Optimal Auction Design

Context

- Mechanism design: seller maximizing profit.
- How? A nash equilibrium among buyers that leads to highest profit for seller.
- This leads to two fundamental results we have covered in class:
  - Revelation Principle
  - Revenue Equivalence Theorum (in a restricted setting)

Everything is pretty familiar. We start with a distribution over values defined on a finite interval, which are I.I.D:

$$f_i : [a_1, b_1] \to \mathbb{R}^+$$

We call this private signal t_i for each player, which gives us a set of all player signals:

$$T = t_1, \ldots, t_n$$

Setup

Expected utility of buyer:

$$U_i(p, x, t_i) = \int_{T_{-i}} (v_i(t)p_i(t) - x_i(t)) f_{-i}(t_{-i}) dt_{-i}$$

Conditional expectation of winning:

$$Q_i(p, t_i) = \int_{T_{-i}} p_i(t) f_{-i}(t_{-i}) dt_{-i}$$

The Revelation Principle

Given any feasible auction mechanism, there exists an equivalent feasible direct revelation mechanism which gives to the seller and all bidders the same expected utilities as in the given mechanism.

The Revelation Principle

One of the core requirements built into the definition of feasibility is the the **incentive-compatibility** conditions:

$$U_i(p, x, t_i) \geq \int_{T_{-i}} (v(t)p_i(t_{-i}, s_i) - x_i(t_{-i}, s_i))f_{-i}(t_{-i})dt_{-i}$$

$$\forall i \in N,\ \forall t_i \in [a_i, b_i],\ \forall s_i \in [a_i, b_i]$$

If the player can't gain from lying to herself, the auctioneer can implement the function that maps the players signal to strategy, $p : T \to \mathbb{R}^n$, into the mechanism of the auction itself, as this function is, by definition, shared by all players.

Core assumption on value function:

$$v_i(t) = t_i + \sum_{j:j \neq i \in N} e_j(t_j)$$

Note that this is not what we would expect, it's not a convex combination of ours and others' signals.

(p, x) is feasible if and only if the following conditions hold:

$$\text{if } s_i \le t_i \text{ then } Q_i(p, s_i) \le Q_i(p, t_i), \forall i \in N, \forall s_i, t_i \in [a_i, b_i]$$

$$U_i(p, x, t_i) = U_i(p, x, a_i) + \int_{a_i}^{t_i} Q_i(p, s_i) ds_i, \forall i \in N, \forall t_i \in [a_i, b_i]$$

$$U_i(p, x, a_i) \ge 0, \forall i \in N$$

Revenue Equivalence - Dependence on Value Function

This implies the following about the relationship between our conditional-expectation-of-winning function and the expected utility function:

$$U_i(p, x, t_i) = U_i(p, x, a_i) + \int_{a_i}^{t_i} \frac{\partial}{\partial s_i} U_i(p, x, s_i) ds_i$$

$$U_i(p, x, t_i) = U_i(p, x, a_i) + \int_{a_i}^{t_i} Q_i(p, s_i) ds_i$$

$$\frac{\partial}{\partial s_i} U_i(p, x, s_i) = Q_i(p, s_i)$$

Revenue Equivalence - Dependence on Value Function

Using $f'(x)$ to denote $\dfrac{\partial}{\partial t_i}$ to ease notation.

$$\frac{\partial}{\partial t_i}\int_{T_{-i}}(v_i(t)p_i(t)-x_i(t))f_{-i}(t_{-i})dt_{-i}=\int_{T_{-i}}p_i(t)f_{-i}(t_{-i})dt_{-i}$$

$$\frac{\partial}{\partial t_i}v_i(t)p_i(t)-x_i(t)=p_i(t)$$

$$v_i(t)p_i'(t)-x_i'(t)=p_i(t)-v_i'(t)p_i(t)$$

Note

We use and rely on $Q_i(p,t)$ in the revenue equivalence proof. But consider the following which would allow us more flexibility to acomodate value functions more inline with common value settings:

$$Q_i' = \frac{\partial}{\partial t_i} v_i'(t) Q_i(p,t)$$

The seller's expected utility from a feasible auction mechanism is completely determined by the probability function p and the numbers $U_i(p, x, a_i)$ for all i. That is, once we know who gets the object in each possible situation (as specified by p) and how much expected utility each bidder would get if his value estimate were at its lowest possible level a_i, then the seller's expected utility from the auction does not depend on the payment function x.

Revenue Equivalence - 2nd Lemma

Previous assumptions allow us to write the seller's utility by:

$$U_o(p,x) = \int_T \left(\sum_{i \in N} \left(t_i - t_0 - e_i(t_i) - \frac{1 - F_i(t_i)}{f_i(t_i)})p_i(t) \right) f(t)dt + \int_T v_0(t)f(t)dt - \sum_{i \in N} U_i(p,x,a) \right.$$

This is maximized with regards to x when:

$$\sum_{i \in N} U_i(p,x,a) = 0$$

Rewriting our previous constraints:

$$\int_{T_{-i}} \left(p_i(t)v_i(t) - \int_{a_i}^{b_i} p_i(t_{-i}, s_i)ds_i - x_i(t) \right) f_{-i}dt_{-i} = U_i(p, x, a_i) \geq 0$$

Choosing x according to:

$$x_i(t) = p_i(t)v_i(t) - \int_{a_i}^{t_i} p_i(t_{-i}, s_i)ds_i$$

Revenue Equivalence - 2nd Lemma

Recall the previous result of our feasible auction:

$$p_i'(t)v_i(t) - x_i'(t) = p_i(t) - p_i(t)v_i'(t)$$

And recall our choice for x_i:

$$x_i(t) = p_i(t)v_i(t) - \int_{a_i}^{t_i} p_i(t_{-i}, s_i)ds_i$$

$$x_i'(t) = p_i'(t)v_i(t) + p_i(t)v_i'(t) - p_i(t)$$