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THE SQUARE ROOT METHOD AND MULTIPLE GROUP METHODS OF FACTOR ANALYSIS

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The square root method for the solution of a set of simultaneous linear equations or the reduction of a matrix has been known for some time under a variety of names. Because of its usefulness in statistical work, especially in factor analysis, the square root method is presented in general terms and an example given. Several independently developed "multiple group methods" for factor analysis are compared and synthesized. Their fundamental concepts are set forth and an appropriate system of notation developed. Detailed computational procedures are outlined, and the square root method is emphasized as a computing aid in multiple group analysis.

There are two principal objectives of this paper: (1) a clarification and coordination of several approaches to multiple group methods of factor analysis; and (2) the application of the square root method to the multiple group method of factoring a correlation matrix. In the process of meeting these objectives, a brief historical sketch of these techniques will be presented and the formal procedures will be developed and illustrated.

1. *Historical Note*

While no attempt is made to give a complete and exhaustive account of the history of the multiple group methods of factor analysis, a short account of the highlights in this development seems to be in order. Recently, Guttman (**8**) called attention to the basic work he had done in multiple group methods of factor analysis, which work apparently was overlooked by Thurstone (**18**) in his account of the similarity between the "simple method of factor analysis" proposed by Holzinger (**11**) and the "multiple group method of factoring the correlation matrix" which he proposed (**16**). A search of the literature discloses definite evidence of work in this direction prior to any of these papers.

It matters little "who got there first," but there are indications that in 1937 Horst (**13**) anticipated the multiple group method of factor analysis. His was a theoretical presentation, however, and the lack of computational procedures apparently was the reason that the method was not adopted and developed. Several other writers dealt with group factors in the early 1930's. Notable among these were Cyril Burt, who considered the "group factor

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method"; R. C. Tryon, who proposed "cluster analysis; and K. J. Holzniger, who developed the "bi-factor" method of analysis. While these methods certainly involve the group factor concept, they are not specifically in the spirit of the multiple group methods of factor analysis under consideration in the present paper. Guttman's presentation (**7**) of the theory in 1944 met with the same fate as Horst's paper, and for the same reasons. However, when Holzinger (**11**), in the same year, and Thurstone (**16**), a year later, presented simple computing procedures for "group factor analysis," there was ready acceptance, even though the similarity of methods was not recognized for several years (**17**, p. 171, **18**, **8**).

The history of the development and use of the "square root method" is even more vague. Certainly, as a formal mathematical procedure for the solution of a set of simultaneous linear equations or the reduction of a matrix, it must have been discovered over and over again, and may go back to the time of Gauss. Perhaps the earliest application of a square root method to the solution of normal equations in least squares theory was made by Commandant A. L. Cholesky of the French Navy around 1915, and published after his death by Commandant Benoit (**3**) in 1924. It was rediscovered by Banachiewicz (**1**, **2**) in 1938 and presented as an efficient means for solving a system of linear equations and for the calculation of determinants and their inverses. The square root method was introduced in the American statistical literature in 1944 by Dwyer (**4**), who emphasized its use in correlation and regression (**5**) and who showed the relationship of this method to other methods of linear computation (**6**).

Concurrent with this development of the square root method as a means of solving formal mathematical and statistical problems, essentially the same technique was being devised specifically for factor analysis. It was recognized that a factor analysis of a set of variables whose intercorrelations constituted a symmetric matrix could always be obtained by a general algebraic procedure known as "completing the square." The method was applied specifically to a correlation matrix by McMahon (**14**) prior to 1923. Then during the rapid development of factor analysis theory in the 1930's, it was independently developed as the "diagonal method" by Thurstone (**15**, p. 78) and as the "solid staircase method" by Holzinger (**10**). Since these methods were designed expressly for factor analysis, they did not present general computing techniques. Nevertheless, they are special instances of the square root method, with the broader implication of a technique for linear computations in general.

2. *The Square Root Method*

While the square root method has been featured in several papers and applied to a variety of statistical problems during the past decade, it still is relatively unfamiliar to many researchers who use the Doolittle worksheets. As pointed out by Dwyer (**6**, p. 115), the advantages of the square root method over the Gauss-Doolittle method are that it is more compact, requiring less recording, and that it permits greater ease in finding the entries to be used. Not only is the square root method more expedient for solving a symmetric set of equations, but it is especially useful in obtaining the inverse matrix in solving problems in statistics.

For the foregoing reasons, the square root method will be presented here in some detail. The procedure will be outlined in general terms and illustrated with a simple problem (9) involving the least-squares prediction of a dependent variable from three independent variables, viz.,

$$z_4 = \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3 \ , \tag{1}$$

where β_i is used for the conventional $\beta_{4i.(2)}$, in which the number in parentheses merely shows how many variables are held fixed. The normal equations in this case are:

$$\begin{aligned} r_{11}\beta_1 + r_{12}\beta_2 + r_{13}\beta_3 &= r_{14} \\ r_{21}\beta_1 + r_{22}\beta_2 + r_{23}\beta_3 &= r_{24} \\ r_{31}\beta_1 + r_{32}\beta_2 + r_{33}\beta_3 &= r_{34} \ , \end{aligned} \tag{2}$$

where, of course, the conditions for symmetry $r_{ij} = r_{ji}$ are satisfied and $r_{ii} = 1$ for $i, j = 1, 2, 3$.

The computing procedure for the determination of the regression coefficients in equation (1) is indicated in general terms in Table 1, and illustrated with specific numerical data. The step-by-step procedure, immediately following, is readily extended to any number of variables.

Step 1. Enter the intercorrelations among the independent variables and their correlations with the dependent variable on the first three lines of the Work Sheet.

Step 2. Obtain the sums by rows, i.e.,

$$t_j = \sum_{i=1}^{4} r_{ji} \ . \qquad (j = 1, 2, 3)$$

Note: Entries in Check column for Lines 1, 2, 3 are described in Step 10.

Step 3. The first step of the square root method is now applied by using r_{11} as a pivot. The first element in Line 4 is given by

$$s_{11} = \sqrt{r_{11}} \ ,$$

while the remaining elements are obtained by the formula:

$$s_{1i} = \frac{r_{1i}}{s_{11}}. \qquad (i > 1)$$

Note: Since $r_{11} = 1$, the elements of Line 4 are equal, respectively, to the elements of Line 1.

TABLE 1

The Square Root Method

Line	Independent Variables z_1	z_2	z_3	Dependent Variable z_4	Total	Check
	General Solution					
1	r_{11}	r_{12}	r_{13}	r_{14}	t_1	r'_{14}
2	*	r_{22}	r_{23}	r_{24}	t_2	r'_{24}
3	*	*	r_{33}	r_{34}	t_3	r'_{34}
4	s_{11}	s_{12}	s_{13}	s_{14}	s_{1t}	s'_{1t}
5		$s_{22\cdot 1}$	$s_{23\cdot 1}$	$s_{24\cdot 1}$	$s_{2t\cdot 1}$	$s'_{2t\cdot 1}$
6			$s_{33\cdot 12}$	$s_{34\cdot 12}$	$s_{3t\cdot 12}$	$s'_{3t\cdot 12}$
7	β_1	β_2	β_3		$R^2_{4\cdot 123}$	$R_{4\cdot 123}$
	Numerical Illustration					
1	1.000	.693	.216	.571	2.480	.571
2	*	1.000	.295	.691	2.679	.691
3	*	*	1.000	.456	1.967	.456
4	1.000	.693	.216	.571	2.480	2.480
5		.721	.202	.410	1.332	1.333
6			.955	.262	1.217	1.217
7	.171	.492	.274		.563	.750

*Terms below the diagonal of a symmetric matrix are deleted for simplicity. Terms below the diagonal of the "square root" matrix are actually zero, and are simply omitted.

Step 4. The calculation in the Total column of Line 4 is carried out as for any other column, yielding s_{1t} . This value should agree, except for rounding errors, with the sum s'_{1t} (Check column) of all elements computed in Step 3.

Step 5. The formulas for the elements of Line 5 are:

$$s_{22\cdot 1} = \sqrt{r_{22} - s_{12}^2} ,$$

$$s_{2i\cdot 1} = \frac{r_{2i} - s_{1i}s_{12}}{s_{22\cdot 1}}. \qquad (i > 2)$$

Step 6. Check Line 5 by comparing the calculated value, $s_{2t\cdot 1}$, with the row sum, $s'_{2t\cdot 1}$.

Step 7. The formulas for the elements of Line 6 are:

$$s_{33\cdot(2)} = \sqrt{r_{33} - s_{13}^2 - s_{23\cdot 1}^2} ,$$

$$s_{3i\cdot(2)} = \frac{r_{3i} - s_{1i}s_{13} - s_{2i\cdot 1}s_{23\cdot 1}}{s_{33\cdot(2)}}, \qquad (i > 3)$$

where the notation $s_{3i\cdot(2)}$ is used instead of the specific $s_{3i\cdot 12}$ to suggest an easy generalization when the number of variables already eliminated is more than 2.

Step 8. Apply row sum check to Line 6.

Step 9. The values of the regression coefficients are obtained by application of the following formulas (back solution):

$$\beta_3 = \frac{s_{34\cdot 12}}{s_{33\cdot 12}},$$

$$\beta_2 = \frac{s_{24\cdot 1} - s_{23\cdot 1}\beta_3}{s_{22\cdot 1}},$$

$$\beta_1 = \frac{s_{14} - s_{13}\beta_3 - s_{12}\beta_2}{s_{11}}.$$

Step 10. A check on the entire computations can be made by substituting the regression coefficients back into the normal equations (2). The results are designated by r'_{14} , r'_{24} , r'_{34} and should agree (except for rounding errors) with the original correlations of independent with dependent variables.

Step 11. The multiple correlation coefficient can be computed by use of the usual formula involving the β's and r's, viz.,

$$R^2_{4\cdot 123} = \beta_1 r_{14} + \beta_2 r_{24} + \beta_3 r_{34} .$$

From the formal solution of the three-variable problem it can be verified that

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} s_{11} & 0 & 0 \\ s_{12} & s_{22\cdot 1} & 0 \\ s_{13} & s_{23\cdot 1} & s_{33\cdot 12} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ 0 & s_{22\cdot 1} & s_{23\cdot 1} \\ 0 & 0 & s_{33\cdot 12} \end{bmatrix}. \tag{3}$$

More generally, the square root method can be formulated in matrix notation, as follows:

$$R = S'S, \tag{4}$$

whence the term "square root of a matrix" is seen to correspond to the ordinary square root of an algebraic expression. [The identity of equation (4) with the fundamental theorem of factor analysis (**12**, p. 19; **15**, p. 70) is a clear indication of why factor analysis independently discovered the square root method, although it was referred to by various names.] In other words, the square root method applied to a matrix R yields a matrix S such that premultiplication by its transpose (i.e., column-by-column multiplication of S by itself) reproduces the matrix R. It is convenient, at times, to refer to the "square root operation," by which is meant $(S')^{-1}$, since $(S')^{-1}$ operating (premultiplying) on R produces S. Then the square root operation can be applied to other matrices than the basic one from which it is derived.

3. *Concepts and Notation in Multiple Group Methods*

The square root method was presented first so that the technique could be employed as needed in the development of the multiple group methods of factor analysis. However, *the notation of the preceding section will not be used in the remainder of this paper.* Instead, a system of notation will be employed which is both clear and, when possible, closely related to that in the existing literature. The notation along with the basic concepts in multiple group factoring methods will be presented in this section.

Of course the fundamental entity in factor analysis, aside from the scores themselves, is the matrix of observed correlation coefficients. In most methods of factor analysis, the reduced correlation matrix is used, meaning that the values in the principal diagonal are reduced from self correlations of unities to (estimates of) communalities.

The basic concept that distinguishes the methods of factor analysis under consideration is that of grouping of variables. Either by arbitrary or carefully selected grouping of variables a number of common factors can be extracted in one operation, and thereby a substantial reduction in the labor of computing residual matrices is realized. Some of the different points of view regarding the selection of groups of variables will be indicated in the final section of this paper. In any event, all the group factor methods have

TABLE 2

Notation used in Multiple-Group Methods of Factoring a Correlation Matrix

Concept	In this Paper	Holzinger (11)	Thurstone (17)	Guttman (8)
Original Correlations (reduced correlation matrix)	R	R	R	G
Grouping of Variables	Groups: G_k $(k = 1, 2, \ldots, m)$	Sections: s	Groups: p	X (s common factors)
Sums of Correlations of Variables with Groups	t_{ik} $(i = 1, 2, \ldots, n)$	T_j's	S	F'_o
Sums of Correlations Among Groups	T_{jk} $(j, k=1, 2, \ldots, m)$	T's	T	L_o
Oblique Factor Structure	$S = [s_{ik}]$	S_{js}	V	F
Correlations Among Factors	$\Phi = [r_{\gamma_j \gamma_k}]$	$r_{c_1 c_2}$	R_{pq}	L
Oblique Factor Pattern	$P = [b_{ij}]$	A_{js}		—
Reproduced Correlations	$R^{\dagger}$	$R^{\dagger}$	R	G_o
Residual Matrix	R_k	R_s	R_p	G_1
Transformation Matrix	Λ^{-1}	..	Λ^{-1}	—
Orthogonal Factor Matrix	$F = [a_{ik}]$		F	—

the common attribute of expediting the total factor analysis by the selection of a number of linearly independent groups approximating the rank of the reduced correlation matrix.

Except in rare circumstances, the common factors extracted in a single operation are oblique to one another. Therefore another basic concept is that of the matrix of correlations among the factors. Also, since the factors are correlated, the immediate results of a group factor analysis must lead to two matrices—a factor pattern and a factor structure (**12**, p. 16). The first of these gives the coefficients of the factors in the linear descriptions of the variables, while the second gives the correlations of the variables with the factors.

These results of the group factor analysis can be used to obtain a matrix of reproduced correlations; and hence the residual matrix can be determined. If the residual matrix is not sufficiently close to the null matrix then the group factor method can be applied, again, to the residual matrix.

After it has been determined that the multiple group factor solution adequately reproduces the observed correlations, some investigators may still consider such a solution as a preliminary step to the rotational problem (**17**, p. 171). In seeking "simple structure" by rotation of axes, the problem is simplified if an orthogonal frame of reference is first obtained. Hence, two additional concepts are introduced—an orthogonal factor matrix and the transformation matrix from the oblique to this orthogonal solution. The implications of stopping with the oblique solution obtained directly by the multiple group method or proceeding to the rotational problem will be brought out in the final section.

The foregoing concepts, which arise in the multiple group methods of factoring, are summarized in Table 2. The symbol associated with each concept as it will be used in the following section is listed for ready reference. Also, the notation employed by the three principal contributors to multiple group methods of factor analysis is presented to assist in making comparisons.

4. *Computations in Multiple Group Methods*

While there are apparent differences in the several presentations of the multiple group method of factor analysis—and several real differences in the generality and breadth of the theory—there is a basic technique underlying all of them. A systematic development of such a basic multiple group method is presented in this section. The procedure is illustrated with a 9-variable example, taken from Holzinger's unpublished notes, "Detailed Outline of Simple Solution," involving a sample of 696 cases, 12 tests, and 4 factors.

a. Reduced correlation matrix. The first problem is that of determining good estimates of communalities. That problem is extraneous to the scope of this paper. [Various methods for estimating communality are discussed in (**12**, pp. 156–159 and **17**, pp. 282–318).] No special justification for the particular choice will be given. Actually, in the 9-variable example, the estimates were obtained by a single triad for each variable, i.e., the quotient of the product of the two highest r's for a given variable by the correlation between the two variables correlating highest with the given variable. The reduced correlation matrix for the nine variables is

$$R = \begin{bmatrix} .81 & .75 & .78 & .44 & .45 & .51 & .21 & .30 & .31 \\ .75 & .69 & .72 & .52 & .53 & .58 & .23 & .32 & .30 \\ .78 & .72 & .75 & .47 & .48 & .54 & .28 & .37 & .37 \\ .44 & .52 & .47 & .91 & .82 & .82 & .33 & .33 & .31 \\ .45 & .53 & .48 & .82 & .74 & .74 & .37 & .36 & .36 \\ .51 & .58 & .54 & .82 & .74 & .74 & .35 & .38 & .38 \\ .21 & .23 & .28 & .33 & .37 & .35 & .35 & .45 & .52 \\ .30 & .32 & .37 & .33 & .36 & .38 & .45 & .58 & .67 \\ .31 & .30 & .37 & .31 & .36 & .38 & .52 & .67 & .77 \end{bmatrix},$$

where the variables are assumed to be in sequence from 1 to 9, and the estimates of the communalities appear in the principal diagonal.

b. Grouping of variables. The analysis begins with an appropriate grouping of variables. Thurstone (**17**) stresses the arbitrariness of grouping the variables, while Holzinger (**11**) and Guttman (**8**) emphasize the desirability of very careful selection of variables in each group according to some *a priori* hypothesis. In the example it is assumed that the nine variables can be placed in three groups such that the common factors corresponding to them will adequately explain the data. The three groups, with their constituent variables, are as follows:

$$G_1 : (1, 2, 3), \qquad G_2 : (4, 5, 6), \qquad G_3 : (7, 8, 9).$$

It may be of some interest to note that the tests in G_1 are of verbal content, G_2 arithmetic, and G_3 spatial relations.

c. Sums of correlations. The factor solution is obtained in several steps, with the computations in this step being preliminary to the actual factorial results. The sums of the correlations of each variable with the respective variables of each group are first required. These are given by the formula:

$$t_{ik} = \sum_{h \epsilon G_k} r_{ih}, \qquad (i = 1, \cdots, n; k = 1, \cdots, m) \tag{5}$$

where the sum is on the index h and the symbol "$h \epsilon G_k$" is read "h is a variable in group G_k." Formula (5) represents nm different sums. For the example, where $n = 9$ and $m = 3$, the 27 sums t_{ik} are given in Table 3.

Next, the sums of correlations among groups are obtained by means of the formula:

$$T_{jk} = \sum_{i \epsilon G_j} t_{ik} , \qquad (j, k = 1, \cdots , m) \tag{6}$$

where the summation is on the index i within each group, in turn. These nine sums appear in Table 4, where an immediate check is available from the symmetry property.

d. Correlations among factors. As indicated above, the common factors obtained in a single operation of a multiple group method of analysis are oblique to one another. The factors are represented by vectors through the centroid (or, more generally, a weighted average) of the respective groups of variables. While the individual variables are in standard measure, the composites are not necessarily so. If the oblique factors are designated by γ_k $(k = 1, \cdots , m)$, this means that the variance of γ_k is not unity but has the value T_{kk} as given in Table 4; and in general, Table 4 consists of the variances and covariances among the $m = 3$ factors. Then, from the theory of correlation between two composites (**12**, pp. 34–37), the correlations among the factors are given by:

$$r_{\gamma_j \gamma_k} = \frac{T_{jk}}{\sqrt{T_{jj}}\,\sqrt{T_{kk}}}, \tag{7}$$

and are recorded in Table 5.

e. Oblique factor structure. The correlations of the tests with the factors—the oblique factor structure—can be obtained by application of the same theory. Any test z_i is in standard measure while a factor γ_k is a composite of such variables and is not in standard form. The structure value s_{ik} is the correlation $r_{z_i \gamma_k}$ and can be computed by the formula (**12**, p. 36):

$$s_{ik} = \frac{t_{ik}}{\sqrt{T_{kk}}}. \tag{8}$$

The structure matrix S for the example is given in Table 6.

f. Oblique factor pattern. To complete the solution in terms of correlated factors, the linear descriptions of the variables in terms of the factors are required as well as their correlations with the factors. The coefficients in these linear equations, i.e., the pattern values, are the coordinates with respect to the oblique (factor) axes of the points representing the variables. The factor pattern can be obtained from the known factor structure S and the correlations among the factors Φ, as follows (**12**, p. 327):

$$P = S\Phi^{-1}. \tag{9}$$

The bulk of work implied in formula (9) is the determination of the inverse of Φ (especially for a large number of factors). Either the Doolittle method or the square root method can be used to obtain the inverse and to syste-

TABLE 3

Sums of Correlations for Variables with Groups

G_k	i	t_{i1}	t_{i2}	t_{i3}
G_1	1	2.34	1.40	.82
	2	2.16	1.63	.85
	3	2.25	1.49	1.02
G_2	4	1.43	2.55	.97
	5	1.46	2.30	1.09
	6	1.63	2.30	1.11
G_3	7	.72	1.05	1.32
	8	.99	1.07	1.70
	9	.98	1.05	1.96

TABLE 4

Sums of Correlations Among Groups: T_{jk}

j \ k	1	2	3
1	6.75	4.52	2.69
2	4.52	7.15	3.17
3	2.69	3.17	4.98
$\sqrt{T_{kk}}$	2.60	2.67	2.23

TABLE 5

Intercorrelations of Factors: Matrix Φ

	γ_1	γ_2	γ_3
γ_1	1.00	.65	.46
γ_2	*	1.00	.53
γ_3	*	*	1.00

TABLE 6

Oblique Factor Structure: Matrix S

Variable i	s_{i1}	s_{i2}	s_{i3}
1	.90	.52	.37
2	.83	.61	.38
3	.87	.56	.46
4	.55	.96	.43
5	.56	.86	.49
6	.63	.86	.50
7	.28	.39	.59
8	.38	.40	.76
9	.38	.39	.88

matically carry out the matrix multiplication to produce the pattern matrix. This explicit computation will not be done for the illustrative example, but will be accomplished along with two other concepts as outlined in the following paragraph.

g. Square root method in multiple group analysis. A very efficient scheme for the calculation of the oblique factor pattern, the transformation matrix, and the orthogonal factor matrix can be developed by application of the square root method. The basic square root operation will be performed on the matrix Φ of factor correlations, and the resulting "square root" matrix will be denoted by Λ. Expressed symbolically,

$$\Phi = \Lambda'\Lambda, \tag{10}$$

and the square root operator is $(\Lambda')^{-1}$. This operator when applied to the transpose of the structure matrix yields the desired orthogonal factor matrix, i.e.,

$$(\Lambda')^{-1}S' = F'. \tag{11}$$

Stated another way, the orthogonal factor matrix is obtained from the oblique factor structure by means of the transformation matrix Λ^{-1}, as follows:

$$F = S\Lambda^{-1}. \tag{12}$$

The mathematical proof of the foregoing expression as the transformation from an oblique to a rectangular coordinate system will not be presented here. A brief description of this special type of transformation is in order, however. The given set of variables may be considered as points in the common factor space, for which sets of coordinates are $F = [a_{ik}]$ and $P = [b_{ij}]$ with respect to an orthogonal and an oblique frame of reference, respectively. Now, if the oblique coordinates are known, the special transformation desired is one in which the first axis of the new system coincides with the first oblique factor axis, the second is in the plane of the first two oblique axes and orthogonal to the first, etc. This transformation is accomplished directly by the square root method applied to the matrix of cosines of angular separations of oblique axes (namely, the correlation matrix Φ). The resulting transformation from oblique to the specified orthogonal coordinates is given by the matrix equation:

$$F = P\Lambda'. \tag{13}$$

But, upon substituting the expression for P from equation (9) into equation (13), it follows that

$$F = S\Phi^{-1}\Lambda' = S\Lambda^{-1}(\Lambda')^{-1}\Lambda' = S\Lambda^{-1},$$

which is precisely formula (12). Hence the transformation to the orthogonal frame of reference (F) is described in terms of projections (s_{ik}) on the oblique axes instead of the coordinates (b_{ik}) in the original oblique reference system.

As indicated above, the orthogonal factor matrix F as expressed in equation (11) can be obtained by application of the square root method. Also, by means of the square root method, the oblique pattern matrix P can be obtained without explicit computation of the inverse of Φ, which is required in formula (9). It follows from formula (13) that

$$P' = \Lambda^{-1}F', \tag{14}$$

and since each element in the right-hand member of this expression comes from the application of the square root method, the oblique pattern results from a simple matrix multiplication.

The schematic formulation for the square root method to get the transformation matrix, the orthogonal factor matrix, and the oblique factor pattern is presented in Table 7A, and the application to the 9-variable example appears in Table 7B. In applying this procedure to a larger set of variables, the worksheet might more conveniently be arranged to list the matrices S, F, and P in adjacent vertical blocks instead of their transposes in horizontal sections.

h. Reproduced correlations. If the multiple group method of analysis is carried to the stage of an orthogonal factor solution, then the reproduced correlations can be obtained by the fundamental theorem of factor analysis, viz.,

$$R^{\dagger} = FF'. \tag{15}$$

On the other hand, if the multiple group method of analysis is employed to get an oblique solution without rotation of axes, then equation (15) is not applicable. There are several formulas for getting the reproduced correlations directly from the component parts of an oblique solution. The basic formula (**12**, p. 19), involving the multiplication of three matrices, follows:

$$R^{\dagger} = P\Phi P'. \tag{16}$$

Two other expressions for the reproduced correlations (**12**, p. 327) are:

$$R^{\dagger} = PS' = SP', \tag{17}$$

which involve precisely the same kind of computation as formula (15). Still another formula is

$$R^{\dagger} = S\Phi^{-1}S', \tag{18}$$

which is essentially the computation described by Guttman (**8**, p. 215).

Since both the orthogonal and oblique solutions were obtained for the illustrative example, the reproduced correlations were computed by formula

TABLE 7

Square Root Method in Multiple Group Analysis

A. Schematic Form

Factors		Variables	Instructions
Factor Correlations Φ $(\Phi = \Lambda'\Lambda)$	Identity Matrix I	Transpose of Oblique Factor Structure S'	Original (Known) Martices
Square Root Matrix Λ	Transpose of Transformation Matrix $(\Lambda')^{-1}$	Transpose of Orthogonal Factor Matrix $(\Lambda')^{-1} S' = F'$	Square Root Operation $(\Lambda')^{-1}$
	Inverse $\Lambda^{-1} (\Lambda')^{-1} = \Phi^{-1}$ (not required)	Transpose of Oblique Factor Pattern $\Lambda^{-1} F' = P'$ (also $\Lambda^{-1} (\Lambda')^{-1} S' = \Phi^{-1} S' = P'$)	Col. - by - Col. Mult. with $(\Lambda')^{-1}$ (Premult. by Λ^{-1})

B. Illustrative Example

	γ_1	γ_2	γ_3				1	2	3	4	5	6	7	8	9	Total	9-Var. Total	Check
1	1.0000	.6511	.4640	1.0000	0	0	.90	.83	.87	.55	.56	.63	.28	.38	.38	8.48		
2	*	1.0000	.5324	0	1.0000	0	.52	.61	.56	.96	.86	.86	.39	.40	.39	8.74		
3	*	*	1.0000	0	0	1.0000	.37	.38	.46	.44	.49	.50	.59	.76	.88	7.86		
	1.0000	.6511	.4640	1.0000	0	0	.90	.83	.87	.55	.56	.63	.28	.38	.38	8.48	5.37	8.48
		.7590	.3034	-.8578	1.3175	0	-.08	.09	-.01	.79	.65	.60	.28	.20	.19	4.24	2.72	4.24
			.8323	-.2448	-.4803	1.2015	-.03	-.04	.07	-.07	.04	.03	.45	.63	.77	3.17	1.86	3.17
				1.7957	-1.0126	-.2941	.98	.76	.85	-.11	-.01	.11	-.08	.05	.02		2.58	2.58
				*	1.9665	-.5771	-.09	.14	-.04	1.07	.84	.77	.15	-.04	-.12		2.69	2.69
				*	*	1.4436	-.04	-.05	.08	-.08	.04	.04	.55	.76	.93		2.23	2.23

(15) and checked by the second of formulas (17). The reproduced correlation matrix, including reproduced communalities, follows:

$$R^{\dagger} = \begin{bmatrix} .82 & .74 & .78 & .43 & .45 & .51 & .21 & .31 & .30 \\ & .70 & .72 & .53 & .52 & .57 & .24 & .31 & .30 \\ & & .75 & .47 & .48 & .54 & .27 & .37 & .38 \\ & & & .93 & .82 & .81 & .34 & .32 & .31 \\ & & & & .74 & .74 & .36 & .37 & .37 \\ & & & & & .75 & .36 & .38 & .38 \\ & & & & & & .36 & .45 & .51 \\ & & & & & & & .58 & .67 \\ & & & & & & & & .78 \end{bmatrix},$$

where the terms below the diagonal of the symmetric matrix were deleted for simplicity.

i. Residual matrix. The residual matrix, with k factors removed, is defined by

$$R_k = R - R^{\dagger}. \tag{19}$$

For the 9-variable example, the residual matrix is

$$R_3 = \begin{bmatrix} -.01 & .01 & .00 & .01 & .00 & .00 & .00 & -.01 & .01 \\ & -.01 & .00 & -.01 & .01 & .01 & -.01 & .01 & .00 \\ & & .00 & .00 & .00 & .00 & .01 & .00 & -.01 \\ & & & -.02 & .00 & .01 & -.01 & .01 & .00 \\ & & & & .00 & .00 & .01 & -.01 & -.01 \\ & & & & & -.01 & -.01 & .00 & .00 \\ & & & & & & -.01 & .00 & .01 \\ & & & & & & & .00 & .00 \\ & & & & & & & & -.01 \end{bmatrix}.$$

5. *Comparison of Multiple Group Methods*

One of the principal differences that appears in the several developments of multiple group methods of factor analysis really is concerned with the formulation of scientific hypotheses rather than with the *method* of analysis. Guttman (8) and Holzinger (11) suggest that the multiple group methods be used in conjunction with some *a priori* psychological theory. Guttman emphasizes the fact that the computational procedures of multiple group methods can be applied in any event, but most psychological meaning can be gained only through the testing by the data of preconceived hypotheses. These hypotheses are reflected in the specific manner of grouping the variables and in the resulting common factors (usually oblique).

On the other hand, Thurstone (**17**, pp. 171, 173) emphasizes that the multiple group method of factoring is quite independent of the manner of grouping the variables, and in an example, deliberately sets up groups in an arbitrary fashion, with little reference to the correlations. This thought is evident when Thurstone (**18**) calls attention to the unnecessary restrictions that Holzinger (**11**) placed upon the matrix of correlations in order to use his "simple method of factor analysis." These restrictions *are* unnecessary when the object is simply to get a reduction of the correlation matrix to a factor matrix by the expedient multiple group method; they are *not* unnecessary when the object is to test some specified hypothesis by use of multiple group analysis.

Thus, while Holzinger considers the multiple group method suitable only if the correlation matrix is amenable to sectioning into portions of approximate unit rank, and Guttman prefers to group the variables so as to avoid or reduce the problem of rotation of axes, Thurstone conceives of the multiple group method primarily as another (efficient) technique for initial factoring to provide an orthogonal factor matrix, "which is the starting-point for the rotational problem" (**17**, p. 171).

There is no doubt about the effectiveness of the multiple group methods in reducing the labor of computing residual matrices. Instead of extracting one factor at a time, and computing a residual matrix after each, the basic theorem (**7**, p. 12) underlying multiple group methods implies that only one residual matrix need be computed after extracting several factors at one time. If a number of linearly independent groups is selected equal to the dimension of the common factor space then only one residual matrix will be necessary; otherwise, if the first estimate of the number of clusters is too small, the process has to be repeated again. If too many clusters should be selected then the case of multi-collinearity will be evident in the matrix of correlations among the group factors (and the inverse will not exist).

It has been pointed out that if the total number of common factors is not extracted in a single operation, then the multiple group method can be applied again to the residual matrix obtained after the first operation. And this can be repeated as many times as necessary to bring the residuals down to negligible values. In the successive application of the multiple group method, the common factors extracted at each stage are oblique to one another, but the factors obtained in any single stage are orthogonal to all factors extracted in other stages. The implication of this is that if an *a priori* hypothesis involves an oblique structure then all the common factors must be extracted in one operation, or else subsequent rotation might be necessitated.

Both Holzinger and Thurstone have considered only simple (unit-weighted) composites of variables in essentially non-overlapping groups. Guttman (**8**, p. 216) considers this case as the simplest, and usually quite adequate, manner of grouping variables. However, by use of the weight matrix X, he presents the most general approach to multiple group methods of factor analysis.

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