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Relating Factors Between Studies Based Upon Different Individuals

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RELATING FACTORS BETWEEN STUDIES BASED UPON DIFFERENT INDIVIDUALS[1]

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ABSTRACT

A method for relating factors between studies based upon different individuals is developed. This approach yields a measure of relationship between all factors under consideration—a measure which may be interpreted as a correlation coefficient.

A problem of major interest in the application of factor analysis is to determine the relationship between factors of two different studies. Four cases may be distinguished, depending upon whether the same or different variables and upon whether the same or different individuals have been observed. For the two cases where the same individuals have participated in both studies, the correlation between factors (or their estimates) may be computed directly because of the common individuals. On the other hand, when the factors of two studies based upon different individuals are to be compared, no such obvious solution exists. Thus, how could it be possible to relate the factors of a study done in Alaska with the factors of another study done 47 years later in Alabama? For the case of "different individuals, different variables," an objective solution is surely impossible. However, if the same variables were observed in Alaska and Alabama, a basis for comparison exists.

It is the purpose of this paper to propose a solution to the problem of relating factors between studies based upon different

1. We are indebted to Professor Julius M. Sassenrath for suggesting this problem.

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individuals with some of the same variables. The method presented below yields a measure of relationship between all factors of the two studies simultaneously—a measure which may be interpreted as a correlation coefficient.

The notion of "relating" factors should be distinguished from the notion of "matching" factors. For the problem of this paper—relating factors—we are interested only in assessing the degree of relationship between the factors for two completed studies taken as they stand—without disturbing the definition of the factors as defined within each study. On the other hand, when matching factors the problem is to rotate two sets of arbitrary reference factors, such as principal axes, to find two new sets of factors which are as close together as possible, thus matching pairs of factors. Cattell (1944) and Tucker (1951) have attacked elegantly the problem of matching factors, but their methods seem to be irrelevant for the problem of relating factors.

The statement of the solution is first motivated in the following section by an appeal to geometric intuition. Next, the technical details of obtaining a numerical solution are outlined. As an example, the method is then applied to relating the factors between two anthropometric studies. Certain limitations of the procedure are discussed.

METHOD

For each of the two studies being compared, consider the geometric configuration of the test vectors and the factor vectors in the space where cosines of angles between vectors signify correlations, the space Holzinger and Harman (1941, Chapter 3) call the "vector representation." Thus for the first study with p_1 tests and q_1 factors, $(p_1 + q_1)$ vectors, all with a common origin, lie in a q_1-dimensional vector space, where the q_1 factor vectors are of unit length and the p_1 test vectors are of length equal to the square-root of their communalities. The cosine of the angle between any two vectors gives the correlation between the two variables represented. (Note that the test vectors do not represent the observed tests, but the common parts of the observed tests, i.e., the projections of the original test vectors onto the common factor space.) For the second study, a similar configuration of $(p_2 + q_2)$ vectors lying in a q_2-dimensional vector space may be visualized. For convenience designate the studies "first" and "second" so that $q_1 \geq q_2$.

Now place both configurations of vectors in the same space with the same origin. Therefore, all $(p_1 + q_1 + p_2 + q_2)$ test and factor vectors now lie in a q_1-dimensional vector space, a space in which the cosines of the angles between vectors within, but not between, studies represent observable correlations. A placement of this sort is not unique, for the correlations between the tests and factors of the first study with the tests and factors of the second study are not known. The problem is to find a unique, meaningful way of placing these two sets of vectors. Consider the $p, p \leq p_1$, $p \leq p_2$, tests which are common to the two studies. (p must be greater than or equal to q_2.) The method of this paper is based upon the following principle: place the two sets of p vectors representing the p common tests pairwise as close together as possible, matching the common part of each test in the first study with its mate in the second. This may be quantified by requiring that the sum of the p inner products be maximized. It is made operational by rotating the complete set of $p + q_2$ vectors for the second study rigidly until the condition is met. After this rotation we measure the cosines of the angles between all pairs of the $q_1 + q_2$ factor vectors. Within each of the two studies, these cosines of course give the actual correlations between the factors within the study. Between studies, however, these cosines are not correlations in the usual sense (because there are no common individuals on which to base a correlation) but they may surely be taken as a measure of relationship between the factors represented, and a measure which certainly can be interpreted in the same way as a correlation coefficient.

The mathematics of the method will now be undertaken. First throw away the $(p_1 - p) + (p_2 - p)$ non-common test vectors, as they are of no concern. This leaves $2p + q_1 + q_2$ vectors under consideration. Extend each of the $2p$ common test vectors to unit length, as the procedure will be applied with normalized test vectors: both in the senior author's alpha factor analysis (Kaiser and Caffrey, 1965) and his varimax rotational criterion (Kaiser, 1958), it has proved desirable to proceed with the common parts of tests normalized.

Because of overwhelming advantages in simplicity, all vectors will be referred to an orthonormal basis, or coordinate system, with unit-length, mutually perpendicular axes. Four distinct situations arise in establishing the projections of the test and factor vectors on the axes of the orthonormal coordinate system. We review these four possibilities in some detail:

(a) Uncorrelated (orthonormal) factors. Simply take the final (probably rotated) factors as the coordinate axes. For a given study (the first, say), a matrix of order $(p + q_1) \times q_1$ may be written:

$$[1] \qquad \begin{bmatrix} \mathbf{H}_1^{-1}\mathbf{F}_1 \\ \mathbf{I} \end{bmatrix},$$

where $\mathbf{F}_1$ is the $p \times q_1$ factor matrix for the p common tests and $\mathbf{H}^2 = \text{diag}(\mathbf{F}_1\mathbf{F}_1')$, the diagonal matrix of reproduced communalities; thus $\mathbf{H}_1^{-1}\mathbf{F}_1$ is simply the row normalized factor matrix. The $q_1 \times q_1$ identity matrix below $\mathbf{H}_1^{-1}\mathbf{F}_1$ in [1] gives the loadings of the uncorrelated factors on themselves. Matrix [1] then gives the loadings of all common tests and factors on the (same) factors.

(b) Correlated primary factors, transformed (rotated) from known uncorrelated factors by a known primary factor transformation $\mathbf{T}_1$. Take the known uncorrelated factors represented by $\mathbf{F}_1$, e.g., principal axes, as the coordinate system. Then, for the first study, $\mathbf{H}_1^{-1}\mathbf{F}_1$ again gives the normalized loadings of the common tests on these factors; however, the loadings of the rotated primary factors on the unrotated factors are given by $\mathbf{T}_1'$, where $\mathbf{T}_1$ is the transformation matrix which transforms $\mathbf{F}_1$ into the oblique primary factors under consideration. Therefore, if it is the first study, the loadings of all $p + q_1$ common tests and (rotated) oblique factors on the (unrotated) uncorrelated factors appear as the $(p + q_1) \times q_1$ matrix:

$$[2] \qquad \begin{bmatrix} \mathbf{H}_1^{-1}\mathbf{F}_1 \\ \mathbf{T}_1' \end{bmatrix}.$$

Typically this instance is rare, for the matrix $\mathbf{T}_1$ is usually not known. (Operationally oblique rotation most often is carried out with respect to the computationally intermediate reference vectors.)

(c) Correlated reference vectors, transformed (rotated) from known uncorrelated factors by a known reference vector transformation $\mathbf{Y}_1$. Since of course we want to relate primary factors, not reference vectors, we must obtain the loadings of the primary factors implied by the known reference vectors on the known uncorrelated factors, again represented by $\mathbf{F}_1$. These loadings are again given by a $\mathbf{T}_1'$, where

$$[3] \qquad \mathbf{T}_1' = \mathbf{D}_1\mathbf{Y}_1^{-1}.$$

$\mathbf{Y}_1$ is the nonsymmetric reference vector transformation (often called "lambda") and $\mathbf{D}_1$ is the diagonal matrix which normalizes the rows of $\mathbf{Y}_1^{-1}$. The complete matrix of projections of tests and factors are set out again as in [2], with $\mathbf{T}_1'$ calculated by [3].

(d) Correlated primary factors, no known uncorrelated factors from which the primary factors were derived. Here an $\mathbf{F}_1$ and a $\mathbf{T}_1$ may be "made up" by the multiple group method (Guttman, 1952). Let $\mathbf{L}_{11}$ be the known intercorrelation matrix of the primary factors (if $\mathbf{L}_{11}$ is not known, or cannot be determined from other matrices, insufficient information is available to proceed). Factor $\mathbf{L}_{11}$ (most conveniently by the square-root method) into a $\mathbf{T}_1$ so that

$$[4] \qquad \mathbf{L}_{11} = \mathbf{T}_1'\mathbf{T}_1 \ ,$$

and find the corresponding $\mathbf{F}_1$ by

$$[5] \qquad \mathbf{F}_1 = \mathbf{B}_1\mathbf{T}_1^{-1}$$

or

$$[6] \qquad \mathbf{F}_1 = \mathbf{A}_1\mathbf{T}_1' \ ,$$

where $\mathbf{B}_1$ is the (known) primary factor *structure* matrix, or $\mathbf{A}_1$ is the (known) primary factor *pattern* matrix, for the common tests. Again, set out the projections of the common tests and factors, as in [2], using the $\mathbf{T}_1'$ from [4] and $\mathbf{H}_1^{-1}\mathbf{F}_1$, the row-normalized $\mathbf{F}_1$, from [5] or [6].

Situations (a) and (c) above are conventional. Situation (b) will arise only in the rare instance of direct rotation, bypassing the reference vectors, and (d) occurs when direct solutions for correlated factors are made, e.g., in direct cluster analysis. In the taxonomy above note that our $\mathbf{T}_1$ is Thurstone's (1947) $\mathbf{T}'$, but Harman's (1960) $\mathbf{T}$; our $\mathbf{L}_{11}$ is Thurstone's $\mathbf{R}_{pq}$ and Harman's Φ. Finally, for more detail regarding the matrix gymnastics of the interrelationships of primary factors and reference vectors, see Harman (1960, Chapter 13).

We can now present algebraically an arbitrary placement of the $2p + q_1 + q_2$ vectors of interest as the $(2p + q_1 + q_2) \times q_1$ matrix:

$$
[7] \qquad \begin{bmatrix} \mathbf{H}_1^{-1}\mathbf{F}_1 & \\ \mathbf{T}_1' & \\ \mathbf{H}_2^{-1}\mathbf{F}_2 & 0 \\ \mathbf{T}_2' & 0 \end{bmatrix},
$$

where the subscripts 1 and 2 refer to matrices for the first and second study, respectively, and the zero matrices, for $q_1 > q_2$, indicate $(q_1 - q_2)$ columns of zeros; remember also that $\mathbf{T}_1$ and/or $\mathbf{T}_2$ are identity matrices when $\mathbf{F}_1$ and/or $\mathbf{F}_2$ represent the (uncorrelated) factors being related.

To find the necessary and sufficient conditions for rotating the test vectors of the second study as close as possible to the test vectors of the first study consider any two columns in the supermatrix [7]. Designate the two $p \times 1$ vectors of tests' loadings for the first study by $\mathbf{u}$ and $\mathbf{v}$ and the two $p \times 1$ vectors of tests' loadings of the second study by $\mathbf{x}$ and $\mathbf{y}$. An orthonormal rotation—to keep our orthonormal basis—of these two columns of the second study may be made, yielding new loadings X and Y, given by the standard formulas

$$
[8] \qquad X = x \cos \varphi + y \sin \varphi \ ,
$$

$$
[9] \qquad Y = -x \sin \varphi + y \cos \varphi \ ,
$$

where φ is the angle of rotation.

The criterion function for rotation—maximizing the pairwise sum of inner products between the two studies—is then

$$
[10] \qquad \mathbf{u}'X + \mathbf{v}'Y \ .
$$

Upon differentiating [10] with respect to φ, using [8] and [9], and setting the derivative equal to zero, after a good deal of algebra, it is seen that the desired angle of rotation is given by

$$
[11] \qquad \varphi = \tan^{-1} \left(\frac{\mathbf{u}'\mathbf{y} - \mathbf{v}'\mathbf{x}}{\mathbf{u}'\mathbf{x} + \mathbf{v}'\mathbf{y}} \right) .
$$

Thus, φ will be zero only if $\mathbf{u}'\mathbf{y} - \mathbf{v}'\mathbf{x}$, the numerator of the fraction in [11], is zero. By inspecting the second derivative of [10] it may be shown that when $\mathbf{u}'\mathbf{x} + \mathbf{v}'\mathbf{y}$, the denominator of the fraction in [11], is positive, a maximum for the criterion function for the plane under consideration has been attained.

The above conditions are for any pair of columns in [7]; we now express these results for the matrix as a whole. Let $\mathbf{K}'$ be a $q_2 \times q_1$ transformation matrix,[2] with $\mathbf{K}$ orthonormal by columns—$\mathbf{K}'\mathbf{K} = \mathbf{I}$—with which to postmultiply the $p \times q_2$ matrix $\mathbf{H}_2^{-1}\mathbf{F}_2$. This $p \times q_1$ product, $\mathbf{H}_2^{-1}\mathbf{F}_2\mathbf{K}'$, is to have rows which "match up" as well as possible with the rows of $\mathbf{H}_1^{-1}\mathbf{F}_1$ according to the conditions of the preceding paragraph, which, for the entire matrix, are that

$$[12] \qquad (\mathbf{H}_1^{-1}\mathbf{F}_1)'(\mathbf{H}_2^{-1}\mathbf{F}_2\mathbf{K}')$$

is to be symmetric, with positive diagonals.

To accomplish this let

$$[13] \qquad \mathbf{C} = (\mathbf{H}_1^{-1}\mathbf{F}_1)'(\mathbf{H}_2^{-1}\mathbf{F}_2) \ ,$$

a matrix of order $q_1 \times q_2$. According to [12], $\mathbf{CK}'$ is to be symmetric. Consider

$$[14] \qquad \mathbf{G} = \mathbf{CC}' \ ,$$

a square symmetric matrix of order q_1 but of rank q_2. Now write $\mathbf{G}$ as

$$[15] \qquad \mathbf{G} = \mathbf{WM}^2\mathbf{W}' \ ,$$

where $\mathbf{W}$ is the $q_1 \times q_2$ matrix of unit-length column eigenvectors of $\mathbf{G}$, and $\mathbf{M}^2$ is the $q_2 \times q_2$ diagonal matrix of positive eigenvalues of $\mathbf{G}$. Since $\mathbf{W}'\mathbf{W} = \mathbf{I}$, $\mathbf{G}$ may be written further as

2. In a preliminary dittoed draft of this paper, which had wide circulation, we defined this matrix as $\mathbf{K}$ rather than $\mathbf{K}'$. Consistent with the convention of having the untransposed version of matrices "tall," we have changed the former $\mathbf{K}$ to $\mathbf{K}'$.

$$[16] \qquad G = (WMW')(WMW')' = (WMW')(WMW') \ ,$$

where $\mathbf{M}$ is the diagonal matrix of positive square roots of $\mathbf{M}^2$ (positive to insure meeting the sufficient conditions for a maximum). This factoring $\mathbf{WMW'}$ of $\mathbf{G}$ obviously is symmetric. Since $\mathbf{K'K} = \mathbf{I}$, we may write

$$[17] \qquad (CK')(CK')' = CC' = G = (WMW')(WMW') \ ,$$

and consequently, we want to transform $\mathbf{C}$ with a $\mathbf{K'}$ so that it equals the symmetric matrix $\mathbf{WMW'}$:

$$[18] \qquad CK' = WMW' \ .$$

This $\mathbf{K'}$, then, is the desired transformation for rotating the common test vectors of the second study as close as possible to the corresponding common test vectors of the first study. Solving [18] for $\mathbf{K'}$ explicitly

$$[19] \qquad K' = (C'C)^{-1}C'WMW' \ .$$

The solution may be simplified[3] by rewriting [19]:

$$[20] \qquad K' = [(C'C)^{-1}C'][WM^2W'][WM^{-1}W']$$

since $\mathbf{W'W} = \mathbf{I}$, and upon noting from [14] and [15] that $\mathbf{WM^2W'} = \mathbf{CC'}$, then $\mathbf{K'}$ becomes

$$[21] \qquad \begin{aligned} K' &= [(C'C)^{-1}C'][CC'][WM^{-1}W'] \\ &= C'WM^{-1}W' \ . \end{aligned}$$

We have shown that the unit-length row vectors $\mathbf{H}_2^{-1}\mathbf{F}_2\mathbf{K'}$ are as close as possible to their mates in $\mathbf{H}_1^{-1}\mathbf{F}_1$. However, finding

3. We are indebted to Professor Ledyard R. Tucker for providing this *simplification.*

$H_2^{-1}F_2K'$ is not of primary concern. Rather, in performing the rigid rotation of all $p + q_2$ row vectors of

$$\text{[22]} \qquad \begin{bmatrix} H_2^{-1}F_2 \\ T_2' \end{bmatrix}$$

with K', the most basic interest is in the resulting factor vectors, the rows of $T_2'K'$—and their relationships to the factor vectors of the first study, the rows of T_1'. To measure the cosines of the angles between these two sets of factor vectors in this space where cosines mean correlations, inner products are taken. Designating the matrix of these inner products between studies by L_{12},

$$\text{[23]} \qquad \begin{aligned} L_{12} &= T_1'(T_2'K')' \\ &= T_1'KT_2 \end{aligned}$$

the rows of L_{12} represent the factors of the first study and the columns of L_{12} the factors of the second study. Remember again that T_1 and/or T_2, as in [7], will be identity matrices when the factors being related are represented directly by F_1 and/or F_2, rather than rotations of F_1, and/or F_2.

The cosines between all $(q_1 + q_2)$ factor vectors may be exhibited as

$$\text{[24]} \qquad \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} T_1' \\ T_2'K' \end{bmatrix} \begin{bmatrix} T_1' \\ T_2'K' \end{bmatrix}'$$

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} T_1'T_1 & T_1'KT_2 \\ T_2'K'T_1 & T_2'T_2 \end{bmatrix}.$$

$L_{11} = T_1'T_1$ and $L_{22} = T_2'T_2$, of course, are actual intercorrelations within the two studies while L_{12} and its transpose L_{21} contain the proposed measure of relationship between factors between studies.

Example

For illustrative purposes, a small problem (but complicated because $q_1 > q_2$ and one set of factors is correlated and the other uncorrelated) will now be solved. In Table 1 is given the complete analysis reported by Thurstone (1946) of 12 anthropometric measurements on 100 adult North Ireland males. (The correlation matrix first appeared in a paper by Hammond (1942). In Table 2 is given the orthonormally rotated varimax factors of the first two centroids of a correlation matrix of 14 anthropometric measurements on 50 London University male students. (These centroids were calculated by Cohen (1939-1941.) The four Thurstone-Hammond rotated primary factors will now be related to the two uncorrelated varimax factors by the method of this paper.

Table 1
Thurstone's Factor Analysis of 12 Anthropometric Measurements on 100 North Ireland Males

	Centroid F				Reference Vector Simple Structure $V = FY$			
Stature[a]	72	–55	–15	–11	–04	76	–08	45
Sitting Height	60	–34	–25	–48	–03	80	00	01
Shoulder Breadth[a]	51	17	22	–06	05	03	49	19
Hip Breadth	59	21	36	–11	–02	01	64	22
Span	79	–35	02	03	02	52	15	55
Chest Breadth[a]	60	32	24	–16	10	01	65	10
Chest Depth[a]	34	34	29	–26	–03	–07	59	–09
Head Length[a]	45	31	–30	04	51	09	23	00
Head Breadth[a]	28	29	–36	32	61	–08	05	11
Head Height	23	16	–49	19	57	10	–11	01
Hand Length	78	–41	01	23	07	46	07	71
Hand Breadth	58	–15	38	31	–07	01	34	67

Reference Vector Transformation Y				Primary Factor Transformation $T = (Y')^{-1}D$				Primary Factor Intercorrelations $L = T'T$			
27	40	47	46	40	69	70	50	100	16	06	07
54	–64	64	–45	45	–36	45	–42	16	100	38	03
–70	–42	58	32	–72	–32	42	27	06	38	100	02
39	–50	–17	70	35	–54	–36	71	07	03	02	100

[a]These variables are common to the two studies being related.

There are $p = 6$ variables in common to the two studies: stature, shoulder breadth, chest breadth, chest depth, head length, and head breadth. In Table 3 is outlined in detail the computations for obtaining the transformation K' and the final matrix L_{12} (or L_{21}) of our measure of relationship between the factors of the two studies.

It may be of interest to ascertain the quality of the fit of the two sets of test vectors after rotation with K' as an indication of

Table 2
Varimax Factors from Cohen's Centroids for 14 Anthropometric Measurements on 50 London University Male Students

Waist Circumference	94	02
Pelvic Circumference	91	07
Pelvic Breadth	78	20
Head Diagonal	70	27
Chest Circumference	84	07
Chest Depth[a]	71	18
Shoulder Breadth[a]	52	44
Chest Breadth[a]	70	15
Head Length	58	29
Stature[a]	15	93
Leg Length	16	89
Arm Length	11	85
Head Breadth[a]	64	04
Trunk Length	10	73

[a]These variables are common to the two studies being related. In Table 3, these variables from this study are reordered to be paired properly with their mates from Thurstone's study.

the confidence one may have in the solution $\mathbf{L}_{12}$. A seemingly reasonable index, varying between zero and one for all problems, would be the mean cosine between the p pairs, given by

$$[25] \qquad \frac{1}{p} \operatorname{trace} [\mathbf{H}_1^{-1}\mathbf{F}_1\mathbf{K}\mathbf{F}_2'\mathbf{H}_2^{-1}] \ .$$

For the present example, this index is .85, suggesting that the pairing of the six variables was reasonable. It may be wise also to look at the individual diagonals of $\mathbf{H}_1^{-1}\mathbf{F}_1\mathbf{K}\mathbf{F}_2'\mathbf{H}_2^{-1}$ to see if any particular pair of purported common tests is badly matched—casting doubt on the validity of the assumption of the two tests being the same.

Cautions

It is easy to fall in the trap of calling two variables from different studies the same merely because they have the same name. Even in the relatively clearcut universe of anthropometric measurements, the fallacy of calling the variable *height* (for adult men) the same variable as *height* (for teen-age girls)—simply because they are both designated "height"—is possibly easily fallen into. In psychological testing this pitfall is undoubtedly worse because of the additional difficulty of having physically different tests[4]

4. E.g., different items, or different time limits, or different scoring procedures, etc.

Table 3
Outline of Computations for Relating Rotated Primary Factors of Tables 1 and 2

$$\begin{bmatrix} H_1^{-1}F_1 & \\ T_1' & \\ H_2^{-1}F_2 & 0 \\ T_2' & 0 \end{bmatrix} = \begin{bmatrix} 78 & -59 & -16 & -12 \\ 87 & 29 & 38 & -10 \\ 81 & 43 & 32 & -22 \\ 55 & 55 & 47 & -42 \\ 72 & 50 & -48 & 06 \\ 45 & 46 & -57 & 51 \\ & & & \\ 40 & 45 & -72 & 35 \\ 69 & -36 & -32 & -54 \\ 70 & 45 & 42 & -36 \\ 50 & -42 & 27 & 71 \\ & & & \\ 16 & 99 & 00 & 00 \\ 76 & 65 & 00 & 00 \\ 98 & 21 & 00 & 00 \\ 97 & 25 & 00 & 00 \\ 89 & 45 & 00 & 00 \\ 100 & 06 & 00 & 00 \\ & & & \\ 100 & 00 & 00 & 00 \\ 00 & 100 & 00 & 00 \end{bmatrix}$$

$$C = (H_1^{-1}F_1)'(H_2^{-1}F_2) = \begin{bmatrix} 321 & 199 \\ 199 & 08 \\ 03 & 02 \\ -15 & -27 \end{bmatrix}$$

$$G = CC' = \begin{bmatrix} 1424 & 653 & 13 & -102 \\ 653 & 396 & 07 & -32 \\ 13 & 07 & 00 & -01 \\ -102 & -32 & -01 & 10 \end{bmatrix}$$

$$W = \begin{bmatrix} 90 & 42 \\ 44 & -89 \\ 01 & 00 \\ -06 & -20 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 1747 & 00 \\ 00 & 82 \end{bmatrix}$$

$$K' = C'WM^{-1}W = \begin{bmatrix} 62 & 78 & 01 & 03 \\ 77 & -61 & 00 & -20 \end{bmatrix}$$

$$L_{12} = T_1'KT_2 = \begin{bmatrix} 61 & -04 \\ 13 & 86 \\ 78 & 34 \\ 01 & 49 \end{bmatrix}$$

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} 100 & 16 & 06 & 07 & 61 & -04 \\ 16 & 100 & 38 & 03 & 13 & 86 \\ 06 & 38 & 100 & 02 & 78 & 34 \\ 07 & 03 & 02 & 100 & 01 & 49 \\ & & & & & \\ 61 & 13 & 78 & 01 & 100 & 00 \\ -04 & 86 & 34 & 49 & 00 & 100 \end{bmatrix}$$

Note.—Decimal points omitted. To conserve space the entire table has been rounded to two places.

purporting to measure the same thing: who is to say that one psycologist's tests of *ego-strength* is another's—or perhaps yet another's test of *emotional stability*? Because of this consideration, pairing two variables from different studies may be somewhat dubious unless the two sets of observations are samples from the same population and, if psychological tests, the observations are on exactly the same test. It is suggested that, except for a highly restricted and thus perhaps sterile class of problems for which the pairing of common tests is absolutely indisputable, there is an element of judgment in the application of the method developed in this paper.

More technical, but closely related, is the consideration that pairing two observed variables may be inappropriate because we are not matching the original variables, but rather their common parts (projections onto the common factor space). This difficulty will arise particularly when one study is based upon more variables than the other, allowing the first to have more factors. The common part of a variable from the first study could then be substantially different from its mate in the second study because of the opportunity of its converting specificity to commonness. Some serious effort indicates that a completely satisfactory resolution of this difficulty is not easy. While it is not suggested that one be so compulsively careful as to require only the same variables in the two studies being related (so that $p = p_1 = p_2$), gross differences in p_1 and p_2 and particularly in q_1 and q_2 are to be viewed carefully to check the appropriateness of pairing the common parts of two variables. In the above example where $q_1 = 4$ and $q_2 = 2$, it is comforting to note that the two sets of communalities are approximately equal.

The above limitations regarding the present technique have been pursued at some length because it would appear that the present method is particularly susceptible to being applied indiscriminantly to yield a substantial amount of nonsense by thoughtless investigators. However, this criticism seems inevitably to apply to any proposed solution to the problem of relating factors between studies based upon different individuals.

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