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AN EQUILIBRIUM THEORY OF LEARNING, SEARCH, AND WAGES

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We examine the labor market effects of incomplete information about the workers' own job-finding process. Search outcomes convey valuable information, and learning from search generates endogenous heterogeneity in workers' beliefs about their job-finding probability. We characterize this process and analyze its interactions with job creation and wage determination. Our theory sheds new light on how unemployment can affect workers' labor market outcomes and wage determination, providing a rational explanation for discouragement as the consequence of negative search outcomes. In particular, longer unemployment durations are likely to be followed by lower reemployment wages because a worker's beliefs about his job-finding process deteriorate with unemployment duration. Moreover, our analysis provides a set of useful results on dynamic programming with optimal learning.

KEYWORDS: Learning, wages, unemployment, directed search, monotone comparative statics.

1. INTRODUCTION

WHEN WORKERS HAVE INCOMPLETE INFORMATION about their own job-finding process, search outcomes convey valuable information. Differences in search outcomes that may initially be caused by luck can induce different updating of workers' beliefs about their own job-finding process, which will influence workers' search behavior in the future and lead to further differences in their reemployment rates and wages. In this paper, we develop an equilibrium framework to characterize this endogenous heterogeneity generated by learning from search, and we analyze its interactions with job creation and wage determination.

Our theory sheds new light on how unemployment can affect workers' labor market outcomes and wage determination. As a particular illustration, our theory provides a novel explanation for why longer unemployment durations are likely to be followed by lower reemployment rates and wages (see Addison and Portugal (1989)). It thus complements common human-capital explanations, which emphasize that workers' skills depreciate during unemployment

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(Pissarides (1992)) or that unemployment durations may signal differences in labor productivity (Lockwood (1991)). These explanations alone are unlikely to explain the effects of unemployment on workers' labor market outcomes. For instance, Addison and Portugal (1989) found that reemployment wages and rates fall significantly over short unemployment durations, and they do so for low-skilled as well as high-skilled workers, even after trying to control for observed and unobserved heterogeneity.[2] Our broader view of human capital emphasizes a distinction between a worker's matching ability and labor productivity, and a distinction between exogenous and endogenous heterogeneity. These distinctions can be useful for devising new empirical strategies to discriminate between duration dependence in workers' search behavior and the effect of uncontrolled worker heterogeneity (Heckman and Borjas (1980)).

One contribution of our paper is to integrate search and learning into an equilibrium framework. The need for an equilibrium framework arises because when workers change their search behavior as a result of learning, firms have an incentive to adjust vacancies and wage offers to respond to these changes. Thus, learning affects the wage distribution. In turn, the availability of vacancies and the wage distribution can affect workers' search behavior and, hence, the information contained in a worker's search outcomes. The equilibrium interactions between workers' search, firms' vacancy creation, and the wage distribution are important for understanding the tensions between aggregate and individual behavior, as reflected, for instance, in the relationship between wages and the duration of vacancies as well as unemployment. Indeed, our analysis uses the properties of the equilibrium wage function to establish a central result that a worker's desired wages are a strictly increasing function of the worker's beliefs.

In our model, a worker's ability is either high or low permanently. A high ability implies that the worker has a higher probability of forming a productive match with a random job. A worker has incomplete information about his ability and, hence, does not precisely know his matching probability. We model search as a directed process as in Moen (1997) and Acemoglu and Shimer (1999). That is, workers know the wage offers before choosing where to apply.[3] Directed search allows for sorting of the workers into jobs, which makes an equilibrium block recursive in the sense that individuals' decisions and market tightness are independent of the distribution of workers. Block recursivity

[2]Changes in wealth and search intensity during unemployment can also play a role. However, even after trying to control for wealth effects and search intensity, Alexopoulos and Gladden (2007) found that unemployment duration still has strong negative effects on a worker's labor market outcomes.

[3]See Peters (1984, 1991), Burdett, Shi, and Wright (2001), and Shi (2001) for analyses of directed search as a strategic problem that leads to the competitive search equilibrium outcome as the market becomes large.

allows for a tractable analysis of the equilibrium interactions between equilibrium wages and learning.[4]

Success and failure to find a match both convey useful information about a worker's type. Success in getting a match is good news about a worker's ability. Failure is bad news, which induces a worker to search for jobs that are easier to get. Those jobs come with lower wages as part of the equilibrium trade-off between wages and market tightness. Thus, learning from search induces not only reservation wages, but also desired wages, to increase with beliefs. Firms offer different wages to cater to these workers, who sort according to beliefs, resulting in a nondegenerate distribution of equilibrium wages among ex post equally productive workers.

Endogenous heterogeneity in workers' beliefs provides a rational explanation for discouragement as the consequence of negative search outcomes. This is a natural explanation for the negative effect of unemployment duration on future wages found by Addison and Portugal (1989). As a worker becomes pessimistic, he searches for lower wages so as to raise his job-finding probability. The flip side of this result is that vacancies that offer high wages to target optimistic workers are filled more rapidly than low-wage vacancies, consistent with the evidence in Barron, Bishop, and Dunkelberg (1985) and Holzer, Katz, and Krueger (1991). Moreover, despite workers' attempt to search for lower wages as the unemployment spell continues, the average job-finding probability can fall with unemployment duration, as the evidence indicates (e.g., Shimer (2008)). This is because the ability composition of workers in any given cohort worsens with unemployment duration.

Our analysis provides a sharp characterization of learning from search, resolving a number of problems inherent to the analysis of optimal learning from experience. These problems arise because, as search outcomes generate variations in a worker's posterior beliefs about his ability, these variations are valuable to the worker only if the worker's value function is convex in beliefs. Because such convexity can make optimal decisions not unique and the value function not differentiable, standard techniques in dynamic programming (Stokey, Lucas, and Prescott (1989)) cannot be used to study the policy function which is the key object in our analysis.[5] We resolve this difficulty by exploiting a connection between convexity of the value function and standard monotone comparative statics results (e.g., Topkis (1998) and Milgrom and Shannon (1994)). The connection is not immediately obvious and, to our knowledge, has not been examined.

[4]Shi (2009) first formalized this notion of block recursive equilibria and proved existence of such equilibria in the context of on-the-job search where firms offer wage–tenure contracts to direct workers' search. Menzio and Shi (2009) established existence of block recursive equilibria in a dynamic, stochastic environment with on-the-job search.

[5]Although the literature on optimal learning (e.g., Easley and Kiefer (1988)) recognizes the analytical difficulty caused by a convex value function, it has either ignored the difficulty or focused on corner solutions (e.g., Balvers and Cosimano (1993)).

Because a worker's decision problem is formulated with dynamic programming, the objective function involves the future value function, which is endogenous. Moreover, we cannot presume properties of the objective function such as concavity, in contrast with other applications of lattice-theoretic techniques to dynamic programming (e.g., Amir, Mirman, and Perkins (1991), Mirman, Morand, and Reffett (2008)). In the end, we establish a set of useful results in dynamic programming with optimal learning. First, convexity of the value function and monotonicity of the policy function are closely related. Second, under a mild condition, the value function is strictly convex and the policy function is strictly monotone. Third, under the same condition, optimal decisions obey the first-order condition and a general version of the envelope theorem is valid. Finally, optimal decisions are unique if the worker's search history has ever contained a match failure.

Our emphasis on learning from search is close in spirit to that of Burdett and Vishwanath (1988). They considered the case in which workers learn about the unknown distribution of wages from the random arrival of wage offers and showed that learning from search can induce reservation wages to decline with unemployment duration. In contrast, we analyze workers' learning about their ability, study an environment where wages and vacancies are endogenously determined, and focus on desired wages rather than reservation wages.

2. THE MODEL ENVIRONMENT

Time is discrete. All agents are risk neutral and discount the future at a rate $r > 0$. There is a unit measure of workers, divided between employment and unemployment. The measure of firms is determined endogenously by free entry. An employed worker produces, after which a separation shock makes him unemployed with probability $\delta > 0$. An unemployed worker searches for a job and receives the unemployment benefit per period, $b \geq 0$.

Each worker has unknown ability i that is either high (H) or low (L). Ability is a worker's permanent characteristic, determined at the time when the worker first enters the market. A new worker has ability i with probability p_i, where $p_H = p \in (0, 1)$ and $p_L = 1 - p$. Ability determines a worker's productivity as follows.[6] Upon meeting a randomly drawn firm, the productivity of a worker with ability i is realized to be $y > 0$ with probability a_i, and $y' \leq 0$ with probability $(1 - a_i)$, while the cost of production is normalized to 0. We refer to a_i as a type-i worker's *productive units* and we assume $0 < a_L < a_H < 1$ so that a high-ability worker is more likely to be productive than a low-ability worker. Clearly, a firm hires a worker only when the worker is productive, and labor productivity of every employed worker is $y > 0$.

[6]We are very grateful to Daron Acemoglu and the referees for directing us toward this formulation. In a previous version of the paper (Gonzalez and Shi (2007)), we formulated the problem as one of incomplete information about the characteristics of local labor markets rather than individuals.

A natural interpretation of a worker's ability in our model is in terms of the worker's skill bundle, in the spirit of recent literature on human capital (see Lazear (2004)). According to this view, workers are heterogeneous with respect to the specific composition of their skill bundle, and different firms demand different skill bundles. A firm must review a worker's application to determine whether a worker's skill bundle fits the firm. However, to focus on workers' learning about their human capital, we abstract from the actual formation of heterogeneous matches by assuming, as above, that the skill bundle of high-ability workers is relatively more likely to fit a random firm.

A worker learns about his ability from his labor market experience. After an infinitely long history in the market, a worker eventually learns his true ability. To rule out this uninteresting case, we assume that, with probability $\sigma > 0$, an exit shock forces a worker (employed or unemployed) out of the market at the end of each period. An exiting worker's payoff is normalized to zero, and the worker is replaced with a new worker who enters the market through unemployment so that the labor force remains constant.

The events in a period unfold as follows. First, new workers enter the market through unemployment, replacing the workers who exited the market in the previous period. Nature determines a new worker's ability. Second, an employed worker produces and gets the wage, after which the job separation shock is realized. Meanwhile, unemployed workers search for jobs and new matches are formed. Finally, the exit shock is realized.

There is a continuum of *submarkets* indexed by x that are linked to matching rates in that submarket. The domain of x is $X = [0, 1/a_H]$. A submarket x is characterized by a wage level, $W(x)$, and a tightness, $\lambda(x)$. The functions $W(\cdot)$ and $\lambda(\cdot)$ are public information, taken as given by agents and determined in equilibrium. A worker's or a firm's search decision in each period is to choose x, that is, the submarket to search.[7] Search is directed in the sense that an agent explicitly takes into account the trade-off that a submarket with a high wage has relatively fewer vacancies per worker in the equilibrium. As in Moen (1997) and Acemoglu and Shimer (1999), a firm does not directly set wages; rather, it chooses a pair (W, λ) from the menu $\{(W(x), \lambda(x)) : x \in X\}$.[8]

In each submarket, the number of matches is given by a matching function. Since a firm hires a worker only when the worker is productive at the job, it is useful to specify the matching function to determine the number of productive matches rather than the number of contacts. Let $v(x)$ denote the number of

[7]Workers who differ in beliefs may also choose different levels of search intensity and labor market participation. Although our analysis can shed light on such differences, we abstract from them for simplicity.

[8]It is inadequate to index the submarkets by the length of unemployment duration of the participating workers. First, workers with the same unemployment duration can be heterogeneous in beliefs about their ability if they have different employment histories. Indexing submarkets by unemployment duration alone does not allow these workers to optimally make different search choices.

vacancies created in submarket x, and let $u_i(x)$ denote the number of type-i unemployed workers in submarket x, where $i \in \{H, L\}$. We define the total productive units of workers searching in submarket x as

$$u_e(x) = a_H u_H(x) + a_L u_L(x). \tag{2.1}$$

A function $F(u_e(x), v(x))$ gives the number of productive matches in the submarket. The index x is the matching rate for each productive unit in submarket x; that is,

$$x = \frac{F(u_e(x), v(x))}{u_e(x)}.$$

For a type-i worker in submarket x, the probability of getting a productive match is $a_i x$. Thus, given x, the lower a worker's ability, the lower his matching probability. The matching probability of a vacancy in submarket x is $F/v = x/\lambda(x)$, where $\lambda(x) \equiv v(x)/u_e(x)$ is the effective tightness in the submarket.

The above specification of the matching function uses workers' productive units as an argument, which are similar to the efficiency units of search commonly used in the literature where workers are heterogeneous. This specification enables us to focus on productive matches by combining the process of making contacts (i.e., receiving applications) and the process of evaluating the applicants. The only relevant information for worker's learning is contained in the matching probabilities $a_H x$ and $a_L x$.[9] This formulation significantly simplifies the analysis of the learning problem, because neither workers nor firms need to learn about the composition of high- versus low-ability workers in a submarket. In any submarket x, a worker's matching probability depends only on his own ability and x, while a vacancy's matching probability depends only on x. Thus, given the choice x, an agent's expected payoff is independent of the level and the composition of the productive units in the submarket. Accordingly, free entry of firms into the submarket ensures that the effective tightness and the wage in the submarket are functions only of x.

We impose the following standard assumption on the matching function.[10]

ASSUMPTION 1: (i) $F(u_e, v) \leq \min\{u_H + u_L, v\}$; (ii) *$F$ is strictly increasing, strictly concave, and twice differentiable in each argument whenever* $x < 1/a_H$; (iii) *F is linearly homogeneous*; (iv) $F(1, 0) = 0$, $F(1, \infty) \geq 1/a_H$, *and* $x/\lambda(x) \leq 1$ *for all* $x \leq 1/a_H$.

[9]In this sense, our matching function implicitly assumes that a worker who fails to get a job does not know whether he has made a contact, that is, whether his application has been considered by a firm.

[10]An example that satisfies the assumption is $F(u_e, v) = u_e v/(u_e + Bv)$ if $v/u_e \leq 1/(a_H - B)$, and $F(u_e, v) = u_e/a_H$ otherwise, where $B \in (0, a_H)$ is a constant. In this example, $\lambda(x) = x/(1 - Bx)$.

Since $F(1, \lambda) = x$, we can solve λ and verify that Assumption 1 implies

$$\lambda'(x) > \lambda(x)/x > 0, \quad \lambda''(x) > 0 \quad \text{for all } x \in (0, 1/a_H]. \tag{2.2}$$

Moreover, $x/\lambda(x)$ strictly decreases in x. That is, if it is easy for a worker to find a match in a submarket, it must be difficult for a firm to find a match there.

The key feature of the model is the incomplete information about worker ability, which implies that workers face a signal extraction problem. Search histories are informative because low-ability workers are more likely to fail to get matches in any given submarket. As we show below, self-selection of workers into submarkets according to their own information implies that firms do not need to know the workers' histories.

3. LEARNING IN DIRECTED SEARCH EQUILIBRIUM

3.1. *Learning From Search*

A worker learns about his a, the probability that he will be productive with a randomly selected job. We refer to a worker's expectation of a as his belief and denote it as μ. The domain of μ is $M \equiv [a_L, a_H]$. When a new worker first enters the market, the initial belief is $\mu_0 = pa_H + (1-p)a_L$, where $p \in (0, 1)$. This initial belief is common to all new workers and it is public information.[11]

The updating of beliefs depends on the particular submarket into which the worker just searched. Consider an arbitrary period. The worker enters the period with P_i as the prior probability of $a = a_i$, where $a_i \in \{a_H, a_L\}$, and with μ as the prior belief computed from these prior probabilities. After searching in the period, the worker either gets a match (denoted as $k = 1$) or fails to get a match (denoted as $k = 0$). Bayesian updating yields the posterior probabilities

$$P(a_i|x, k=1) = P_i a_i/\mu, \quad P(a_i|x, k=0) = P_i(1 - xa_i)/(1 - x\mu). \tag{3.1}$$

The posterior belief is $\mathbb{E}(a|x, k) = a_H P(a_H|x, k) + a_L[1 - P(a_H|x, k)]$. Using the relationship $\mu = P_H a_H + (1 - P_H)a_L$, we can solve P_i in terms of μ:

$$P_H = (\mu - a_L)/(a_H - a_L), \quad P_L = (a_H - \mu)/(a_H - a_L). \tag{3.2}$$

Substituting (3.1) and (3.2), we express posterior beliefs as $\mathbb{E}(a|x, k=1) = \phi(\mu)$ and $\mathbb{E}(a|x, k=0) = H(x, \mu)$, where

$$\phi(\mu) \equiv a_H + a_L - a_H a_L/\mu, \tag{3.3}$$

$$H(x, \mu) \equiv a_H - (a_H - \mu)(1 - xa_L)/(1 - x\mu). \tag{3.4}$$

[11] For simplicity we abstract from heterogeneity in the initial beliefs among new workers. Note that our model does generate heterogeneous beliefs among workers with different employment histories.

If $\mu > a_L$, then $\mathbb{E}(a|x, k) > a_L$ for both $k = 0$ and $k = 1$. Also, $\phi(\mu) > \mu > H(x, \mu)$ for all $\mu \in (a_L, a_H)$, $\phi'(\mu) > 0$, and $\phi''(\mu) < 0$. The sequence of beliefs, μ, is a Markov process, and a worker's belief is a sufficient statistic for the worker's unemployment history. Note that $H(x, \mu)$ is decreasing in x; that is, a higher x reduces the worker's posterior belief after the worker fails to find a match. However, ϕ is independent of x, because x does not affect the likelihood ratio of a match success between the two types.

The value of x measures the informativeness of search. Intuitively, search outcomes in a market with a higher x are more informative because such a market has a relatively higher matching probability for a worker; if a worker fails to find a match in such a market, the worker more likely attributes the failure to low ability. This relationship between x and the informativeness of search can be made precise using Blackwell's (1951) criterion. Consider the information revealed by search in two different submarkets, with $x > x'$. Let K and K' be the random number of matches associated with x and x'. Intuitively, one can construct the random variable K' by "adding noise" to K as follows. First, let the worker randomize with probability of success ax, where $a \in \{a_L, a_H\}$; then, whenever the realization is a success, randomize again with success probability x'/x. The result is a Bernoulli trial with probability of success equal to ax'. In other words, if $x > x'$, the random variable, or experiment, K is sufficient for K' (see DeGroot (1970, pp. 433–439)).

3.2. *A Worker's Value Function*

Consider first a worker with belief μ who is employed at wage w in a period. Denote the worker's value function, discounted to the *end of the previous period*, as $J_e(\mu, w)$. After producing and obtaining the wage w, the separation shock forces the worker into unemployment with probability δ and then, independently, the exit shock forces the worker out of the market with probability σ. If the worker remains employed after the two shocks, the continuation value is $J_e(\mu, w)$. If the worker is separated from the job but remains in the market, the continuation value is denoted $V(\mu)$. If the worker is out of the market, the continuation value is 0. Thus, the Bellman equation for J_e is

$$(1 + r)J_e(\mu, w) = w + (1 - \sigma)[(1 - \delta)J_e(\mu, w) + \delta V(\mu)].$$

The above equation yields

$$J_e(\mu, w) = \frac{1}{A}\left[\frac{w}{1-\sigma} + \delta V(\mu)\right], \quad \text{where} \quad A \equiv \frac{r+\sigma}{1-\sigma} + \delta. \tag{3.5}$$

Now consider an unemployed worker who enters a period with belief μ. If he chooses to search in submarket x, the expected probability of finding a (productive) match is $x\mu$. If he fails to find a match, his belief is updated downward to $H(x, \mu)$ as defined by (3.4). In this case, his continuation value is

$(1-\sigma)V(H(x,\mu))$, which takes into account the probability of exogenous exit. If the worker succeeds in finding a match in the current period, his belief is updated upward to $\phi(\mu)$ as defined by (3.3). In this case, the worker can choose whether or not to accept the match. We impose Assumption 2 below to guarantee that a worker always accepts a match, and so the worker's continuation value after finding a match is $(1-\sigma)J_e(\phi(\mu), W(x))$. Thus, under Assumption 2, the worker's expected return to searching in submarket x, excluding the unemployment benefit, is $(1-\sigma)R(x,\mu)$, where

$$R(x,\mu) \equiv x\mu J_e(\phi(\mu), W(x)) + (1-x\mu)V(H(x,\mu)). \tag{3.6}$$

Since the value functions are discounted to the end of the previous period, then

$$(1+r)V(\mu) = b + (1-\sigma)\max_{x\in X} R(x,\mu). \tag{3.7}$$

Denote the set of optimal decisions in (3.7) as $G(\mu)$ and a selection from $G(\mu)$ as $g(\mu)$.

When choosing a submarket x, the worker faces two considerations. One is the familiar trade-off between the wage and the matching probability in models of directed search. That is, a submarket with a higher x has a higher job-finding probability and a lower wage. Another consideration is learning from the search outcome. As discussed earlier, search in a submarket with a high x (i.e., a low wage) is more informative than search in a submarket with a low x. The value of this information is captured by the features of the value function, to be described later in Theorems 3.1 and 4.1.

It is useful to note that the set of solutions $G(\mu)$ generically contains only a finite number of values. That is, given beliefs μ, a worker prefers to search in only a few submarkets and possibly only one submarket. Over time, the worker switches from one submarket to another not because he is indifferent between these submarkets, but because search outcomes induce the worker to update beliefs.

In principle, workers may have incentive to engage in the following "experimentation": searching during a period solely to gather information and, thus, refusing to enter a match once they learn that a match has occurred. This may occur because a worker who finds a match revises his belief upward to $\phi(\mu)$. We do not think that this form of experimentation is important in practice, unless it is associated with heterogeneity among productive matches, which does not exist here. Thus, we rule out such experimentation by focusing on the case in which employment is sufficiently valuable to a worker so that the worker always prefers to accept a match that he searches for.

ASSUMPTION 2: *Define x^* by the solution to $\lambda'(x^*) = a_H\lambda(a_H^{-1})$ and note that $x^* \in (0, 1/a_H)$. Assume that labor productivity satisfies*

$$(y-b)/c > [A + a_H x^*]\lambda'(x^*) - a_H\lambda(x^*).$$

This sufficient condition implies that a worker prefers getting the lowest equilibrium wage every period starting now to remaining unemployed in the current period and then getting the highest possible wage from a match starting next period (see Appendix A). Intuitively, the condition requires that the opportunity cost of rejecting a match, as reflected by $(y - b)$, should be sufficiently high to a worker.[12] Stronger than necessary, this condition significantly simplifies the analysis and the exposition of our main results. As in Burdett and Vishwanath (1988), one can relax the condition by introducing a direct cost of search per period, which further increases a worker's opportunity cost of rejecting an offer. For simplicity, however, we have not included such a cost of search.

REMARK 1: Since $x^*\lambda'(x^*) > \lambda(x^*)$, Assumption 2 implies $y - b > cA \times \lambda(x^*)/x^*$, which in turn implies $y - b > cA\lambda'(0)$. The last inequality says that there are feasible wages at which employment is better than unemployment for a worker.

3.3. *Free Entry of Firms and the Equilibrium Definition*

There is free entry of firms into the market. After incurring a cost $c \in (0, y)$, a firm can post a vacancy for a period in any one of the submarkets. Denote the value of a job filled at wage w, discounted to the end of the previous period, as $J_f(w)$. Then

$$(1 + r)J_f(w) = y - w + (1 - \sigma)(1 - \delta)J_f(w). \tag{3.8}$$

The matching probability for a vacancy in submarket x is $x/\lambda(x)$, and the continuation value of a match is $(1 - \sigma)J_f(W(x))$. Solving J_f from (3.8) and using A defined in (3.5), we can express a firm's value of a vacancy in submarket x as

$$J_v(x) = -c + \frac{x}{\lambda(x)} \frac{y - W(x)}{A}. \tag{3.9}$$

A recruiting firm chooses x to maximize $J_v(x)$. In equilibrium, a firm is willing to enter any submarket, provided that the wage in the submarket is consistent with the free-entry condition. Precisely, $J_v(x)$ and the number of vacancies, $v(x)$, satisfy $J_v(x) \le 0$ and $v(x) \ge 0$ for all $x \in X$, where the two inequali-

[12] The discount rate in Assumption 2, appearing through A, reflects both workers' and firms' discount rate. For a worker, a higher discount rate lowers the benefit from experimentation for any given wage. However, when firms discount future at a higher rate, the present value of a filled job falls and wages in all submarkets must be lower to induce firms to enter. In this case, the loss of the current wage from experimentation falls. With a common discount rate, the effect through firms' discount rate dominates.

ties hold with complementary slackness. Thus, for all x such that $v(x) > 0$, the wage function is

$$W(x) = y - cA\lambda(x)/x. \tag{3.10}$$

Conversely, for any feasible wage level specified in (3.11)(i) below, we require the number of vacancies to be positive. The wage function has the properties

$$\begin{aligned} &\text{(i)} \quad b + ca_H[x^*\lambda'(x^*) - \lambda(x^*)] \le W(x) \le y - cA\lambda'(0), \\ &\text{(ii)} \quad W'(x) < 0, \quad \text{(iii)} \quad 2W'(x) + xW''(x) < 0. \end{aligned} \tag{3.11}$$

Part (i) specifies the interval of feasible wages, where x^* is defined in Assumption 2. The upper bound on wages comes from the fact that $\lambda(x)/x \ge \lambda'(0)$. The lower bound on wages comes from Assumption 2 and the fact that $\lambda(x)/x \le a_H\lambda(a_H^{-1}) = \lambda'(x^*)$. The lower bound on wages is strictly greater than b because $x^*\lambda'(x^*) > \lambda(x^*)$. Also, Assumption 2 is sufficient for the wage interval in (i) to be nonempty.

Parts (ii) and (iii) of (3.11) are implied by (2.2), which is in turn implied by Assumption 1 on the matching function. Part (ii) says that a higher employment probability occurs together with a lower wage. This negative relationship is necessary for providing a meaningful trade-off between the two variables in directed search. As such, part (ii) is necessary to induce firms to enter the submarket. Part (iii) is implied by $\lambda''(x) > 0$, and it says that the function $xW(x)$ is strictly concave in x. In general, $[xW(x)]$ is nonmonotone because there is a trade-off between the matching probability and the wage in a submarket.

Focus on stationary symmetric equilibria. Such an *equilibrium* consists of workers' choices of x, a wage function $W(x)$, value functions (J_e, V, J_f, J_v), and a sequence of beliefs that meet the following requirements. (i) Given the wage function, all workers with the same belief μ use the same optimal search policy $x = g(\mu) \in G(\mu)$ that solves (3.7). (ii) A worker with beliefs μ updates beliefs according to $\phi(\mu)$ upon getting a match and according to $H(g(\mu), \mu)$ upon failing to get a match. (iii) The value functions satisfy (3.5), (3.7), (3.8), and (3.9). (iv) Free entry: The wage function $W(x)$ satisfies (3.10). (v) Consistency: For every submarket x with positive entry, the mass of all vacancies in x divided by the productive units of workers who choose x is equal to $\lambda(x)$.

In the above definition, we have left out the distributions of workers and wages, which is characterized in Section 6. We deliberately do so to emphasize the property that individuals' decisions and matching probabilities are independent of such distributions. For analyzing the former, it is sufficient to know the wage function $W(\cdot)$ and the tightness function $\lambda(\cdot)$, which are determined by firms' free-entry condition and the matching function. After completing this analysis, we can simply aggregate individuals' decisions to find equilibrium distributions of workers and wages. This property, referred to as block recursivity of an equilibrium (see Shi (2009)), makes the analysis tractable by significantly reducing the dimensionality of the state variables in individuals' decisions. Block recursivity is a consequence of directed search. In our model, directed search allows the workers to sort according to beliefs about their ability. Since each submarket attracts only the workers with particular beliefs, firms that post vacancies in that submarket calculate the expected profit with only such workers in mind—they do not need to consider how other workers with different beliefs are distributed. Free entry of firms guarantees that each submarket has exactly the effective tightness specified for that submarket. If search were undirected, instead, an individual's search decision would depend on the wage distribution which, in turn, would evolve as individuals learn about their ability.

3.4. *Existence of an Equilibrium*

Let us analyze a worker's problem, (3.7). It is easy to see that the right-hand side of (3.7) is a contraction mapping on V. Using (3.11), standard arguments show that a unique value function V exists, which is positive, bounded, and continuous on $M = [a_L, a_H]$ (see Stokey, Lucas, and Prescott (1989, p. 79)). Moreover, the set of maximizers, G, is nonempty, closed, and upper hemicontinuous. The following theorem summarizes the existence result and some other features of the equilibrium (see Appendix A for a proof).

THEOREM 3.1: *Under Assumptions 1 and 2, there exists an equilibrium where all matches are accepted. In the equilibrium, $g(\mu) > 0$ for all $g(\mu) \in G(\mu)$ and all $\mu \in M$. Moreover, V is strictly increasing, (weakly) convex and almost everywhere differentiable.*

Let us explain the results in the theorem. First, optimal choices of x are strictly positive. A worker who chooses $x = 0$ never finds a match and does not learn anything from search (i.e., $H(0, \mu) = \mu$). Since there are feasible wages at which employment is strictly better than unemployment (see Remark 1), a worker will choose $x > 0$. Second, the value function of an unemployed worker is strictly increasing in the worker's beliefs. Because a worker with higher beliefs can always choose to enter the same submarket as does a worker with lower beliefs and, thereby, can obtain a match with a higher expected probability, the former gets a higher expected payoff. Third, the value function is (weakly) convex in beliefs, as is standard in optimal learning problems (see Nyarko (1994)). Search generates information by creating variations in the worker's posterior beliefs. Such variations can never be harmful to the worker because the worker can always choose to ignore the information. Weak convexity of the value function reflects this fact.

A worker's *reservation wage* can be defined in the conventional way as the lowest permanent income that a worker will accept to forego search. This is given as $(r + \sigma)V(\mu)$. Monotonicity of the value function determines the behavior of reservation wages. Because $V(\mu)$ is strictly increasing, the reservation wage strictly falls over each unemployment spell as the worker's beliefs about his own ability deteriorate. Put differently, a worker's permanent income strictly declines over each unemployment spell. Similarly, with strict monotonicity of V, (3.7) implies that a worker's reservation wage is always strictly lower than the desired wage, that is, $(r+\sigma)V(\mu) < W(g(\mu))$ for all $\mu > a_L$.

Our focus is on a worker's *desired wage*, which is defined as $w(\mu) = W(g(\mu))$. Desired wages are much more difficult to analyze than reservation wages, because they depend on optimal learning from search. As it will become clear in the next section, monotonicity of the optimal search decision relies crucially on convexity of the value function.

4. MONOTONICITY OF WORKERS' DESIRED WAGES

In this section, we establish the result that a worker's desired wage, $w(\mu)$, is an increasing function of beliefs. Because $w(\mu) = W(g(\mu))$, where $W(\cdot)$ is decreasing, it is equivalent to establish the result that a worker's policy function for the submarket to search, $x = g(\mu)$, is a decreasing function. For what follows, we define $z = -x$ and refer to z, rather than x, as the worker's search decision. Then the objective function in (3.7) becomes $R(-z, \mu)$, and the feasible set of choices is $-X = [-a_H^{-1}, 0]$. The domain of μ is $M = [a_L, a_H]$.

A difficulty in proving monotonicity of the policy function arises from the feature inherent to learning that the value function is convex in beliefs. This feature implies that optimal choices may not be unique or interior, and the value function may not be differentiable.[13] In this context, standard techniques in dynamic programming for proving policy functions to be monotone are not applicable (e.g., Stokey, Lucas, and Prescott (1989, pp. 80–87)).[14] A natural approach is to use lattice-theoretic techniques associated with a supermodular objective function (see Topkis (1998)). In our model, R is supermodular in (z, μ) if and only if R has increasing differences in (z, μ), because the latter variables lie in closed intervals of the real line.[15]

[13]One can attempt to impose strong assumptions to make R concave in z and then use the first-order condition to characterize the optimal choice of z. However, such assumptions invariably require restrictions on the degree of convexity of the value function. It is difficult to verify that these restrictions can be satisfied by the fixed point of the Bellman equation, (3.7).

[14]In different modeling environments, there are techniques to establish differentiability of value functions and optimal choices in dynamic programming, for example, Santos (1991). However, those techniques also require the value function to be concave. On the other hand, the literature on optimal learning (e.g., Easley and Kiefer (1988)) has either ignored the difficulty arising from a convex value function or focused on corner solutions (e.g., Balvers and Cosimano (1993)).

[15]Let $z \in Z$ and $\mu \in M$, where Z and M are partially ordered sets. A function $f(z, \mu)$ has increasing differences in (z, μ) if $f(z_1, \mu_1) - f(z_1, \mu_2) \geq f(z_2, \mu_1) - f(z_2, \mu_2)$ for all $z_1 > z_2$ and

The connection between lattice-theoretic techniques and the dynamic programming problem in (3.7) is far from obvious. First, because $[-zW(-z)]$ is nonmonotone in z, the current payoff in the objective function, $[-\mu zW(-z)]$, is not supermodular in (μ, z). Second, whatever features one imposes on the value function to make the objective function supermodular must be confirmed as those of the fixed point of the Bellman equation, (3.7). In other applications of lattice-theoretic techniques to dynamic programming, this confirmation is achieved by assuming supermodularity of the current payoff function and using concavity of the value function recursively via the Bellman equation (e.g., Amir, Mirman, and Perkins (1991) and Mirman, Morand, and Reffett (2008)). This approach is not applicable here, because the current payoff is not supermodular and the value function is convex.

To proceed, we note that optimal choices remain unchanged if we divide the objective function R by the state variable μ. This transformation eliminates the first difficulty above, that is, the ambiguous effect of the nonmonotone function $[-zW(-z)]$ on modularity of the objective function. Accordingly, we express (3.7) as

$$(1+r)V(\mu) = b + (1-\sigma)\mu \max_{z\in -X} \hat{R}(z, \mu),$$

where $\hat{R}$ is defined as $\hat{R}(z, \mu) \equiv \mu^{-1}R(-z, \mu)$, that is,

$$\hat{R}(z, \mu) = -\frac{zW(-z)}{A(1-\sigma)} - \frac{\delta}{A} zV(\phi(\mu)) + (z + \mu^{-1})V(H(-z, \mu)). \tag{4.1}$$

Next, note that the second term on the right-hand side of (4.1), with negative sign, is associated with the value of the information contained in a match success. This term is strictly submodular in (z, μ) because searching in a high-z market has a relatively low chance of success, while the payoff of success increases in a worker's belief. To ensure that $\hat{R}$ is supermodular, it is necessary to restrict this submodular term. The value of the information contained in a match success is regulated by the job separation rate δ, because a worker uses such information only when he becomes unemployed again in the future. Accordingly, we impose an upper bound on δ:

ASSUMPTION 3: *The job separation rate satisfies* $0 < \delta \leq \bar{\delta}$, *where* $\bar{\delta} > 0$ *is defined in part* (iii) *of Lemma* A.1 *in Appendix* A.

Denote $Z(\mu) = \arg\max_{z\in -X} \hat{R}(z, \mu)$ and $z(\mu) \in Z(\mu)$. Clearly, $G(\mu) = -Z(\mu)$ and $g(\mu) = -z(\mu)$. Denote the greatest selection of $Z(\mu)$ as $\bar{z}(\mu)$ and

$\mu_1 > \mu_2$. If the inequality is strict, then f has strictly increasing differences. Because Z, M, and $Z \times M$ are all lattices in our model, the feature of increasing differences implies supermodularity (see Topkis (1998, p. 45)).

the smallest selection as $\underline{z}(\mu)$. Every selection $z(\mu)$ is an increasing function if for all μ_a and μ_b in M, with $\mu_a > \mu_b$, it is true that $z(\mu_a) \geq z(\mu_b)$ for all $z(\mu_a) \in Z(\mu_a)$ and all $z(\mu_b) \in Z(\mu_b)$. If $z(\mu_a) > z(\mu_b)$ for all $\mu_a > \mu_b$ in the preceding definition, then every selection $z(\mu)$ is strictly increasing. We state the following theorem (see Appendix B for a proof).

THEOREM 4.1: *Maintain Assumptions* 1, 2, *and* 3. *Part* 1: $\hat{R}(z, \mu)$ *is strictly supermodular in* (z, μ), *and so every selection* $z(\mu)$ *is an increasing function. Part* 2: *The following statements are all equivalent to each other*: (i) $V(\mu)$ *is strictly convex for all* μ. (ii) *Every selection* $z(\mu) \in Z(\mu)$ *is strictly increasing in* μ. (iii) *For all* $\mu > a_L$, $\{-a_H^{-1}\} \notin Z(\mu)$ *and so* $Z(\mu)$ *is interior*. (iv) $\{-a_H^{-1}\} \notin Z(a_H)$. (v) *The following condition holds*:

$$(y - b)/c < (A + 1)\lambda'(a_H^{-1}) - a_H \lambda(a_H^{-1}). \tag{4.2}$$

Part 1 states that the policy function, $z(\mu)$, is a (weakly) increasing function and so desired wages are an increasing function of beliefs. These results are implied by strict supermodularity of $\hat{R}$ (see Topkis (1998, p. 79)). $\hat{R}$ is strictly supermodular because a worker's expected value in the case of a match failure, $(1 + \mu z)V(H(-z, \mu))$, is strictly supermodular and Assumption 3 ensures that this supermodular component dominates the submodular component, $\frac{-\delta}{A}\mu z V(\phi(\mu))$, that is associated with a match success.

To gain further intuition, consider a hypothetical lottery that gives a "prize" of $\mathbb{E}(a|z, k)$ when $k = 0$ (i.e., match failure) and 0 when $k = 1$ (i.e., match success), where $k = 0$ occurs with probability $(1 + z\mu)$. The expected value of the lottery conditional on $k = 0$ is $\alpha \equiv (1 + \mu z)H(-z, \mu)$. Note that α increases in z and is strictly supermodular in (z, μ). Lowering z (i.e., increasing $x = -z$) always lowers α, because it reduces the probability and the expected size of the prize. This effect of a lower z has the flavor of a winner's curse. In addition, the curse gets worse as μ is higher, because the marginal impact of lowering z on α increases in μ. That is, with a higher μ, lowering z reduces the probability of winning the prize by a larger amount, in which case a match failure indicates that the expected prize is even more likely to be low. Now note that supermodularity of α translates into supermodularity of $(1+\mu z)V(H(-z, \mu))$, because V is convex and strictly increasing.

Let us make three remarks on Part 1. First, convexity of the value function plays an important role in the proof of strict supermodularity of $\hat{R}$, as explained above. Second, because $\hat{R}$ is strictly supermodular, every selection of z, rather than just the greatest or the smallest selection, is an increasing function. Third, it can be verified that strict supermodularity of $\hat{R}$ is sufficient but not necessary for the original function $R(-z, \mu)$ to have strict single crossing in (z, μ), as defined by Milgrom and Shannon (1994) (see the Supplemental Material (Gonzalez and Shi (2010))). Although strict single crossing is enough to prove that the policy function is weakly increasing, we need strict supermodularity of $\hat{R}$ to establish strict monotonicity.

Part 2 states the necessary and sufficient conditions for the policy function to be strictly increasing, and establishes the equivalence between strict monotonicity of the policy function and strict convexity of the value function. In general, strong conditions are required for the policy function to be strictly monotone.[16] In our model, only a very mild condition, (4.2), is necessary and sufficient for every selection $z(\mu)$ to be a strictly increasing function. Condition (4.2) is equivalent to statement (iv), which requires that a worker with the most optimistic belief $\mu = a_H$ finds it not optimal to search for the lowest wage, that is, to search in the submarket with the lowest z. If (4.2) is not satisfied, then it is optimal for all workers to search for the lowest wage, regardless of their beliefs. In this sense, (4.2) can be viewed as a regularity condition for learning to be a useful explanation for the fact that wage losses upon reemployment increase with unemployment duration.[17]

To see the role of (4.2), let us first explain statements (i)–(iii) in Theorem 4.1. The equivalence between (i) and (ii) relies on the following standard property of optimal learning: The value function V is strictly convex in beliefs if and only if there do not exist μ_a and μ_b in M, with $\mu_a > \mu_b$, and a choice x_0 such that x_0 is optimal for all $\mu \in [\mu_b, \mu_a]$ (see Nyarko (1994)). Because every selection $z(\mu)$ is weakly increasing, as established earlier, this standard property implies that the value function is strictly convex if and only if every selection $z(\mu)$ is strictly increasing. It is easy to see that statement (ii) in Theorem 4.1 implies (iii) which, in turn, implies (iv).

The key step in the proof of Part 2 is to show that (iv) implies (i). That is, if the value function is not strictly convex, then a worker with the most optimistic belief should search for the lowest wage. To understand this result, suppose that the value function is not strictly convex. In this case, there is an interval of beliefs $[\mu_b, \mu_a]$, with $\mu_a > \mu_b$, such that the optimal choice is the same under such beliefs. This must mean that under such beliefs, local variations in the positive or negative signal are not valuable to the worker. In particular, the value function must be linear in the interval of beliefs induced by a match success, $[\phi(\mu_b), \phi(\mu_a)]$. In this case, strict concavity of the function $[-zW(-z)]$ implies that the payoff function is strictly concave in z and, hence, the optimal

[16]See Amir (1996) and Edlin and Shannon (1998). Their methods require the value function to be continuously differentiable. In particular, Edlin and Shannon (1998) assumed that the objective function, $\hat{R}(z, \mu)$, has increasing marginal differences. To compute marginal differences, $\hat{R}(z, \mu)$ must be continuously differentiable with respect to z. Because $\hat{R}$ depends on z through the future value function, as well as W, it is differentiable with respect to z only if the value function is so.

[17]The condition in the theorem can hold simultaneously with Assumption 2. To see this, note that the right-hand side of the condition in Assumption 2 is strictly increasing in x^* and, hence, is less than the right-hand side of the condition given in the theorem (since $x^* < 1/a_H$).

choice of z is unique for all $\mu \in (\mu_b, \mu_a)$. For this unique choice to be constant for all $\mu \in (\mu_b, \mu_a)$, it must be at the corner $z = -a_H^{-1}$; otherwise, strict supermodularity of $\hat{R}$ implies that the optimal choice strictly increases in μ. Repeating the above argument, we know that for any positive integer i, the value function must be linear over beliefs in $[\phi^i(\mu_b), \phi^i(\mu_a)]$ and that the optimal choice under such beliefs must be the singleton $\{-a_H^{-1}\}$, where ϕ^i is defined as $\phi^i(\cdot) = \phi(\phi^{i-1}(\cdot))$. Because $\phi^i(\mu)$ converges to a_H for all $\mu \in (a_L, a_H)$, the choice $\{-a_H^{-1}\}$ must also be optimal when $\mu = a_H$.

The above explanation for why (iv) implies (i) relies on the assumption $\delta > 0$.[18] If $\delta = 0$, instead, the information revealed by a match success is not valuable to the worker, because the worker will never be unemployed again. In this case, the above induction does not apply and so, for some belief $\mu = \mu_a > a_L$, the worker may find it optimal to choose $z = -a_H^{-1}$. Once this happens, it is optimal for the worker to choose $z = -a_H^{-1}$ for all $\mu \leq \mu_a$, in which case the value function is linear in the subinterval $[a_L, \mu_a]$.

It is useful to clarify the role of the equilibrium wage function for the results obtained so far. In contrast to a model of decision theory, our model requires the wage in each submarket to be consistent with free entry of firms. This equilibrium requirement results in the wage function $W(-z)$, as given by (3.10). Given the standard assumptions on the matching technology in Assumption 1, the wage function has the properties listed in (3.11). These properties are not important for the policy functions $z(\mu)$ and $w(\mu)$ to be weakly increasing. The latter relies on strict supermodularity of $\hat{R}$, which requires only that the value function V be weakly convex. However, the equilibrium wage function is critical for the policy functions to be *strictly* increasing. In particular, in the above explanation for why statement (iv) implies (i), we have explicitly used property (iii) in (3.11) that the function $[-zW'(-z)]$ is strictly concave. If the wage function were exogenous or if it had no connection to the matching technology, it would not be clear how it should satisfy (3.11). In this sense, the equilibrium structure of the model is essential for our analysis to capture the intuitive link between learning from search and discouragement.

5. FURTHER CHARACTERIZATION OF EQUILIBRIUM PATHS

Condition (4.2) is necessary and sufficient for optimal choices to be interior for all $\mu > a_L$ (see Theorem 4.1). We explore this feature to provide a sharper characterization of an equilibrium than in the previous section. In particular, we establish the validity of the first-order condition, a generalized version of the envelope theorem, and a discipline on the set of paths of optimal choices. Together with monotonicity of the policy function, these results provide an operational way to do dynamic programming when the value function is convex.

[18]The equivalence between statements (i) and (ii) in Theorem 4.1 does not require $\delta > 0$. However, the equivalence between these two statements and other statements does require $\delta > 0$.

Since we maintain Assumptions 1–3 and condition (4.2) in this section, the results rely on strict supermodularity of $\hat{R}$. Since we focus on symmetric equilibria, all workers with beliefs μ use the same selection $z(\mu)$.

Let us introduce some notation. For any μ, let μ^+ denote the limit to μ from the right, μ^- denote the limit from the left, $f'(\mu^+)$ denote the right-hand derivative of any function f, and $f'(\mu^-)$ denote the left-hand derivative. Recall that $\phi(\mu)$ is the posterior belief reached from the prior μ through a match success. Denote the posterior belief reached through a match failure with an optimal choice as $h(\mu) \equiv H(-z(\mu), \mu)$. For any S in the σ-algebra of M, denote $\phi(S) = \{\phi(\mu) : \mu \in S\}$ and $h(S) = \{h(\mu) : \mu \in S\}$. For any $\mu \in M$, construct $Y(\mu) = \{Y_i(\mu)\}_{i=0}^{\infty}$ by $Y_0(\mu) = \{\mu\}$ and $Y_{i+1}(\mu) = \{\phi(Y_i(\mu)), h(Y_i(\mu))\}$ for $i = 0, 1, \dots, \infty$. We call $Y(\mu)$ the tree of equilibrium beliefs generated from μ and call $Y_i(\mu)$ the ith layer of the tree. Given μ and the optimal choice $z(\mu)$, beliefs in the next period will be $\phi(\mu)$ with mean probability $-z(\mu)\mu$, and $h(\mu)$ with mean probability $[1 + z(\mu)\mu]$.

In Appendix C, we establish the following theorem.

THEOREM 5.1: *Let μ be any arbitrary value in the interior of (a_L, a_H). The following results hold*: (i) *$V'(h(\mu))$ exists for all $z(\mu) \in Z(\mu)$, and so optimal choices in every period obey the first-order condition, $\hat{R}_1(z(\mu), \mu) = 0$.* (ii) *$\bar{z}(\mu)$ is right-continuous and $\underline{z}(\mu)$ is left-continuous.* (iii) *V' satisfies the envelope conditions*

$$(1 + r)V'(\mu^+) = (1 - \sigma)R_2(-\bar{z}(\mu), \mu^+),$$
$$(1 + r)V'(\mu^-) = (1 - \sigma)R_2(-\underline{z}(\mu), \mu^-).$$

(iv) *$V'(\mu)$ exists if and only if $V'(\phi(\mu))$ exists and $\bar{z}(\mu) = \underline{z}(\mu)$.* (v) *If $V'(\mu_a)$ exists for a particular* (*interior*) *μ_a, such as $\mu_a = h(\mu)$ for any interior μ, then the optimal choice $z(\mu')$ is unique and the value function $V(\mu')$ is differentiable at all $\mu' \in Y(\mu_a)$.*

Recall that the value function is differentiable almost everywhere (see Theorem 3.1). Part (i) above states that a match failure induces posterior beliefs at which the value function is differentiable, regardless of whether the value function is differentiable at the prior belief. To explain this result, consider an arbitrary (interior) prior belief μ and let the posterior belief following a match failure be $\mu' = h(\mu)$. If the value function is not differentiable at μ', the left-hand derivative of $V(\mu')$ must be strictly lower than the right-hand derivative. This implies that by searching for a wage slightly lower than $w(\mu)$ (i.e., in a submarket slightly lower than $z(\mu)$), the worker's future marginal value falls by a discrete amount even though the worker learns only slightly more about his ability when he fails to find a match. The worker can avoid this discretely larger marginal loss by choosing z slightly above $z(\mu)$, which keeps the posterior slightly above μ'. Since the cost to increasing z is a marginal reduction in the matching probability, the net marginal gain from increasing z slightly above $z(\mu)$ is positive. This contradicts the optimality of $z(\mu)$.

The above limited sense of differentiability of the value function is sufficient to ensure that optimal choices obey the first-order condition, as stated in part (i) of Theorem 5.1. Although the value function may fail to be differentiable if a worker has never experienced a match failure, this potential failure does not invalidate the first-order condition. The reason is that for any given prior belief, μ, search choices in the current period do not affect the posterior belief in the case of a match success, $\phi(\mu)$. Thus, optimal search decisions are independent of whether or not the value function is differentiable at $\phi(\mu)$. As long as the value function is differentiable at $h(\mu)$, the worker's objective function is differentiable at optimal choices, and so the first-order condition applies.

Part (ii) of Theorem 5.1 describes one-sided continuity of the highest and the smallest selection of optimal choices. Such continuity is needed for part (iii), which is a generalized version of the envelope theorem. Part (iv) states that uniqueness of the optimal choice under a belief μ is necessary, but not sufficient, for the derivative $V'(\mu)$ to exist. For the latter, the value function must also be differentiable at the posterior belief $\phi(\mu)$.

Part (v) of Theorem 5.1 puts discipline on equilibrium paths. If initial beliefs lie outside the measure-zero set where the value function is not differentiable, then the value function remains differentiable on the entire tree of beliefs generated by the equilibrium, in which case the optimal choice is unique. Even if a worker's initial beliefs lie in this measure-zero set, the first match failure takes the worker out of this set, after which the value function is differentiable and the optimal choice is unique.

6. STEADY STATE DISTRIBUTIONS AND WORKER FLOWS

We now determine the distribution of workers and discuss how current unemployment durations and past unemployment spells can influence reemployment rates and wages.

Immediately before the labor market opens in a period, measure employed workers with beliefs μ and type $i \in \{H, L\}$ as $e_i(\mu)$, and similarly, $\hat{u}_i(\mu)$ for the unemployed. The stationary distribution of workers over beliefs is $\{(e_H(\mu), e_L(\mu), \hat{u}_H(\mu), \hat{u}_L(\mu)) : \mu \in Y(\mu_0)\}$, where $Y(\mu_0)$ is the tree of equilibrium beliefs generated from μ_0.

Consider unemployed workers of type $i \in \{H, L\}$. There are three cases. One is that the unemployed workers are newborns. The measure of newborns with type i is

$$\hat{u}_i(\mu_0) = \sigma p_i, \tag{6.1}$$

where $p_H = p$ and $p_L = 1 - p$. The outflow from and inflow to this group are both equal to σp_i, and so stationarity always holds for this group.

The second case of unemployed workers of type i is that these workers were unemployed in the previous period, in which case their beliefs in the current period are $h(\mu) = H(-z(\mu), \mu)$ for some $\mu \in Y(\mu_0)$. All of these workers move out of the group in the period. The inflow is type-i unemployed workers with beliefs μ who survive exogenous exit and fail to find a match in the current period; the probability of this joint event is $(1-\sigma)[1-a_i g(\mu)]$. Thus, stationarity requires

$$\hat{u}_i(h(\mu)) = (1-\sigma)[1-a_i g(\mu)]\hat{u}_i(\mu), \quad \mu \in Y(\mu_0). \tag{6.2}$$

The third case of unemployed workers of type i is that these workers separated from their jobs in the previous period. These workers' beliefs in the current period are $\phi(\mu)$ for some $\mu \in Y(\mu_0)$. Again, all of these workers move out of the group in the period. The inflow is type-i employed workers with beliefs $\phi(\mu)$ who exogenously separate from jobs and survive exogenous exit. Thus, stationarity requires

$$\hat{u}_i(\phi(\mu)) = (1-\sigma)\delta e_i(\phi(\mu)), \quad \mu \in Y(\mu_0). \tag{6.3}$$

Similarly, consider employed workers of type i with beliefs $\phi(\mu)$. The outflow from the group in the period is $[\sigma + (1-\sigma)\delta]e_i(\phi(\mu))$, which is generated by exogenous exit from the market and exogenous job separation. The inflow is type-i unemployed workers with beliefs μ who find a match in the current period and survive exogenous exist; the probability of this joint event is $(1-\sigma)a_i g(\mu)$. Thus, stationarity requires

$$[\sigma + (1-\sigma)\delta]e_i(\phi(\mu)) = (1-\sigma)a_i g(\mu)\hat{u}_i(\mu), \quad \mu \in Y(\mu_0). \tag{6.4}$$

The stationary distribution is determined by (6.1)–(6.4), together with the requirement that the total measure of workers is one. Because the equilibrium is block recursive, optimal choices are independent of the distribution, and so (6.1)–(6.4) are linear equations of the measures of workers. It is straightforward to solve for these equations by going through the nodes of the tree, starting at the root, μ_0. Given the equilibrium tree of beliefs, $Y(\mu_0)$, the stationary distribution of workers over such beliefs is unique.

In the stationary equilibrium, the set of active submarkets is $\{g(\mu) : \mu \in Y(\mu_0)\}$. In submarket $g(\mu)$, the measure of type-i workers is $u_i(g(\mu)) = \hat{u}_i(\mu)$, where $i \in \{H, L\}$. The total number of matches in this submarket is $[a_H \hat{u}_H(\mu) + a_L \hat{u}_L(\mu)]g(\mu)$. The average job-finding probability in submarket $g(\mu)$ is

$$f(g(\mu)) = \frac{a_H \hat{u}_H(\mu) + a_L \hat{u}_L(\mu)}{\hat{u}_H(\mu) + \hat{u}_L(\mu)} g(\mu). \tag{6.5}$$

Given μ, this probability is stationary over time because the composition of workers in the submarket is constant in the stationary equilibrium. Similarly, the average job-finding probability in the entire economy is constant over time.

To see how unemployment duration influences reemployment rates and wages, let us follow a given cohort of unemployed workers with beliefs μ. As established in Theorem 4.1, workers search for lower wages as their beliefs about their ability deteriorate with unemployment duration. Accordingly, discouragement is reflected in wage losses at reemployment, providing a natural explanation for the negative effect of unemployment duration on future wages found by Addison and Portugal (1989).[19] This mechanism can also help to explain why similar workers are paid different wages (see Burdett and Mortensen (1998) and Mortensen (2003)). On reemployment rates, learning from search has two effects. First, for any given ability a_i, the job-finding probability $a_i g(\mu)$ increases in the course of unemployment as workers search for lower wages that are easier to get. Second, as high-ability workers are more successful in getting jobs and exiting from unemployment, the average ability in the cohort remaining unemployed decreases with unemployment duration, which reduces the average job-finding probability in the cohort. More precisely, the average job-finding probability in the cohort, given by (6.5), is an increasing function of the ratio of high- to low-ability workers in the cohort, $\hat{u}_H(\mu)/\hat{u}_L(\mu)$, which decreases as μ decreases with unemployment duration. When this composition effect dominates the effect of $g(\mu)$, the average job-finding probability falls with unemployment duration.

Confounding the above composition effect, workers who become unemployed at the same time can differ in their beliefs μ because their histories of past unemployment can differ. This implication naturally suggests that an empirical investigation of job-finding probabilities and reemployment wages should take into account not only the worker's most recent unemployment spell, as it is typically done in the empirical literature, but also the history of the worker's previous unemployment spells. Our theory suggests a simple empirical strategy to take into account a worker's labor market history. Because a worker's beliefs follow a Markov process, the effect of past labor market history is summarized by the worker's beliefs when entering the most recent unemployment spell. In turn, the latter beliefs have a monotone relationship to the worker's wage at the most recent job. Thus, a worker's pre-unemployment wage serves the role of summarizing the worker's previous experience in the labor market. This role complements conventional human-capital explanations that view wages as a summary of the workers' human capital.

The previous argument also provides a novel explanation for Addison and Portugal's (1989) finding that unemployment duration increases with pre-unemployment wages after controlling for skills. Workers with higher pre-unemployment wages are those who had relatively shorter durations in previous unemployment spells and, hence, are more optimistic about their ability

[19]Addison and Portugal (1989) controled for observed heterogeneity by including, for example, schooling, age, race, location, experience, and industry dummy. They also controled for unobserved heterogeneity by estimating a predisplacement wage equation first and then imposing the resulting restrictions in the postdisplacement wage equation.

when entering the current unemployment spell. These workers will search for jobs offering higher wages, which are relatively harder to get.

7. CONCLUSION

In this paper, we proposed an equilibrium theory of learning from search in the labor market. The main assumption is that unemployed workers have incomplete information about their job-finding ability and learn about their ability from search outcomes. Success and failure of search both convey useful information about a worker's type. As workers experience different search outcomes, their labor market histories and, hence, their beliefs about their ability diverge. The theory formalizes a notion akin to discouragement. That is, over each unemployment spell, unemployed workers update their beliefs about their job-finding ability downward and reduce not only reservation wages, but also desired wages. Firms cater to these workers by offering different wages. Thus, learning from search generates endogenous heterogeneity in workers' histories that can be useful for understanding how unemployment can affect workers' labor market outcomes and wage determination.

Our paper integrates learnings from search into an equilibrium framework to determine jointly the workers' search behavior, the incentives to create jobs, and the wage distribution. The equilibrium analysis was made tractable with directed search, which made the equilibrium block recursive in the sense that search behavior and market tightness are independent of the wage distribution. Another contribution of the paper is to provide a set of results in dynamic programming when the value function is convex. We identified a connection between convexity of a worker's value function in beliefs and the property of supermodularity, established the property that the policy functions are monotone, and provided conditions under which the first-order condition and the envelope condition are valid. These results are likely to be useful in other learning problems, because convexity of the value function in beliefs is inherent to optimal learning from experience.

The equilibrium theory of learning from search provides a novel mechanism for generating endogenous heterogeneity among unemployed workers. The learning process turns ex ante identical workers into ex post heterogeneous workers who differ in posterior beliefs about their job-finding probabilities. Such endogenous heterogeneity makes a worker's entire labor market history relevant for his future labor market outcomes. With block recursivity, it will be feasible and interesting to examine the interactions between such endogenous heterogeneity and ex ante heterogeneity among workers and firms.

APPENDIX

See the Supplemental Material (Gonzalez and Shi (2010)) for complete proofs.

A. *Proof of Theorem 3.1*

First, we prove existence of the equilibrium. Given the analysis leading to Theorem 3.1, it suffices to show that Assumption 2 is sufficient for all matches to be accepted, in which case V indeed obeys (3.7). Consider a worker with beliefs $\mu \in M$ who gets a match in submarket $x \in X$. The worker strictly prefers to accept the match if and only if $J_e(\phi(\mu), W(x)) > V(\phi(\mu))$, which is equivalent to $W(x) > (r+\sigma)V(\phi(\mu))$. A sufficient condition is that the inequality holds for $x = 1/a_H$ and $\mu = a_H$. Substituting $V(a_H)$ from (A.1) in Lemma A.1, we rewrite this sufficient condition as $(y-b)/c > [A + a_H x_H]\lambda'(x^*) - a_H\lambda(x_H)$, where x^* is defined by $\lambda'(x^*) = a_H\lambda(a_H^{-1})$ and $x_H = g(a_H)$. Since the right-hand side of the inequality is maximized at $x_H = x^*$, the inequality is ensured by Assumption 2.

Second, we prove that $g(\mu) > 0$ for all $g(\mu) \in G(\mu)$ and all $\mu \in M$. Suppose that $g(\mu) = 0$ for some $\mu \in M$, contrary to the theorem. In this case, (3.6) and (3.7) yield $R(0, \mu) = V(\mu) = b/(r+\sigma)$. Substituting this value of V for the future value function, we obtain a lower bound on the payoff R, say, $\tilde{R}(x, \mu)$. Using Remark 1, we can prove that some $x_0 > 0$ maximizes $\tilde{R}(x, \mu)$ and achieves $\tilde{R}(x_0, \mu) > R(0, \mu)$—contradiction.

Third, we prove that V is strictly increasing. Let $TV(\mu)$ denote the right-hand side of (3.7). Since T is a contraction mapping, it suffices to prove that $TV(\mu_a) > TV(\mu_b)$ for any continuous and increasing function V and any $\mu_a, \mu_b \in M$, with $\mu_a > \mu_b$ (see Stokey, Lucas, and Prescott (1989)). Denote $g_i = g(\mu_i) \in G(\mu_i)$, where $i \in \{a, b\}$. We have

$$\begin{aligned}
&R(g_a, \mu_a) - R(g_b, \mu_b) \\
&\quad \geq R(g_b, \mu_a) - R(g_b, \mu_b) \\
&\quad \geq g_b(\mu_a - \mu_b)\left\{\frac{W(g_b)}{A(1-\sigma)} + \frac{\delta}{A}V(\phi(\mu_b)) - V(H(g_b, \mu_b))\right\} \\
&\quad > g_b(\mu_a - \mu_b)\big[V(\phi(\mu_b)) - V(H(g_b, \mu_b))\big] \geq 0.
\end{aligned}$$

The first inequality comes from the fact that $g_i \in \arg\max_x R(x, \mu_i)$ and the second inequality comes from $V(H(g_b, \mu_a)) \geq V(H(g_b, \mu_b))$. The strict inequality uses the fact that $g_b > 0$ and $W(x) > (r+\sigma)V(\phi(\mu))$ for all x and μ (see above proof). The last inequality comes from $\phi(\mu_b) \geq H(g_b, \mu_b)$. Hence, $TV(\mu_a) > TV(\mu_b)$.

Finally, (weak) convexity of V follows from standard arguments (e.g., Nyarko (1994, Proposition 3.2)). Because a convex function is almost everywhere differentiable (see Royden (1988, pp. 113–114)), V is almost everywhere differentiable. *Q.E.D.*

The proofs of the following lemmas are omitted:

LEMMA A.1: *Denote* $x_i = g(a_i)$, *where* $i \in \{H, L\}$. *The following results hold*: (i) *The optimal choice* x_i *is unique and satisfies* $R_1(x_i, a_i) \geq 0$, *with strict inequality only if* $x_i = 1/a_H$. *The value function satisfies*

$$V(a_i) = \frac{Ab + a_i x_i W(x_i)}{(r+\sigma)[A + a_i x_i]}. \tag{A.1}$$

(ii) *Condition* (4.2) *is necessary and sufficient for* $x_H < 1/a_H$. *Also*, $x_L \geq x_H$ *with strict inequality if* $x_H < 1/a_H$. (iii) $\delta/A < V'(a_L^+)/V'(a_H^-)$ *for all* $\delta \leq \bar{\delta}$, *where* $\bar{\delta}$ *is the smallest positive solution to* $\Omega(\delta) = 0$ *and* Ω *is defined as*

$$\Omega(\delta) = \frac{r+\sigma}{1-\sigma}\left(\frac{r+\sigma}{1-\sigma} + \delta\right)^2 - \delta\left[\left(1 + \frac{a_L}{a_H}\right)\left(\frac{r+\sigma}{1-\sigma} + \delta\right) + \frac{a_L}{a_H}\right]. \tag{A.2}$$

LEMMA A.2: *For any given* z, *the functions* $\mu V(\phi(\mu))$ *and* $(1 + z\mu)V(H(-z, \mu))$ *are convex in* μ *if* $V(\cdot)$ *is convex, and strictly convex in* μ *if* $V(\cdot)$ *is strictly convex.*

B. *Proof of Theorem 4.1*

First, we prove that $\hat{R}(z, \mu)$ is strictly supermodular. Once this is done, the monotone selection theorem in Topkis (1998, Theorem 2.8.4, p. 79) implies that every selection from $Z(\mu)$ is increasing. To prove that $\hat{R}$ is strictly supermodular, take arbitrary $z_a, z_b \in -X$, and arbitrary $\mu_a, \mu_b \in M$, with $z_a > z_b$ and $\mu_a > \mu_b$. Denote $D = [\hat{R}(z_a, \mu_a) - \hat{R}(z_a, \mu_b)] - [\hat{R}(z_b, \mu_a) - \hat{R}(z_b, \mu_b)]$. We need to show $D > 0$. Temporarily denote $\phi_j = \phi(\mu_j)$, $H_{ij} = H(-z_i, \mu_j)$, and $V_{ij} = V(H_{ij})$, where $i, j \in \{a, b\}$. Computing D, we have

$$\begin{aligned} D &= D_1 - [V(\phi_a) - V(\phi_b)](z_a - z_b)\delta/A \\ &\geq D_1 - V'(a_H^+)[\phi_a - \phi_b](z_a - z_b)\delta/A, \end{aligned}$$

where the inequality follows from convexity of V and where D_1 denotes

$$D_1 = (z_a + \mu_a^{-1})V_{aa} - (z_b + \mu_a^{-1})V_{ba} - (z_a + \mu_b^{-1})V_{ab} + (z_b + \mu_b^{-1})V_{bb}.$$

Denote $\tilde{H} = \min\{H_{ba}, H_{ab}\}$. Because $H(-z, \mu)$ is a strictly increasing function of z and μ for all $\mu \in (a_L, a_H)$, then $H_{aa} > \tilde{H} \geq H_{bb}$. Because V is convex, we have

$$\min\left\{\frac{V_{aa} - V_{ba}}{H_{aa} - H_{ba}}, \frac{V_{aa} - V_{ab}}{H_{aa} - H_{ab}}\right\} \geq \frac{V_{aa} - V(\tilde{H})}{H_{aa} - \tilde{H}} \geq \frac{V_{aa} - V_{bb}}{H_{aa} - H_{bb}}.$$

Substituting V_{ba}, V_{ab}, and V_{bb} from these inequalities, and substituting H, we have

$$\begin{aligned} D_1 &\geq (z_a - z_b)(\phi_a - \phi_b)[V_{aa} - V(\tilde{H})]/[H_{aa} - \tilde{H}] \\ &\geq V'(a_L^+)(z_a - z_b)(\phi_a - \phi_b), \end{aligned}$$

where the second inequality follows from convexity. Thus, a sufficient condition for $D > 0$ is $\delta/A < V'(a_L^+)/V'(a_H^-)$, which is implied by Assumption 3 (see Lemma A.1).

We next establish that the five statements (i)–(v) in Theorem 4.1 are equivalent.

(i) $\Longleftrightarrow$ (ii) Optimal learning has the following standard property (see Nyarko (1994, Proposition 4.1)): The value function is strictly convex in beliefs if and only if there do not exist μ_a and μ_b in M, with $\mu_a > \mu_b$, and a choice z_0 such that $z_0 \in Z(\mu)$ for all $\mu \in [\mu_b, \mu_a]$. Since $z(\mu)$ is an increasing function, as proven above, the standard property implies that V is strictly convex if and only if every selection $z(\mu)$ is strictly increasing for all μ.

(ii) $\Longrightarrow$ (iii) Suppose $\{-a_H^{-1}\} \in Z(\mu_a)$ for some $\mu_a > a_L$ so that (iii) is violated. Because every selection $z(\mu)$ is increasing, $Z(\mu)$ contains only the singleton $\{-a_H^{-1}\}$ for all $\mu < \mu_a$. In this case, (ii) does not hold for $\mu \leq \mu_a$. Note that since $z(\mu) < 0$ by Theorem 3.1, the result $\{-a_H^{-1}\} \notin Z(\mu)$ implies that $Z(\mu)$ is interior.

(iii) $\Longrightarrow$ (iv) This follows from $a_H > a_L$.

(iv) $\Longleftrightarrow$ (v) See part (ii) of Lemma A.1.

(iv) $\Longrightarrow$ (i) We prove that a violation of (i) implies that $\{-a_H^{-1}\} \in Z(a_H)$, which violates (iv). Suppose that V is not strictly convex. Proposition 4.1 in Nyarko (1994) implies that there exist μ_a and μ_b in M, with $\mu_a > \mu_b$, and a choice z_0 such that $z_0 \in Z(\mu)$ and $V(\mu)$ is linear for all $\mu \in [\mu_b, \mu_a]$. Since $\mu_a > \mu_b$, let $\mu_b > a_L$ and $\mu_a < a_H$ without loss of generality. We deduce that $V(\mu)$ is linear for all $\mu \in [\phi(\mu_b), \phi(\mu_a)]$: If $V(\mu)$ were strictly convex in any subinterval of $[\phi(\mu_b), \phi(\mu_a)]$, Lemma A.2 above would imply that $R(-z_0, \mu)$ is strictly convex μ in some subinterval of $[\mu_b, \mu_a]$. Similarly, $V(\mu)$ is linear for all $\mu \in [H_b, H_a]$, where H_i denotes $H(-z_0, \mu_i)$ for $i \in \{a, b\}$. Denote the slope of V as $V'(\phi_b)$ for $\mu \in [\phi(\mu_b), \phi(\mu_a)]$ and $V'(H_b)$ for $\mu \in [H_b, H_b]$. For all $\mu \in [\mu_b, \mu_a]$, we have

$$\begin{aligned}\hat{R}(z, \mu) = {} & -\frac{zW(-z)}{(1-\sigma)A} - \frac{\delta z}{A}\big\{V(\phi(\mu_b)) + V'(\phi_b)[\phi(\mu) - \phi(\mu_b)]\big\} \\ & + \frac{1}{\mu}(1 + z\mu)\big\{V(H_b) + V'(H_b)[H(-z, \mu) - H_b]\big\}.\end{aligned}$$

Because $(1 + z\mu)H(-z, \mu)$ is linear in z, the last two terms in the above expression are linear in z. In this case, part (iii) in (3.11) implies that $\hat{R}(z, \mu)$ is strictly concave in z and twice continuously differentiable in z and μ for all $\mu \in [\mu_b, \mu_a]$. Thus, the optimal choice $z(\mu)$ is unique and, by the supposition, equal to z_0. Since $z_0 < 0$ (see Theorem 3.1), z_0 satisfies the complementary slackness condition, $\hat{R}_1(z_0, \mu) \leq 0$ and $z_0 \geq -1/a_H$. Moreover, in this case, strict supermodularity of $\hat{R}$ implies $\hat{R}_{12}(z, \mu) > 0$ and strictly concavity of $\hat{R}$ in z implies $\hat{R}_{11}(z, \mu) < 0$ for all $\mu \in [\mu_b, \mu_a]$. If $z_0 > -1/a_H$, then $\hat{R}_1(z_0, \mu) = 0$, which implies $dz_0/d\mu = -\hat{R}_{12}/\hat{R}_{11} > 0$. This contradicts the supposition that z_0 is constant for all $\mu \in [\mu_b, \mu_a]$. Thus, $z_0 = -1/a_H$.

Repeat the above argument for all $\mu \in [\phi^i(\mu_b), \phi^i(\mu_a)]$, where $\phi^i(\mu) = \phi(\phi^{i-1}(\mu))$ and $i = 1, 2, \ldots$. For such μ, V is linear and $Z(\mu)$ is the singleton, $\{-a_H^{-1}\}$.

Take an arbitrary $\mu_c \in (\mu_b, \mu_a)$. Since $Z(\phi^i(\mu_c)) = \{-a_H^{-1}\}$ for all positive integers i, then $\lim_{i\to\infty} Z(\phi^i(\mu_c)) = \{-a_H^{-1}\}$. From the definition of $\phi(\mu)$, it is clear that $\phi(a_H) = a_H$, $\phi(a_L) = a_L$, and $\phi(\mu) > \mu$ for all $\mu \in (a_L, a_H)$. Thus, $\lim_{i\to\infty} \phi^i(\mu) = a_H$ for every $\mu \in (a_L, a_H)$ and, particularly, for $\mu = \mu_c$. Because Z is upper hemicontinuous, we conclude that $\{-a_H^{-1}\} \in Z(a_H)$. *Q.E.D.*

C. *Proof of Theorem 5.1*

Fix $\mu \in (a_L, a_H)$ and use the notation $h(\mu) = H(-z(\mu), \mu)$.

(i) Because $H(-z, \mu)$ is increasing in z, $H(-z^+(\mu), \mu) = h^+(\mu)$ and $H(-z^-(\mu), \mu) = h^-(\mu)$. Since $V'(h^+(\mu)) \geq V'(h^-(\mu))$, we can prove that $\hat{R}_1(z^+(\mu), \mu) \geq \hat{R}_1(z^-(\mu), \mu)$ (see the Supplemental Material (Gonzalez and Shi (2010))). However, the optimality of $z(\mu)$ requires $\hat{R}_1(z^+(\mu), \mu) \leq 0 \leq \hat{R}_1(z^-(\mu), \mu)$. It must be true that $\hat{R}_1(z^-(\mu), \mu) = \hat{R}_1(z^+(\mu), \mu) = 0$, which requires that $V'(h^-(\mu)) = V'(h^+(\mu)) = V'(h(\mu))$.

(ii) Let $\{\mu_i\}$ be a sequence with $\mu_i \to \mu$ and $\mu_i \geq \mu_{i+1} \geq \mu$ for all i. Because $\bar{z}(\mu)$ is an increasing function, $\{\bar{z}(\mu_i)\}$ is a decreasing sequence, and $\bar{z}(\mu_i) \geq \bar{z}(\mu)$ for all i. Thus, $\bar{z}(\mu_i) \downarrow z_c$ for some $z_c \geq \bar{z}(\mu)$. On the other hand, the theorem of the maximum implies that the correspondence $Z(\mu)$ is upper hemicontinuous (see Stokey, Lucas, and Prescott (1989, p. 62)). Because $\mu_i \to \mu$, and $\bar{z}(\mu_i) \in Z(\mu_i)$ for each i, upper hemicontinuity of Z implies that there is a subsequence of $\{\bar{z}(\mu_i)\}$ that converges to an element in $Z(\mu)$. This element must be z_c, because all convergent subsequences of a convergent sequence must have the same limit. Thus, $z_c \in Z(\mu)$, and so $z_c \leq \max Z(\mu) = \bar{z}(\mu)$. Therefore, $\bar{z}(\mu_i) \downarrow z_c = \bar{z}(\mu)$, which shows that $\bar{z}(\mu)$ is right-continuous. Similarly, by examining the sequence $\{\mu_i\}$ with $\mu_i \to \mu$ and $\mu \geq \mu_{i+1} \geq \mu_i$ for all i, we can show that $\underline{z}$ is left-continuous.

(iii) Let μ_a be another arbitrary value in the interior of (a_L, a_H). Because $\bar{z}(\mu)$ maximizes $R(-z, \mu)$ for each given μ, then

$$\begin{aligned}(1+r)V(\mu_a) &= b + (1-\sigma)R(-\bar{z}(\mu_a), \mu_a)\\ &\geq b + (1-\sigma)R(-\bar{z}(\mu), \mu_a),\\ (1+r)V(\mu) &= b + (1-\sigma)R(-\bar{z}(\mu), \mu) \geq b + (1-\sigma)R(-\bar{z}(\mu_a), \mu).\end{aligned}$$

For $\mu_a > \mu$, we have

$$\begin{aligned}\frac{R(-\bar{z}(\mu), \mu_a) - R(-\bar{z}(\mu), \mu)}{(1+r)(\mu_a - \mu)} &\leq \frac{V(\mu_a) - V(\mu)}{(1-\sigma)(\mu_a - \mu)}\\ &\leq \frac{R(-\bar{z}(\mu_a), \mu_a) - R(-\bar{z}(\mu_a), \mu)}{(1+r)(\mu_a - \mu)}.\end{aligned}$$

Take the limit $\mu_a \downarrow \mu$. Under (4.2), $V'(H(-\bar{z}(\mu_a), \mu_a))$ exists for each μ_a (see part (i)). Because $\bar{z}(\mu_a)$ is right-continuous, $\lim_{\mu_a \downarrow \mu} \bar{z}(\mu_a) = \bar{z}(\mu)$. Thus, all three ratios above converge to the same limit, $\frac{1}{1-\sigma} V'(\mu^+) = \frac{1}{1+r} R_2(-\bar{z}(\mu), \mu^+)$, where $R_2(-\bar{z}(\mu), \mu^+)$ is given as

$$\begin{aligned}
&\bar{z}(\mu)\left[-\frac{W(-\bar{z}(\mu))}{(1-\sigma)A} - \frac{\delta}{A} V(\phi(\mu)) + V\big(H(-\bar{z}(\mu), \mu)\big)\right] \\
&\quad - \frac{\mu \bar{z}(\mu)\delta}{A} V'(\phi^+(\mu))\phi'(\mu) \\
&\quad + [\mu \bar{z}(\mu) + 1] V'\big(H(-\bar{z}(\mu), \mu)\big) H_2(-\bar{z}(\mu), \mu).
\end{aligned}$$

Similarly, using left-continuity of $\underline{z}(\mu_a)$, we can prove that $V'(\mu^-) = \frac{1-\sigma}{1+r} \times R_2(-\underline{z}(\mu), \mu^-)$.

(iv) From the above expression for R_2 and the relation $R = \mu \hat{R}$, we can verify

$$\begin{aligned}
R_2(-\bar{z}(\mu), \mu^+) &\geq R_2(-\bar{z}(\mu), \mu^-) \\
&\geq \hat{R}(\underline{z}(\mu), \mu^-) + \mu \hat{R}_2(\underline{z}(\mu), \mu^-) = R_2(-\underline{z}(\mu), \mu^-).
\end{aligned}$$

The first inequality comes from strict convexity of V, and it is strict if and only if $V'(\phi^+(\mu)) > V'(\phi^-(\mu))$. The second inequality comes from strict supermodularity of $\hat{R}(z, \mu)$, and it is strict if and only if $\bar{z}(\mu) > \underline{z}(\mu)$. Therefore, $V'(\mu^+) = V'(\mu^-)$ if and only if $V'(\phi(\mu))$ exists and $\bar{z}(\mu) = \underline{z}(\mu)$.

(v) Assume that $V'(\mu_a)$ exists for a particular (interior) μ_a, such as $\mu_a = h(\mu)$ for any arbitrary interior μ. By part (iv), $z(\mu_a)$ is unique and $V'(\phi(\mu_a))$ exists. Recall that $V'(h(\mu_a))$ always exists, by part (i). Since V is now differentiable at all posterior beliefs reached from μ_a under the optimal choice, we can take each of these subsequent nodes and repeat the argument. This shows that the optimal choice is unique and the value function is differentiable at all nodes on the tree generated from μ_a in the equilibrium. *Q.E.D.*

REFERENCES

ACEMOGLU, D., AND R. SHIMER (1999): "Efficient Unemployment Insurance," *Journal of Political Economy*, 107, 893–928. [510,513]

ADDISON, J. T., AND P. PORTUGAL (1989): "Job Displacement, Relative Wage Changes, and Duration of Unemployment," *Journal of Labor Economics*, 7, 281–302. [509-511,529]

ALEXOPOULOS, M., AND T. GLADDEN (2007): "The Effects of Wealth and Unemployment Benefits on Search Behavior and Labor Market Transitions," Manuscript, University of Toronto. [510]

AMIR, R. (1996): "Sensitivity Analysis of Multisector Optimal Economic Dynamics," *Journal of Mathematical Economics*, 25, 123–141. [524]

AMIR, R., L. J. MIRMAN, AND W. R. PERKINS (1991): "One-Sector Nonclassical Optimal Growth: Optimality Conditions and Comparative Dynamics," *International Economic Review*, 32, 625–644. [512,522]

BALVERS, R. J., AND T. F. COSIMANO (1993): "Periodic Learning About a Hidden State Variable," *Journal of Economic Dynamics and Control*, 17, 805–827. [511,521]

BARRON, J. M., J. BISHOP, AND W. C. DUNKELBERG (1985): "Employer Search: The Interviewing and Hiring of New Employees," *The Review of Economics and Statistics*, 67, 43–52. [511]

BLACKWELL, D. (1951): "The Comparison of Experiments," in *Proceedings of the Second Berkeley Symposium on Statistics and Probability*. Berkeley: University of California Press. [516]

BURDETT, K., AND D. T. MORTENSEN (1998): "Wage Differentials, Employer Size, and Unemployment," *International Economic Review*, 39, 257–273. [529]

BURDETT, K., AND T. VISHWANATH (1988): "Declining Reservation Wages and Learning," *Review of Economic Studies*, 55, 655–665. [512,518]

BURDETT, K., S. SHI, AND R. WRIGHT (2001): "Pricing and Matching With Frictions," *Journal of Political Economy*, 109, 1060–1085. [510]

DEGROOT, M. H. (1970): *Optimal Statistical Decisions*. New York: McGraw-Hill. [516]

EASLEY, D., AND N. M. KIEFER (1988): "Controlling a Stochastic Process With Unknown Parameters," *Econometrica*, 56, 1045–1064. [511,521]

EDLIN, A. S., AND C. SHANNON (1998): "Strict Monotonicity in Comparative Statics," *Journal of Economic Theory*, 81, 201–219. [524]

GONZALEZ, F. M., AND S. SHI (2007): "An Equilibrium Theory of Declining Reservation Wages and Learning," Working Paper 292, University of Toronto. [512]

——— (2010): "Supplement to 'An Equilibrium Theory of Learning, Search and Wages'," *Econometrica Supplemental Material*, 78, http://www.econometricsociety.org/ecta/Supmat/8061_proofs.pdf. [523,530,534]

HECKMAN, J. T., AND G. J. BORJAS (1980): "Does Unemployment Cause Future Unemployment? Definitions, Questions and Answers From a Continuous Time Model of Heterogeneity and State Dependence," *Economica*, 47, 247–283. [510]

HOLZER, H. J., L. F. KATZ, AND A. B. KRUEGER (1991): "Job Queues and Wages," *Quarterly Journal of Economics*, 106, 739–768. [511]

LAZEAR, E. (2004): "Firm-Specific Human Capital: A Skill-Weights Approach," Manuscript, Stanford University. [513]

LOCKWOOD, B. (1991): "Information Externalities in the Labour Market and the Duration of Unemployment," *Review of Economic Studies*, 58, 733–753. [510]

MENZIO, G., AND S. SHI (2009): "Block Recursive Equilibria for Stochastic Models of Search on the Job," *Journal of Economic Theory* (forthcoming). [511]

MILGROM, P., AND C. SHANNON (1994): "Monotone Comparative Statistics," *Econometrica*, 62, 157–180. [511,523]

MIRMAN, L. J., O. F. MORAND, AND K. L. REFFETT (2008): "A Qualitative Approach to Markovian Equilibrium in Infinite Horizon Economies With Capital," *Journal of Economic Theory*, 139, 75–98. [512,522]

MOEN, E. R. (1997): "Competitive Search Equilibrium," *Journal of Political Economy*, 105, 385–411. [510,513]

MORTENSEN, D. T. (2003): *Wage Dispersion: Why Are Similar Workers Paid Differently*? Cambridge: MIT Press. [529]

NYARKO, Y. (1994): "On the Convexity of the Value Function in Bayesian Optimal Control Problems," *Economic Theory*, 4, 303–309. [520,524,531,533]

PETERS, M. (1984): "Bertrand Equilibrium With Capacity Constraints and Restricted Mobility," *Econometrica*, 52, 1117–1129. [510]

——— (1991): "Ex ante Price Offers in Matching Games: Non-Steady State," *Econometrica*, 59, 1425–1454. [510]

PISSARIDES, C. A. (1992): "Loss of Skill During Unemployment and the Persistence of Employment Shocks," *Quarterly Journal of Economics*, 107, 1371–1391. [510]

ROYDEN, H. L. (1988): *Real Analysis*. New York: Macmillan Co. [531]

SANTOS, M. (1991): "Smoothness of the Policy Function in Discrete Time Economic Models," *Econometrica*, 59, 1365–1382. [521]

SHI, S. (2001): "Frictional Assignment I: Efficiency," *Journal of Economic Theory*, 98, 232–260. [510]

——— (2009): "Directed Search for Equilibrium Wage-Tenure Contracts," *Econometrica*, 77, 561–584. [511,519]

SHIMER, R. (2008): "The Probability of Finding a Job," *American Economic Review: Papers and Proceedings*, 98, 268–273. [511]

STOKEY, N., R. E. LUCAS, JR., AND E. PRESCOTT (1989): *Recursive Methods in Economic Dynamics*. Cambridge, MA: Harvard University Press. [511,520,521,531,534]

TOPKIS, D. M. (1998): *Supermodularity and Complementarity*. Princeton, NJ: Princeton University Press. [511,521-523,532]

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