1 Pricing and Hedging Asian Options

1.1

1.1.1

Differentiating the discounted portfolio value with respect to time:

$$\tilde{V}_t = e^{-rt} V_t \tag{1}$$
$$\frac{d}{d_t}\tilde{V}_t = -r\tilde{V}_t + e^{-rt}(\frac{d}{d_t}V_t) \tag{2}$$

We apply Itô's formula to find the derivative with respect to time of our portfolio value. Plugging in the definitions of dY_t, our asian option price, and dS_t, our stock price which follows the Black-Scholes. We use $v = v(t, S_t, Y_t)$ for ease of notation for the value function:

$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + \frac{dY_t}{d_t}\frac{d}{d_y}v + \frac{dS_t}{d_t}\frac{d}{d_x}v + \frac{1}{2}\frac{d^2}{d_{xx}}v$$
$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + \frac{S_t d_t}{d_t}\frac{d}{d_y}v + \frac{S_t r d_t + S_t\sigma dW_t}{d_t}\frac{d}{d_x}v + \frac{1}{2}\sigma^2 S_t^2\frac{d^2}{d_{xx}}v$$
$$\frac{d}{d_t}V_t = \frac{d}{d_t}v + S_t\frac{d}{d_y}v + S_t r\frac{d}{d_x}v + S_t\sigma\frac{d}{d_t}dW_t\frac{d}{d_x}v + \frac{1}{2}\sigma^2 S_t^2\frac{d^2}{d_{xx}}v$$

Simplifying notation by using $\partial_t = \frac{d}{d_t}$ and $x = S_t$:

$$\partial_t V_t = \partial_t V_t + x\partial_y V_t + rx\partial_x V_t + S_t\sigma\partial_t W_t\partial_x V_t + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 V_t \tag{3}$$

Substituting (3) into (2) and setting the change in discounted portfolio value to zero to satisfy the self-financing condition:

$$0 = -r\tilde{V}_t + e^{-rt}(\frac{d}{d_t}V_t)$$
$$r\tilde{V}_t e^{rt} = \partial_t v + x\partial_y v + rx\partial_x v + S_t\sigma\partial_t W_t\partial_x v + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 v$$

Taking advantage of the martingale property and (1) for the portfolio value:

$$rV_t = \partial_t v(t, S_t, Y_t) + x\partial_y v(t, S_t, Y_t) + rx\partial_x v(t, S_t, Y_t) + \frac{1}{2}\sigma^2 S_t^2\partial_{xx}^2 v(t, S_t, Y_t)$$

1.1.2 Boundary Conditions

We begin by more fully parameterizing our Y function and making use of the fact that our asset price is the result of geometric brownian motion steps:

$$Y_T = \int_0^T S_u du$$
$$Y_{t,x,T} = \int_t^T x \int_t^u W_x dx\ du$$

We will re-write our payoff as a function of two variables by utilizing the fact that Y is an integral, and as such can be decomposed into the sum of two parts:

$$H(Y_T) = \left(\frac{1}{T}Y_T - K\right)^+$$
$$H(Y_{0,S_0,t}, Y_{t,S_t,T}) = \left(\frac{1}{T}Y_{0,S_0,t} + \frac{1}{T}Y_{t,S_t,T} - K\right)^+$$

This allows us to use the fact that $Y_{0,S_0,t}$ is $\mathcal{F}_t$ measurable, S_t is $\mathcal{F}_t$ is measurable, and $Y_{t,s,T}$ is independent of $\mathcal{F}_t$:

$$\tilde{V}_t = e^{-r(T-t)}\ \mathbb{E}\big[H(y, Y_{t,x,T})|_{x=S_t,y=Y_t}\big]$$

And we solve with $y = Y_t$ and $S_t = x$ for our boundary conditions. The first when $x = 0$, it is easy to see that:

$$Y_{t,0,T} = \int_t^T 0 \int_t^u W_x dx\ du = 0$$

Therefore:

$$v(t,0,y) = e^{-r(T-t)}\ \mathbb{E}\Big[\left(\frac{y}{T} + \frac{Y_{t,0,T}}{T} - K\right)^+|_{x=0,y=y}\Big]$$
$$v(t,0,y) = e^{-r(T-t)}\ \mathbb{E}\Big[\left(\frac{y}{T} - K\right)^+|_{x=0,y=y}\Big]$$
$$v(t,0,y) = e^{-r(T-t)}\ \left(\frac{y}{T} - K\right)^+$$

Similarly for the condition when $t = T$, the second parameter of our payoff function is the integral over 0 distance and therefore is equal to 0:

$$v(T, x, y) = e^{-r(T-t)}\ \mathbb{E}\Big[\left(\frac{y}{T} + \frac{Y_{T,x,T}}{T} - K\right)^{+} |_{x=x, y=y}\Big]$$

$$v(T, x, y) = e^{-r(T-t)}\ \mathbb{E}\Big[\left(\frac{y}{T} + 0 - K\right)^{+} |_{x=x, y=y}\Big]$$

$$v(T, x, y) = e^{-r(T-t)}\ \left(\frac{y}{T} - K\right)^{+}$$