2

a

Firstly, note that in the provided formulation our control variable is consumption(c_t), the assets (a_t) is an endogenous state variable, and the interest rate (R_t) is an exogenous state variable. Consumption can be written, by using the constraint, as a function of our state variable a_t:

$$c_t = a_t - \frac{a_{t+1}}{R_t}$$

To simplify the formulation of the problem, we create a new endogenous state variable that will also be our control variable, called *savings*:

$$s_t = \frac{a_{t+1}}{R_t}$$

This allows us to express consumption at time t as an expression of savings, interest rate, and future savings:

$$c_t = s_{t-1}R_{t-1} - \frac{s_t R_t}{R_t} = s_{t-1}R_{t-1} - s_t$$

With this formulation, our Bellman equation becomes:

$$V(s_t, R_t) = \max_{s_{t+1}=\pi(s_t,R_t)} U(s_t, R_t, s_{t+1}) + \beta\mathbb{E}[V(s_{t+1}, R_{t+1})]$$

where

$$U(s_t, R_t, s_{t+1}) = u(s_t R_t - s_{t+1}) = u(c_{t+1})$$

and our state consists of endogenous *savings* (s_t) and exogenous *interest rate* (R_t), and our control variable is the next period's endogenous state (s_{t+1}).

c

Notation We use f'_x to denote the first partial derivative of f with regards to x ($\frac{df}{dx}$) for ease of notation. Similarly, D_x denotes the partial derivative of an expression with regards to x.

FOC First-order conditions are given by maximizing our value function under our control variable (the output of our policy function, s_{t+1}).

$$V'_{s_{t+1}}(s_t, R_t) = U'_{s_{t+1}}(s_t, R_t, s_{t+1}) + D_{s_{t+1}}\beta\mathbb{E}[V(s_{t+1}, R_{t+1})] = 0$$

Because our only source of stochasticity is our exogenous state variable, R_t, we can rewrite the above equation, moving the partial derivative into the expectation:

$$V'_{s_{t+1}}(s_t, R_1) = U'_{s_{t+1}}(s_t, R_t, s_{t+1}) + \beta\mathbb{E}[V'_{s_{t+1}}(s_{t+1}, R_{t+1})] = 0$$

Envelope We evaluate $V'_{s_{t+1}}(s_{t+1}, R_{t+1})$ directly to obtain the "forwarded" envelope conditions:

$$V'_{s_{t+1}}(s_{t+1}, R_{t+1}) = \\ U'_{s_{t+1}}(s_{t+1}, R_{t+1}, s_{t+2}) + \\ \underbrace{U'_{s_{t+2}}(s_{t+1}, R_{t+1}, s_{t+1})\frac{ds_{t+2}}{ds_{t+1}} + \beta\mathbb{E}[V'_{s_{t+2}}(s_{t+2}, R_{t+1})\frac{ds_{t+2}}{ds_{t+1}}]}_{=0 \text{ by FOC}}$$

Euler With the forwarded envelope equation, we plug it into the first-order conditions to get our Euler equation:

$$-U'_{s_{t+1}}(s_t, R_t, s_{t+1}) = \beta\mathbb{E}[U'_{s_{t+1}}(s_{t+1}, R_{t+1}, s_{t+2})]$$

Tranversality A "terminal" condition to ensure a solution to our equation is required, we can get this by ensuring a corner solution (*savings*, and therefore *assets* at the end of time are zero), or an interior solution (our utility function "saturates", which is to say, flattens with respect to our *savings*):

$$\lim_{t\to\infty} \beta^t \; \mathbb{E}[U'_{s_t}(s_t, R, s_{t+1})] \; s_t = 0$$