1 Pricing and Hedging Asian Options

1.1

1.1.1

Differentiating the discounted portfolio value with respect to time:

$$\tilde{V_t} = e^{-rt}V_t \tag{1}$$

$$\frac{d}{d_t}\tilde{V_t} = -r\tilde{V_t} + e^{-rt}(\frac{d}{d_t}V_t) \tag{2}$$

We apply Itô's formula to find the derivative with respect to time of our portfolio value. Plugging in the definitions of dY_t, our asian option price, and dS_t, our stock price which follows the Black-Scholes. We use $v = v(t, S_t, Y_t)$ for ease of notation for the value function:

$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + \frac{dY_t}{d_t}\frac{d}{d_y}v + \frac{dS_t}{d_t}\frac{d}{d_x}v + \frac{1}{2}\frac{d^2}{d_{xx}}v$$
$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + \frac{S_t d_t}{d_t}\frac{d}{d_y}v + \frac{S_t r d_t + S_t\sigma dW_t}{d_t}\frac{d}{d_x}v + \frac{1}{2}\sigma^2 S_t^2\frac{d^2}{d_{xx}}v$$
$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + S_t\frac{d}{d_y}v + S_t r\frac{d}{d_x}v + S_t\sigma\frac{d}{d_t}dW_t\frac{d}{d_x}v + \frac{1}{2}\sigma^2 S_t^2\frac{d^2}{d_{xx}}v$$

Simplifying notation by using $\partial_t = \frac{d}{d_t}$ and $x = S_t$:

$$\partial_t V_t = \partial_t V_t + x\partial_y V_t + rx\partial_x V_t + S_t\sigma\partial_t W_t\partial_x V_t + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 V_t \tag{3}$$

Substituting (3) into (2) and setting the change in discounted portfolio value to zero to satisfy the self-financing condition:

$$0 = -r\tilde{V_t} + e^{-rt}(\frac{d}{d_t}V_t)$$
$$r\tilde{V_t}e^{rt} = \partial_t v + x\partial_y v + rx\partial_x v + S_t\sigma\partial_t W_t\partial_x v + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 v$$

Taking advantage of the martingale property and (1) for the portfolio value:

$$rV_t = \partial_t v(t, S_t, Y_t) + x\partial_y v(t, S_t, Y_t) + rx\partial_x v(t, S_t, Y_t) + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 v(t, S_t, Y_t)$$

1.1.2 Boundary Conditions

We begin by more fully parameterizing our Y function and and using the integral form of our asset price with initial price equal to x:

$$
\begin{aligned}
Y_T &= \int_0^T S_u du \\
Y_{t,x,T} &= \int_t^T \left(x + \int_t^u dS_s ds \right) du
\end{aligned}
$$

We will re-write our payoff as a function of two variables by utilizing the fact that Y is an integral, and as such can be decomposed into the sum of two parts:

$$
\begin{aligned}
H(Y_T) &= \left(\frac{1}{T} Y_T - K \right)^+ \\
H(Y_{0,S_0,t}, Y_{t,S_t,T}) &= \left(\frac{1}{T} Y_{0,S_0,t} + \frac{1}{T} Y_{t,S_t,T} - K \right)^+
\end{aligned}
$$

This allows us to use the fact that $Y_{0,S_0,t}$ is $\mathcal{F}_t$ measurable, S_t is $\mathcal{F}_t$ is measurable, and $Y_{t,s,T}$ is independent of $\mathcal{F}_t$:

$$
\tilde{V}_t = e^{-r(T-t)} \ \mathbb{E}\big[H(y, Y_{t,x,T})|_{x=S_t, y=Y_t} \big]
$$

For the condition when $t = T$, the second parameter of our payoff function is the integral over 0 distance and therefore is equal to 0:

$$
\begin{aligned}
v(T,x,y) &= e^{-r(T-T)} \ \mathbb{E}\big[H(y, 0)|_{x=S_t, y=Y_t} \big] \\
v(T,x,y) &= e^{0} \ \mathbb{E}\Big[\left(\frac{y}{T} + 0 - K \right)^+ |_{x=x, y=y} \Big] \\
v(T,x,y) &= \left(\frac{y}{T} - K \right)^+
\end{aligned}
$$

We look at the condition when $x = 0$:

$$
v(T,x,y) = e^{-r(T-t)} \ \mathbb{E}\big[H(y, Y_{t,0,T})|_{x=0, y=Y_t} \big]
$$

Recall the formula for Y and use the differential equation for the Black-Scholes model to see what happens if an asset price is ever at zero, it will from then on always stay at zero:

$$Y_{t,0,T} = \int_t^T \left(0 + \int_t^u dS_s ds\right) du$$
$$Y_{t,0,T} = \int_t^T \left(\int_t^u S_s(rdt + \sigma dW_t) ds\right) du$$
$$Y_{t,0,T} = \int_t^T \left(\int_t^u 0(rdt + \sigma dW_t) ds\right) du$$
$$Y_{t,0,T} = 0$$

As before, we have a deterministic payoff function:

$$v(t,0,y) = e^{-r(T-t)}\ \mathbb{E}\big[\left(\frac{y}{T} + 0 - K\right)^+ |_{x=x,y=y}\big]$$
$$v(t,0,y) = e^{-r(T-t)}\ \left(\frac{y}{T} - K\right)^+$$

1.2 K = 0

Using the fact that $K = 0$ and by construction $Y_T \geq 0$ we can remove the maximum:

$$V_t = e^{-r(T-t)}\tilde{\mathbb{E}}\left[\left(\frac{1}{T}Y_T - K\right)^+ |\mathcal{F}_t\right]$$
$$V_t = e^{-r(T-t)}\tilde{\mathbb{E}}\big[\frac{1}{T}Y_T|\mathcal{F}_t\big]$$

We split Y_T into its two components:

$$V_t = e^{-r(T-t)}\tilde{\mathbb{E}}\left[\frac{1}{T}Y_t + \frac{1}{T}\int_t^T S_u du\ |\mathcal{F}_t\right]$$
$$V_t = e^{-r(T-t)}\left(\frac{1}{T}Y_t + \tilde{\mathbb{E}}\left[\frac{T-t}{T}\left(\frac{1}{T-t}\right)\int_t^T S_u du\ |\mathcal{F}_t\right]\right)$$
$$V_t = e^{-r(T-t)}\left(\frac{1}{T}Y_t + \frac{T-t}{T}\tilde{\mathbb{E}}\left[\mathbb{E}[S_{i\in[t,T]}]\ |\mathcal{F}_t\right]\right)$$

Given nested expectations we use the most binding, and using the fact that under the risk-nuetral probability our asset price is a martingale, we know that $\mathbb{E}[S_{i\in[t,T]}|\mathcal{F}_t] = S_t$:

$$V_t = e^{-r(T-t)}\left(\frac{1}{T}Y_t + \frac{T-t}{T}S_t\right)$$
$$V_t = e^{-r(T-t)}\left(\frac{1}{T}Y_t + S_t - \frac{t}{T}S_t\right)$$