

1 Distribution of P-value Under the Null

1.1

$$\begin{aligned}\alpha &= 1 - F_H(C_\alpha) \\ F_H(C_\alpha) &= 1 - \alpha \\ C_\alpha &= F_H^{-1}(1 - \alpha)\end{aligned}$$

1.2

Given $\alpha = 1 - F_H(C_\alpha)$ and the definition of the p-value as the smallest α for which $T(x) > C_\alpha$, and the knowledge that F_H is continuous and increasing:

$$\begin{aligned}\alpha &= 1 - F_H(C_\alpha) \\ p(X) &= 1 - F_H(T(X))\end{aligned}$$

1.3

Given a random variable y , drawn from an unknown distribution with cumulative distribution function F , we want to find the probability distribution of $F(y)$. We will find the CDF of $F(y)$, which we will call G :

$$G(x) = \mathbb{P}(F(y) \leq x)$$

Because F is a CDF, we know it is continuous, and therefore invertible with a unique solution when $x \in (0, 1)$, which we will restrict ourselves to from now on:

$$G(x) = \mathbb{P}(y \leq F^{-1}(x))$$

And the probability that a random variable y is less than or equal to some other random variable, is by definition the CDF of y . In this case, that random variable, is $F^{-1}(x)$:

$$G(x) = F(F^{-1}(x))$$

Which leads us to the CDF for a uniform distribution, which is by definition $F_{uni}(X) = X$ within our interval $[0, 1]$:

$$G(x) = x$$

We can repeat the same for $1 - F(y)$, restricting again $x \in (0, 1)$, and again we end up with another uniform distribution function, this time over the interval $[1, 0]$:

$$\begin{aligned} G(x) &= \mathbb{P}(1 - F(y) \leq x) \\ G(x) &= \mathbb{P}(y \leq F^{-1}(1 - x)) \\ G(x) &= F(F^{-1}(1 - x)) \\ G(x) &= 1 - x \end{aligned}$$

1.4

We have proven that for any univariate random variable y with distribution F_y , $G(x)$ is uniformly distributed:

$$G(x) = 1 - F_y(y)$$

F_H is defined as the distribution of $T(x)$ under H . The p-value can be shown to be directly plugged into the above definition, proving that the p-value is uniformly distributed:

$$\begin{aligned} G(x) &= 1 - F_y(y) \\ p\text{-value} &= 1 - F_H(T(x)) \end{aligned}$$

2 MT Under Independence Assumptions

2.1

Using the formula for distribution complements $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$, along with the definition for a given random variable y drawn from a uniform distribution spanning 0 to 1, and $\alpha \in [0, 1]$:

$$\begin{aligned} \mathbb{P}(y \leq \alpha) &= \alpha \\ \mathbb{P}(y > \alpha) &= 1 - \alpha \end{aligned}$$

The joint probability of a series of independent events is given by the product of their individual probabilities. If $\mathbb{P}(A_m)$ for all values of m are equal, this is equivalent to:

$$\mathbb{P}(\cap_{i=1}^m A_i) = \mathbb{P}(A)^m$$

From which follows that for any series of independent draws from a uniform distribution, y_1, \dots, y_m :

$$\mathbb{P}(\cap_{i=1}^m \{y_i > \alpha\}) = \mathbb{P}(y_1 > \alpha)^m$$

$$\mathbb{P}(\cap_{i=1}^m \{y_i > \alpha\}) = (1 - \alpha)^m$$

2.2

We reject a single null hypothesis at significance level α with probability α . The complement of rejecting a single null hypothesis is to accept a single null hypothesis, which we therefore do with probability $1 - \alpha$. To accept ALL null hypotheses, from a set of m of hypothesis tests with I.I.D. uniformly distributed p-values, is therefore given by:

$$\mathbb{P}(\text{accepting all null hypotheses}) = (1 - \alpha)^m$$

The complement of accepting every null hypothesis, would be to reject one or more null hypothesis:

$$\mathbb{P}(\text{reject one or more null hypotheses}) = 1 - \mathbb{P}(\text{accepting all null hypotheses})$$

$$\mathbb{P}(\text{reject one or more null hypotheses}) = 1 - (1 - \alpha)^m$$

2.3

Let α_0 represent the significance level with which we reject each individual null hypothesis and α represent the probability that we reject one or more null hypotheses from a set of hypotheses with I.I.D. uniform p-values. Starting from our equation above, we can find α_0 in terms of α :

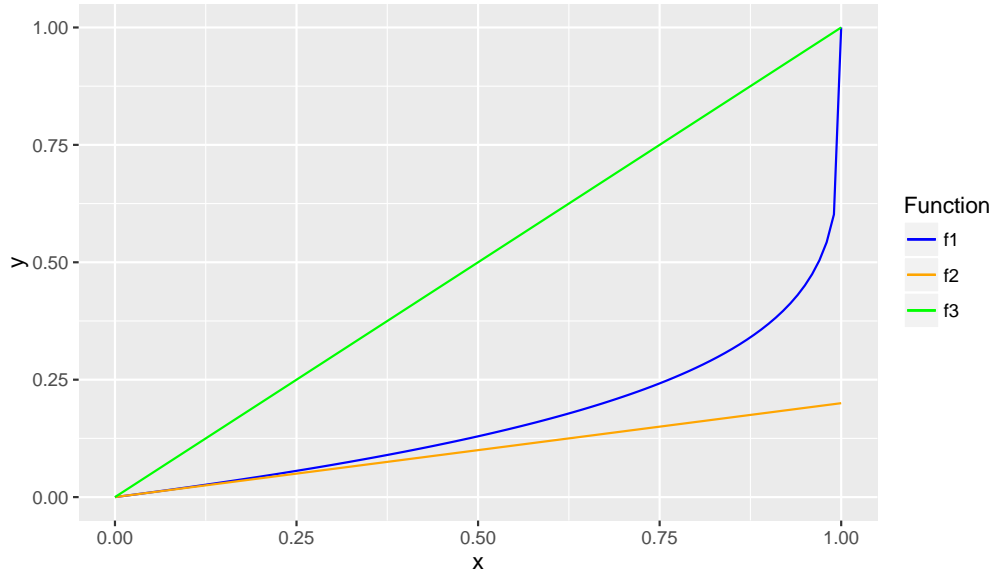
$$\alpha = 1 - (1 - \alpha_0)^m$$

$$1 - \alpha = (1 - \alpha_0)^m$$

$$(1 - \alpha)^{\frac{1}{m}} = 1 - \alpha_0$$

$$\alpha_0 = 1 - (1 - \alpha)^{\frac{1}{m}}$$

2.4



2.5

We consider all situations where $\alpha \in [0, 1]$ and all $m \geq 1$, where for all positive real values for x , we can say that $x \geq \frac{x}{m}$. From this it follows that $f_3(\alpha) \geq f_2(\alpha)$. We can say that for all $x \in [0, 1]$, $x^{\frac{1}{m}} \geq x$ for all $m \geq 1$. Reliant on this, we can easily prove our remaining inequalities:

$$\begin{aligned}
 f_3(\alpha) &\geq f_1(\alpha) \\
 \alpha &\geq 1 - (1 - \alpha)^{\frac{1}{m}} \\
 (1 - \alpha)^{\frac{1}{m}} &\geq 1 - \alpha
 \end{aligned}$$

And similarly for $f_2(\alpha) \leq f_1(\alpha)$:

$$\begin{aligned}
 f_1(\alpha) &\geq f_2(\alpha) \\
 1 - (1 - \alpha)^{\frac{1}{m}} &\geq \frac{\alpha}{m} \\
 1 - \frac{\alpha}{m} &\geq (1 - \alpha)^{\frac{1}{m}}
 \end{aligned}$$

2.6

Under independence assumptions, as shown in (2.3), the significance level needed for each test, α_0 , is shown to be $1 - (1 - \alpha)^{\frac{1}{m}}$. Making now correction for multiple testing, we follow the generic per-comparisson error rate (PCER), and accept or reject each individual test at $\alpha_0 = \alpha$. As shown in (2.5):

$$\alpha \geq 1 - (1 - \alpha)^{\frac{1}{m}}$$

And from that we can see that the significance level needed for each individual test if we consider each test p-value as being I.I.D. from a uniform distribution, is much lower if we consider the joint probability of all our tests and aim to keep the probability of rejecting one or more null hypothesis at our significance level, then if we make no correction for the joint probability of our multiple tests.

3 Proof of Bonferroni Correction

Using Boole's Inequality, we can rewrite the probability of rejecting one or more null hypotheses to be:

$$\mathbb{P}_{C-H}(\cup_{i=1}^m \{p_i(\mathbf{Y}_i) \leq \alpha\}) \leq \sum_{i=1}^m \mathbb{P}(\mathbf{Y}_i \leq \alpha)$$

If we consider $\mathbf{Y}_1, \dots, \mathbf{Y}_m$ to be p-values drawn from a uniform distribution from 0 to 1, we can state that the probability of any given $\mathbf{Y}_i \leq \alpha$, where $\alpha \in [0, 1]$, is equal to α . This leads to the following:

$$\begin{aligned} \mathbb{P}_{C-H}(\cup_{i=1}^m \{p_i(\mathbf{Y}_i) \leq \alpha\}) &\leq \sum_{i=1}^m \mathbb{P}(\mathbf{Y}_i \leq \alpha) \\ \mathbb{P}_{C-H}(\cup_{i=1}^m \{p_i(\mathbf{Y}_i) \leq \alpha\}) &\leq \sum_{i=1}^m \alpha \\ \mathbb{P}_{C-H}(\cup_{i=1}^m \{p_i(\mathbf{Y}_i) \leq \alpha\}) &\leq m\alpha \end{aligned}$$

The lower bound is easily understood with the previous statements regarding $\mathbf{Y}_1, \dots, \mathbf{Y}_m$ to be drawn from uniform (0,1). We can consider the case of a single i , where we know the probability of a single rejection, $\mathbb{P}(\mathbf{Y}_i \leq \alpha)$ will be exactly α .

$$\mathbb{P}(\mathbf{Y}_i \leq \alpha) = \alpha$$

If we increase the number of trials, the number of i 's, the probability of rejecting a single hypothesis cannot possibly be LESS than the probability of rejecting a single hypothesis with one trial. Therefore, it is lower-bounded by the above α :

$$\alpha \leq \mathbb{P}_{C-H}(\cup_{i=1}^m \{p_i(\mathbf{Y}_i) \leq \alpha\})$$

4 Ordered P-values and Controlling the FDR

We've been given that for any set of ordered variables y_i drawn from a uniform distribution with range (0,1):

$$\mathbb{P}\left(\cap_{i=1}^m \left\{p_i\left(y_i > \frac{i\alpha}{m}\right)\right\}\right) = 1 - \alpha$$

As shown in 2.2, the probability of at least one false rejection under the complete null is the probability of any given p-value falling under our rejection threshold for that p-value. This is the complement of the probability of the above formula (which is the probability of all ordered uniform random variable falling above our threshold), and therefore can be calculated as following:

$$\mathbb{P}\left(\cup_{i=1}^m \left\{p_i\left(y_i < \frac{i\alpha}{m}\right)\right\}\right) = 1 - (1 - \alpha)$$

$$\mathbb{P}\left(\cup_{i=1}^m \left\{p_i\left(y_i < \frac{i\alpha}{m}\right)\right\}\right) = \alpha$$