<u>MAE 232A: Finite elements for Solid mechanics</u>

<u>Final project report</u>: FEM simulation for a 2D-boundary value problem using MATLAB.

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PROBLEM STATEMENT

Consider the following 2-dimensional boundary value problem in Fig. 1:

$$u_{,ii} = u_{,xx} + u_{,yy} = (x^2 + y^2)e^{xy} \quad in\ \Omega$$
$$u = 1\ on\ \Gamma^1$$
$$u = 1\ on\ \Gamma^2$$
$$u_{,i}\, n_i = ye^y\ on\ \Gamma^3$$
$$u_{,i}\, n_i = xe^x\ on\ \Gamma^4$$

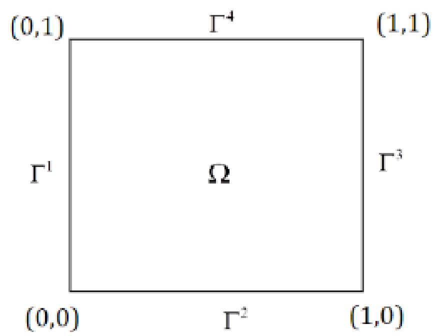


Figure 1. Problem Domain

The analytical solution of this problem is $u = e^{xy}$.

(1) Obtain the finite element solution u^h <u>using 6x6, 12x12, and 24x24 4-node elements</u>, and compare the finite element solution u^h for all discretizations with the analytical solution u along the line $y=0.5$ in a plot.

(2) Compare the finite element solution $u^h_{,x}$ obtained by using the 3 discretizations with the analytical solution $u_{,x}$ along the line $y=0.5$ in a plot.

(3) Plot the errors of finite element solutions $e = \left[\int_\Omega \left(u - u^h\right)^2 d\Omega\right]^{1/2}$ vs element dimensions "h" in a log-log plot and obtain the rate of convergence of this error measure.

(4) Discuss how the finite element solution errors are reduced as the model is refined and how the numerical rate of convergence compares with the theoretical rate of convergence.

FINITE ELEMENT FORMULATION

To obtain the finite element solution, we must reduce the strong form provided in the problem statement to the weak form.

Thus, weak form (W): Find u ∈ S = {u|u ∈ H^1, u = 1 on Γ_1, u = 1 on Γ_2} and w ∈ V = {w|w ∈ H_1, w = 0 on $\Gamma_1 \cup \Gamma_2$}.

We are then pre-multiplying our test function w and integrate over the domain to obtain -

$$\int_{\Omega} \omega(u_{,ii} - (x^2 + y^2)e^{xy}) d\Omega = 0$$

Further, integrating by parts gives –

$$\int_{\Omega} \omega(u_{,ii}) d\Omega - \int_{\Omega} \omega_{,i}(u_{,i}) d\Omega = \int_{\Omega} (x^2 + y^2)e^{xy}\, d\Omega$$

Applying divergence theorem and applying w = 0 on $(\Gamma_1 \cup \Gamma_2)$ as our essential boundary conditions on the domain in the first term: (Ω: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$)

- $\int_{\Gamma_1} \omega(u_{,ii}) d\Gamma + \int_{\Gamma_2} \omega(u_{,ii}) d\Gamma + \int_{\Gamma_3} \omega(u_{,ii}) d\Gamma + \int_{\Gamma_4} \omega(u_{,ii}) d\Gamma$
  $- \int_{\Omega} \omega_{,i}(u_{,i}) d\Omega = \int_{\Omega} \omega(x^2 + y^2)e^{xy}\, d\Omega$

  Which is rewritten as –

- $\int_{\Omega} \omega_{,i}(u_{,i}) d\Omega = \int_{\Gamma_3} \omega(ye^{y}) d\Gamma + \int_{\Gamma_4} \omega(xe^{x}) d\Gamma - \int_{\Omega} \omega(x^2 + y^2)e^{xy}\, d\Omega$

Rearranging the above terms gives us the weak form –

(W): Find u $\in$ S = $\{u | u \in H^1$, u = 1 on Γ_1, u = 1 on $\Gamma_2\}$ and

w $\in$ V = $\{w | w \in H_1$, w = 0 on $\Gamma_1 \cup \Gamma_2\}$.

$$\boxed{\int_{\Omega} \omega_{,i}(u_{,i}) d\Omega = \int_{\Gamma_3} \omega(ye^{y}) d\Gamma + \int_{\Gamma_4} \omega(xe^{x}) d\Gamma - \int_{\Omega} \omega(x^2 + y^2)e^{xy}\, d\Omega}$$

Now to obtain the stiffness matrices and force vectors to solve for the FEM solution, we need to obtain the Galerkin form –

For our Galerkin approximation, we restrict our function spaces:

(G): Find $u^h \in S^h \subset$ s.t. $\forall\, w^h \in V^h \subset V$.

$$\boxed{\int_{\Omega} \omega^h_{,i}(u^h_{,i}) d\Omega = \int_{\Gamma_3} \omega^h(u^h_{,ii}) d\Gamma + \int_{\Gamma_4} \omega^h(u^h_{,ii}) d\Gamma - \int_{\Omega} \omega^h(x^2 + y^2)e^{xy}\, d\Omega}$$

We then obtain-

$$u^h = \sum_1^4 N_I\, d_I = Nd,$$

$$w^h = \sum_1^4 N_I\, d_I = Nc = c^T N^T,$$

$$u_{,i}^h = \sum_1^4 N_{I,i}\, d_I, = Bd = d^T B^T,$$

$$w_{,i}^h = \sum_1^4 N_{I,i}\, c_I, = Bc = c^T B^T,$$

$$x = \sum_1^4 N_I\, x_I$$

$$y = \sum_1^4 N_I\, y_I$$

Then, the terms in the Galerkin form becomes –

$$\sum_e \int_{\Omega^e} (\mathrm{c}^e)^T\, B^T B d\Omega d^e$$

$$= \sum_e \int_{\Gamma_3^e} (c^e)^T\, N^T N y e^y d\Gamma \;+\; \sum_e \int_{\Gamma_4^e} (c^e)^T\, N^T N x e^x d\Gamma$$

$$- \sum_e \int_{\Omega_4^e} (c^e)^T\, N^T N (x^2 + y^2) e^{xy} d\Omega$$

Pulling out $(\mathrm{c}^e)^T$ and cancelling it out gives –

$$\sum_e \int_{\Omega^e} B^T B d\Omega d^e$$

$$= \sum_e \int_{\Gamma_3^e} N^T N y e^y d\Gamma \;+\; \sum_e \int_{\Gamma_4^e} N^T N x e^x d\Gamma$$

$$- \sum_e \int_{\Omega_4^e} N^T N (x^2 + y^2) e^{xy} d\Omega$$

Thus,

> Stiffness matrix, $K^e = \int_{\Omega^e} B^T\, B d\Omega$
> Force vectors, $f^e = \int_{\Gamma_3^e} N^T\, N y e^y d \int_{\Gamma_4^e} N^T\, N x e^x d\Gamma$
> $- \int_{\Omega_4^e} N^T\, N(x^2 + y^2)e^{xy} d\Omega$

Where:

$$N^T = \begin{bmatrix} \frac{1}{4}(1-\xi)(1-\eta) \\ \frac{1}{4}(1+\xi)(1-\eta) \\ \frac{1}{4}(1+\xi)(1+\eta) \\ \frac{1}{4}(1-\xi)(1+\eta) \end{bmatrix} = \begin{bmatrix} N1 \\ N2 \\ N3 \\ N4 \end{bmatrix}, N_{,\xi}^{\ T} = \begin{bmatrix} -\frac{1}{4}(1-\eta) \\ \frac{1}{4}(1-\eta) \\ \frac{1}{4}(1+\eta) \\ -\frac{1}{4}(1+\eta) \end{bmatrix}, N_{,\eta}^{\ T} = \begin{bmatrix} -\frac{1}{4}(1-\xi) \\ \frac{1}{4}(1+\xi) \\ \frac{1}{4}(1+\xi) \\ \frac{1}{4}(1+\xi) \end{bmatrix}$$

Thus, the gradient of the shape functions will be –

$$B = \begin{bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{bmatrix} = A^{-1} = \begin{bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{bmatrix}$$

Where –

$$A = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & x_{,\eta} \end{bmatrix}$$

Thus, to obtain the values of A, we need to perform isoparametric mapping –

$$\mathbf{x}_{,\xi} = \frac{\mathbf{1}}{\mathbf{4}}\left(\mathbf{N}_{\mathbf{1},\xi}\,\mathbf{x}_\mathbf{1} + \mathbf{N}_{\mathbf{2},\xi}\,\mathbf{x}_\mathbf{2} + \mathbf{N}_{\mathbf{3},\xi}\,\mathbf{x}_\mathbf{3} + \mathbf{N}_{\mathbf{4},\xi}\,\mathbf{x}_\mathbf{4}\right)$$
$$\mathbf{y}_{,\xi} = \frac{\mathbf{1}}{\mathbf{4}}\left(\mathbf{N}_{\mathbf{1},\xi}\,\mathbf{y}_\mathbf{1} + \mathbf{N}_{\mathbf{2},\xi}\,\mathbf{y}_\mathbf{2} + \mathbf{N}_{\mathbf{3},\xi}\,\mathbf{y}_\mathbf{3} + \mathbf{N}_{\mathbf{4},\xi}\,\mathbf{y}_\mathbf{4}\right)$$
$$\mathbf{x}_{,\eta} = \frac{\mathbf{1}}{\mathbf{4}}\left(\mathbf{N}_{\mathbf{1},\eta}\,\mathbf{x}_\mathbf{1} + \mathbf{N}_{\mathbf{2},\eta}\,\mathbf{x}_\mathbf{2} + \mathbf{N}_{\mathbf{3},\eta}\,\mathbf{x}_\mathbf{3} + \mathbf{N}_{\mathbf{4},\eta}\,\mathbf{x}_\mathbf{4}\right)$$
$$\mathbf{y}_{,\eta} = \frac{\mathbf{1}}{\mathbf{4}}\left(\mathbf{N}_{\mathbf{1},\eta}\,\mathbf{y}_\mathbf{1} + \mathbf{N}_{\mathbf{2},\eta}\,\mathbf{y}_\mathbf{2} + \mathbf{N}_{\mathbf{3},\eta}\,\mathbf{y}_\mathbf{3} + \mathbf{N}_{\mathbf{4},\eta}\,\mathbf{y}_\mathbf{4}\right)$$

While the volume Jacobian is given as – $J^V = \det(A)$

Thus, rewriting our stiffness matrix –

$$\boxed{K^{(1)} = \int_{-1}^{+1}\int_{-1}^{+1} B^T B J^v d\xi d\eta}$$

and our force vector is –

$$\boxed{\begin{aligned} f^{(1)} = \int_{-1}^{+1} N^T\, N y^{(1)} e^{y^{(1)}} J^{s|}_{\xi=1} d\eta + \int_{-1}^{+1} N^T\, N x^{(1)} e^{x^{(1)}} J^{s|}_{\eta=1} d\xi \\ - \int_{-1}^{+1} N^T\, N \left(x^{2^{(1)}} + y^{2^{(1)}}\right) e^{x^{(1)} y^{(1)}} J^{v} d\xi d\eta \end{aligned}}$$

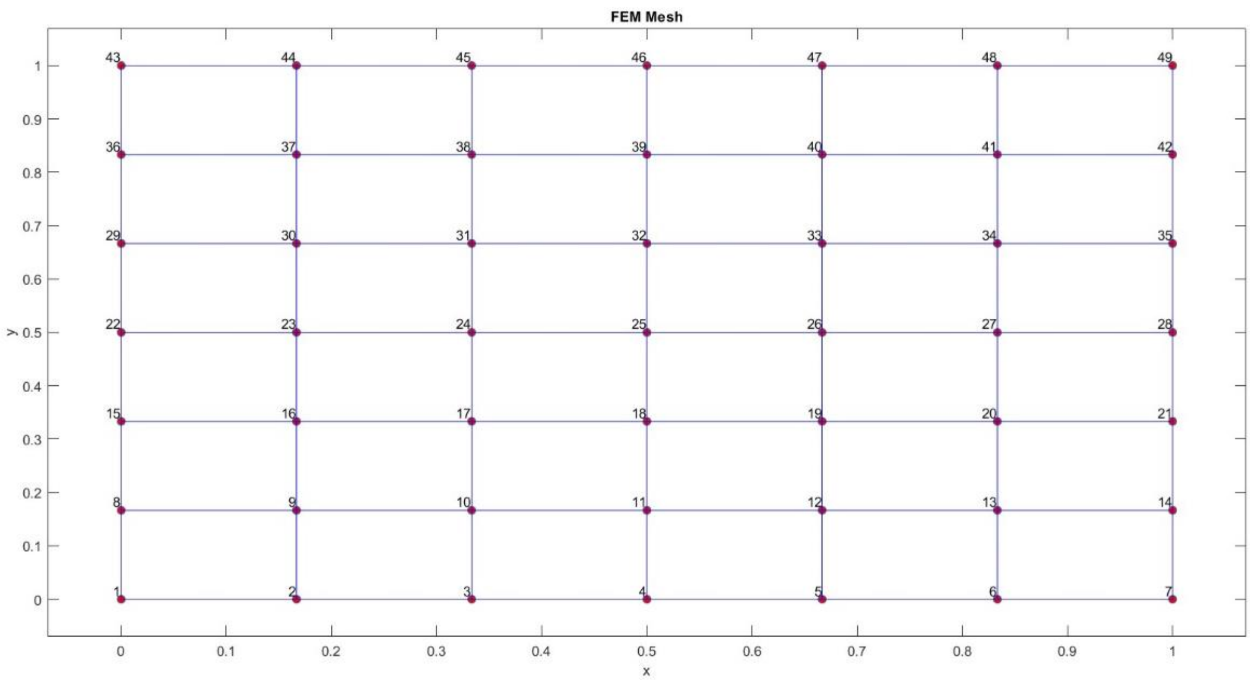
The surface Jacobian is denoted as –

$$J^s|_{\xi=1} = \sqrt{(\frac{\partial x}{\partial \eta})^2 + (\frac{\partial y}{\partial \eta})^2};\ J^s|_{\eta=1} = \sqrt{(\frac{\partial x}{\partial \xi})^2 + (\frac{\partial y}{\partial \xi})^2}$$

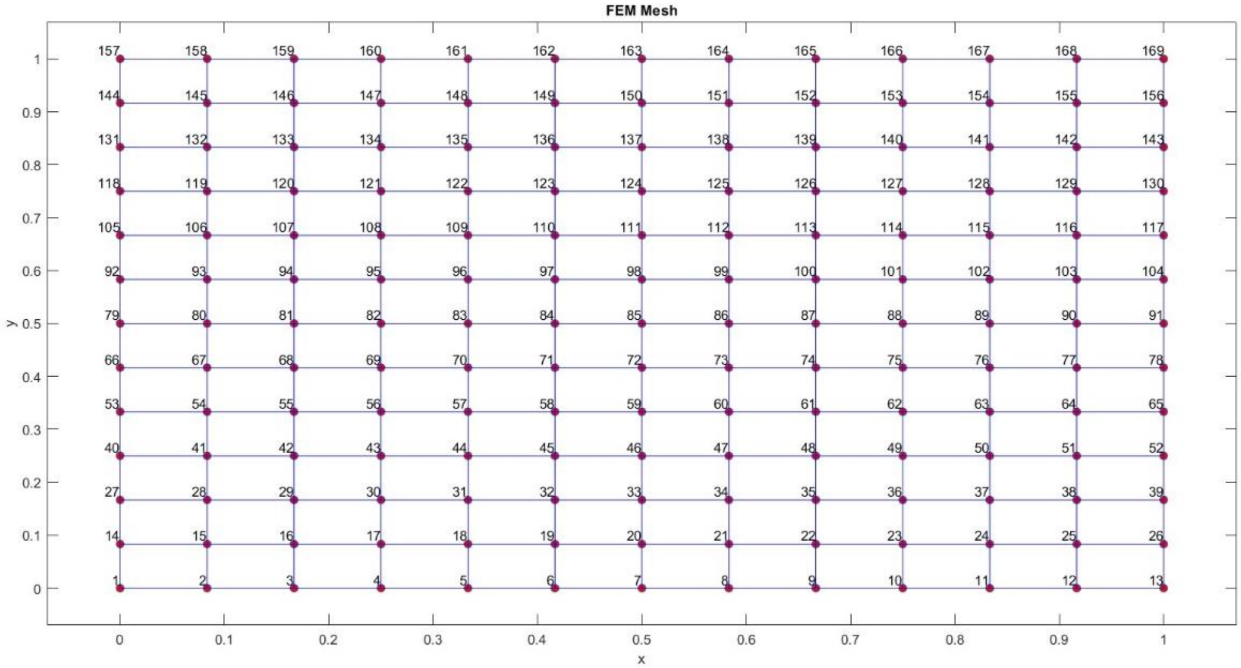
NUMERICAL RESULTS

Mesh grid:

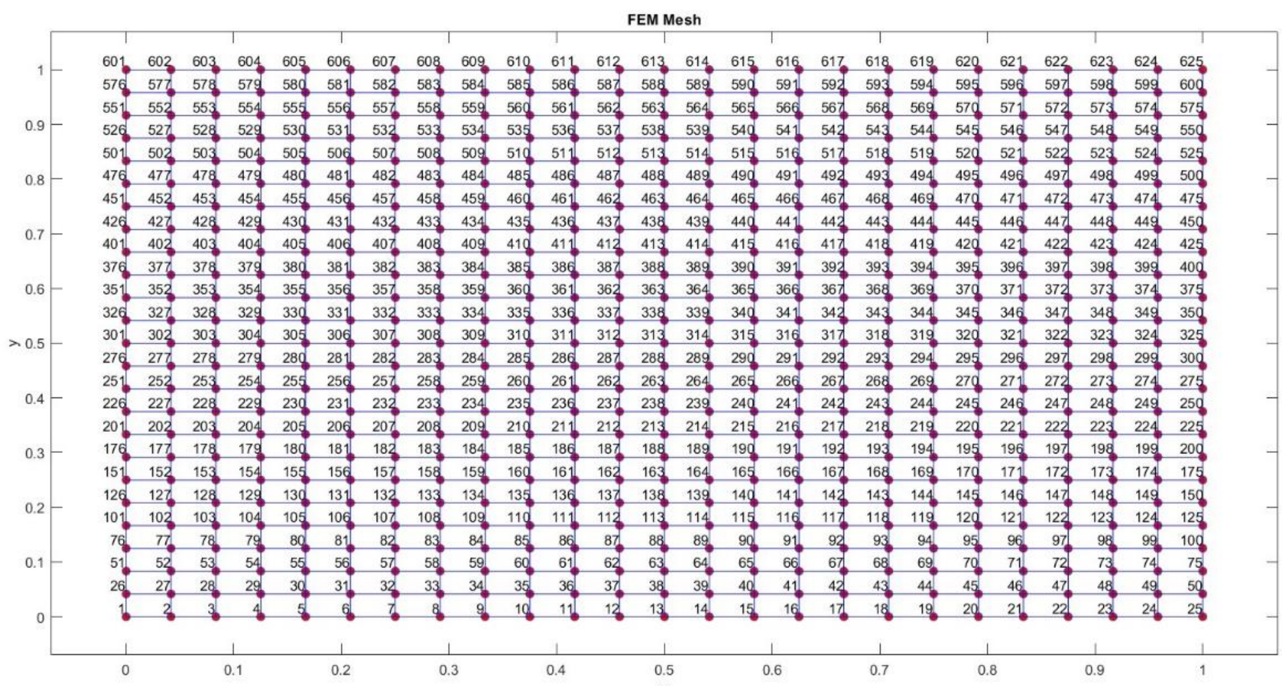
<u>For 6x6 4-node elements:</u>



For 12x12 4-node elements:



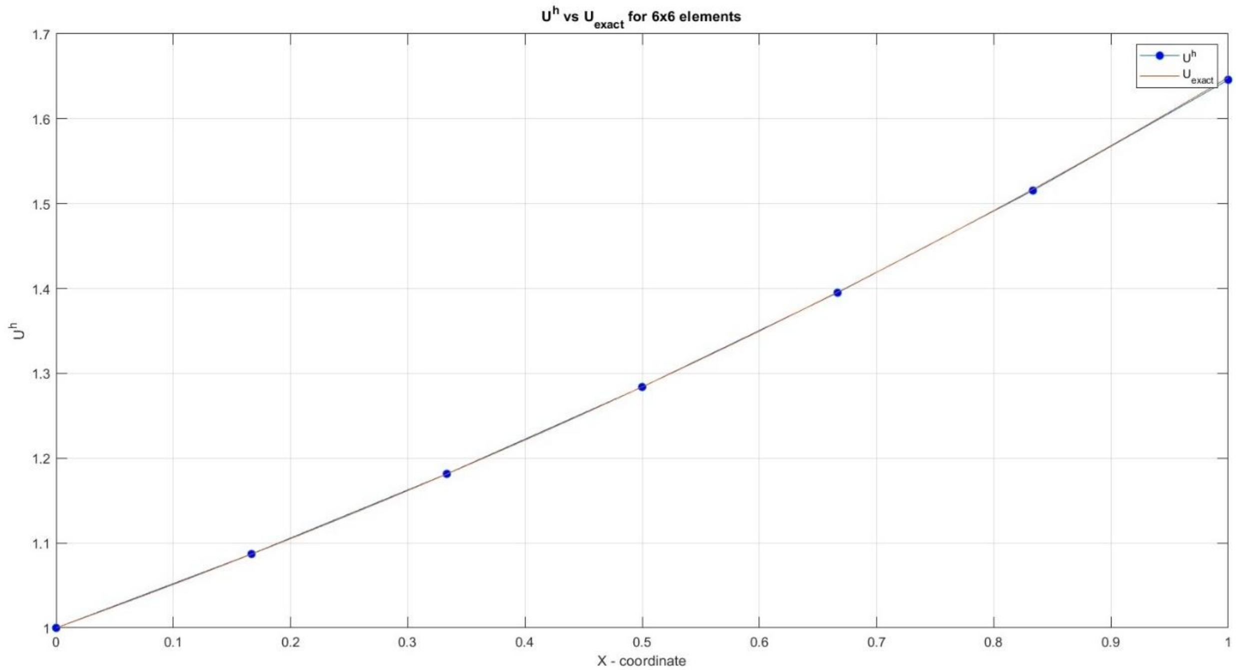
For 24x24 4-node elements:



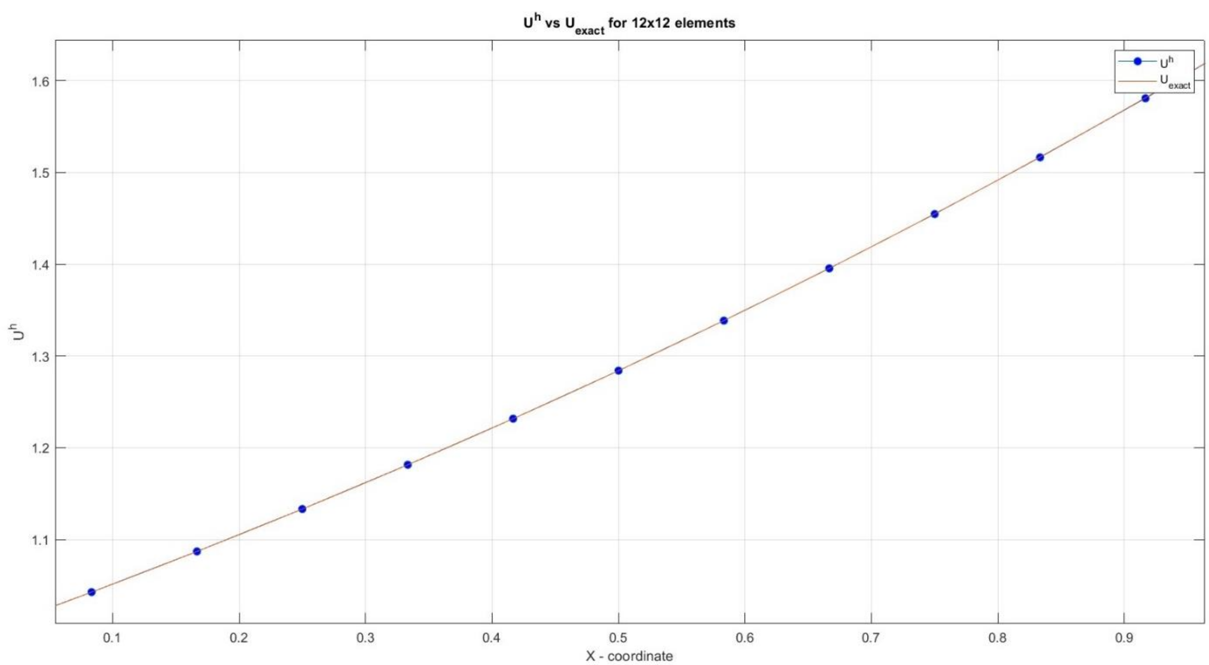
PART (1): <u>Comparison between the FEM solution and the exact solution</u>

FEM solutions are compared by plotting it against an exact solution of $\boldsymbol{u} = \boldsymbol{e}^{\boldsymbol{xy}}$

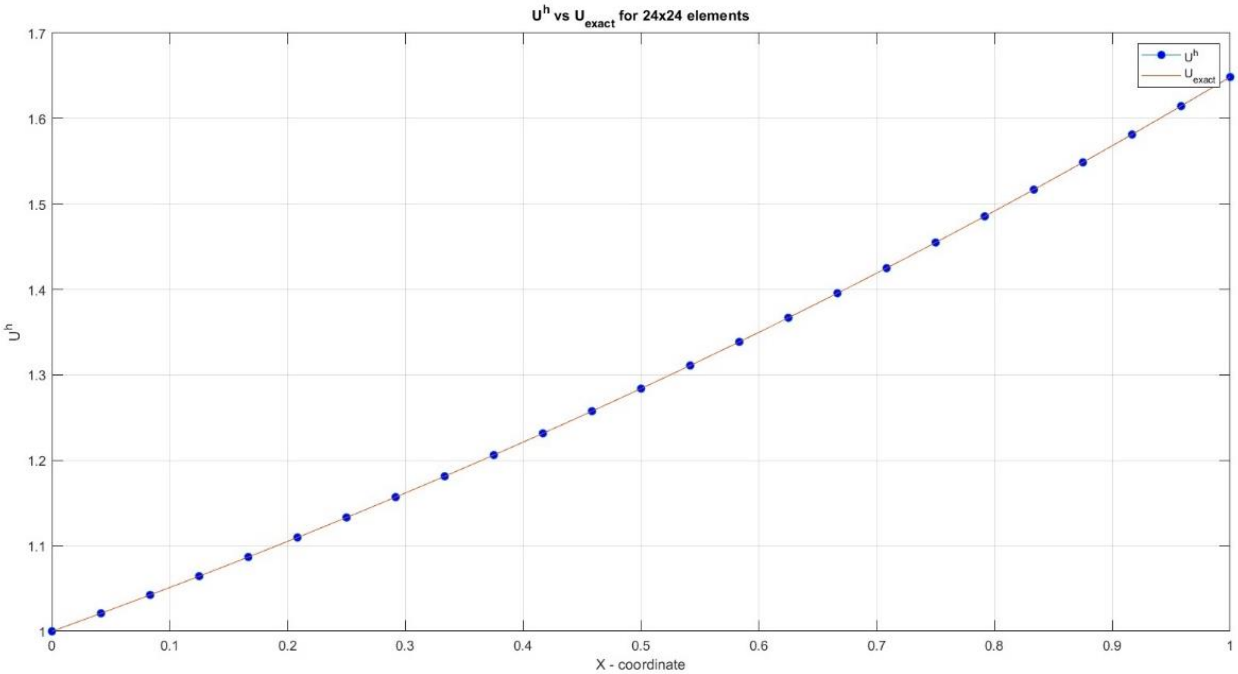
For 6x6 4-node element:



For 12x12 4-node element:

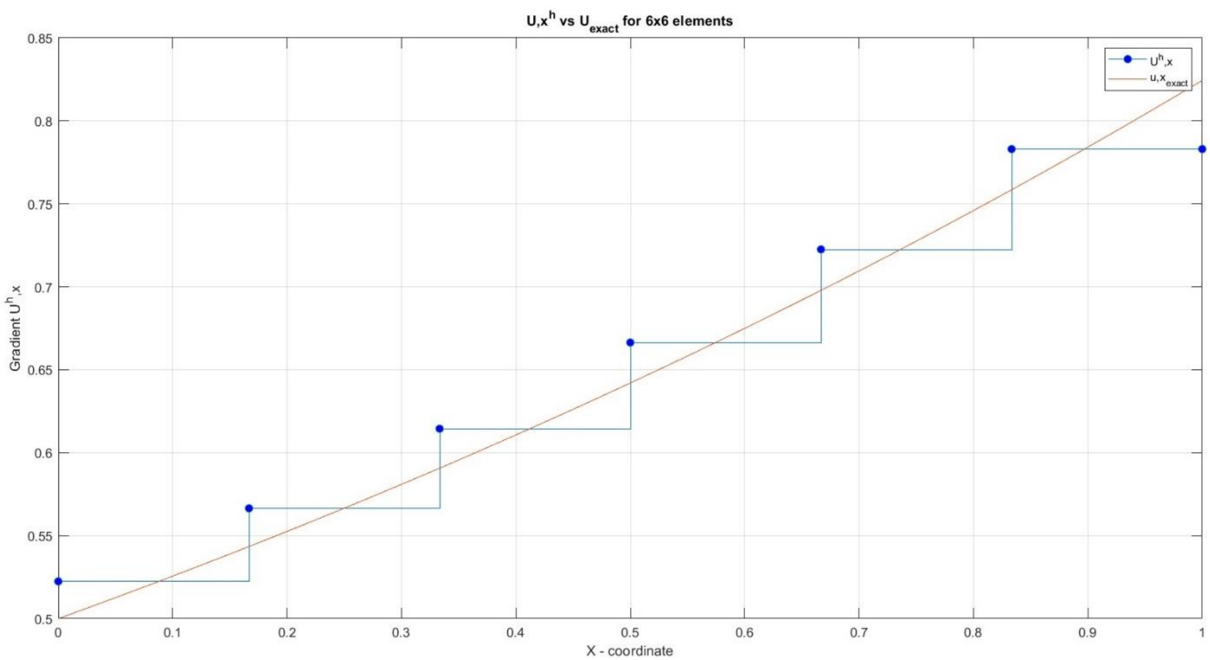


For 24x24 4-node element:

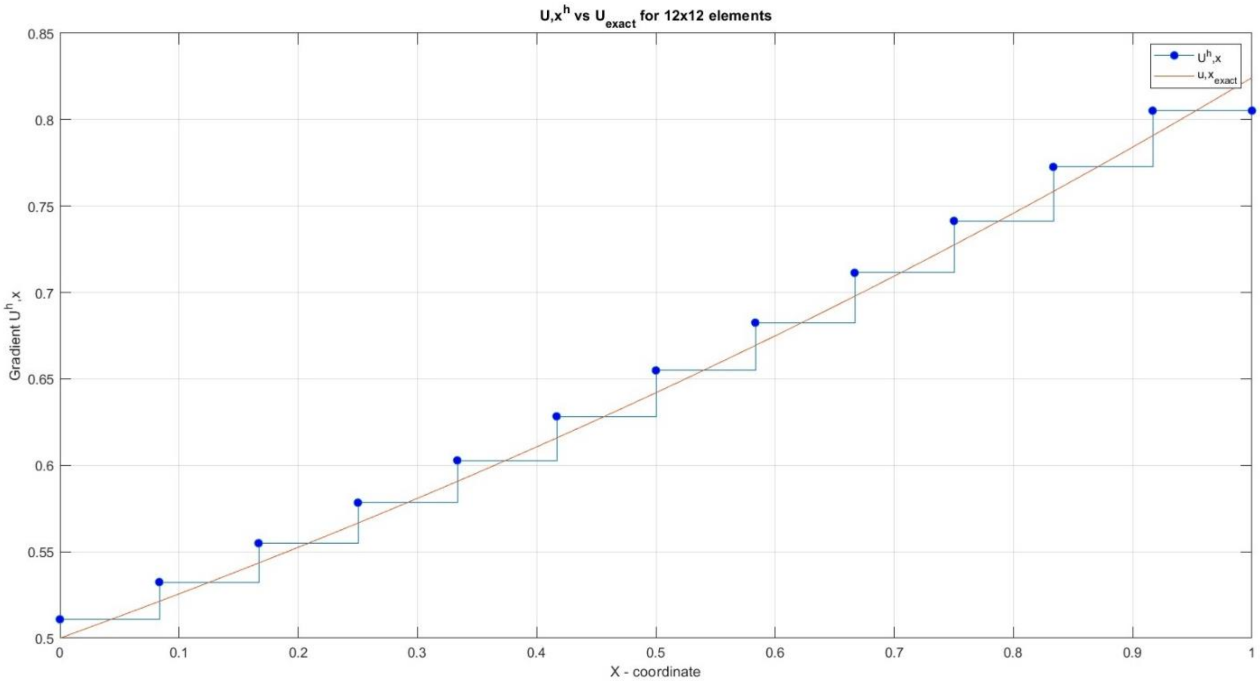


Part (2): Comparison between the FEM solution derivative and the exact solution derivative

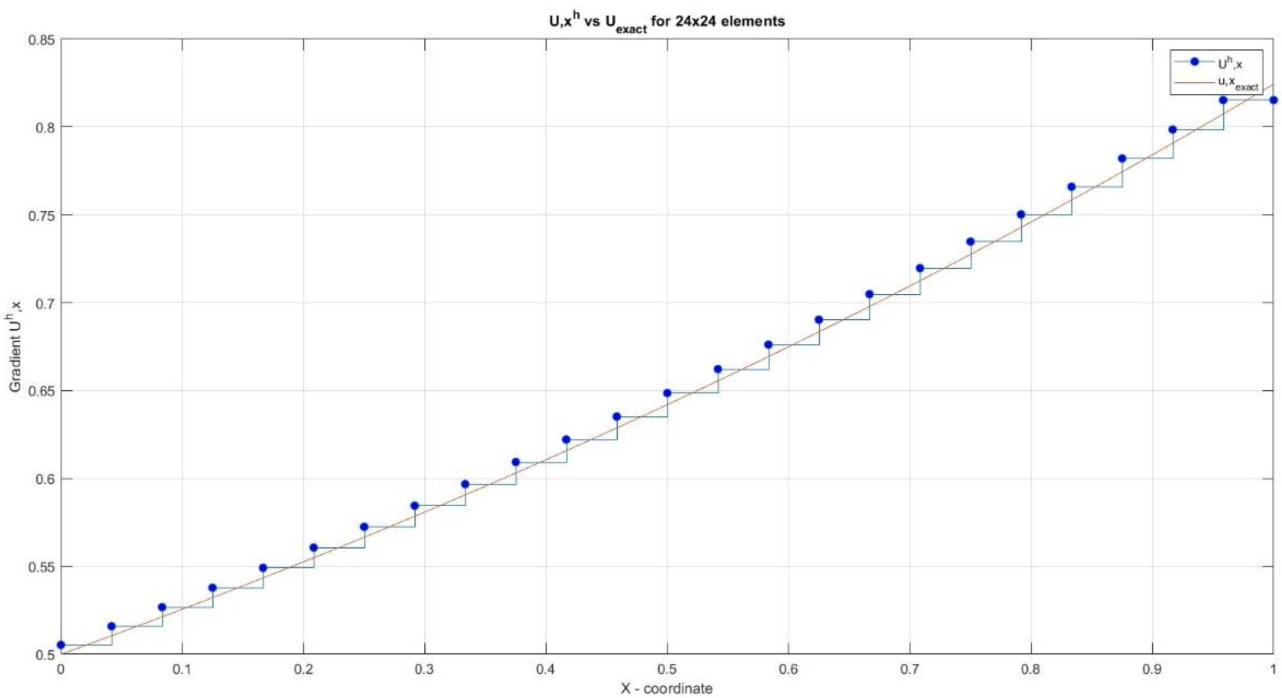
For 6x6 4-node elements



For 12x12 4-node elements:

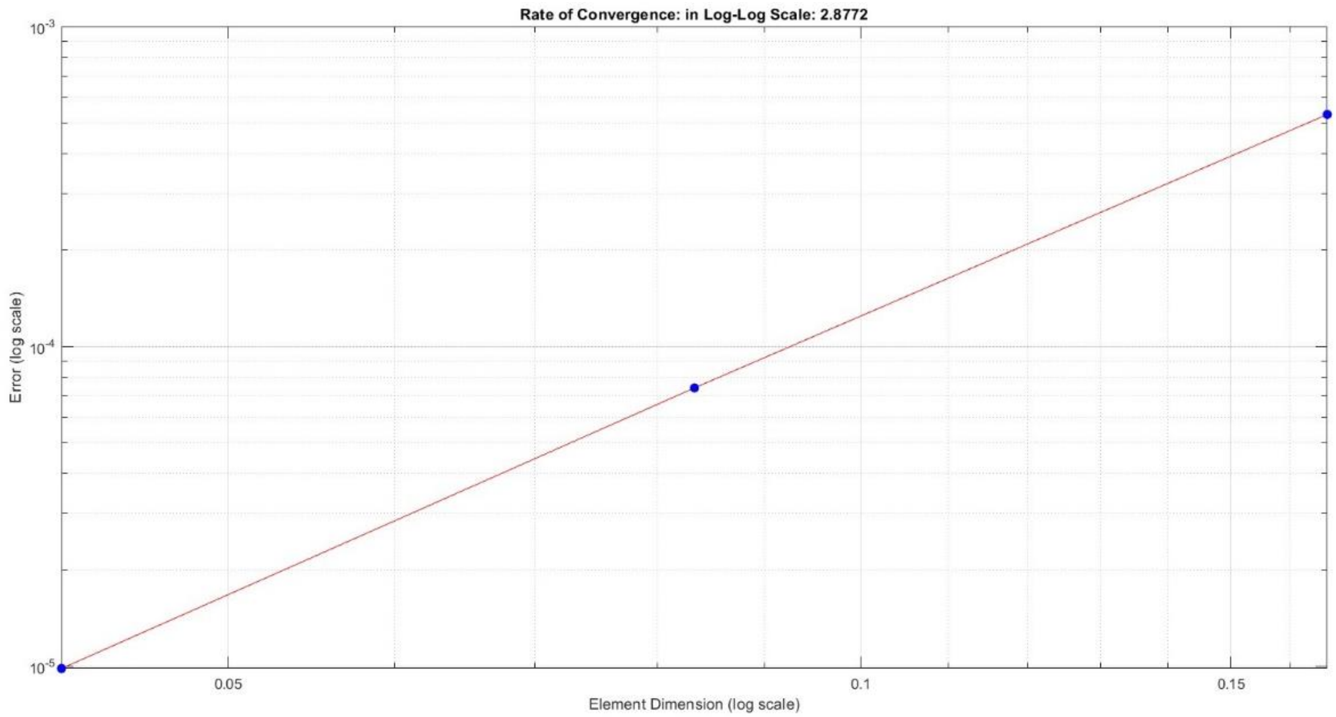


For 24x24 4-node elements:



Part (3): <u>Rate of convergence</u>

- The rate of convergence obtained is <u>**2.87**</u> which is obtained by calculating the slop of the curve.
- It is obtained by plotting errors = [5.3084e-04, 7.4139e-05, 9.8342e-06] for the L-2 error norm of the FEM solutions, $\boldsymbol{e} = \left[\int_{\Omega}(\boldsymbol{u} - \boldsymbol{u^h})^2 \, \boldsymbol{d\Omega}\right]^{\frac{1}{2}}$ against element dimensions, [6, 12, 24].



Rate of convergance :

Since rate of convergence is given by the slope of the curve,

$$\therefore \ ROC = \frac{Log(e_{12\times12}) - Log(e_{6\times6})}{Log(h_{12\times12}) - Log(h_{6\times6})}$$

$$= \frac{7.42e-05 - 5.31e-04}{1/12 - 1/6} = \boxed{2.84}$$

$$ROC = \frac{Log(e_{24\times24}) - Log(e_{12\times12})}{Log(h_{24\times24}) - Log(h_{12\times12})}$$

$$= \frac{9.83e-06 - 7.41e-05}{1/24 - 1/12} = \boxed{2.91}$$

Part (4): DISCUSSIONS

Procedure to compare numerical rate of convergence and theoretical rate of convergence are –

1. Comparison of FEM and Analytical Solutions: Primary way to see if the numerical rate of convergence matches theoretical rate of convergence is to see if the FEM solution and the derivative of the FEM solution matches the exact solution and the derivative of the exact solution which matches as per requirement in our case as seen in the above plots for elements (6x6, 12x12, 24x24).
2. Error Analysis: The errors are then computed using the L-2 error norm is given by $\boldsymbol{|u - u^h|_0 \leq ch^{n+1-r}|u|_{n+1}}$ where n is order of approximation of the finite element, u is the exact solution and u^h is the FEM solution. In our case, n = 1 because a 4-node linear element is used. Hence the theoretical convergence rate is supposed to be 2.
3. Rate of Convergence: A log-log plot of finite element solution is obtained by plotting errors versus element dimensions. The numerical convergence rate in our case is 2.87.
4. Impact of Model Refinement: The errors can be reduced by increasing the number of elements or by increasing the order of the shape function can influence the finite element solution errors. The above assumptions can be validated by observing our error vector as there is a decrease in the error as the number of elements increases.
5. Theoretical vs. Numerical Convergence: Finaly, we can conclude by saying the obtained numerical rate of convergence of the finite element method is within the range of the theoretical convergence rate you observed. Due to discrepancies in computational abilities of the FEM manpower can cause deviations from the desired convergence rate while other flaws in the nature of the problem and improper boundary conditions.