - 27$

$= (27) \cdot (3^{3n+3}) - 26(n+1) - 27$

$= (26+1) \cdot (3^{3n+3}) - 26n - 26 - 27$

$= 26 \cdot (3^{3n+3} - 1) + 3^{3n+3} - 26n - 27.$

We note that,

$27 \equiv 1 (\mod 26)$

$\implies 27^{n+1} \equiv 1 (\mod 26)$

$\implies 26 \mid 3^{3n+3} - 1$

$\implies 26^2 \mid 26 \cdot (3^{3n+3} - 1)$ But $169 \mid 26^2 \implies 169 \mid 26 \cdot (3^{3n+3-1})$

Also, by our assumption

$169 \mid (3^{3n+3} - 26n - 27)$

Hence, $169 \mid 3^{3r+3} - 26r - 27$ for r=n $\implies 169 \mid 3^{3r+3} - 26r - 27$ for r=n+1.

We note that, for $n = 1$,

$3^{3+3} - 26 - 27 = 676 = 169(4)$

$\implies 169 \mid 3^{3n+3} - 26n - 27$ at $n = 1$.

Hence, by induction, $169 \mid 3^{3n+3} - 26n - 27 \ \forall \ n \in \mathbb{N}$. **Proved.**

2. Prove that for positive integer n we have, $n^2 \mid (n+1)^n - 1$

Sol.

With Binomial Theorem, we have,

$(1+x)^n = \binom{n}{0} + \binom{n}{1} \cdot x^1 + \binom{n}{2} \cdot x^2 + ... + \binom{n}{n} \cdot x^n$

$\implies \exists \ k \in \mathbb{Z}$ such that

$(1+x)^n = 1 + n \cdot x + k \cdot x^2$

Substituting x = n,

$(1+n)^n = 1 + n^2 + k \cdot n^2$

$\implies (1+n)^n - 1 = l \cdot (n^2)$ for $l \in \mathbb{Z}$ and $l = k + 1$

3. Prove that if for integers a and b we have, $7 \mid a^2 + b^2$ then $7 \mid a$ and $7 \mid b$.

Let $a = 7Q + R$

and $b = 7q + r$ for Q,R,q,r $\in \mathbb{N}$ and $0 \leq R, r \leq 6$

Given $7 \mid a^2 + b^2$

$a^2 + b^2 = (7Q + R)^2 + (7q + r)^2$

$\implies a^2 + b^2 = 49(Q^2 + q^2) + 14(QR + qr) + R^2 + r^2$

$\implies 7 \mid a^2 + b^2 \iff 7 \mid R^2 + r^2$

The only possible pair is R=0 and r=0.

$\implies 7 \mid a$ and $7 \mid b$.

4. For numbers $2k - 1$ and $9k + 4$, find their greatest common divisor as a function of k.

We see that $9k + 4 > 2k - 1$

Starting the Division Algorithm,

$9k + 4 = 4 \cdot ((2k - 1)) + (k + 8)$

Now, if $k + 8 < 2k - 1$, we have our first remainder.

this happens when $k > 9$.

if $k = 9$, then remainder is 0. So gcd(9k+4,2k-1) at k=9 is 17 (i.e 2k-1 itself).

if $k < 9 \implies k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$

then 2k-1 is prime for all k less than 9 except at k =5.

when k=5, we note that gcd(49,9)=1.

Hence, we conclude that, gcd(9k+4,2k-1) = 1 for all $k < 9$.

Next, if k¿9, first remainder is k+8.

Then $2k - 1 = (k + 8) + (k - 9)$

Also,$k + 8 = 1 \cdot ((k - 9)) + 17$

Now, if 17 | (k-9), then gcd=17.

if not, then gcd = gcd(k-9,17) = 1 (since 17 is prime.)

Hence, we conclude that,

gcd(9k+4,2k-1) = {17 if $k = 17n + 9$ and 1 otherwise }

5. Find the remainder when 2^{81} is divided by 17.

Sol. $2^{81} = 2 \cdot 16^{20}$

$16 \equiv -1 \pmod{17} \implies 16^{20} \equiv 1 \pmod{17}$

$\implies 2^{81} \equiv 2 * 1 \pmod{17}$

Hence, the remainder when 2^{81} divides 17 is 2.

6. Prove that $2^n + 6 \cdot 9^n$ is always divisible by 7 for any positive integer n.

Sol.

Let the relation hold for n=k.

$\implies 7 \mid 2^k + 6 \cdot 9^k$

Then, for n=k+1, we have,

$2^{k+1} + 6 \cdot 9^{k+1} = 2 \cdot 2^k + 9 \cdot 6 \cdot 9^k = (9-7) \cdot 2^k + 9 \cdot 6 \cdot 9^k$

$= 9 \cdot (2^k + 6 \cdot 9^k) - 7 \cdot 2^k$

Since, $7 \mid 2^k + 6 \cdot 9^k$,

$\implies 7 \mid 9 \cdot (2^k + 6 \cdot 9^k) - 7 \cdot 2^k$

$\implies 7 \mid 2^{k+1} + 6 \cdot 9^{k+1}$

Hence, $7 \mid 2^k + 6 \cdot 9^k \implies 7 \mid 2^{k+1} + 6 \cdot 9^{k+1}$.

But, $7 \mid 2^1 + 6 \cdot 9^1$.

By Induction, $7 \mid 2^k + 6 \cdot 9^k$ For k=2,3,4....

Hence **Proved.**

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sol. Given N=19202122232425.....909192

For divisibility by 3 or 9, we need the sum of the digits.

First, excluding 19,90,91,92.

Ones place, (0+1+2+3 ... 9)$\cdot$(7) $= 45 \cdot 7$

This is clearly divisible by 9.

On tens place, we have $2 \cdot 10 + 3 \cdot 10 + 4 \cdot 10 + 5 \cdot 10 .... 8 \cdot 10 = 350$

$350 \equiv 8 \pmod{9}$ Now, adding the remaining numbers,

$19 \equiv 1 \pmod{9}$

$90 \equiv 0 \pmod{9}$

$91 \equiv 1 \pmod{9}$

$92 \equiv 2 \pmod{9}$

$Sum \equiv 3 \pmod{9}$

Hence, the number is not divisible by 9.

It is divisible by 3. **k=1**

8 Integer solutions (x,y,z) of $x^2 + y^2 = 10^z - 1$

for $z \geq 2$, $10^z - 1 \equiv 3 (\mod 4)$

For any integer x,

One of the following cases hold:

$x \equiv 0 (\mod 4) \implies x^2 \equiv 0 (\mod 4)$

$x \equiv 1 (\mod 4) \implies x^2 \equiv 1 (\mod 4)$

$x \equiv 2 (\mod 4) \implies x^2 \equiv 0 (\mod 4)$

$x \equiv 3 (\mod 4) \implies x^2 \equiv 1 (\mod 4)$ Hence, we can say,

$x^2 + y^2 \equiv r (\mod 4)$

where r can be 0,1 or 2.

Hence, RHS can never be equivalent to RHS for integers x,y,z.

Hence, no integral solution exists.

9 and **10**

Used Euclidean Algorithm (extended) to find gcd of two numbers and express it as their linear combination.

Following Code in C language.

GCD