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Directed search with multiple job applications

Manolis Galenianos [a,*], Philipp Kircher [b]

[a] Department of Economics, Pennsylvania State University, 522 Kern Graduate Building, State College, PA 16802, USA
[b] Department of Economics, University of Pennsylvania, 3718 Locust Walk, Philadelphia, PA 19104, USA

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Abstract

We develop an equilibrium directed search model of the labor market where workers can simultaneously apply for multiple jobs. Our main theoretical contribution is to integrate the portfolio choice problem faced by workers into an equilibrium framework. All equilibria of our model exhibit wage dispersion. Consistent with stylized facts, the density of wages is decreasing and higher wage firms receive more applications per vacancy. Unlike most models of directed search, the equilibria are not constrained efficient.

JEL classification: C78; J41; J64

Keywords: Directed search; Multiple applications; Portfolio choice

1. Introduction

We develop an equilibrium directed search model of the labor market where workers simultaneously apply for multiple jobs. Our main theoretical contribution is to integrate the portfolio choice problem faced by workers into an equilibrium framework. Our model yields a number of interesting results. First, all equilibria exhibit wage dispersion despite the assumption of agent homogeneity. This is empirically relevant because a large part of wage variation cannot be explained by productivity differences (Abowd, Kramarz and Margolis [1]).[1] Second, the density of posted wages is declining and firms that post higher wages receive more applicants. These pre-

* Corresponding author. Fax: +1 814 863 4775.
E-mail addresses: manolis@psu.edu (M. Galenianos), kircher@econ.upenn.edu (P. Kircher).

[1] Abowd, Kramarz and Margolis [1] find that observable worker characteristics explain only 30 percent of wage differentials and controlling for unobserved worker heterogeneity can account for only half of the residual variation. Similarly,

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dictions are consistent with the evidence in Mortensen [19] and Holzer, Katz and Krueger [14], respectively, and they arise precisely because we model search to be directed, as opposed to random.[2] Third, the number of matches is inefficiently low in contrast to most directed search models where constrained efficiency obtains.

Models of directed search combine the presence of frictions, which appear to be pervasive in the labor market, with a guiding role for prices, which is mostly absent in random search models. In directed search, every firm publicly posts and commits to a wage and each worker chooses the job to which he applies. Frictions are introduced by assuming that workers cannot coordinate their search decisions. A single wage is posted in the unique equilibrium of a homogeneous agent environment (Burdett, Shi and Wright [6]). Constrained efficiency obtains due to the firms' ability to price their hiring probability (Moen [18]).[3]

In this paper each worker simultaneously applies for N jobs. Sending multiple applications has two effects. First, it increases the probability of getting a job. Second, it introduces a portfolio choice element to the worker's optimization problem: The worker's expected utility is a non-trivial function of the *combination* of firms where he applies because his payoffs only depend on the most attractive offer that he receives. As a result, despite risk neutrality, he cares about the probability of success over and above the expected payoff of each individual application.[4] Loosely speaking, a worker's optimal strategy is to apply to both "safe" low wage and "risky" high wage jobs: the former provide a high probability of getting a job offer but for low pay; the latter provide high payoff conditional on success while the downside risk is limited by the possibility of getting the low wage job. This decision rule is a special case of the marginal improvement algorithm proposed in Chade and Smith [7].

The main theoretical contribution of our paper is to integrate this portfolio choice problem into an equilibrium framework where both the success probabilities and the payoffs (wages) are equilibrium outcomes. The willingness of workers to send each application to a separate wage level creates an incentive for firms to post different wages. In equilibrium, exactly N wages are posted and every worker applies once to each distinct wage level. Since high wage firms receive more applicants, our characterization implies that the wage density is declining. The firms' expected profits are equal at all wage levels because the lower margins of high wages are balanced with a higher probability of filling the vacancy. It is important to reiterate, however, that this intuition fails in the single application case. The incentives for different wages to be posted arise only because every worker applies for multiple jobs.

Our paper is related to Peters [22] and Burdett, Shi and Wright [6] who solve different versions of the single application, homogeneous agent, directed search model. Shi [24] and Shimer [25] introduce firm and worker heterogeneity leading to wage dispersion which is, however, driven by the underlying dispersion in productivity. In Albrecht, Gautier and Vroman [3] workers apply for multiple jobs in a directed search framework but with two crucial differences from our paper: first, when two or more firms make a job offer to the same worker they engage in Bertrand

Krueger and Summers [17] and Gibbons and Katz [13] conclude that observed and unobserved productivity differences cannot account for the full extent of wage variation.

[2] Models of random search, such as Burdett and Mortensen [5], typically predict an increasing density when workers and firms are homogeneous which is generally seen as a failing of the basic model (see Mortensen [19]). In random search models the arrival rate of workers does not depend on the wage the firm is offering.

[3] By contrast, in random search models workers looking for employment have no prior information about the characteristics of the firms that they sample and efficiency is a non-generic outcome (Hosios [15]).

[4] Two assumptions are crucial for the portfolio choice problem: firms commit to the wages that they post and workers receive the responses from all applications before deciding which offer to accept.

competition, while in this paper we assume commitment to posted wages; second, they only examine equilibria where a single wage is posted which means that there is no portfolio choice problem for the workers. We discuss these differences in the conclusions.

Delacroix and Shi [9] develop a single-application directed search model with on-the-job search. The focus of their paper is on worker flows across jobs, as well as wage dispersion, but their equilibrium exhibits many similarities to ours. In their model the worker faces a portfolio choice problem over time as opposed to at a single instance, which is the case here. The outside option of employed job seekers depends on the wage that they are currently receiving and hence highly-paid workers are willing to tolerate a lower probability of getting a job than low-paid or unemployed workers. This endogenous heterogeneity in outside options leads to wage dispersion with a declining density for reasons similar to our paper.

Chade and Smith [7] provide the optimal algorithm for solving the following general portfolio choice problem: a decision maker faces a number of exogenous (payoff, probability-of-success) pairs and he has to determine how many applications to send and where to apply given that only the best realized alternative is exercised. That algorithm is an important building block in our analysis. Chade, Lewis and Smith [8] and Nagypal [21] develop equilibrium models of directed college choice where applicants can simultaneously apply to many colleges. In both papers the payoffs of attending a particular college are exogenous and the focus is on whether there is assortative matching between students and colleges in the context of incomplete information.

The rest of the paper is structured as follows. Section 2 presents the model. Section 3 discusses the special case when workers send two applications which provides many important insights. The following section extends the results to an arbitrary (but finite) number of applications. Section 5 evaluates the distribution of wages and illustrates the reason why our directed search model generates a very different shape compared to random search models. Section 6 concludes.

2. The model

In this section we introduce the main features of the model, and define outcomes, payoffs, and equilibrium. At the end we state the main theorem and prove a preliminary result.

2.1. Environment and strategies

There is a continuum of workers of measure b and a continuum of firms of measure 1. Each firm has one vacancy. All workers and all firms are identical, risk neutral, and they produce one unit of output when matched and zero otherwise. The utility of an employed worker is equal to his wage and the profits of a firm that employs a worker at wage w are given by $1 - w$. The payoffs of unmatched agents are normalized to zero.

The matching process has four distinct stages. Firms start by simultaneously posting (and committing to) wages. Then, all postings are observed by the workers and each worker sends N applications to N different firms. Firms follow by making a job offer to one of the applicants they have received, if any. Last, workers that get one or more offers choose which job to accept. Importantly, an applicant receives a response from all the firms that he has applied to before having to make a decision.[5] If a firm's chosen applicant rejects the job offer then the firm remains

[5] For instance, if responses arrived sequentially and the worker had to decide whether to accept or reject an offer before receiving additional responses, the portfolio choice problem would be very different because risky applications would lose part of their option value.

unmatched.[6] Firms therefore compete for workers in two separate stages: they want to attract at least one applicant in the second stage and they try to keep that applicant in the last stage; we label these "ex ante" and "ex post" competition, respectively.

As is common in the directed search literature, trading frictions are introduced by focusing on symmetric mixed strategies for workers. The main idea is that asymmetric strategies require a lot of coordination since each worker has to know his personal strategy. Therefore, a single symmetric strategy appears to be a more plausible outcome in a large market where coordination among workers is difficult to achieve. For simplicity, it is also assumed that workers' strategies are anonymous, i.e. a worker treats identically all the firms that post the same wage. This assumption, however, is not necessary: it is possible to let workers condition on the firms' names (say, a real number in [0, 1]) but this would clutter the exposition without changing the results. Last, the firms also follow anonymous strategies, meaning that they treat all workers the same in the event that they receive multiple applicants. This is the standard environment in the directed search literature, such as Peters [22] or Burdett, Shi and Wright [6], except for the innocuous assumption of the anonymity of workers' strategies and the key difference that we allow multiple applications.

Before describing the agents' strategies, observe that the last two stages of the game can be immediately solved. In the fourth stage, workers with multiple job offers choose the highest wage and randomize with equal probabilities in the case of a tie. In the third stage, firms with multiple applicants choose one at random. Therefore we only need to consider the strategies for the first two stages.

A strategy for the firm is a wage w that it posts in the beginning of the game. Denote the distribution of posted wages by F with support $\mathcal{W}_F$. In the second stage, workers observe F and decide where to apply. By anonymity, the strategy of a worker can be summarized by the wages where he applies (in particular, the name of the firm that receives each application does not matter). Therefore, a pure strategy for a worker is an N-tuple of wages to which he applies and a mixed strategy is a randomization over different N-tuples. We denote the workers' strategy by G_F, which is a cumulative distribution function on $[0,1]^N$ conditional on the posted distribution F. Let G_F^i be the marginal distribution of the ith application with support $\mathcal{W}_G^i$ and define $\mathcal{W}_G \equiv \bigcup_{i=1}^N \mathcal{W}_G^i$. That is, $\mathcal{W}_G$ denotes the set of the wages that receive some application with positive probability.

Given any N-tuple $\mathbf{w} = (w_1, w_2, \dots, w_N)$ chosen by the worker, we assume that $w_N \geqslant w_{N-1} \geqslant \cdots \geqslant w_1$ and that the worker chooses the wage with the higher index in the case of a tie. Both assumptions are without loss of generality. This is clear in the former assumption. Concerning the latter, the randomization between tied wages can occur before the applications are actually sent and therefore a higher index can be assigned to the "preferred" wage.

2.2. Outcomes and equilibrium

We define $q(w)$ to be the probability that a firm posting w receives at least one application and $\psi(w)$ to be the conditional probability that a randomly chosen worker who has applied to

[6] Kircher [16] relaxes this assumption and allows the recall of all applicants in the case a firm's offer is rejected. Though the matching process is quite different, the unique equilibrium exhibits an N-point distribution of posted wages suggesting that the qualitative features of our model are robust. However, the predictions of that model are different concerning the shape of the distribution of wages. Furthermore, constrained efficiency is recovered in Kircher [16] as discussed in Section 3.3.

such a firm accepts a *different* job offer (i.e. the probability that the firm does *not* get the worker). Let $p(w)$ be the probability that a worker applying to wage w gets an offer. When a wage is not posted by any firm ($w \notin \mathcal{W}_F$) we have $p(w) = 0$ which immediately implies that $\mathcal{W}_G \subset \mathcal{W}_F$. Last, we define the *value* of an *individual* application to some wage w to be $p(w)w$.

The expected profits of a firm that posts w and the expected utility of a worker who applies to $\mathbf{w}$ are given by

$$\pi(w) = q(w)\big(1 - \psi(w)\big)(1 - w), \tag{1}$$

$$U(\mathbf{w}) = p(w_N)w_N + \big(1 - p(w_N)\big)p(w_{N-1})w_{N-1} + \cdots + \prod_{i=2}^{N}\big(1 - p(w_i)\big)p(w_1)w_1. \tag{2}$$

The expected profits of a firm are equal to the probability that at least one worker applies for the job times the retention probability times $(1 - w)$. A worker gets utility w_N from his highest application, which is successful with probability $p(w_N)$. With the complementary probability that application fails and with probability $p(w_{N-1})$ he receives w_{N-1}. And so on.

We now relate p, q and ψ to the agents' strategies. On $\mathcal{W}_F$, both $p(w)$ and $q(w)$ depend on the average *queue length* at w, which is denoted by $\lambda(w)$. Intuitively, the queue length faced by a firm offering wage w is given by the number of applications sent to w divided by the number of firms who post that wage. Formally, $\lambda(w)$ is defined by the integral equation:

$$\int_0^w \lambda(\tilde{w})\,dF(\tilde{w}) = b\hat{G}_F(w), \tag{3}$$

where $\hat{G}_F(w)$ is the expected number of applications that a worker sends to wages no greater than w, i.e. $\hat{G}_F(w) = \sum_{i=1}^{N} G_F^i(w)$. The right-hand side of Eq. (3) gives the number of applications that are sent up to wage w by *all* workers, while the left-hand side gives the number of firms that post a wage up to w multiplied by the average number of applications they receive.

Anonymity implies that a worker who sends an application to some wage w randomizes over all the firms offering that wage. As a result, the number of applications received by a firm posting w is random and follows a Poisson distribution with mean $\lambda(w)$.[7] Therefore the probability that a firm posting w receives at least one application is $q(w) = 1 - e^{-\lambda(w)}$ and the probability that a worker who applies to such a firm gets an offer is $p(w) = (1 - e^{-\lambda(w)})/\lambda(w)$, where $p(\tilde{w}) = 1$ when $\lambda(\tilde{w}) = 0$.[8]

In order to evaluate $\psi(w)$ for some $w \in \mathcal{W}_F$ we need to find the probability that, after applying to w, a worker rejects an offer from that firm in favor of a different job. As described earlier, a worker who receives multiple offers chooses by construction the wage with the higher index. Therefore the probability that a worker accepts an offer from w_j conditional on having applied to (w_j, w_{-j}) is given by $R_j(w_j, w_{-j}) \equiv \prod_{k>j}(1 - p(w_k))$. We can now integrate over all the possible wages that a worker applies to. Let $\Pr[j \mid w]$ be the conditional probability that a worker

[7] Suppose that n applications are randomly allocated to m firms. The number of applications received by a particular firm follows a binomial distribution with probability $1/m$ and sample size n. As $n, m \to \infty$ keeping $n/m = \lambda$ the binomial distribution converges to a Poisson distribution with mean λ.

[8] This matching process does not depend on the anonymity of the worker strategies. Symmetry and optimality clearly imply that firms with the same wage must have the same expected queue length. Poisson matching follows.

who applied to $w \in \mathcal{W}_F$ did so with his jth application. Furthermore, let $G_F^j(w_{-j} \mid w)$ be the conditional distribution over the other applications, given that the jth application was sent to wage w. Then $\psi(w)$ is given by

$$\psi(w) = 1 - \sum_{j=1}^{N} \Pr[j \mid w] \int R_j(w, w_{-j})\, dG_F^j(w_{-j} \mid w). \tag{4}$$

We have defined $\lambda(w)$ and $\psi(w)$ for wages on the support of F which means that the workers' optimization problem can be solved for a given distribution of posted wages. To solve the firm's optimization problem, $\lambda(w)$ and $\psi(w)$ need to be well-defined on the full domain $[0, 1]$ since a firm needs to know the queue length and retention probability that it would face at *any* wage it could post. Although no one actually applies to wages that are not posted, the queue lengths at such wages could be positive since they represent the firms' beliefs about how many workers *would* apply *if* these wage were offered; and similarly for $\psi(w)$.

It turns out that determining off-equilibrium beliefs presents a challenge. Given the sequential structure of the model, the most natural approach would be to require subgame perfection for the firms' off-equilibrium beliefs. However, the fact that this is a continuum economy means that the symmetric response of a mass of workers to the deviation of a single (zero measure) firm is not well-defined. Formally, $\lambda(w)$ and $\psi(w)$ cannot be determined using Eqs. (3) and (4) for $w \notin \mathcal{W}_F$ because both F and G_F have zero density at those wages.

To get around this issue we define λ and ψ as if "many" firms post every wage in $[0, 1]$ so that the reaction of workers can be meaningfully evaluated. We introduce a fraction of noise firms of measure ϵ that post a wage at random from some distribution $\tilde{F}$ with full support. An alternative interpretation is that firms make a mistake with probability ϵ. Given a candidate F, the distribution of posted wages becomes $F_\epsilon(w) = (1 - \epsilon)F(w) + \epsilon\tilde{F}(w)$ and the game can be analyzed from the second stage onwards. Let G_{F_ϵ} denote the equilibrium response of workers when facing F_ϵ. The outcomes λ_ϵ and ψ_ϵ can be calculated on the entire domain $[0, 1]$ using F_ϵ, G_{F_ϵ}, and Eqs. (3) and (4). As $\epsilon \to 0$ the perturbed distribution converges to F, and we define $\lambda(w) = \lim_{\epsilon\to 0} \lambda_\epsilon(w)$ and $\psi(w) = \lim_{\epsilon\to 0} \psi_\epsilon(w)$ for all $w \in [0, 1]$. We should emphasize that noise firms are simply a means of evaluating the profits that a firm would obtain when deviating and, as we will show in the next section, none of our results depend on the exact choice of $\tilde{F}$. The only crucial requirement is that the noise distribution has full support because otherwise the same problem would recur at any wage which is outside the support of $\tilde{F}$.[9]

We can now define an equilibrium, given a distribution with full support $\tilde{F}$.

[9] We explored two further alternatives, both of which lead to the same results (the proof is available upon request). The first is to introduce trembles on a finite but collapsing grid. The second alternative is the market utility approach used in Moen [18], Acemoğlu and Shimer [2], Shi [24] and Shimer [25] for the $N = 1$ case. It posits that workers' response to deviations is such that they are indifferent between applying to any wage. Our multiple application framework makes this concept less appealing due to the notational and expositional complexity of specifying indifferences over sets of wages (see Kircher [16] for that specification in a related model). A third potential approach is to solve for the subgame perfect Nash equilibrium of a finite version of the same model and then take the limit of that equilibrium as the number of agents goes to infinity, as in Peters [23] and Burdett, Shi and Wright [6]. While arguably the correct (or most reasonable) approach, this problem becomes intractable when introducing multiple applications because the probability of success is correlated across applications (see Albrecht, Gautier, Tan, and Vroman [4]).

Definition 2.1. An equilibrium is a set of strategies $\{F, G_F\}$ such that

1. $\pi(w) \geqslant \pi(w')$ for all $w \in \mathcal{W}_F$ and $w' \in [0, 1]$.
2. $U(\mathbf{w}) \geqslant U(\mathbf{w}')$ for all $\mathbf{w} \in \operatorname{supp} G_F$ and $\mathbf{w}' \in [0, 1]^N$.

The first condition ensures that no firm can increase its profits by posting a different wage than prescribed by F. The second condition ensures that no worker can increase his expected utility by applying to a different set of wages.

We now state the main theorem of this paper.

Theorem 2.1. *An equilibrium exists for all N and is unique when $N = 2$. N different wages are posted by firms and every worker sends one application to each distinct wage. The expected number of applicants is increasing with the wage. The number of firms that post a given wage is decreasing with the wage. The equilibria are not constrained efficient.*

2.3. A preliminary result

The next lemma establishes some immediate conditions on the expected queue of applications which will be useful in the following sections. Let $\underline{w}$ be the lowest wage where some worker applies, i.e. $\underline{w} = \inf\{w \in \mathcal{W}_G\}$.

Lemma 2.1. *Given any distribution of posted wages, worker optimization implies that $\lambda(w)$ is continuous and strictly increasing on $[\underline{w}, 1] \cap \mathcal{W}_F$.*

Proof. Consider the maximization problem of a single worker. Recall that the probability of getting a job is given by $p(w) = (1 - e^{-\lambda(w)})/\lambda(w)$ for $w \in \mathcal{W}_F$ which a strictly decreasing function of $\lambda(w)$. If $\lambda(w)$ is not strictly increasing there exist $w, w' \in \mathcal{W}_F$ such that $w > w'$, $p(w) \geqslant p(w')$ with $w' \in \mathcal{W}_G$. A worker who applies to w' with positive probability can profitably deviate by switching to w since that wage is higher and the probability of getting an offer is at least as high. Therefore, applying to w' is inconsistent with the worker's optimizing behavior and hence, when considering the fact that all workers optimize, any equilibrium $\lambda(w)$ has to be strictly increasing above the lowest wage where workers apply, i.e. on $[\underline{w}, 1] \cap \mathcal{W}_F$. Next, suppose that $\lambda(w)$ is discontinuous at some $\hat{w} \in [\underline{w}, 1] \cap \mathcal{W}_F$. Then the probability of getting a job offer is also discontinuous at $\hat{w}$ and a worker applying in a neighborhood of that wage has an obvious profitable deviation. This implies that $\lambda(w)$ has to be continuous in equilibrium. □

The properties described in the lemma are very natural. The expected number of applicants increases with the wage that a firm posts, which also implies that the probability of getting an offer for that job is strictly decreasing. Moreover, any discontinuity in $p(w)$ leads to the possibility of a profitable deviation for some worker since he can discretely increase his chances of an offer by slightly changing the wage that he applies for. This means that in equilibrium $\lambda(w)$ is continuous regardless of the underlying F. In particular, even if a positive measure of firms post some wage, the optimal response of workers is to send a positive measure of applications to that wage and hence there are no jumps in the queue length. These results hold for any perturbation and hence they hold for the unperturbed game as well, which implies that the queue length that a firm expects is continuously increasing in w regardless of whether that wage is posted or not.[10] Last, note that *any* noise distribution with full support leads to monotonicity and continuity. Using Lemma 2.1 we restrict attention to $\lambda(w)$ that are continuous and strictly increasing in the relevant range for the remainder of the paper.

3. A special case: $N = 2$

We now look at the special case where workers send only two applications which provides many of the main insights. The case of a general N is discussed in the next section. We start by solving the workers' optimization problem given an arbitrary distribution of posted wages. We then characterize the wages that firms post in equilibrium, establish existence and uniqueness, and evaluate efficiency.

3.1. Worker optimization

We start by characterizing the equilibrium response of workers that face an arbitrary distribution of posted wages F. This distribution could be the result of a perturbation but in that case the subscript ϵ is omitted to keep notation simple.

We consider first the problem of a worker who optimizes given F and some strategy of other workers, G_F.[11] The queue length at the offered wages, and hence the probability of success, is determined by Eq. (3). Thus, the worker takes the menu of wage and probability pairs as given when contemplating where to apply. The individual worker's problem is a special case of that analyzed by Chade and Smith [7]. The main difference is that in this paper a worker can send both applications to firms with the same wage, which turns out to simplify the analysis considerably and allows the following derivation. The worker solves

$$\max_{(w_2,w_1)\in\mathcal{W}_F^2} p(w_2)w_2 + \big(1 - p(w_2)\big)p(w_1)w_1, \tag{5}$$

where $w_2 \geqslant w_1$ by convention.[12] Differentiability of $p(w)$ is not guaranteed so the problem cannot be solved by taking the first order conditions. Even though this is a simultaneous choice problem, it can be simplified by evaluating the low wage application separately from the high wage application. That is, the problem admits a convenient recursive solution.

The low wage application is exercised only if w_2 fails, which means that the optimal choice for w_1 solves

$$\max_{w\in\mathcal{W}_F} p(w)w. \tag{6}$$

[10] It is not hard to show that the functions $\lambda_\epsilon(w)$ are equicontinuous and hence $\lambda(w) \equiv \lim_{\epsilon\to 0} \lambda_\epsilon(w)$ is continuous in w. Furthermore, the gradient of $\lambda_\epsilon(w)$ is bounded away from zero at any w where the queue length converges to a strictly positive limit (by an argument similar to Lemma 2.1) which means that $\lambda(w)$ is strictly increasing in $(\underline{w}, 1)$. The only requirement for the above statements to hold is that at least some firms post a non-zero wage (i.e. $F(0) < 1$) which arises in any equilibrium as shown in Proposition 3.2.

[11] The only restriction on G_F is that the resulting queue length is continuous and strictly increasing on the relevant domain, since Lemma 2.1 proves that these properties are necessary for equilibrium.

[12] The maximum is well-defined because $\lambda(w)$ is continuous and $\mathcal{W}_F$ is a closed set. Similarly for the rest of the paper.

Let u_1 denote this maximum value. Given that the worker sends his low wage application to a particular w_1 that solves (6), his optimal choice for the high wage application solves

$$\max_{w \in \mathcal{W}_F} p(w)w + \big(1 - p(w)\big)u_1. \tag{7}$$

Let u_2 denote the highest utility a worker can receive when sending two applications. One can readily verify that a pair of wages is a solution to (5) if and only if it solves (6) and (7), and therefore the recursive procedure yields the optimal decision.

We now exploit the structure of our model to highlight a useful feature of the solution to the worker's portfolio choice problem. Let $\bar{w}$ be the highest wage that yields value equal to u_1, i.e. $\bar{w} = \max\{w \in \mathcal{W}_F \mid p(w)w = u_1\}$. The next proposition shows that the worker can solve the two problems independently of each other.

Proposition 3.1. *Given any distribution of posted wages, a necessary condition for optimization is that $w_1 \leqslant \bar{w} \leqslant w_2$ holds for every (w_1, w_2) to which the worker is willing to apply.*

Proof. Suppose this is not true. Since $w_1 \leqslant w_2$ the only other possibilities are $\bar{w} < w_1$ or $w_2 < \bar{w}$. By construction $w_1 > \bar{w}$ implies that $p(w_1)w_1 < u_1$ which cannot be optimal. If $w_2 < \bar{w}$ then the worker can deviate and send his high wage application to $\bar{w}$ instead of w_2. This deviation is profitable because

$$\begin{aligned} &p(\bar{w})\bar{w} + \big(1 - p(\bar{w})\big)p(w_1)w_1 - \big[p(w_2)w_2 + \big(1 - p(w_2)\big)p(w_1)w_1\big] \\ &\quad = \big[p(\bar{w})\bar{w} - p(w_2)w_2\big] + \big[p(w_2) - p(\bar{w})\big]p(w_1)w_1 > 0. \end{aligned} \tag{8}$$

The first term of (8) is non-negative since $\bar{w}$ provides the highest possible value by definition. The second term is strictly positive because $\bar{w} > w_2 \Rightarrow p(\bar{w}) < p(w_2)$, by Lemma 2.1. □

The fact that, given F, *all* workers solve the same optimization problem means that we can use Proposition 3.1 to determine some of the outcomes of interest. Any (symmetric) equilibrium strategy by workers, G_F, gives rise to the objects u_1, u_2, and $\bar{w}$ which determine $\lambda(w)$ in the following way. A firm that posts a wage in $[0, u_1)$ does not receive any applications since the value of that job is too low regardless of the queue length. A firm that posts a wage in $[u_1, \bar{w}]$ receives a worker's low application and hence its queue length is such that the job's value is given by u_1. A wage in $[\bar{w}, 1]$ receives a worker's high application and therefore it provides utility u_2 when coupled with a wage in $[u_1, \bar{w}]$. This implies that a worker is indifferent about which combination of wages to apply for so long as the wages are on opposite sides of $\bar{w}$. These results hold for any perturbed distribution of wages and hence they hold in the limit as $\epsilon \to 0$. To summarize this discussion, the queue length that a firm faces when it posts a wage w is uniquely defined by the following conditions:

$$p(w)w = u_1, \quad \forall w \in [u_1, \bar{w}], \tag{9}$$

$$p(w)w + \big(1 - p(w)\big)u_1 = u_2, \quad \forall w \in [\bar{w}, 1]. \tag{10}$$

These observations are illustrated in Fig. 1. The high indifference curve, IC-H, traces the wage and queue length pairs where workers are willing to send a high wage application, while IC-L is the indifference curve for the low wage applications. The two curves intersect at $\bar{w}$ where workers are indifferent about whether they apply with a high or a low application. The equilibrium queue length for any wage is given by the upper envelope of the two indifference

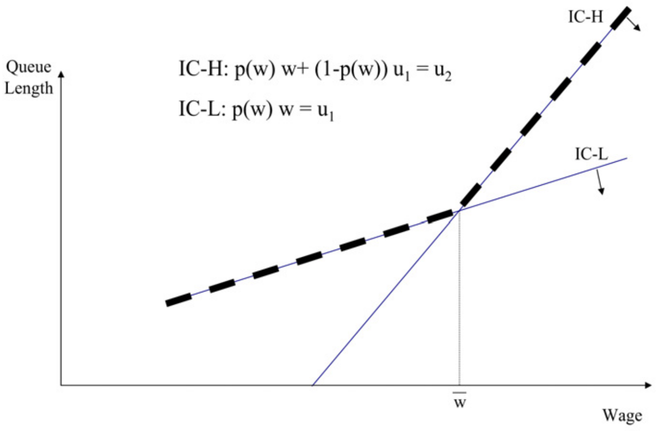

Fig. 1. Workers' application behavior. IC-H and IC-L are the workers' indifference curves for 'high' and 'low' applications. Linearity is only used for illustration.

curves: if the queue length is below the dashed line at some wage $\hat{w}$, then a worker can move to a higher indifference curve by applying to $\hat{w}$ instead of some other w (note that utility increases in the southeast direction). In other words, the queue length is 'bid up' to IC-H for $w > \bar{w}$ and to IC-L for $w < \bar{w}$. Hence the dashed line is the indifference curve that firms anticipate when contemplating which wage to post.

It is worth noting that while the *total* utility of any pair of wages is always equal to u_2, wages that are strictly above $\bar{w}$ give value that is *strictly lower* than u_1. Workers nevertheless apply there which may appear to be counterintuitive at first sight: if a worker can apply to wages that offer value u_1, why would he choose some wage with a strictly lower individual value? The answer is that the return to failure in the high wage application is not zero: it is equal to the value that the next application brings in, as can be seen in Eq. (10). As a result, when the worker chooses where to send his high wage application he faces a tradeoff between the value that he can get from that particular application and the probability of exercising his fallback option, i.e. the low wage application. Since the low wage provides with insurance against the possible failure of w_2, it is profitable for the worker to try a risky application that has high returns conditional on success (i.e., the wage is high) and also offers a high probability of continuing to the next application. Therefore, the low wage application goes to a relatively 'safe' region and the high application is sent to a 'risky' part of the wage distribution.[13]

The next result proves that wage dispersion is present in all equilibria.

Proposition 3.2. *There does not exist an equilibrium in which only one wage is posted.*

Proof. Assume that an equilibrium exists where all firms post the same wage w^*. A firm's expected profits are given by $\pi(w^*) = q(w^*)(1 - w^*)(1 - \psi(w^*))$. We proceed to show that firms have a profitable deviation.

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[13] In contrast, in models of directed search where the wage dispersion is driven by firms' productivity heterogeneity every application yields the same value to identical workers (e.g. Shi [24], Shimer [25]). The reason is that in those models every worker has one application to send and hence there is no portfolio choice problem, which is at the heart of the distinction between 'safe' and 'risky' applications.

Consider $w^* \in (0,1)$ first. A worker sends both his applications to w^* and, with positive probability, he receives two equally good offers and randomizes between them. $\psi(w^*) > 0$ follows. Proposition 3.1 implies that $\bar{w} = w^*$ when trembles are sufficiently small, since otherwise all workers would send one of their applications to the arbitrarily few noise firms which is clearly suboptimal. Therefore $\psi(w) = 0\ \forall w > w^*$ for ϵ small enough and hence this property holds in the limit as $\epsilon \to 0$. Since the queue length (and $q(w)$) is increasing in w, the profits of a firm that posts a wage just above w^* are equal to $\lim_{w \searrow w^*} \pi(w) = q(w^*)(1-w^*) > q(w^*)(1-w^*)(1-\psi(w^*)) = \pi(w^*)$. Therefore offering a wage just above w^* is a profitable deviation.

If $w^* = 1$, firms make zero expected profits. Eq. (9) implies that there is some $\tilde{w} < 1$ close enough to 1 which has strictly positive queue length in the unperturbed game, yielding a profitable deviation. If $w^* = 0$, a worker receives zero expected utility and so for any $\epsilon > 0$ he sends both applications to some of the positive wages. As the trembles become smaller the hiring probability of a firm with a strictly positive wage converges to one and hence $\pi(\tilde{w}) = 1 - \tilde{w} > q(0)(1-\psi(0))$ for $\tilde{w}$ close enough to zero. □

The intuition of the proof is straightforward. If a single wage is posted, workers are indifferent about which firm to work for and hence they randomize when receiving multiple job offers. When posting a slightly higher wage, a deviant firm hires its preferred applicant for sure even if that worker receives other offers (the deviant firm also has a slightly higher expected queue length). This deviation raises profits since the increase in the hiring probability is discrete, while the increase in the wage can be arbitrarily small.

It is important to note that workers respond to wage differentials in different ways depending on whether they are at the stage of sending their initial applications (ex ante competition for the firms) or whether they are deciding which of their offers to accept (ex post competition). In the first case they respond in a continuous way, since a slightly higher wage comes with a slightly lower probability of acceptance due to the market frictions. This force, present in all models of directed search, allows firms to post interior wages and prevents a Bertrand outcome. In the last stage, however, there is no possibility of being rationed and workers accept the highest wage offer with probability one no matter how small the difference. As a result, in this event their strategy is discontinuous in the wage level: arbitrarily small differences in wages lead to very pronounced changes in behavior. It is therefore the ex post competition among firms that precludes a single wage equilibrium.

3.2. *Firm optimization*

We now turn to the analysis of the first stage of the model. We prove that in equilibrium exactly two wages are posted when each worker sends two applications. We proceed to characterize them and prove the existence and uniqueness of equilibrium.

A firm chooses what wage to post given the strategies of other firms, F, and the workers' response, G_F which determine the equilibrium objects $\{\bar{w}, u_1, u_2\}$. It solves

$$\max_{w \in [0,1]} q(w)\big[1 - \psi(w)\big](1-w). \tag{11}$$

The probability that the firm receives at least one applicant, $q(w)$, depends on the average queue length according to $q(w) = 1 - e^{-\lambda(w)}$ and the queue length is determined by Eqs. (9) and (10). The probability of losing a worker after making an offer, $\psi(w)$, depends on whether a wage is above or below the cutoff $\bar{w}$ which determines the type of application received (high or low).

We label the firms that attract high (low) wage applications as high (low) wage firms. While this problem looks complicated, the results of the previous section help to simplify it considerably. In what follows, we provide the main characterization result of this section: all high wage firms post w_2^* and all low wage firms post w_1^*, where $w_2^* > w_1^*$.

An offer by a high wage firm is never rejected since it is an applicant's best alternative. Therefore $\psi(w) = 0$ when $w > \bar{w}$ and the problem of a high wage firm is given by[14]:

$$\max_{w\in[\bar{w},1]} \left[1 - e^{-\lambda(w)}\right](1 - w) \tag{12}$$

$$\text{s.t. } p(w)w + \big(1 - p(w)\big)u_1 = u_2. \tag{13}$$

Rearranging (13) yields $w = u_1 + (u_2 - u_1)/p(w)$, where $p(w) = [1 - e^{-\lambda(w)}]/\lambda(w)$. Since the mapping between the wage and the queue length is one-to-one we can substitute this expression into the objective function and maximize over λ rather than w:

$$\max_{\lambda\geqslant\bar{\lambda}} \big(1 - e^{-\lambda}\big)(1 - u_1) - \lambda(u_2 - u_1), \tag{14}$$

where $\bar{\lambda} = \lambda(\bar{w})$. This is a strictly concave function since $u_1 < u_2 < 1$. Strict concavity implies that the profit maximization problem of a high wage firm has a unique solution, λ_2^*, which corresponds to some wage w_2^*. That wage is either characterized by the first order conditions, in which case it is in the interior of the domain, $\hat{w}_2 \in (\bar{w}, 1)$ with $w_2^* = \hat{\lambda}_2 = \lambda(\hat{w}_2)$, or it lies at the lower boundary of the high wage range, $\lambda_2^* = \bar{w}$. Also, note that all high wage firms post the same wage in equilibrium since they all face the same problem.

The next step is to show that $w_2^* = \hat{w}_2$ is inconsistent with equilibrium, leaving $w_2^* = \bar{w}$ as the only candidate. We prove this by contradiction: we first assume that posting $\hat{w}_2$ is the outcome of high wage firms' profit maximization; we then prove that in that event low wage firms make lower profits which cannot happen in equilibrium. Setting the first order condition of Eq. (14) to zero yields $u_2 - u_1 = e^{-\hat{\lambda}_2}(1 - u_1)$. Substituting this expression into the profit function and rearranging results in the following expression for the profits of high wage firms:

$$\pi(\hat{w}_2) = \big(1 - e^{-\hat{\lambda}_2}\big)(1 - u_1)\left(1 - \frac{\hat{\lambda}_2 e^{-\hat{\lambda}_2}}{1 - e^{-\hat{\lambda}_2}}\right). \tag{15}$$

Now consider low wage firms. A low wage firm retains an applicant only if he does not have a high wage offer. The probability of that event is $1 - p(\hat{w}_2)$ since $\hat{w}_2$ is the only high wage that is posted. The profits of a low wage firm, posting some $w_1 < \bar{w}$, are given by

$$\pi(w_1) = \big(1 - e^{-\lambda(w_1)}\big)(1 - w_1)\left(1 - \frac{1 - e^{-\hat{\lambda}_2}}{\hat{\lambda}_2}\right). \tag{16}$$

A term-by-term comparison shows that $\pi(\hat{w}_2) > \pi(w_1)$ for any $w_1 \leqslant \bar{w}$ since $\hat{\lambda}_2 \geqslant \lambda(w_1)$, $w_1 > u_1$ and the third term follows after some algebra. The preceding argument proves that there is no equilibrium when $w_2^* = \hat{w}_2$. As a result, for an equilibrium to exist, high wage firms need to post $w_2^* = \bar{w}$. Furthermore, that wage is profit maximizing only if $\bar{w} > \hat{w}_2$.

[14] In principle, $\psi(\bar{w}) > 0$ is a possibility since $\bar{w}$ might also attract low wage applications. However, if $\bar{w} \notin \operatorname{argmax} q(w)[1 - \psi(w)](1 - w)$ then the value of $\psi(\bar{w})$ is irrelevant; if $\bar{w} \in \operatorname{argmax} q(w)[1 - \psi(w)](1 - w)$ then $\psi(\bar{w}) > 0$ contradicts optimality since wages arbitrarily close but higher than $\bar{w}$ would be preferable to $\bar{w}$ by an argument similar to Proposition 3.2. Therefore the maximization problem in (12) is specified correctly.

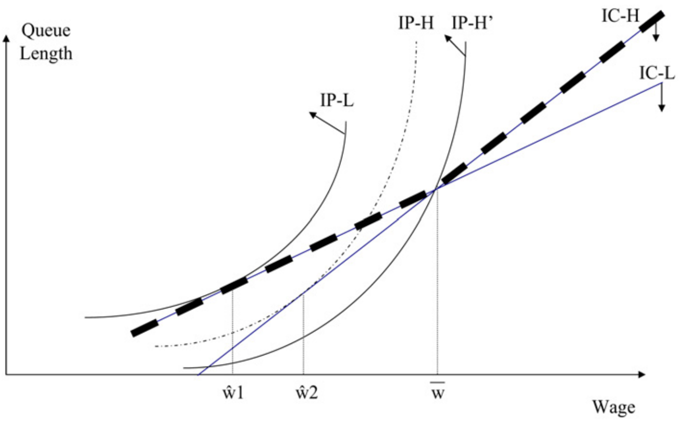

Fig. 2. Firms' equilibrium behavior. IP-H and IP-L are the isoprofit curves for high and low wage firms.

Turning to the problem of a low wage firm we first calculate its retention probability. Eq. (4) implies that $\psi(w) = p(\bar{w})$ for wages that are posted in equilibrium ($w \in \mathcal{W}_F$). For wages outside the support of F it is determined by the response of workers to the firms' trembles, as discussed in Section 2, which could potentially yield a different number. However, it can be (and will be) shown that the set of wages solving the low wage firm's problem can be characterized completely by the case $\psi(w) = p(\bar{w})\ \forall w < \bar{w}$. Since these complications are of a technical nature and do not help to understand the underlying trade-offs we deal with them in the proof of the proposition below. The problem of a low wage firm is

$$\max_{w\in[0,\bar{w}]}\left[1-e^{-\lambda(w)}\right]\left[1-p(\bar{w})\right](1-w) \tag{17}$$

$$\text{s.t. } p(w)w = u_1. \tag{18}$$

Note that the retention probability enters the maximization problem as a constant, and hence it does not affect any decision of a low wage firm. As before, we rearrange (18) to get $w = u_1/p(w)$ and substitute it into the objective function which becomes

$$\max_{\lambda\leqslant\bar{\lambda}}\left(1-e^{-\lambda}-\lambda u_1\right)\left[1-p(\bar{w})\right]. \tag{19}$$

Once more, the problem is strictly concave and admits a unique solution, λ_1^*, corresponding to some w_1^*. Proposition 3.2 ensures that low wage firms cannot be posting the same wage as high wage firms in equilibrium and hence $w_1^* < \bar{w} = w_2^*$. As a result, w_1^* is characterized by the low wage firm's first order conditions. Furthermore, since every low wage firm faces the same problem they all post w_1^*.

Fig. 2 presents a graphical illustration of the above results. The two isoprofit curves yield the same expected profits to high and low wage firms. Note that they do not need to intersect since the retention probability is different for the two types of firms. For profits to be equalized across the two types of firms $\hat{w}_2$ is necessarily below $\bar{w}$ which means that a high wage firm would like to post $\hat{w}_2$, but this would place it in the low application area. Therefore, the strict concavity of the profit function implies that the optimal strategy for a high wage firm is to post the lowest wage that allows it to receive a high wage application, i.e. $w_2^* = \bar{w}$.

The result that all firms of the same type post the same wage extends existing results in a natural way: conditional on attracting a particular type of applications, firms compete with each other in the same way as in the single application case (e.g. Burdett, Shi and Wright [6]) which exhibits a unique equilibrium wage.[15] The only difference is that now there are additional boundary conditions which delineate the type of applications (high or low) that a firm receives. The following proposition summarizes the result and deals with the technical issue described earlier.

Proposition 3.3. *In equilibrium, all high wage firms post $\bar{w}$ and all low wage firms post $\hat{w}_1 \in (u_1, \bar{w})$ which is characterized by the first order conditions.*

Proof. We only need to show that assuming $\psi(w) = p(\bar{w})\ \forall w < \bar{w}$ was without loss of generality. Recall that $\psi(w) = p(w_2^*)$ for $w \in [0, \bar{w}) \cap \mathcal{W}_F$. Now, consider the case where the worker strategies are such that $\psi(w)$ takes different values in $[0, \bar{w})$. An example of why this could happen is the following. Suppose that one of the pairs of wages that the workers randomize over in response to every perturbed distribution is $(\tilde{w}_1, \tilde{w}_2)$ where $\tilde{w}_2 = 1$. If workers applying to $\tilde{w}_1$ send their high wage application to $\tilde{w}_2$ only, then the retention probability at $\tilde{w}_1$ is very high since $\tilde{w}_2 = 1$ implies that $p(\tilde{w}_2)$ has to be very low. As the trembles become smaller, the probability that this particular pair is chosen converges to zero if $\tilde{w}_1$ or $\tilde{w}_2$ are not offered in the limit, however $\psi_\epsilon(\tilde{w}_1)$ remains equal to $p(\tilde{w}_2)$ and so it converges to a relatively low value. This would be troublesome if a different equilibrium could be supported in the way described. Suppose that there is such an equilibrium in which low wage firms post some $\tilde{w} \neq \hat{w}_1$. For $\tilde{w}$ to be posted it needs to provide the highest possible profits, implying in particular that $\pi(\tilde{w}) \geqslant \pi(\hat{w}_1)$. The last inequality can only hold if $\psi(\hat{w}_1) > \psi(\tilde{w})$ since $\{\hat{w}_1\} = \operatorname{argmax}(1 - e^{-\lambda(w)})(1 - w)$. However, the fact that $\tilde{w}$ is actually posted means that $\psi(\tilde{w}) = p(w_2^*)$. Moreover, $w_2^* = \bar{w}$ implies that $p(w) \leqslant p(w_2^*)$ for all wages w in the high region and hence $\psi(\tilde{w}) = p(w_2^*) \geqslant \psi(\hat{w}_1)$, yielding a contradiction. Therefore no other equilibrium can be supported. This completes the proof of Proposition 3.3. □

It is now easy to see that the density of posted wages is falling. Each wage level receives one application per worker so $\lambda(w_i^*) = b/d_i$ where d_i is the fraction of firms posting w_i^*. $d_1 > d_2$ follows from Lemma 2.1 which established that the queue length is strictly increasing with the wage rate. Note that having an increasing queue length is not sufficient for a decreasing density: we also use the result that each wage level receives the same number of applications. If workers would send more applications to high wage than to low wage firms, then one could end up with an increasing queue length *and* an increasing density. In Section 5.2 we discuss using directed search leads to a decreasing density, in contrast to models of random search.

Proposition 3.4. *The distribution of posted wages is decreasing, i.e., $d_1 > d_2$.*

Proof. See above. □

Turning to the existence of equilibrium, we need to find the 'correct' fraction of firms to post each wage so that profits are equalized across types of firms and the necessary conditions we derived earlier are satisfied. Formally, an equilibrium exists if there is $\{d_1, d_2\}$ such that

[15] We should add that Burdett, Shi and Wright [6] only look for equilibria in which a single wage is posted, i.e. equilibria in symmetric strategies for firms. However, in Galenianos and Kircher [11] we show that even when considering the possibility of firms following asymmetric strategies, there do not exist equilibria with wage dispersion when agents are homogeneous.

$d_1 + d_2 = 1$ and there is no profitable deviation when w_i^* is posted by d_i firms, where $w_1^* = \hat{w}_1$ and $w_2^* = \bar{w}$. The equilibrium is unique when there is a single pair of d_i's that satisfies these conditions.

Proposition 3.5. *An equilibrium exists and it is unique.*

Proof. We first show that $\hat{w}_1$ and $\bar{w}$ maximize the profits of the two types of firms. We then fix $w_1^* = \hat{w}_1$ and $w_2^* = \bar{w}$ and find the d_is that lead to equal profits across firms.

To prove that $w_2^* = \bar{w}$ we need to show that $\bar{w} > \hat{w}_2$ holds. Suppose that all high firms post $\bar{w}$ and compare their profits with what they would earn had they all posted $\hat{w}_2$, disregarding feasibility and optimality for now. Under both possible wages each firm receives the same number of applications (one per worker) and hence the level of profits is completely determined by level of wages. If $\bar{w} < \hat{w}_2$, then high wage firms would make higher profits if they could coordinate to post the lower wage (of course, this will not occur in equilibrium since each individual firm has an incentive to deviate to $\hat{w}_2$). Similarly, if $\bar{w} > \hat{w}_2$, then high wage firms make lower profits by posting $\bar{w}$. Recall that when high wage firms post $\hat{w}_2$ they earn higher profits than low wage firms and note that their profits would be lowered by posting $\bar{w}$ only if $\bar{w} > \hat{w}_2$. This means that if we can show that profits are equalized across the two types of firms when all high wage firms post $\bar{w}$ then it must be the case that $\bar{w} > \hat{w}_2$. As a result, equalizing profits also proves that $\hat{w}_2$ is not a feasible wage for high wage firms. For low wage firms, it is easy to see that since $\hat{w}_1$ is derived by their first order condition it also maximizes their profits.

The next step is to prove that profits can be equalized across the different types of firms. To simplify notation let $\pi_i = \pi(w_i^*)$, $p_i = p(w_i^*)$, $\tilde{\pi}_1 = \pi_1/(1-p_2)$, and $\lambda_i^* = b/d_i$. We first show that given some arbitrary $d_1 \in (0,1)$ we can find a $d_2 \in (0, d_1)$ such that $\Delta\pi(d_2; d_1) \equiv \tilde{\pi}_1 - \pi_2/(1-p_2) = 0$. Setting the first order conditions of (19) to zero yields $w_1^* = \lambda_1^* e^{-\lambda_1^*}/(1-e^{-\lambda_1^*})$ leading to $\tilde{\pi}_1 = 1 - e^{-\lambda_1^*} - \lambda_1^* e^{-\lambda_1^*}$. Furthermore, $w_2^* = \bar{w}$ means that $p_2 w_2^* = u_1 \Rightarrow w_2^* = w_1^* p_1/p_2$. Inserting that expression into the profit equation we get $\pi_2 = 1 - e^{-\lambda_2^*} - \lambda_2^* e^{-\lambda_1^*}$. Evaluating $\Delta\pi(d_2; d_1)$ at the two limits of its domain yields different signs. Note that the queue lengths are the same when $d_2 = d_1$, which means that $p_1 = p_2$, $w_1^* = w_2^*$, and $\tilde{\pi}_1 = \pi_2$ leading to $\Delta\pi(d_1; d_1) < 0$. On the other hand, $\lambda_2 \to \infty$ when $d_2 \to 0$ which means that $p_2 \to 0$ and therefore $w_2^* \to \infty$ leading to $\pi_2 < 0$ (this occurs because the high wage firm is assumed to post $\bar{w}$). As a result $\Delta\pi(d_2; d_1) > 0$ when $d_2 \approx 0$, and by the intermediate value theorem there exists a $d_2(d_1)$ such that high and low wage firms make the same profits.

To prove the existence and uniqueness of equilibrium when $N = 2$ we show that d_1 and $d_2(d_1)$ are positively related along the isoprofit curve, and hence there is a unique pair that equalizes profits and sums up to one. Implicit differentiation of d_2 with respect to d_1 while keeping profits equal yields $\partial d_2/\partial d_1 = -(\partial\Delta\pi/\partial d_1)/(\partial\Delta\pi/\partial d_2)$. Some algebra shows that the numerator is given by $\partial\Delta\pi/\partial d_1 = (\partial\lambda_1^*/\partial d_1)e^{-\lambda_1^*}(\lambda_1^* - \lambda_2^*/(1-p_2))$, which is positive since the queue length is inversely related to the number of firms and $\lambda_1^* < \lambda_2^*$. The denominator is equal to $\partial\Delta\pi/\partial d_2 = -\pi_2\partial[1/(1-p_2)]/\partial d_2 - [\partial\pi_2/\partial d_2]/(1-p_2)$. When d_i increases the queue length decreases and hence the probability of getting a job increases. Therefore the first partial is positive and the first term as a whole is strictly negative. The second partial is also negative because the constraint of high wage firms binds and hence $\partial\pi_2/\partial\lambda_2 < 0$. This proves that $\partial d_2/\partial d_1 > 0$. Hence if we start with $d_1 < 1/2$ we have $d_1 + d_2(d_1) < 1$ and by increasing d_1 we eventually find the unique $\{d_1, d_2\}$ pair such that profits are equal and $d_1 + d_2 = 1$. $\square$

At this point it should be remarked that the only property of the trembling distribution, $\tilde{F}$, that we used in solving the model is that it has full support. As a result, the unique equilibrium that was constructed survives any choice of $\tilde{F}$.

3.3. *Efficiency*

We now examine the efficiency properties of the equilibrium. We ask whether a planner can generate higher output by providing instructions to the workers about which jobs to apply for *subject to the matching frictions* in the market.[16] As described in Section 2, frictions are introduced by restricting attention to symmetric strategies for workers so we constrain the planner to do the same. The main result of this section is that constrained efficiency does not obtain.

Maximizing output in our environment is equivalent to maximizing the number of matches. By definition, the number of matches is equal to the number of workers that become employed, as well as the number of vacancies that are filled. The probability that a worker becomes employed equals the probability of receiving at least one job offer. The difference to the standard analysis with one application is that we have to account for the possibility that a single worker obtains several offers.

We showed that in equilibrium a worker sends each of his applications to a different group of firms, which was identified by its distinct wage. A simple way to prove that the matching process is inefficient is to show that output can be increased by reallocating firms between the two groups. We ignore wages since they have no bearing on aggregate welfare. Consider an arbitrary b and let d be the fraction of firms in the first group and $1 - d$ the fraction in the second group. The following proposition states the result.

Proposition 3.6. *When $N = 2$ the number of matches is maximized only if $d = 1/2$ or $d \in \{0, 1\}$.*

Proof. See Appendix A. $\square$

The proposition shows that it may be optimal for workers to send only *one* application due to congestion. In that event the planner's solution is to place all firms in one group ($d \in \{0, 1\}$). If it is optimal to send two applications, then the number of firms should be equal in both groups. However, we know that in equilibrium the number of firms posting the low wage is larger and hence this efficiency condition is never met. Moreover, since the lack of efficiency arises from the matching process it carries over even if the number of applications is endogenized or if the ratio of workers to firms is determined by free entry subject to a fixed cost.

It is worthwhile to mention that efficiency *does* obtain in the usual directed search environment with one application. The reason is that firms can price the arrival rate (in essence, the queue length) of workers through the wages they post.[17] When workers send multiple applications firms care about the probability of retaining a worker, as well as the arrival rate of applicants. The arrival rate can still be priced using the posted wage, but the probability of retaining a worker does not depend on how many applications a firm has received: if a firm's chosen applicant has a better offer, the firm remains idle regardless of how many other workers it attracted. Therefore, the arrival rate of applicants does not change the probability of hiring at the second stage, once at least

[16] For further discussion of this approach see e.g. Shimer [25].

[17] See Mortensen and Wright [20] for a discussion.

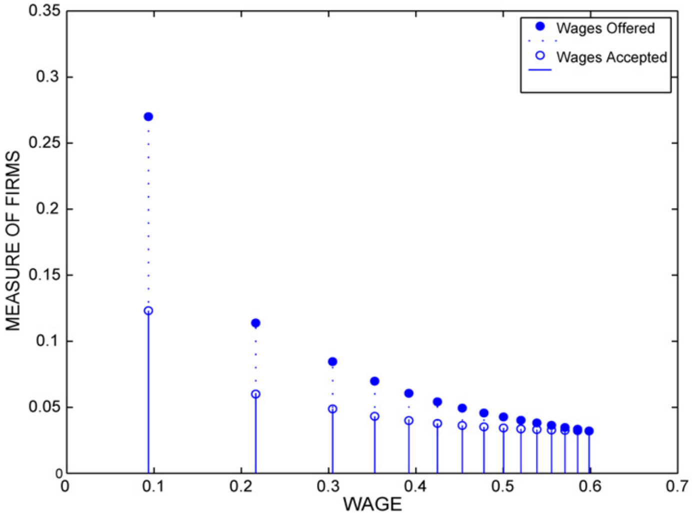

Fig. 3. Equilibrium wage dispersion for $N = 15$ and $b = 1$.

one worker has applied. Since the firm can only influence the arrival rate of workers but not the retention probability, it cannot fully price its hiring probability and hence efficiency does not obtain.

Interestingly, Kircher [16] finds that constrained efficiency is restored when firms can recall all the applicants they receive, in an otherwise similar model. In that environment, the second phase of the hiring process also depends on the queue length since a firm can offer the job to all the applicants it receives, until one of them accepts (or all of them reject it). This is consistent with the intuition that if firms are able to price their full hiring probability then efficiency obtains. However, if firms can only recall up to a certain (finite) number of applicants, the queue length will only partially influence the retention probability. Hence it is our conjecture that efficiency fails when recall is imperfect. Therefore, we believe that our inefficiency result can be seen as a general feature of limited recall.

4. The general case: $N \geqslant 2$

We turn to the model with a general N. The analysis mirrors the one of Section 3 and we prove that all results except for uniqueness generalize in a straightforward manner. While we believe that the equilibrium is unique we have been unable to prove so and we describe the difficulties involved in Section 4.2. In our working paper version (Galenianos and Kircher [10]) we present some numerical evidence of uniqueness.

Fig. 3 illustrates the distribution of posted and received wages for an economy with equal number of workers and firms and $N = 15$. Properties of the distribution of wages are explored in the next section.

4.1. *Worker optimization*

Recall that $\mathcal{W}_G^i$ is the set of wages that receive the ith application of workers. As before, the utility of the lowest i applications has to be the same in any N-tuple of wages which defines the following recursive relationship

$$u_i = p(w_i)w_i + \big(1 - p(w_i)\big)u_{i-1}, \quad \forall w_i \in \mathcal{W}_G^i,\ i \in \{1, 2, \dots, N\}, \tag{20}$$

where $u_0 \equiv 0$. The fact that $p(w)$ is strictly decreasing together with the convention $w_i \geqslant w_{i-1}$ imply that $u_i > u_{i-1}$. Moreover, u_i is the highest possible utility a worker can get from i applications when his fallback option is u_{i-1}. Let $\bar{w}_i$ be the highest wage that provides total utility equal to u_i when the fallback option is u_{i-1}, i.e. $\bar{w}_i = \max\{w \mid p(w)w + (1-p(w))u_{i-1} = u_i\}$. Let $\bar{w}_0$ be the lowest wage that receives applications with positive probability. Proposition 3.1 is generalized as follows.

Proposition 4.1. *When a worker sends N applications optimally, $w \in \mathcal{W}_G^i$ implies that $w \in [\bar{w}_{i-1}, \bar{w}_i]$ for $i \in \{1, 2, \ldots, N\}$.*

Proof. The proof is by induction. It is sufficient to show that the following property holds for all i: $w < \bar{w}_i \ \Rightarrow\ w \notin \mathcal{W}_G^k$ for $k \geqslant i+1$. The initial step for $i = 1$ was proven in Section 3, where $\bar{w}_1 = \bar{w}$. Define $v(w, u_{i-1}) = p(w)w + (1-p(w))u_{i-1}$ to be the utility of applying to a particular wage w when the fallback option is u_{i-1}. We want to show that $v(\bar{w}_i, u_i) > v(\tilde{w}, u_i)$ for all $\tilde{w} < \bar{w}_i$. Note that

$$v(\tilde{w}, u_{i-1}) = p(\tilde{w})\tilde{w} + \big(1 - p(\tilde{w})\big)u_{i-1} \leqslant p(\bar{w}_i)\bar{w}_i + \big(1 - p(\bar{w}_i)\big)u_{i-1} = v(\bar{w}_i, u_{i-1}),$$

as the rightmost equation is the optimal choice when u_{i-1} is the fallback option which provides with the maximum level of utility. Replacing u_{i-1} with u_i in both sides of the inequality we get the term that we want to compare. Since $\bar{w}_i > \tilde{w} \ \Rightarrow\ (1 - p(\bar{w}_i)) > (1 - p(\tilde{w}))$ the second term increases by more and the inequality becomes strict which proves the result. □

An implication of the proposition is that the queue lengths facing the firms attracting the ith application are given by the following expression

$$p(w)w + \big(1 - p(w)\big)u_{i-1} = u_i, \quad \forall w \in [\bar{w}_{i-1}, \bar{w}_i], \tag{21}$$

which is a straightforward generalization of Eqs. (9) and (10).

4.2. Firm optimization

We now turn to the first stage of the model. For the remainder of the paper firms that receive the ith lowest application of workers are referred to as *type i firms*. The profit maximization problem of each type of firm is solved and profits are then equalized across types.

When posting a wage, a firm takes as given the cutoffs $\{\bar{w}_k\}_{k=0}^{N-1}$ and the equilibrium utility levels $\{u_k\}_{k=1}^{N}$, which determine the utility provided to workers for their lowest k applications. A firm of type i solves the following profit maximization problem:

$$\max_{w \in [\bar{w}_{i-1}, \bar{w}_i]} q(w)\big[1 - \psi(w)\big](1 - w), \tag{22}$$

where the queue lengths are determined by Eqs. (21).

The problem of a type N firm is identical to (12) with $\bar{w}_{N-1}$ replacing $\bar{w}$ and $\{u_{N-1}, u_N\}$ replacing $\{u_1, u_2\}$. Following the same logic it is shown that $\bar{w}_{N-1}$ is the solution to the problem of type N firms. This means that $\psi(w) = p(\bar{w}_{N-1})$ for type $N-1$ firms and their maximization problem is, in turn, the same but with $\bar{w}_{N-2}$ as the lower cutoff and $\{u_{N-2}, u_{N-1}\}$ as the relevant utility levels that determine the queue lengths. Again, this leads to $w_{N-1}^* = \bar{w}_{N-2}$ which implies that $\psi(w) = [1 - p(\bar{w}_{N-1})][1 - p(\bar{w}_{N-2})]$ for type $N-3$ firms and so on. In general, it will be shown that the retention probability of a type i firms is $1 - \psi(w) = \prod_{n=i+1}^{N}[1 - p(w_n^*)] \equiv 1 - \psi_i$ and, given ψ_i, the maximization problem for a type i firm becomes

$$\max_{w\in[\bar{w}_{i-1},\bar{w}_i]} q(w)[1-\psi_i](1-w) \tag{23}$$

$$\text{s.t. } p(w)w+\big(1-p(w)\big)u_{i-1}=u_i. \tag{24}$$

To generalize Proposition 3.3 to any N it is sufficient to show that type $i\geqslant 2$ firms make strictly higher profits than firms of type $i-1$ unless $w_i^*=\bar{w}_{i-1}$. Using the constraint (24) to substitute for the wage in (23) and taking the first order conditions with respect to the queue length, the profits of a type i firm are given by

$$\pi(\hat{w}_i)=\big(1-e^{-\lambda_i^*}\big)(1-u_{i-1})\left(1-\frac{\lambda_i^* e^{-\lambda_i^*}}{1-e^{-\lambda_i^*}}\right)(1-\psi_i). \tag{25}$$

The profit of a type $i-1$ firm is given by

$$\pi(w_{i-1})=\big(1-e^{-\lambda_{i-1}}\big)(1-w_{i-1})\left(1-\frac{1-e^{-\lambda_i^*}}{\lambda_i^*}\right)(1-\psi_i), \tag{26}$$

and it is strictly lower than $\pi(\hat{w}_i)$ for the same reasons as in Section 3.

Proposition 4.2. *In equilibrium, all type i firms post the same wage $w_i^*=\bar{w}_{i-1}$ for $i\geqslant 2$. All type 1 firms post $\hat{w}_1$ which it is determined by the first order conditions.*

Proof. See above. □

It is straightforward to show that the density of posted wages is falling.

Proposition 4.3. *The distribution of posted wages is decreasing, i.e., $d_i>d_{i+1}$ for all $i\in\{1,\dots,N-1\}$.*

Proof. We established that $\lambda(w_i^*)=b/d_i$, which is strictly increasing by Lemma 2.1. □

To establish existence of an equilibrium, we show that there is a sequence $\{d_1,d_2,\dots,d_N\}$ such that $d_1+d_2+\cdots+d_N=1$ and there is no profitable deviation when wage w_i^* is posted by exactly d_i firms, where $w_1^*=\hat{w}_1$ and $w_i^*=\bar{w}_{i-1}$ for $i\geqslant 2$.

Proposition 4.4. *An equilibrium exists for any N.*

Proof. See Appendix A. □

Showing that the equilibrium is unique has proved elusive. In Galenianos and Kircher [10] we reduce the uniqueness problem to the analysis of a one-dimensional function and provide numerical evidence that the equilibrium is unique for a wide range of parameter values. Furthermore, we have not encountered any multiplicities in any parameter regions. The difficulties in proving uniqueness arises because the equilibrium wages are determined by the endogenous constraints imposed by lower wage firms rather than the first order conditions of the profit function. This complicates the task of finding the equilibrium measures of firms across types since we cannot rely on envelop-type arguments. For instance, increasing the measure of low wage firms has ambiguous effects on the profits of high wage firms because it affects the constraints under which they are operating and these are a complicated function of the changes at the lower wages. These difficulties are manageable when $N=2$ but the lack of tractability of the general case does not allow us to prove uniqueness in general.

4.3. Efficiency

The efficiency results of Section 3.3 generalize to any N.

Proposition 4.5. *Constrained efficiency does not obtain in the general N case.*

Proof. See Appendix A. □

5. The distribution of wages

In this section we investigate some properties of the distribution of wages. We examine the shape of the empirical distribution and how its tail varies with labor market tightness. We also contrast our results with those of random search models.

5.1. The empirical distribution

A well-known stylized fact of the labor market is that the empirical density of wages is decreasing. So far we have shown that the density of posted wages is decreasing. We prove that the density of received wages is decreasing everywhere so long as the worker-firm ratio is not too small. This is not an immediate result because higher wages are accepted more often.

Proposition 5.1. *The distribution of received wages is decreasing when the ratio of workers to firms is large enough.*

Proof. The measure of workers who are employed at wage w_i^* is given by $b(1-\psi_{i+1})p_i \equiv E_i$. Moreover, $E_{i-1} = b(1-\psi_i)p_{i-1} = b(1-\psi_{i+1})(1-p_i)p_{i-1}$. The density to be declining, i.e. $E_i < E_{i-1}$, if and only if $p_i < (1-p_i)p_{i-1}$ for all i. Equal profits imply that $q(w_i^*)(1-w_i^*) = q(w_{i-1}^*)(1-p_i)(1-w_{i-1}^*)$ or $p_i\lambda_i(1-w_i^*) = p_{i-1}\lambda_{i-1}(1-p_i)(1-w_{i-1}^*)$ yielding the condition $\lambda_i(1-w_i^*) > \lambda_{i-1}(1-w_{i-1}^*)$. Using the equilibrium conditions $w_i^* = (u_{i-1}-u_{i-2})/p_i + u_{i-1}$ and $w_{i-1}^* = (u_{i-1}-u_{i-2})/p_{i-1} + u_{i-1}$ the inequality becomes $\lambda_i(1-x/p_i) > \lambda_{i-1}(1-x/p_{i-1})$ where $x \equiv (u_{i-1}-u_{i-2})/(1-u_{i-2})$. Therefore, the empirical distribution is decreasing if $g(\lambda) \equiv \lambda(1-\lambda x/(1-e^{-\lambda}))$ is increasing with respect to the queue length. The first derivative yields $\partial g/\partial\lambda = (1-e^{-\lambda})(1-e^{-\lambda}-2\lambda x) + \lambda^2 x e^{-\lambda}$ which is positive if λx is small. Noting that $\lambda_i x = (1-e^{-\lambda_i})(w_i^* - u_{i-2})/(1-u_{i-2})$ and that the right-hand side goes to zero for b large enough establishes the result. □

A further observation of empirical interest is that the tail of the received wage distribution is thicker when b is lower. This is an immediate implication of the previous result and the following observation that the density in the tail of the received wage distribution is increasing when b is small. Since small b (few workers relative to firms) implies that the unemployment rate at the end of the interaction is low, this also implies that the thickness of the tail decreases in the unemployment level.

Proposition 5.2. *The tail of the distribution of received wages is increasing when the ratio of workers to firms is sufficiently small.*

Proof. Recalling the notation from the previous proof, increasing distribution in the tail means $E_N > E_{N-1}$ or $p_N > (1 - p_N)p_{N-1}$. When b is small $p_i > (1 - p_i)p_{i-1}$ has to hold for some i, as it is implied by p_i larger $1/2$. We use this as the induction anchor. If this implies $p_{i+1} > (1 - p_{i+1})p_i$, the result holds by induction. The proof proceeds by contradiction. Assume $p_{i+1} \leqslant (1 - p_{i+1})p_i$. Dividing each side of the induction anchor by the respective side of this inequality yields after rearranging $p_i > p_{i-1}(1 - p_i)/p_i/(1 - p_{i+1})/p_{i+1}$. We know $p_i < p_{i-1}$, and $p_{i+1} < p_i$ implies $(1 - p_i)/p_i < (1 - p_{i+1})/p_{i+1}$, which yield the desired contradiction. □

5.2. *The wage density under directed vs. random search*

We now compare the predictions of our model concerning the shape of the wage density with those of random search models. As already noted, our model predicts a decreasing density of posted and (under certain parameter restrictions) received wages, which is in accordance with data (see Mortensen [19]). Models of random search typically predict that the density of wages is increasing. We argue that it is precisely the directedness of the search process that leads to the desirable results about the shape of the wage distribution and we use a random search version of our model to illustrate this point.

Consider a version of our model where search is random rather than directed. Firms post wages, but they cannot communicate them to workers. A worker sends N applications at random to as many firms. Each firm chooses one of its applicants at random to make a job offer at the posted wage, and workers with multiple offers accept the most desirable job. In our terminology, there is ex post but not ex ante competition among firms. This environment is examined in Gautier and Moraga-González [12] and it leads to wage dispersion, since posting a higher wage results in hiring a worker who has additional offers with greater probability.

Denote the probability that a worker gets an offer by $\bar{p}$, and the probability that a firm has at least one applicant by $\bar{q}$. Note that since the arrival rate of workers is independent of the posted wage, these outcomes do not depend on the wage but only on b. Let F be the distribution of posted wages and first consider the $N = 2$ case. The probability that a firm hires a worker is given by $\bar{q}[1 - \bar{p} + \bar{p}F(w)]$. The first term is the probability that at least one worker applies, while the second term is the probability that this worker has no other offer *or* his other offer is for a lower wage. Equal profits imply that the following condition has to hold for all $w \in \operatorname{supp} F$:

$$\begin{aligned} &\bar{q}\big(1 - \bar{p} + \bar{p}F(w)\big)(1 - w) = \Pi \\ &\Leftrightarrow \quad \bar{q}\big(1 - \bar{p} + \bar{p}F(w)\big) = \frac{\Pi}{1 - w}, \end{aligned}$$

where $1 - w$ is the margin of the firm and Π denotes the equilibrium level of profits. Note that $\Pi/(1 - w)$ is a strictly convex function of w and that $F(w)$ is the only non-constant on the left-hand side of the equation above. As a result, equal profits imply that the distribution of posted wages has to be strictly convex.

The intuition behind this result is that when moving upwards on the support of F, the *percentage* decrease in the margin becomes larger at an increasing rate. For profits to remain constant, this requires an equivalent increase in the probability of hiring. In a random search environment, a higher wage firm increases its hiring probability only by getting more workers who might have different offers, i.e. only the ex post competition margin improves. This means that, when moving to the top of the distribution, a firm needs to 'overtake' an increasing number of competing firms or, in other words, the distribution of wages needs to be convex. The same reasoning holds

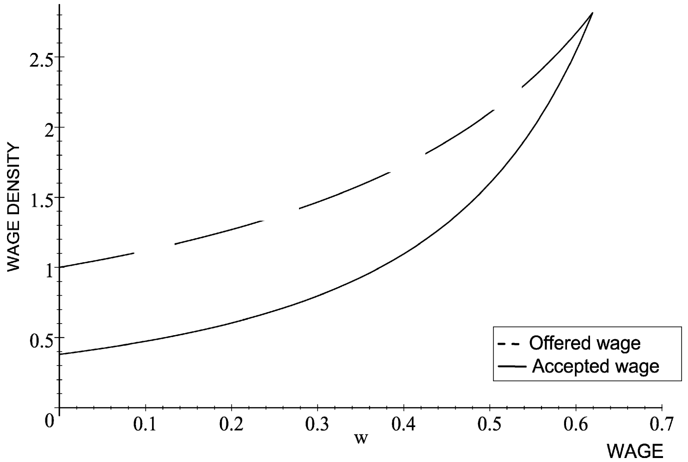

Fig. 4. Wage densities with random search, for $N = 15$ and $b = 1$.

for arbitrary N, in which case the profits are given by $\bar{q}(1-\bar{p}-\bar{p}F(w))^{N-1}(1-w)$ (see Gautier and Moraga-González [12]).

Fig. 4 shows the wage density under random search for $N = 15$ and equal number of workers and firms, which allows a comparison with Fig. 3 for directed search.

In a directed search environment a decreasing wage profile is possible due to the presence of ex ante competition: the ability of firms to attract more applicants by posting a higher wage gives an additional channel through which they increase the probability of hiring. Therefore, high wage firms need not 'overtake' as many of their competitors to guarantee equal profits.

6. Conclusions

We develop a directed search model where workers apply simultaneously for N jobs. We find that all equilibria exhibit wage dispersion, with firms posting N different wages and workers sending one application to each distinct wage. The dispersion is driven by the portfolio choice that workers face, and integrating this problem in an equilibrium framework is our main theoretical contribution. The matching process is a source of inefficiency because higher paying firms fill their vacancies too often. This model delivers some potentially testable predictions. In line with stylized facts, the density of posted and, for suitable parameter values, received wages is decreasing, a result which is due to the directedness of the search process. Firms that post high wages receive more applications per vacancy than lower wage firms. A firm's job offer is not necessarily accepted, but higher wage offers are accepted more often. While wage dispersion has been repeatedly examined in the literature, the last two implications have not received much attention.

As noted in the introduction, Albrecht, Gautier and Vroman [3] develop a directed search model where workers simultaneously apply for multiple jobs in an environment where firms cannot commit to their wage offers. They characterize the unique equilibrium where a single wage is posted and show that it is equal to the workers' reservation value. Furthermore, they show that the entry of firms is excessive from an efficiency viewpoint. We see our paper as complementary to theirs for a number of reasons. First, one might conjecture that their inefficiency result is due to the inability to price workers' applications appropriately under lack of commitment. The analysis of our alternative formulation shows that commitment alone is not the reason for the inefficiencies; rather they stem from the fact that a firm's wage only affects the applications it receives but not where workers additionally apply. Second, our formulation leads to wage dispersion with the desirable qualitative features described above and, additionally, the extent of dispersion depends on the number of applications that workers send in a non-trivial way. Finally, in terms of modeling assumptions, commitment to posted wages is based on the presumption that, in certain environments, a firm's offer to a worker are non-verifiable by third parties which reduces the incentives to compete against other (potentially fictitious) offers.

Our model is easy to extend in a number of ways. The main insights developed above carry over when we allow for free entry, for endogenous decisions concerning the number of applications and for a dynamic labor market interaction, as we describe in the working paper version, Galenianos and Kircher [10]. Other potentially interesting extensions, which we leave for future research, include allowing firms to post more general mechanisms, such as lotteries over the wage, and introducing heterogeneity and risk aversion. The results on the separation of applications do not hinge on firm homogeneity and hence they extend to the case of productivity differentials among firms. The firms' optimization problem will be different, of course, and we conjecture that more productive firms will post higher wages since they have a higher opportunity cost of remaining idle. Moderate risk aversion of workers can be easily accommodated in our framework by replacing w with a concave function $\nu(w)$ when specifying the worker's utility, leaving the worker's problem virtually unchanged and affecting the firms only by slightly modifying the equations that determine the queue lengths.

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Appendix A

Proof of Proposition 3.6. The planner solves the following problem: $\max_{d\in[0,1]} m(d) = p_1 + p_2 - p_1 p_2$. The technical difficulty arises because this problem is for some parameters neither globally convex nor concave. We will establish that there is at most one interior maximum, and it arises at $d = 1/2$. If the problem has an interior solution, the first order conditions yield

$$\frac{\partial p_2}{\partial d_1}(1-p_1) + \frac{\partial p_1}{\partial d_1}(1-p_2) = 0. \tag{A.1}$$

Recalling that $\lambda_1 = b/(1-d)$ and $\lambda_2 = b/d$ it is easy to see that Eq. (A.1) can be rewritten as $(1 - e^{-\lambda_2} - \lambda_2 e^{-\lambda_2})(1 - \frac{1-e^{-\lambda_1}}{\lambda_1}) = (1 - e^{-\lambda_1} - \lambda_1 e^{-\lambda_1})(1 - \frac{1-e^{-\lambda_2}}{\lambda_2})$ because $\partial p_i/\partial d = -\partial\lambda_i/\partial d(1 - e^{\lambda_i} - \lambda_i e^{-\lambda_i})/\lambda_i^2$, $\partial\lambda_1/\partial d = b/(1-d)^2 = \lambda_1^2/b$, and $\partial\lambda_2/\partial d = -b/d^2 = -\lambda_2^2/b$. It is immediate that one extremum occurs when $\lambda_1 = \lambda_2$, or $d = 1/2$. The second derivative is given by

$$\begin{aligned}\frac{\partial^2 m}{\partial d^2} &= \frac{1}{b^2}\big(1 - e^{-\lambda_2} - \lambda_2 e^{-\lambda_2}\big)\big(1 - e^{-\lambda_1} - \lambda_1 e^{-\lambda_1}\big) - \frac{1}{b^2}\lambda_2^3 e^{-\lambda_2}(1-p_1) \\ &\quad + \frac{1}{b^2}\big(1 - e^{-\lambda_2} - \lambda_2 e^{-\lambda_2}\big)\big(1 - e^{-\lambda_1} - \lambda_1 e^{-\lambda_1}\big) - \frac{1}{b^2}\lambda_1^3 e^{-\lambda_1}(1-p_2).\end{aligned} \tag{A.2}$$

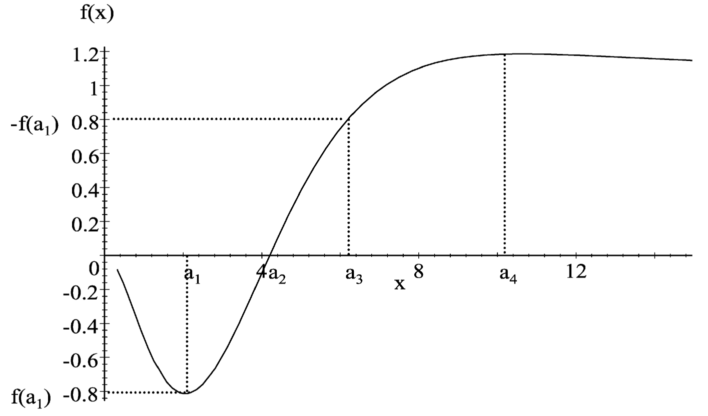

Fig. 5. $f(x)$ for $x \geqslant 0$.

Using the relations derived above and dividing by $(1-p_1)(1-p_2)/b^2$ establishes that the sign of the second derivative is given by $\operatorname{sign}(\partial^2 m/\partial d^2) = \operatorname{sign}(f(\lambda_2)+f(\lambda_1))$ at all candidate extreme points, where

$$f(\lambda) = \frac{(1-e^{-\lambda}-\lambda e^{-\lambda})^2}{(1-(1-e^{\lambda})/\lambda)^2} - \frac{\lambda^3 e^{-\lambda}}{1-(1-e^{\lambda})/\lambda}. \tag{A.3}$$

Therefore, we want to show that there is no $b>0$ such that there exists $d \in (1/2, 1)$ where (A.1) holds and

$$f\left(\frac{b}{d}\right) + f\left(\frac{b}{1-d}\right) \leqslant 0. \tag{A.4}$$

Fig. 5 shows $f(\lambda)$ for $\lambda \geqslant 0$. The function is strictly decreasing on $(0, a_1)$, strictly increasing on (a_1, a_4), again strictly decreasing on (a_4, ∞) and converges to 1 for $\lambda \to \infty$. The only roots of the function are 0 and a_2. We will discuss this function in order to establish the result. Note that for any b, the specific value of d defines $\lambda_1 = b/d$ and $\lambda_2 = b/(1-d)$. Note that for $\lambda_2 > a_3$ it is not possible to fulfill (A.4), where a_3 is such that $f(a_3) = -f(a_1)$. Therefore we will restrict the discussion to $\lambda_2 < a_3$. This also implies that we do not have to discuss any b where $2b > a_3$. For $d = 1/2$ we know that $\lambda_1 = \lambda_2$, and therefore the first order condition holds and $\operatorname{sign}(\partial^2 m/\partial d^2) = \operatorname{sign} f(2b)$.

Case 1. $b \geqslant a_2/2$. Then at $d = 1/2$ we have $2f(2b) \geqslant 0$. Starting from $d = 1/2$, i.e. $\lambda_1 = \lambda_2$, we will increase d and thus spread λ_1 and λ_2 apart. We will show that there does not exist $d > 1/2$ such that (A.4) holds. Assume that (A.4) holds for the given b at some $d > 1/2$. Then for any $b' \in [a_2/2, b)$ there exists a $d' > 1/2$ such that (A.4) holds. This is easy to see if there exists $d' > 1/2$ such that $\lambda_1 = b/d = b'/d' = \lambda_1'$. Then $f(\lambda_1) = f(\lambda_1')$. Since $\lambda_2 = b/(1-d) > b'/(1-d') = \lambda_2'$, $f(\lambda_2) > f(\lambda_2')$. But then $f(\lambda_1) + f(\lambda_2) \leqslant 0$ implies $f(\lambda_1') + f(\lambda_2') < 0$. If for some $b' \in [a_2/2, b)$ no such $d' > 1/2$ exists, we reach a contradiction: There is some $b'' \in [b', b)$ such that at $d'' = 1/2$ it holds that $\lambda_1 = b/d = b''/d'' = \lambda_1'$. By the prior argument $f(\lambda_1'') + f(\lambda_2'') < 0$, but this violates $2f(2b) = f(\lambda_1'') + f(\lambda_2'') \geqslant 0$. Therefore, if we know that (A.4) does not hold at $\tilde{b} = a_2/2$, then we know that (A.4) does not hold for any $b > a_2/2$. Fig. 6 shows $f(a_2/2d) + f(a_2/(2(1-d)))$ for all $d \geqslant 1/2$, which is strictly positive for all $d > 1/2$. Therefore, (A.4) does not hold for any $b \geqslant a_2/2$.

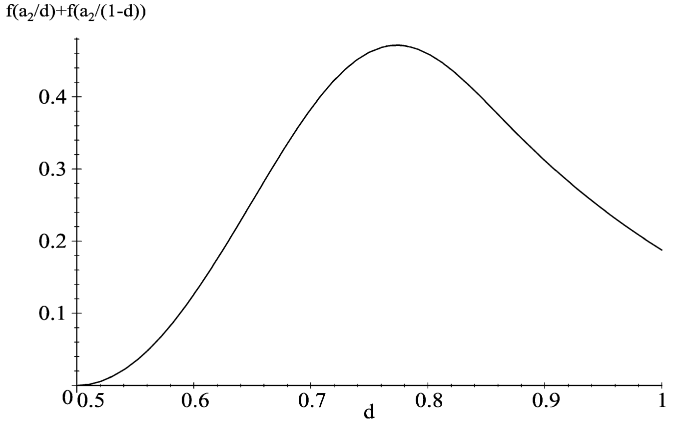

Fig. 6. $f(\frac{a_2}{2d}) + f(\frac{a_2}{2(1-d)})$ for $d \in [0, 1]$.

Case 2. $b < a_2/2$. In this case we have at $d = 1/2$ that $2f(2b) < 0$, i.e. we are in a local maximum. If there exist any other local maxima at $d > 1/2$, there has to be some $d' \in (1/2, d)$ that constitutes a local minimum. Therefore, if for some d conditions (A.4) and (A.1) hold simultaneously, then there exists $1/2 < d' < d$ such that $f(b/d') + f(b/1 - d') > 0$. At d' it has to hold $\lambda_2' = b/(1 - d') > a_2$, otherwise $f(\lambda_1') + f(\lambda_2') > 0$ would not be possible. We also know that $\lambda_1' < b/2 < a_2$. Since $d' < d$, we know that $\lambda_1 < \lambda_1'$ and $\lambda_2' < \lambda_2$. Now consider a d' at which $f(\lambda_1') + f(\lambda_2') > 0$. If we increase d to values above d', the derivative of $f(\lambda_1) + f(\lambda_2)$ is

$$\frac{\partial(f(\lambda_1) + f(\lambda_2))}{\partial d} = f'(\lambda_1)\frac{\partial \lambda_1}{\partial d} + f'(\lambda_2)\frac{\partial \lambda_2}{\partial d} \tag{A.5}$$

$$= \frac{1}{b}\bigl[-f'(\lambda_1)\lambda_1^2 + f'(\lambda_2)\lambda_2^2\bigr]. \tag{A.6}$$

If the term in square brackets is positive, then $f(\lambda_1) + f(\lambda_2)$ is increasing as we increase d further. So if we can show that the part in the square brackets is positive for all $(\lambda_1, \lambda_2) \in [0, a_2] \times [a_2, a_3]$, then it is not possible to increase d starting from any d' and achieve a negative value of $f(\lambda_1) + f(\lambda_2)$ (which we would need to arrive at another maximum). Since $\max_{[0,a_2]} f'(\lambda)\lambda^2 \leqslant \min_{[a_2,a_3]} f'(\lambda)\lambda^2$, as can be seen in Fig. 7, it is not possible to have another local maximum in the interior apart from $d = 1/2$. □

Proof of Proposition 4.4. An argument similar to Proposition 3.5 shows that w_i^* is profit maximizing when w_i^* is posted by all type i firms. Define $\tilde{\pi}_i \equiv \pi_i/(1 - \psi_i)$ and $\Delta\pi_i(d_i \mid d_{i-1}) \equiv \tilde{\pi}_{i-1} - \tilde{\pi}_i/(1 - p_i)$. For equal profits across types it is sufficient to show that $\pi_i = \pi_{i-1}$ for all i, which is the same as $\Delta\pi_i(d_i \mid d_{i-1}) = 0$ since the term $(1 - \psi_i)$ is common to both sides.

By the definition of u_{i-1} we know that $u_{i-1} = p_{i-1}w_{i-1}^* + (1 - p_{i-1})u_{i-2}$. The fact that $w_i^* = \bar{w}_{i-1}$ implies $u_{i-1} = p_i w_i^* + (1 - p_i)u_{i-2}$. Using these two equations one can show that $\Delta\pi_i(d_{i-1}; d_{i-1}) < 0$ and $\Delta\pi_i(0; d_{i-1}) > 0$ for the same reasons as in Proposition 3.5. Furthermore, note that $u_i - u_{i-1} = (1 - p_i)(u_{i-1} - u_{i-2})$. Therefore there is a $d_i(d_{i-1})$ such that type i and $i - 1$ firms make the same profits. Moreover, the solution $d_i(d_{i-1})$ is unique if

$$\frac{\partial \Delta\pi_i}{\partial d_i} = -\tilde{\pi}_i \frac{\partial(1/(1 - p_i))}{\partial d_i} - \frac{1}{1 - p_i}\frac{\partial \tilde{\pi}_i}{\partial d_i} < 0. \tag{A.7}$$

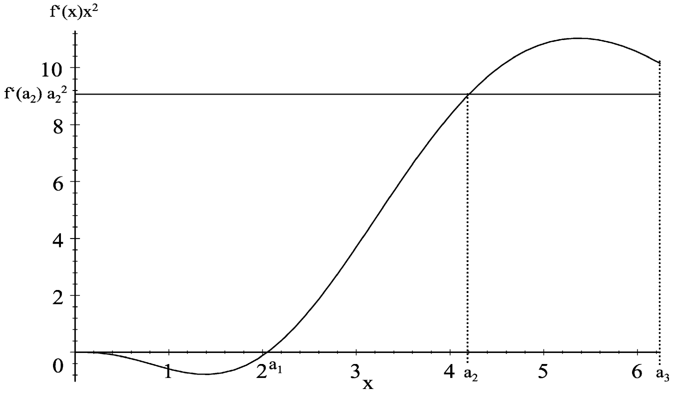

Fig. 7. $f'(x)x^2$ for $x \in [0, a_2]$.

To show this inequality holds note that the first term is strictly negative because a higher d_i leads to lower queue length and hence a higher probability of receiving a job offer.

To sign the second term, rewrite the profits of a type i firm as $\pi_i = (1-\psi_i)[(1-e^{-\lambda_i})(1-u_{i-2}) - \lambda_i(u_{i-1}-u_{i-2})]$ using Proposition 4.2 and Eq. (23). Differentiating that expression with respect to d_i yields

$$\frac{\partial \pi_i}{\partial d_i} = (1-\psi_i)\frac{\partial \lambda_i}{\partial d_i}\big[e^{-\lambda_i}(1-u_{i-2}) - (u_{i-1}-u_{i-2})\big]. \tag{A.8}$$

Using the first order conditions of an individual firm with respect to its queue length yields $(1-\psi_i)[e^{-\lambda_i}(1-u_{i-1}) - (u_i - u_{i-1})] < 0$, where the sign results from Proposition 4.2. Putting these two expressions together yields

$$e^{-\lambda_i}(1-u_{i-2}) - (u_{i-1}-u_{i-2}) < e^{-\lambda_i}(1-u_{i-1}) - (u_i - u_{i-1}) < 0 \tag{A.9}$$

where the first inequality results from noting that $u_i - u_{i-1} = (1-p_i)(u_{i-1}-u_{i-2})$. Then (A.8) yields $\partial\pi_i/\partial d_i > 0$ since $\partial\lambda_i/\partial d_i < 0$, which proves the inequality in Eq. (A.7).

Since $d_i(d_{i-1})$ is unique, for a given d_1 the sequence $d_2(d_1), d_3(d_2(d_1)), \ldots$ can be uniquely constructed such that all types of firms make the same profits. To find the sequence whose elements sum up to one define $S(d_1) \equiv \sum_{i=1}^N d_i(d_1)$ and note that it is continuous since all of its components vary continuously with d_1. Moreover, $S(1/N) < 1$ since $d_i(d_{i-1}) < d_{i-1}$ and $S(1) > 1$ so there is some d_1^* such that $S(d_1^*) = 1$ and an equilibrium exists for any N. □

Proof of Proposition 4.5. Let $\mathbf{d} = \{d_1, d_2, \ldots, d_N\}$ be the vector of the equilibrium fractions of firms and let $p_i = p(w_i)$ and $m(\mathbf{d}) \equiv 1 - \prod_{i=1}^N (1-p_i)$. Given any two groups of firms, k and l, observe that $m(\mathbf{d}) = 1 - (1-p_k)(1-p_l)\prod_{i\neq k,l}(1-p_i)$ which means that an equilibrium is constrained efficient only if d_k and d_l minimize $(1-p_k)(1-p_l)$. This is equivalent to $\max_{d_k,d_l \geqslant 0}(p_k + p_l - p_k p_l)$ subject to $d_k + d_l = 1 - \sum_{i\neq k,l} d_i$ which is identical to the two application case when the measure of firms is equal to $1 - \sum_{i\neq k,l} d_i$. Applying the results of Proposition 3.6 completes the proof. □

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