End Semester Exam MA3.101: Linear Algebra
Spring 2022

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Instructions:

1. Full Marks 100, Time- 3hrs
2. All questions of Section A are compulsory
3. Answer any five from Section B and any six from Section C.
4. It is a closed book exam, no sharing of notes and books
5. Notations has their usual meaning.
6. Go though the question paper before start attempting so that you do not miss out any questions

1 Section A: Answer all of them $\quad 10\times 2$

1. Show that the eigen values of Hermitian matrix are real
2. If A is an $m \times n$ matrix, then find out whether $A^T A$ have positive eigen values .
3. If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ find the eigen values of the matrix $\sqrt{A}$.
4. Use Cramer's rule to solve the equation:
   2x-y=5 $\longrightarrow$ $\frac{32}{5} - \frac{7}{5} = \frac{25}{5}$
   x-3y=-1

$-\frac{6}{5}$

5. What is the quadratic form of the associated matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 5 & 4 \\ -1 & 4 & 3 \end{pmatrix}$
6. Prove that if A is similar to B, then A^T is similar to B^T.
7. Is the singular value decomposition of a matrix A of size $m \times n$ is unique? Justify

8. Find the inverse of the elementary matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

9. Find the dimension of a vector space W of symmetric 2×2 matrices.

10. Determine whether the matrix $A = \begin{pmatrix} 1/3 & 1/2 & 1/3 \\ 1/3 & -1/2 & 1/5 \\ -1/3 & 0 & 2/5 \end{pmatrix}$ is orthogonal or not

2 Section B: Answer any five

5×4

1. Let A and B are similar matrices. Prove that the algebraic multiplicities of eigenvalues of A and B are same

2. Prove that $d(u,v) = \sqrt{||u||^2 + ||v||^2}$ iff u and v are orthogonal.

3. Verify whether the matrix $A = \begin{pmatrix} 2+i & 0 & 3i \\ 0 & 2-i & 5 \\ 3i & 5i & 1-i \end{pmatrix}$ is Hermitian or not.

   $A^* = A$

4. Let A_1, A_2 be sub spaces of a vector space. Find out the condition under which $A_1 \cup A_2$ is a subspace.

5. Solve the system of equation :
   a+ b+ c+ d= 4
   a+ 2b+ 3c+ 4d= 10
   a+ 3b +6c+10d=20
   a+4b+ 10c+20d=35.

   $b+2c+4d=6$
   $b+3c+6d=10$
   $a+2d=4$

6. Prove that if A is a positive definite matrix with SVD, $A = U \sum V^T$ (where U and V are orthogonal matrix), then $U = V$

7. Let F be a field and consider the vector space $V = F^2$. Let T be a linear operator on V defined as $T((x_1, x_2)) = (x_2, x_1)$. . Find out the matrix representation of the linear operator T.

8. Prove that if any upper triangular matrix is orthogonal, then it must be diagonal matrix.

3 Section C: Answer any six

1. Show $||u||^2 + ||v||^2 + 2 < u, v >= ||u + v||^2$. Prove that $||u + v|| = ||u - v||$ if and only if u and v are orthogonal. Show that a square matrix $A = \begin{pmatrix} P & Q \\ O & S \end{pmatrix}$ where P and S are square matrices (O is the null matrix). Prove that $det(A) = det(P)det(S)$(3+4+3)

2. Compute the (a) Characteristic polynomials, (b) eigen values of A and B (c) basis for each eigen spaces of each A and B (d) the algebraic and geometric multiplicity of each eigenvalues of A and B: (i) $A = = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{pmatrix}$

(ii) $B = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}$. If Q is orthogonal matrix show that any matrix obtained by rearranging the rows of Q is also orthogonal.(8+2)

3. Let A be a symmetric positive definite $n \times n$ matrix and let u and v are vectors in R^n. Show that $< u, v >= u^T Av$ defines ~~an inner product.Let~~ $T : P_2 \to P_2$ be the linear transformation defined by $T(p(x)) = p(2x - 1)$. Find the matrix of T with respect to the basis $[1, x, x^2]$. Find a unitary matrix U and a diagonal matrix D for the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ such that $U^* AU = D$ (3+3+4)

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4. Find the singular value decomposition of the following matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(skipped)

Find the pseudo inverse of the matrix $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$(4+6)

5. Use Gram Schmidt process to find an orthogonal basis for the column spaces of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix}$ and find a QR factorization of the matrix. If A and B are orthogonally diagonalizable and $AB = BA$, show that AB is orthogonally diagonalizable. Show that the vectors $B_1 = \{(1, 1, 1), (1, 2, 3), (2, 1, 1)\}$ are linearly independent in R^3.(6+2+2)

skip

6. Find a spectral decomposition of the matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ Classify the quadratic form $f(x, y, z) = 3x^2 + 3y^2 + 3z^2 - 2xy - 2xz - 2yz$. Suppose we are given bases of subspaces U, W of a vector space V. How do you find the basis of the subspace $U \cap W$?(5+3+2)

7. Diagonalize the quadratic forms in the following expressions by finding an orthogonal matrix Q such that the change of variable $x = Qy$ transforms the given form into one with no cross product terms, $\cancel{(a) 2x_1^2 + 5x_2^2 - 4x_1x_2}$ (b)$2xy + 2xz + 2yz$.(5+5)

2x

8. Let (e_1, e_2, e_3) be the canonical basis of R^3, and define $f_1 = e_1 + e_2 + e_3$, $f_2 = e_2 + e_3$, $f_3 = e_3$.. Apply the Gram-Schmidt process to the basis (f_1, f_2, f_3).Find the Kernel and Range of the differential operator D :

$P_3 \to P_2$ defined by $D(p(x) = dp/dx$. Let A be an $n \times n$ matrix. If A is invertible then show that A is a product of elementary matrices. (4+3+3)