1 Pricing and Hedging Asian Options

1.1 Pricing

1.1.1 Proving the PDE

Differentiating the discounted portfolio value with respect to time:

$$\tilde{V}_t = e^{-rt}V_t \tag{1}$$

$$\frac{d}{d_t}\tilde{V}_t = -r\tilde{V}_t + e^{-rt}(\frac{d}{d_t}V_t) \tag{2}$$

We apply Itô's formula to find the derivative with respect to time of our portfolio value. Plugging in the definitions of dY_t, our asian option price, and dS_t, our stock price which follows the Black-Scholes. We use $v = v(t, S_t, Y_t)$ for ease of notation for the value function:

$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + \frac{dY_t}{d_t}\frac{d}{d_y}v + \frac{dS_t}{d_t}\frac{d}{d_x}v + \frac{1}{2}\frac{d^2}{d_{xx}}v$$
$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + \frac{S_t d_t}{d_t}\frac{d}{d_y}v + \frac{S_t r d_t + S_t\sigma dW_t}{d_t}\frac{d}{d_x}v + \frac{1}{2}\sigma^2 S_t^2\frac{d^2}{d_{xx}}v$$
$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + S_t\frac{d}{d_y}v + S_t r\frac{d}{d_x}v + S_t\sigma\frac{d}{d_t}dW_t\frac{d}{d_x}v + \frac{1}{2}\sigma^2 S_t^2\frac{d^2}{d_{xx}}v$$

Simplifying notation by using $\partial_t = \frac{d}{d_t}$ and $x = S_t$:

$$\partial_t V_t = \partial_t V_t + x\partial_y V_t + rx\partial_x V_t + S_t\sigma\partial_t W_t\partial_x V_t + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 V_t \tag{3}$$

Substituting (3) into (2) and setting the change in discounted portfolio value to zero to satisfy the self-financing condition:

$$0 = -r\tilde{V}_t + e^{-rt}(\frac{d}{d_t}V_t)$$
$$r\tilde{V}_t e^{rt} = \partial_t v + x\partial_y v + rx\partial_x v + S_t\sigma\partial_t W_t\partial_x v + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 v$$

Taking advantage of the martingale property and (1) for the portfolio value:

$$rV_t = \partial_t v(t, S_t, Y_t) + x\partial_y v(t, S_t, Y_t) + rx\partial_x v(t, S_t, Y_t) + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 v(t, S_t, Y_t)$$

1.1.2 Boundary Conditions

We begin by more fully parameterizing our Y function and and using the integral form of our asset price with initial price equal to x:

$$\begin{aligned} Y_T &= \int_0^T S_u du \\ Y_{t,x,T} &= \int_t^T \left(x + \int_t^u dS_s ds \right) du \end{aligned}$$

We will re-write our payoff as a function of two variables by utilizing the fact that Y is an integral, and as such can be decomposed into the sum of two parts:

$$\begin{aligned} H(Y_T) &= \left(\frac{1}{T} Y_T - K \right)^+ \\ H(Y_{0,S_0,t}, Y_{t,S_t,T}) &= \left(\frac{1}{T} Y_{0,S_0,t} + \frac{1}{T} Y_{t,S_t,T} - K \right)^+ \end{aligned}$$

This allows us to use the fact that $Y_{0,S_0,t}$ is $\mathcal{F}_t$ measurable, S_t is $\mathcal{F}_t$ is measurable, and $Y_{t,s,T}$ is independent of $\mathcal{F}_t$:

$$\tilde{V}_t = e^{-r(T-t)} \ \mathbb{E}\big[H(y, Y_{t,x,T})|_{x=S_t, y=Y_t} \big]$$

For the condition when $t = T$, the second parameter of our payoff function is the integral over 0 distance and therefore is equal to 0:

$$\begin{aligned} v(T,x,y) &= e^{-r(T-T)} \ \mathbb{E}\big[H(y,0)|_{x=S_t, y=Y_t} \big] \\ v(T,x,y) &= e^0 \ \mathbb{E}\Big[\left(\frac{y}{T} + 0 - K \right)^+ |_{x=x, y=y} \Big] \\ v(T,x,y) &= \left(\frac{y}{T} - K \right)^+ \end{aligned}$$

We look at the condition when $x = 0$:

$$v(T,x,y) = e^{-r(T-t)} \ \mathbb{E}\big[H(y, Y_{t,0,T})|_{x=0, y=Y_t} \big]$$

Recall the formula for Y and use the differential equation for the Black-Scholes model to see what happens if an asset price is ever at zero, it will from then on always stay at zero:

$$
\begin{aligned}
Y_{t,0,T} &= \int_t^T \left(0 + \int_t^u dS_s ds\right) du \\
Y_{t,0,T} &= \int_t^T \left(\int_t^u S_s(rdt + \sigma dW_t) ds\right) du \\
Y_{t,0,T} &= \int_t^T \left(\int_t^u 0(rdt + \sigma dW_t) ds\right) du \\
Y_{t,0,T} &= 0
\end{aligned}
$$

As before, we have a deterministic payoff function:

$$
\begin{aligned}
v(t, 0, y) &= e^{-r(T-t)} \ \mathbb{E}\big[\left(\frac{y}{T} + 0 - K\right)^+ |_{x=x,y=y}\big] \\
v(t, 0, y) &= e^{-r(T-t)} \ \left(\frac{y}{T} - K\right)^+
\end{aligned}
$$

1.2 K = 0

1.2.1 Finding the Closed Form Solution

Using the fact that $K = 0$ and by construction $Y_T \geq 0$ we can remove the maximum:

$$
\begin{aligned}
V_t &= e^{-r(T-t)}\tilde{\mathbb{E}}\left[\left(\frac{1}{T}Y_T - K\right)^+ |\mathcal{F}_t\right] \\
V_t &= e^{-r(T-t)}\tilde{\mathbb{E}}\big[\frac{1}{T}Y_T|\mathcal{F}_t\big]
\end{aligned}
$$

We split Y_T into its two components:

$$
\begin{aligned}
V_t &= e^{-r(T-t)}\tilde{\mathbb{E}}\Big[\frac{1}{T}Y_t + \frac{1}{T}\int_t^T S_u du \ |\mathcal{F}_t\big] \\
V_t &= e^{-r(T-t)}\left(\frac{1}{T}Y_t + \frac{1}{T}\tilde{\mathbb{E}}\Big[\int_t^T S_u du \ |\mathcal{F}_t\big]\right)
\end{aligned}
$$

The expectation and integral can be swapped in this instance, and we use the fact that S_t is a martingale and therefore the discounted expectation $\tilde{\mathbb{E}}[S_u|\mathcal{F}_t] = e^{r(u-t)}S_t$:

$$
V_t = e^{-r(T-t)}\left(\frac{1}{T}Y_t + \frac{S_t}{T}\int_t^T e^{r(u-t)} du \ \right)
$$

At which point we can simply integrate over the change in value due to interest rate:

$$V_t = e^{-r(T-t)}\left(\frac{1}{T}Y_t + \frac{S_t}{T}\left(\frac{1}{r}e^{r(T-t)} - \frac{1}{r}\right)\right)$$

And we obtain the following solution:

$$V_t = \frac{Y_t}{T}e^{-r(T-t)} + \frac{S_t}{rT}(1 - e^{-r(T-t)}) \tag{4}$$

1.2.2 Checking the Solution Against the PDE

We start by creating the partial derivatives of the closed form solution: w

$$\begin{aligned}
\partial_t v(t, S_t, Y_t) &= \frac{rY_t}{T}e^{-r(T-t)} - \frac{S_t}{T}e^{-r(T-t)} \\
\partial_y v(t, S_t, Y_t) &= \frac{1}{T}e^{-r(T-t)} \\
\partial_x v(t, S_t, Y_t) &= \frac{1}{rT} - \frac{1}{rT}e^{-r(T-t)}
\end{aligned}$$

There is clearly no second partial derivative with regards to x.

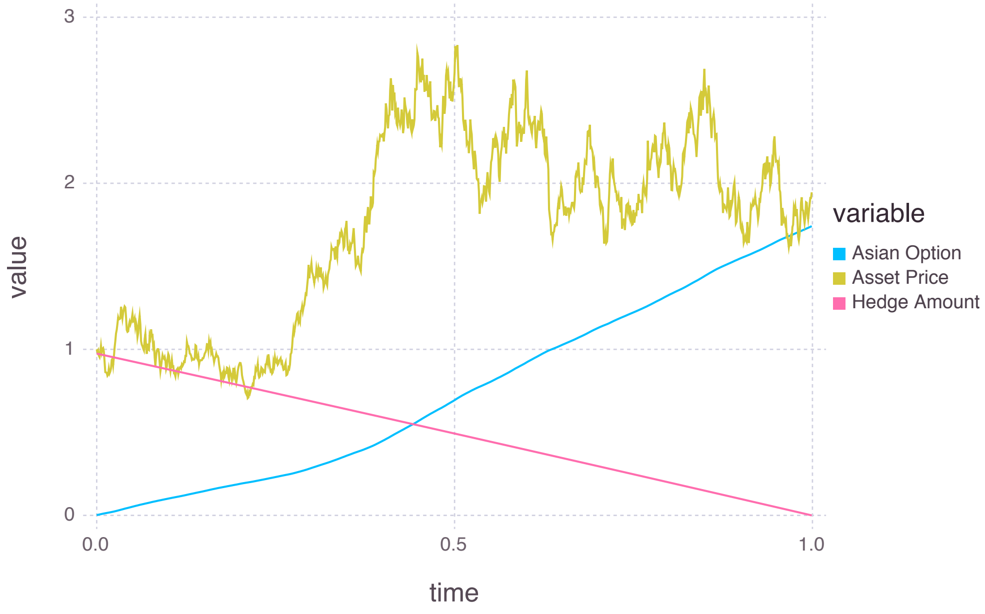
$$\begin{aligned}
rV_t &= \partial_t v(t, S_t, Y_t) + x\partial_y v(t, S_t, Y_t) + rx\partial_x v(t, S_t, Y_t) \\
rV_t &= \frac{rY_t}{T}e^{-r(T-t)} - \frac{S_t}{T}e^{-r(T-t)} + \frac{S_t}{T}e^{-r(T-t)} + \frac{S_t}{T} - \frac{S_t}{T}e^{-r(T-t)} \\
rV_t &= \frac{rY_t}{T}e^{-r(T-t)} + \frac{S_t}{T}(1 - e^{-r(T-t)}) \\
r(\frac{Y_t}{T}e^{-r(T-t)} &+ \frac{S_t}{rT}(1 - e^{-r(T-t)})) = \frac{rY_t}{T}e^{-r(T-t)} + \frac{S_t}{T}(1 - e^{-r(T-t)})
\end{aligned}$$

1.3 Hedging Strategy

We repeat the $\partial_x v(t, S_t, Y_t)$ of (4) from above:

$$\begin{aligned}
b_t &= \partial_x v(t, S_t, Y_t) \\
b_t &= \frac{1}{rT}(1 - e^{-r(T-t)})
\end{aligned}$$

1.4 Simulation



2 Code

```
using Distributions
using Gadfly
using DataFrames

########################################
# Basic Brownian Motion Functions
########################################
function make_walk(steps::AbstractArray{Float64,1}, start = 0.0)
    reduce((a,b) -> append!(a, a[end] + b), [start], steps)
end

make_time(N, T) = range(0, T/N, N)
brownian(N, T, start = 0.0) = make_walk(rand(Normal(0, sqrt(T/N)), N-1), start)

geom(w, mu, sigma, t) = exp(sigma*w + (mu - sigma^2/2)*t)
geometric(brownian, time, mu, sigma) = [geom(b,mu,sigma,t) for (b,t) in zip(brownian, time)]

########################################
# Asians
########################################

# T/N is like 1/T in discrete
# N - i is like (T - t)
discount(r, N, T, i) = exp(-r * T/N * (N - i))

function expected_value(y, s, r, i, N, T)
    d = discount(r, N, T, i)
    y * T/N * d + s/r * T/N * (1 - d)
end

function asian_value(S, r, N, T)
    Y = cumsum(S)
    [expected_value(Y[i], S[i], r, i, N, T) for i in 1:N]
end

hedge(r, i, N, T) = 1/r * (1 - discount(r, N, T, i))
hedge_value(r, N, T) = [hedge(r, i, N, T) for i in 1:N]
```

```
function path_and_price(r, mu, sigma, N, T)
    time = make_time(N, T)
    asset = geometric(brownian(N,T), time, mu, sigma)
    value = asian_value(asset, r, N, T)
    hedge = hedge_value(r, N, T)
    vcat(DataFrame(value = value, time = time, variable = fill("Asian Option", N)),
         DataFrame(value = asset, time = time, variable = fill("Asset Price", N)),
         DataFrame(value = hedge, time = time, variable = fill("Hedge Amount", N)))
end

function plot_asian(r, mu, sigma, N, T)
    d = path_and_price(r, mu, sigma, N, T)
    plot(d, x = :time, y = :value, color = :variable, Geom.line)
end
```