1 Pricing and Hedging Asian Options

1.1 Pricing

1.1.1 Proving the PDE

Differentiating the discounted portfolio value with respect to time:

$$\tilde{V}_t = e^{-rt} V_t \tag{1}$$

$$\frac{d}{d_t}\tilde{V}_t = -r\tilde{V}_t + e^{-rt}(\frac{d}{d_t}V_t) \tag{2}$$

We apply Itô's formula to find the derivative with respect to time of our portfolio value. Plugging in the definitions of dY_t, our asian option price, and dS_t, our stock price which follows the Black-Scholes. We use $v = v(t, S_t, Y_t)$ for ease of notation for the value function:

$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + \frac{dY_t}{d_t}\frac{d}{d_y}v + \frac{dS_t}{d_t}\frac{d}{d_x}v + \frac{1}{2}\frac{d^2}{d_{xx}}v$$

$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + \frac{S_t d_t}{d_t}\frac{d}{d_y}v + \frac{S_t r d_t + S_t\sigma dW_t}{d_t}\frac{d}{d_x}v + \frac{1}{2}\sigma^2 S_t^2\frac{d^2}{d_{xx}}v$$

$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + S_t\frac{d}{d_y}v + S_t r\frac{d}{d_x}v + S_t\sigma\frac{d}{d_t}dW_t\frac{d}{d_x}v + \frac{1}{2}\sigma^2 S_t^2\frac{d^2}{d_{xx}}v$$

Simplifying notation by using $\partial_t = \frac{d}{d_t}$ and $x = S_t$:

$$\partial_t V_t = \partial_t V_t + x\partial_y V_t + rx\partial_x V_t + S_t\sigma\partial_t W_t\partial_x V_t + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 V_t \tag{3}$$

Substituting (3) into (2) and setting the change in discounted portfolio value to zero to satisfy the self-financing condition:

$$0 = -r\tilde{V}_t + e^{-rt}(\frac{d}{d_t}V_t)$$

$$r\tilde{V}_t e^{rt} = \partial_t v + x\partial_y v + rx\partial_x v + S_t\sigma\partial_t W_t\partial_x v + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 v$$

Taking advantage of the martingale property and (1) for the portfolio value:

$$rV_t = \partial_t v(t, S_t, Y_t) + x\partial_y v(t, S_t, Y_t) + rx\partial_x v(t, S_t, Y_t) + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 v(t, S_t, Y_t)$$

1.1.2 Boundary Conditions

We begin by more fully parameterizing our Y function and and using the integral form of our asset price with initial price equal to x:

$$Y_T = \int_0^T S_u du$$
$$Y_{t,x,T} = \int_t^T \left(x + \int_t^u dS_s ds\right) du$$

We will re-write our payoff as a function of two variables by utilizing the fact that Y is an integral, and as such can be decomposed into the sum of two parts:

$$H(Y_T) = \left(\frac{1}{T}Y_T - K\right)^+$$
$$H(Y_{0,S_0,t}, Y_{t,S_t,T}) = \left(\frac{1}{T}Y_{0,S_0,t} + \frac{1}{T}Y_{t,S_t,T} - K\right)^+$$

This allows us to use the fact that $Y_{0,S_0,t}$ is $\mathcal{F}_t$ measurable, S_t is $\mathcal{F}_t$ is measurable, and $Y_{t,s,T}$ is independent of $\mathcal{F}_t$:

$$\tilde{V}_t = e^{-r(T-t)}\ \mathbb{E}\big[H(y, Y_{t,x,T})|_{x=S_t,y=Y_t}\big]$$

For the condition when $t = T$, the second parameter of our payoff function is the integral over 0 distance and therefore is equal to 0:

$$v(T,x,y) = e^{-r(T-T)}\ \mathbb{E}\big[H(y,0)|_{x=S_t,y=Y_t}\big]$$
$$v(T,x,y) = e^0\ \mathbb{E}\Big[\left(\frac{y}{T} + 0 - K\right)^+|_{x=x,y=y}\Big]$$
$$v(T,x,y) = \left(\frac{y}{T} - K\right)^+$$

We look at the condition when $x = 0$:

$$v(T,x,y) = e^{-r(T-t)}\ \mathbb{E}\big[H(y, Y_{t,0,T})|_{x=0,y=Y_t}\big]$$

Recall the formula for Y and use the differential equation for the Black-Scholes model to see what happens if an asset price is ever at zero, it will from then on always stay at zero:

$$
\begin{aligned}
Y_{t,0,T} &= \int_t^T \Big(0 + \int_t^u dS_s ds\Big) du \\
Y_{t,0,T} &= \int_t^T \Big(\int_t^u S_s(rdt + \sigma dW_t) ds\Big) du \\
Y_{t,0,T} &= \int_t^T \Big(\int_t^u 0(rdt + \sigma dW_t) ds\Big) du \\
Y_{t,0,T} &= 0
\end{aligned}
$$

As before, we have a deterministic payoff function:

$$
\begin{aligned}
v(t,0,y) &= e^{-r(T-t)}\ \mathbb{E}\Big[\Big(\frac{y}{T} + 0 - K\Big)^+ |_{x=x,y=y}\Big] \\
v(t,0,y) &= e^{-r(T-t)}\ \Big(\frac{y}{T} - K\Big)^+
\end{aligned}
$$

1.2 K = 0

1.2.1 Finding the Closed Form Solution

Using the fact that $K = 0$ and by construction $Y_T \geq 0$ we can remove the maximum:

$$
\begin{aligned}
V_t &= e^{-r(T-t)}\tilde{\mathbb{E}}\Big[\Big(\frac{1}{T}Y_T - K\Big)^+ |\mathcal{F}_t\Big] \\
V_t &= e^{-r(T-t)}\tilde{\mathbb{E}}\big[\frac{1}{T}Y_T|\mathcal{F}_t\big]
\end{aligned}
$$

We split Y_T into its two components:

$$
\begin{aligned}
V_t &= e^{-r(T-t)}\tilde{\mathbb{E}}\Big[\frac{1}{T}Y_t + \frac{1}{T}\int_t^T S_u du\ |\mathcal{F}_t\big] \\
V_t &= e^{-r(T-t)}\Big(\frac{1}{T}Y_t + \frac{1}{T}\tilde{\mathbb{E}}\Big[\int_t^T S_u du\ |\mathcal{F}_t\big]\Big)
\end{aligned}
$$

The expectation and integral can be swapped in this instance, and we use the fact that S_t is a martingale and therefore the discounted expectation $\tilde{\mathbb{E}}[S_u|\mathcal{F}_t] = e^{r(u-t)}S_t$:

$$
V_t = e^{-r(T-t)}\Big(\frac{1}{T}Y_t + \frac{S_t}{T}\int_t^T e^{r(u-t)} du\ \Big)
$$

At which point we can simply integrate over the change in value due to interest rate:

$$V_t = e^{-r(T-t)}\left(\frac{1}{T}Y_t + \frac{S_t}{T}\left(\frac{1}{r}e^{r(T-t)} - \frac{1}{r}\right)\right)$$

And we obtain the following solution:

$$V_t = \frac{Y_t}{T}e^{-r(T-t)} + \frac{S_t}{rT}(1 - e^{-r(T-t)}) \tag{4}$$

1.2.2 Checking the Solution Against the PDE

We start by creating the partial derivatives of the closed form solution: w

$$\begin{aligned}
\partial_t v(t, S_t, Y_t) &= \frac{rY_t}{T}e^{-r(T-t)} - \frac{S_t}{T}e^{-r(T-t)} \\
\partial_y v(t, S_t, Y_t) &= \frac{1}{T}e^{-r(T-t)} \\
\partial_x v(t, S_t, Y_t) &= \frac{1}{rT} - \frac{1}{rT}e^{-r(T-t)}
\end{aligned}$$

There is clearly no second partial derivative with regards to x.

$$\begin{aligned}
rV_t &= \partial_t v(t, S_t, Y_t) + x\partial_y v(t, S_t, Y_t) + rx\partial_x v(t, S_t, Y_t) \\
rV_t &= \frac{rY_t}{T}e^{-r(T-t)} - \frac{S_t}{T}e^{-r(T-t)} + \frac{S_t}{T}e^{-r(T-t)} + \frac{S_t}{T} - \frac{S_t}{T}e^{-r(T-t)} \\
rV_t &= \frac{rY_t}{T}e^{-r(T-t)} + \frac{S_t}{T}(1 - e^{-r(T-t)}) \\
r(\frac{Y_t}{T}e^{-r(T-t)} &+ \frac{S_t}{rT}(1 - e^{-r(T-t)})) = \frac{rY_t}{T}e^{-r(T-t)} + \frac{S_t}{T}(1 - e^{-r(T-t)})
\end{aligned}$$

1.3 Hedging Strategy

We repeat the $\partial_x v(t, S_t, Y_t)$ of (4) from above:

$$\begin{aligned}
b_t &= \partial_x v(t, S_t, Y_t) \\
b_t &= \frac{1}{rT}(1 - e^{-r(T-t)})
\end{aligned}$$