

1 Pricing and Hedging Asian Options

1.1 Pricing

1.1.1 Proving the PDE

Differentiating the discounted portfolio value with respect to time:

$$\tilde{V}_t = e^{-rt} V_t \quad (1)$$

$$\frac{d}{dt} \tilde{V}_t = -r \tilde{V}_t + e^{-rt} \left(\frac{d}{dt} V_t \right) \quad (2)$$

We apply Itô's formula to find the derivative with respect to time of our portfolio value. Plugging in the definitions of dY_t , our asian option price, and dS_t , our stock price which follows the Black-Scholes. We use $v = v(t, S_t, Y_t)$ for ease of notation for the value function:

$$\begin{aligned} \frac{d}{dt} V_t &= \frac{d}{dt} v + \frac{dY_t}{dt} \frac{d}{dY_t} v + \frac{dS_t}{dt} \frac{d}{dS_t} v + \frac{1}{2} \frac{d^2}{d_{xx}} v \\ \frac{d}{dt} V_t &= \frac{d}{dt} v + \frac{S_t d_t}{dt} \frac{d}{dY_t} v + \frac{S_t r d_t + S_t \sigma dW_t}{dt} \frac{d}{dS_t} v + \frac{1}{2} \sigma^2 S_t^2 \frac{d^2}{d_{xx}} v \\ \frac{d}{dt} V_t &= \frac{d}{dt} v + S_t \frac{d}{dY_t} v + S_t r \frac{d}{dS_t} v + S_t \sigma \frac{d}{dt} dW_t \frac{d}{dS_t} v + \frac{1}{2} \sigma^2 S_t^2 \frac{d^2}{d_{xx}} v \end{aligned}$$

Simplifying notation by using $\partial_t = \frac{d}{dt}$ and $x = S_t$:

$$\partial_t V_t = \partial_t V_t + x \partial_y V_t + r x \partial_x V_t + S_t \sigma \partial_t W_t \partial_x V_t + \frac{1}{2} \sigma^2 S_t^2 \partial_{xx}^2 V_t \quad (3)$$

Substituting (3) into (2) and setting the change in discounted portfolio value to zero to satisfy the self-financing condition:

$$\begin{aligned} 0 &= -r \tilde{V}_t + e^{-rt} \left(\frac{d}{dt} V_t \right) \\ r \tilde{V}_t e^{rt} &= \partial_t v + x \partial_y v + r x \partial_x v + S_t \sigma \partial_t W_t \partial_x v + \frac{1}{2} \sigma^2 S_t^2 \partial_{xx}^2 v \end{aligned}$$

Taking advantage of the martingale property and (1) for the portfolio value:

$$r V_t = \partial_t v(t, S_t, Y_t) + x \partial_y v(t, S_t, Y_t) + r x \partial_x v(t, S_t, Y_t) + \frac{1}{2} \sigma^2 S_t^2 \partial_{xx}^2 v(t, S_t, Y_t)$$

1.1.2 Boundary Conditions

We begin by more fully parameterizing our Y function and using the integral form of our asset price with initial price equal to x :

$$Y_T = \int_0^T S_u du$$

$$Y_{t,x,T} = \int_t^T \left(x + \int_t^u dS_s ds \right) du$$

We will re-write our payoff as a function of two variables by utilizing the fact that Y is an integral, and as such can be decomposed into the sum of two parts:

$$H(Y_T) = \left(\frac{1}{T} Y_T - K \right)^+$$

$$H(Y_{0,S_0,t}, Y_{t,S_t,T}) = \left(\frac{1}{T} Y_{0,S_0,t} + \frac{1}{T} Y_{t,S_t,T} - K \right)^+$$

This allows us to use the fact that $Y_{0,S_0,t}$ is \mathcal{F}_t measurable, S_t is \mathcal{F}_t measurable, and $Y_{t,S_t,T}$ is independent of \mathcal{F}_t :

$$\tilde{V}_t = e^{-r(T-t)} \mathbb{E} [H(y, Y_{t,x,T}) |_{x=S_t, y=Y_t}]$$

For the condition when $t = T$, the second parameter of our payoff function is the integral over 0 distance and therefore is equal to 0:

$$v(T, x, y) = e^{-r(T-T)} \mathbb{E} [H(y, 0) |_{x=S_t, y=Y_t}]$$

$$v(T, x, y) = e^0 \mathbb{E} \left[\left(\frac{y}{T} + 0 - K \right)^+ \mid_{x=x, y=y} \right]$$

$$v(T, x, y) = \left(\frac{y}{T} - K \right)^+$$

We look at the condition when $x = 0$:

$$v(T, x, y) = e^{-r(T-t)} \mathbb{E} [H(y, Y_{t,0,T}) |_{x=0, y=Y_t}]$$

Recall the formula for Y and use the differential equation for the Black-Scholes model to see what happens if an asset price is ever at zero, it will

from then on always stay at zero:

$$\begin{aligned} Y_{t,0,T} &= \int_t^T \left(0 + \int_t^u dS_s ds \right) du \\ Y_{t,0,T} &= \int_t^T \left(\int_t^u S_s (rdt + \sigma dW_t) ds \right) du \\ Y_{t,0,T} &= \int_t^T \left(\int_t^u 0 (rdt + \sigma dW_t) ds \right) du \\ Y_{t,0,T} &= 0 \end{aligned}$$

As before, we have a deterministic payoff function:

$$\begin{aligned} v(t, 0, y) &= e^{-r(T-t)} \mathbb{E} \left[\left(\frac{y}{T} + 0 - K \right)^+ \mid x=x, y=y \right] \\ v(t, 0, y) &= e^{-r(T-t)} \left(\frac{y}{T} - K \right)^+ \end{aligned}$$

1.2 $K = 0$

1.2.1 Finding the Closed Form Solution

Using the fact that $K = 0$ and by construction $Y_T \geq 0$ we can remove the maximum:

$$\begin{aligned} V_t &= e^{-r(T-t)} \tilde{\mathbb{E}} \left[\left(\frac{1}{T} Y_T - K \right)^+ \mid \mathcal{F}_t \right] \\ V_t &= e^{-r(T-t)} \tilde{\mathbb{E}} \left[\frac{1}{T} Y_T \mid \mathcal{F}_t \right] \end{aligned}$$

We split Y_T into its two components:

$$\begin{aligned} V_t &= e^{-r(T-t)} \tilde{\mathbb{E}} \left[\frac{1}{T} Y_t + \frac{1}{T} \int_t^T S_u du \mid \mathcal{F}_t \right] \\ V_t &= e^{-r(T-t)} \left(\frac{1}{T} Y_t + \frac{1}{T} \tilde{\mathbb{E}} \left[\int_t^T S_u du \mid \mathcal{F}_t \right] \right) \end{aligned}$$

The expectation and integral can be swapped in this instance, and we use the fact that S_t is a martingale and therefore the discounted expectation $\tilde{\mathbb{E}}[S_u \mid \mathcal{F}_t] = e^{r(u-t)} S_t$:

$$V_t = e^{-r(T-t)} \left(\frac{1}{T} Y_t + \frac{S_t}{T} \int_t^T e^{r(u-t)} du \right)$$

At which point we can simply integrate over the change in value due to interest rate:

$$V_t = e^{-r(T-t)} \left(\frac{1}{T} Y_t + \frac{S_t}{T} \left(\frac{1}{r} e^{r(T-t)} - \frac{1}{r} \right) \right)$$

And we obtain the following solution:

$$V_t = \frac{Y_t}{T} e^{-r(T-t)} + \frac{S_t}{rT} (1 - e^{-r(T-t)}) \quad (4)$$

1.2.2 Checking the Solution Against the PDE

We start by creating the partial derivatives of the closed form solution: w

$$\partial_t v(t, S_t, Y_t) = \frac{rY_t}{T} e^{-r(T-t)} - \frac{S_t}{T} e^{-r(T-t)}$$

$$\partial_y v(t, S_t, Y_t) = \frac{1}{T} e^{-r(T-t)}$$

$$\partial_x v(t, S_t, Y_t) = \frac{1}{rT} - \frac{1}{rT} e^{-r(T-t)}$$

There is clearly no second partial derivative with regards to x.

$$rV_t = \partial_t v(t, S_t, Y_t) + x \partial_y v(t, S_t, Y_t) + rx \partial_x v(t, S_t, Y_t)$$

$$rV_t = \frac{rY_t}{T} e^{-r(T-t)} - \frac{S_t}{T} e^{-r(T-t)} + \frac{S_t}{T} e^{-r(T-t)} + \frac{S_t}{T} - \frac{S_t}{T} e^{-r(T-t)}$$

$$rV_t = \frac{rY_t}{T} e^{-r(T-t)} + \frac{S_t}{T} (1 - e^{-r(T-t)})$$

$$r \left(\frac{Y_t}{T} e^{-r(T-t)} + \frac{S_t}{rT} (1 - e^{-r(T-t)}) \right) = \frac{rY_t}{T} e^{-r(T-t)} + \frac{S_t}{T} (1 - e^{-r(T-t)})$$

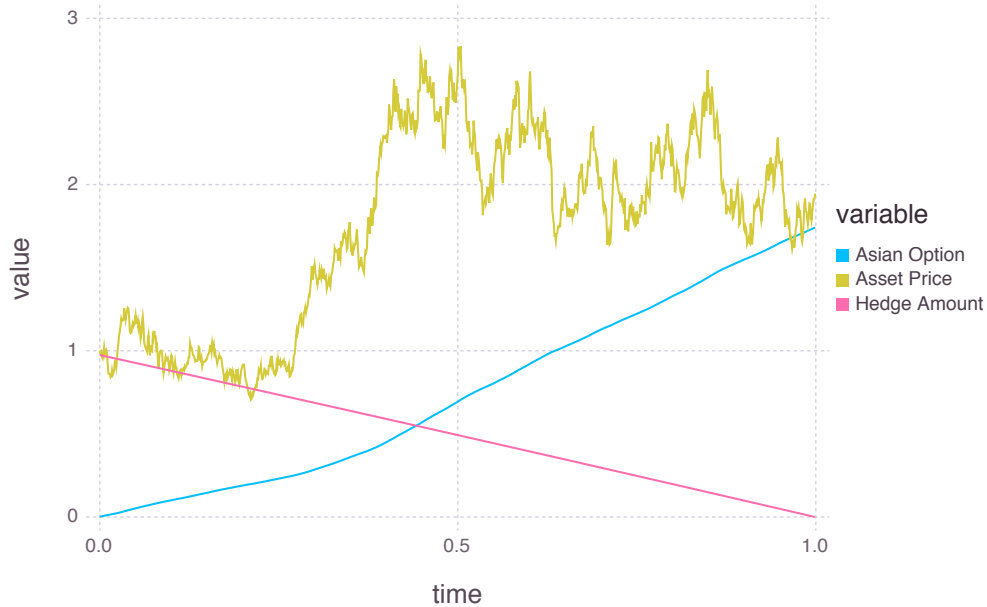
1.3 Hedging Strategy

We repeat the $\partial_x v(t, S_t, Y_t)$ of (4) from above:

$$b_t = \partial_x v(t, S_t, Y_t)$$

$$b_t = \frac{1}{rT} (1 - e^{-r(T-t)})$$

1.4 Simulation



2 Code

```
using Distributions
using Gadfly
using DataFrames

#####
# Basic Brownian Motion Functions
#####
function make_walk(steps::AbstractArray{Float64,1}, start = 0.0)
    reduce((a,b) -> append!(a, a[end] + b), [start], steps)
end

make_time(N, T) = range(0, T/N, N)
brownian(N, T, start = 0.0) = make_walk(rand(Normal(0, sqrt(T/N))), N-1, start)

geom(w, mu, sigma, t) = exp(sigma*w + (mu - sigma^2/2)*t)
geometric(brownian, time, mu, sigma) = [geom(b,mu,sigma,t) for (b,t) in zip(brownian, time)]

#####
# Asians
#####

# T/N is like 1/T in discrete
# N - i is like (T - t)
discount(r, N, T, i) = exp(-r * T/N * (N - i))

function expected_value(y, s, r, i, N, T)

    d = discount(r, N, T, i)
    y * T/N * d + s/r * T/N * (1 - d)
end

function asian_value(S, r, N, T)
    Y = cumsum(S)
    [expected_value(Y[i], S[i], r, i, N, T) for i in 1:N]
end

hedge(r, i, N, T) = 1/r * (1 - discount(r, N, T, i))
```

```
hedge_value(r, N, T) = [hedge(r, i, N, T) for i in 1:N]

function path_and_price(r, mu, sigma, N, T)
    time = make_time(N, T)
    asset = geometric(brownian(N,T), time, mu, sigma)
    value = asian_value(asset, r, N, T)
    hedge = hedge_value(r, N, T)
    vcat(DataFrame(value = value, time = time, variable = fill("Asian Option", N)),
        DataFrame(value = asset, time = time, variable = fill("Asset Price", N)),
        DataFrame(value = hedge, time = time, variable = fill("Hedge Amount", N)))
end

function plot_asian(r, mu, sigma, N, T)
    d = path_and_price(r, mu, sigma, N, T)
    plot(d, x = :time, y = :value, color = :variable, Geom.line)
end
```