1

We prove:

$$|m - M| \leq \sqrt{2}\sigma$$

By using Chebyshev's inequality along with the fact that standard deviation, σ, is the square root of the variance:

$$P(|X - \mathbb{E}[X]| \geq a) \leq \frac{var[X]}{a^2}$$
$$P(|X - \mathbb{E}[X]| \geq \sqrt{2}\sigma) \leq \frac{\sigma^2}{(\sqrt{2}\sigma)^2}$$
$$P(|X - \mathbb{E}[X]| \geq \sqrt{2}\sigma) \leq \frac{1}{2}$$

For a random continuous variable, the median is defined as the point at which the probability of a a realization being greater is exactly one half (the middle of the distribution!). If the probability of any random variable being more than $\sqrt{2}\sigma$ from the mean is less than one half, and the median is the point at which all points greater are have probability 1/2 or less, than the median cannot be more than $\sqrt{2}\sigma$ from the mean!

2

Using the exponent rules $e^x \geq 1 + x$ and the provided $e^-x \leq 1 - x + \frac{x^2}{2}$, we work some magic after multiplying both sides of the inequality in our original probability by -1, so that we can apply Markov's rule, and assuming continuity so that $p(x > y) = p(x \geq y)$:

$$\begin{aligned}
p\left(\frac{-1}{n}\sum X_i > t - m\right) &\leq \frac{\mathbb{E}[\frac{-1}{n}\sum X_i]}{t-m} \\
p\left(e^{\frac{-\lambda}{n}\sum X_i} > e^{\lambda(t-m)}\right) &\leq \frac{\mathbb{E}[e^{\frac{-\lambda}{n}\sum X_i}]}{e^{\lambda(t-m)}} \\
&\leq \frac{\mathbb{E}[e^{\frac{-\lambda}{n}X_i}]^n}{e^{\lambda(t-m)}} \\
&\leq \frac{\mathbb{E}[1+\frac{\lambda^2 X_1^2}{2n^2} - \frac{\lambda X_1}{n^2}]^n}{e^{\lambda(t-m)}} \\
&\leq \frac{\left(1+\frac{\lambda^2 a^2}{2n^2} - \frac{\lambda m}{n^2}\right)^n}{e^{\lambda(t-m)}} \\
&\leq \frac{\left(\exp\left\{\frac{\lambda^2 a^2}{2n^2} - \frac{\lambda m}{n}\right\}\right)^n}{e^{\lambda(t-m)}} \\
&\leq \exp\left\{\frac{\lambda^2 a^2}{2n} - \lambda m - \lambda(t-m)\right\} \\
&\leq \exp\left\{\frac{\lambda^2 a^2}{2n} - \lambda t\right\}
\end{aligned}$$

Here we minimize with respect to λ:

$$\begin{aligned}
\frac{\lambda a^2}{n} - t = 0 \\
\lambda = \frac{tn}{a^2}
\end{aligned}$$

Plugging this back into our equation we solve:

$$\begin{aligned}
p\left(\frac{1}{n}\sum X_i < m - t\right) &\leq \exp\left\{\frac{t^2 n^2 a^2}{2na^4} - \frac{t^2 n}{a^2}\right\} \\
&\leq \exp\left\{\frac{nt^2}{2a^2} - \frac{nt^2}{a^2}\right\} \\
&\leq \exp\left\{-\frac{nt^2}{2a^2}\right\}
\end{aligned}$$