

1 Pricing and Hedging Asian Options

1.1

1.1.1

Differentiating the discounted portfolio value with respect to time:

$$\tilde{V}_t = e^{-rt} V_t \quad (1)$$

$$\frac{d}{dt} \tilde{V}_t = -r \tilde{V}_t + e^{-rt} \left(\frac{d}{dt} V_t \right) \quad (2)$$

We apply Itô's formula to find the derivative with respect to time of our portfolio value. Plugging in the definitions of dY_t , our asian option price, and dS_t , our stock price which follows the Black-Scholes. We use $v = v(t, S_t, Y_t)$ for ease of notation for the value function:

$$\begin{aligned} \frac{d}{dt} V_t &= \frac{d}{dt} v + \frac{dY_t}{dt} \frac{d}{dY} v + \frac{dS_t}{dt} \frac{d}{dS} v + \frac{1}{2} \frac{d^2}{d_{xx}} v \\ \frac{d}{dt} V_t &= \frac{d}{dt} v + \frac{S_t d_t}{dt} \frac{d}{dY} v + \frac{S_t r d_t + S_t \sigma dW_t}{dt} \frac{d}{dS} v + \frac{1}{2} \sigma^2 S_t^2 \frac{d^2}{d_{xx}} v \\ \frac{d}{dt} V_t &= \frac{d}{dt} v + S_t \frac{d}{dY} v + S_t r \frac{d}{dS} v + S_t \sigma \frac{d}{dt} dW_t \frac{d}{dS} v + \frac{1}{2} \sigma^2 S_t^2 \frac{d^2}{d_{xx}} v \end{aligned}$$

Simplifying notation by using $\partial_t = \frac{d}{dt}$ and $x = S_t$:

$$\partial_t V_t = \partial_t V_t + x \partial_y V_t + r x \partial_x V_t + S_t \sigma \partial_t W_t \partial_x V_t + \frac{1}{2} \sigma^2 S_t^2 \partial_{xx}^2 V_t \quad (3)$$

Substituting (3) into (2) and setting the change in discounted portfolio value to zero to satisfy the self-financing condition:

$$\begin{aligned} 0 &= -r \tilde{V}_t + e^{-rt} \left(\frac{d}{dt} V_t \right) \\ r \tilde{V}_t e^{rt} &= \partial_t v + x \partial_y v + r x \partial_x v + S_t \sigma \partial_t W_t \partial_x v + \frac{1}{2} \sigma^2 S_t^2 \partial_{xx}^2 v \end{aligned}$$

Taking advantage of the martingale property and (1) for the portfolio value:

$$r V_t = \partial_t v(t, S_t, Y_t) + x \partial_y v(t, S_t, Y_t) + r x \partial_x v(t, S_t, Y_t) + \frac{1}{2} \sigma^2 S_t^2 \partial_{xx}^2 v(t, S_t, Y_t)$$

1.1.2 Boundary Conditions

We begin by more fully parameterizing our Y function and using the integral form of our asset price with initial price equal to x :

$$Y_T = \int_0^T S_u du$$

$$Y_{t,x,T} = \int_t^T \left(x + \int_t^u dS_s ds \right) du$$

We will re-write our payoff as a function of two variables by utilizing the fact that Y is an integral, and as such can be decomposed into the sum of two parts:

$$H(Y_T) = \left(\frac{1}{T} Y_T - K \right)^+$$

$$H(Y_{0,S_0,t}, Y_{t,S_t,T}) = \left(\frac{1}{T} Y_{0,S_0,t} + \frac{1}{T} Y_{t,S_t,T} - K \right)^+$$

This allows us to use the fact that $Y_{0,S_0,t}$ is \mathcal{F}_t measurable, S_t is \mathcal{F}_t measurable, and $Y_{t,S_t,T}$ is independent of \mathcal{F}_t :

$$\tilde{V}_t = e^{-r(T-t)} \mathbb{E} [H(y, Y_{t,x,T}) |_{x=S_t, y=Y_t}]$$

For the condition when $t = T$, the second parameter of our payoff function is the integral over 0 distance and therefore is equal to 0:

$$v(T, x, y) = e^{-r(T-T)} \mathbb{E} [H(y, 0) |_{x=S_t, y=Y_t}]$$

$$v(T, x, y) = e^0 \mathbb{E} \left[\left(\frac{y}{T} + 0 - K \right)^+ \mid_{x=x, y=y} \right]$$

$$v(T, x, y) = \left(\frac{y}{T} - K \right)^+$$

No idea what to show here, this doesn't seem right:

$$v(T, x, y) = e^{-r(T-t)} \mathbb{E} [H(y, Y_{t,0,T}) |_{x=0, y=Y_t}]$$

1.2 $K = 0$

Using the fact that $K = 0$ and by construction $Y_T \geq 0$ we can remove the maximum:

$$V_t = e^{-r(T-t)} \tilde{\mathbb{E}} \left[\left(\frac{1}{T} Y_T - K \right)^+ | \mathcal{F}_t \right]$$

$$V_t = e^{-r(T-t)} \tilde{\mathbb{E}} \left[\frac{1}{T} Y_T | \mathcal{F}_t \right]$$

We split Y_T into its two components:

$$V_t = e^{-r(T-t)} \tilde{\mathbb{E}} \left[\frac{1}{T} Y_t + \frac{1}{T} \int_t^T S_u du | \mathcal{F}_t \right]$$

$$V_t = e^{-r(T-t)} \left(\frac{1}{T} Y_t + \tilde{\mathbb{E}} \left[\frac{T-t}{T} \left(\frac{1}{T-t} \right) \int_t^T S_u du | \mathcal{F}_t \right] \right)$$

$$V_t = e^{-r(T-t)} \left(\frac{1}{T} Y_t + \frac{T-t}{T} \tilde{\mathbb{E}} \left[\mathbb{E}[S_{i \in [t, T]}] | \mathcal{F}_t \right] \right)$$

Given nested expectations we use the most binding, and using the fact that under the risk-neutral probability our asset price is a martingale, we know that $\mathbb{E}[S_{i \in [t, T]} | \mathcal{F}_t] = S_t$:

$$V_t = e^{-r(T-t)} \left(\frac{1}{T} Y_t + \frac{T-t}{T} S_t \right)$$

$$V_t = e^{-r(T-t)} \left(\frac{1}{T} Y_t + S_t - \frac{t}{T} S_t \right)$$