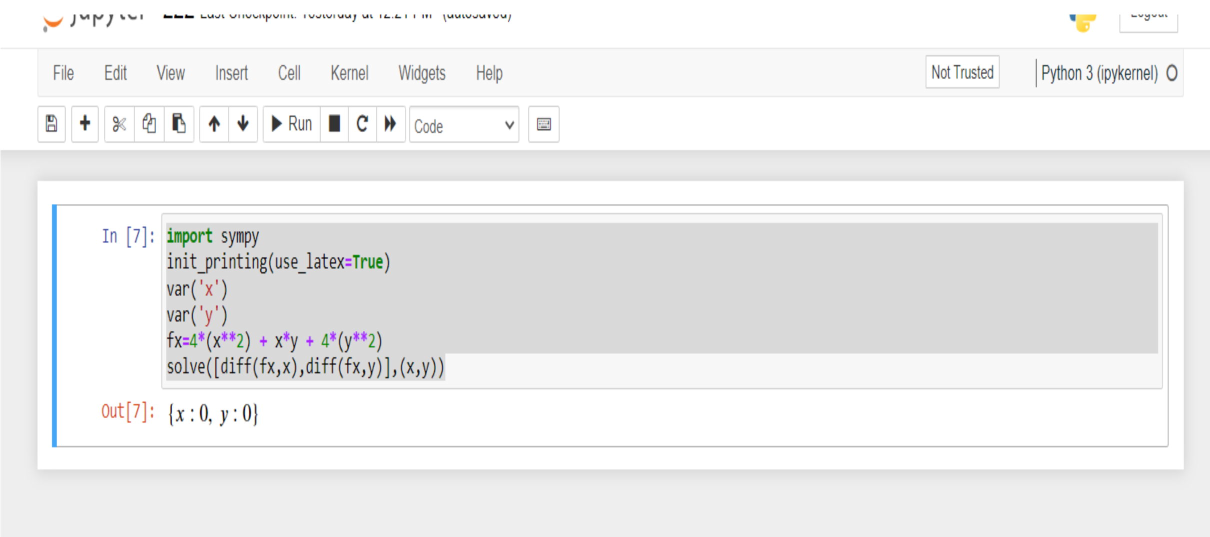
5501-Assignment 2-2

1.

Python code

```
import sympy
init_printing(use_latex=True)
var('x')
var('y')
fx=4*(x**2) + x*y + 4*(y**2)
solve([diff(fx,x),diff(fx,y)],(x,y))
```

output: $\{x:0, y:0\}$



② Given the function below, determine the gradient of the function

$$f(x,y,z) = xe^{xy} + xy + z$$

sol:

$$f(x,y,z) = \frac{df}{dx} i + \frac{df}{dx} y + \frac{df}{dx} z$$

$$= x.e^{xy} + xy + z$$ (Applying the product rule)

$$= u.\frac{d}{dx}(v) + v\frac{d}{dx}(u) + y$$

$$= x\frac{d}{dx}(e^{xy}) + e^{xy}\frac{d}{dx}(x) + y$$

$$= xye^{xy} + e^{xy} + y$$

$$\boxed{\frac{df}{dx} = xye^{xy} + e^{xy} + y}$$

$$\frac{df}{dy} = xe^{xy} + xy + z$$

$$= x\frac{d}{dy}(e^{xy}) + e^{xy}\frac{d}{dx}(x) + x$$

$$= xxe^{xy} + 0 + x$$

$$\boxed{\frac{df}{dy} = x^2e^{xy} + x}$$

$$\boxed{\frac{df}{dz} = 1}$$

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gradient $f(x,y,z) = xye^{xy} + e^{xy} + y,\ x^2e^{xy} + x, 1$

$$= (xye^{xy} + e^{xy} + y)\hat{i} + (x^2e^{xy} + x)\hat{j} + \hat{k}$$

$$= (e^{xy}(xy+1) + y)\hat{i} + (x^2e^{xy} + x)\hat{j} + \hat{k}$$

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③ Determine the directional derivative of the given function in the indicated direction and at the indicated point

$f(x,y) = x\cos(y)$, in the direction of $\vec{v} = \langle 4,2 \rangle$ at $(1, \pi/4)$

Sol:

$f(x,y) = x\cos(y)$

$\nabla f = \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

$= \frac{\partial(x\cos(y))}{\partial x}, \frac{\partial(x\cos(y))}{\partial y}, \frac{\partial(x\cos(y))}{\partial z}$

$= \cos(y), -x\sin(y), 0$

$= \cos(y), -x\sin(y)$

At the point $(1, \pi/4)$

$\nabla f = \cos\left(\frac{\pi}{4}\right), -1\sin\left(\frac{\pi}{4}\right)\rangle$

$= \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

$\vec{v} = \langle 4,2 \rangle$

$|\vec{v}| = \sqrt{4^2+2^2}$

$= \sqrt{20}$

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Direction derivative of a vector field f in the direction of vector $\vec{v}$ is given by

$$\frac{\nabla f \cdot \vec{v}}{|\vec{v}|} = \frac{\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle \langle 4, 2 \rangle}{\sqrt{20}}$$

$$= \frac{\frac{4}{\sqrt{2}} - \frac{2}{\sqrt{2}}}{\sqrt{20}}$$

$$= \frac{2.828 - 1.414}{4.4721}$$

$$\boxed{= 0.316}$$

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(7) Determine the directional derivative of the given function below in the indicated direction of $\vec{V} = \langle V1, V2, V3 \rangle$.

$$f(x,y,z) = x^2y^3 - 4xz$$

Sol:

$$\vec{V} = \langle V1, V2, V3 \rangle$$

$$f(x,y,z) = x^2y^3 \cdot 4xz$$

$$\frac{\partial f}{\partial x} = 2x^2y^3 - 4z$$

$$\frac{\partial f}{\partial y} = 3y^2x^2$$

$$\frac{\partial f}{\partial z} = -4x$$

$$\nabla f = \langle 2x^2y^3 - 4z, 3y^2x^2, -4x \rangle$$

converting vector to unit vector

$$\frac{\vec{V}}{|\vec{V}|} = \left\langle \frac{V_1}{\sqrt{v_1^2+v_2^2+v_3^2}}, \frac{V2}{\sqrt{v_1^2+v_2^2+v_3^2}}, \frac{V3}{\sqrt{v_1^2, v_2^2, v_3^2}} \right\rangle$$

$$\text{Directional derivate} = \frac{\Delta f\ \vec{V}}{|\vec{V}|}$$

$$(2xy^3 - 4z, 3x^2y^2, -4x) \left\langle \frac{V_1}{\sqrt{v_1^2, v_2^2, v_3^2}}, \frac{V2}{\sqrt{v_1^2, v_2^2, v_3^2}}, \frac{V3}{\sqrt{v_1^2, v_2^2, v_3^2}} \right\rangle$$

$$= \frac{(2xy^3 - 4z)V_1 + (3x^2y^2)V_2 - (4x)V_3}{\sqrt{V_1^2 + V_2^2 + V_3^2}}$$

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⑤ Given the function as:

$$f(x,y) = \sqrt{x^2+y^4}$$

ⓐ find the maximum rate of change of the given function at $(-1,1)$

ⓑ find the direction in which this maximum rate of change occurs at $(-1,1)$

sol:

To find the maximum rate of change of the given function at $(-1,1)$ we have to find gradient of $f(x)$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \langle f_x(x,y), f_y(x,y)\rangle$$

$$\nabla f(x,y) = \left\langle \frac{\partial(\sqrt{x^2+y^4})}{\partial x}, \frac{\partial(\sqrt{x^2+y^4})}{\partial y}\right\rangle$$

$$\frac{\partial(\sqrt{x^2+y^4})}{\partial x} = \frac{1}{2\sqrt{x^2+y^4}}\frac{\partial}{\partial x}(x^2+y^4)$$

$$= \frac{1}{2\sqrt{x^2+y^4}}(2x)$$

$$= \frac{2x}{2\sqrt{x^2+y^4}}$$

$$= \frac{x}{\sqrt{x^2+y^4}}$$

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Similarly

$$\frac{\partial\left(\sqrt{x^2+y^4}\right)}{\partial y} = \frac{1}{2\sqrt{x^2+y^4}}\frac{d}{dy}(x^2+y^4)$$

$$= \frac{1}{2\sqrt{x^2+y^4}}\,4y^3$$

$$= \frac{4y^3}{2\sqrt{x^2+y^4}}$$

$$= \frac{2y^3}{\sqrt{x^2+y^4}}$$

$$\nabla f(x,y) = \left< \frac{x}{\sqrt{x^2+y^4}}, \frac{2y^3}{\sqrt{x^2+y^4}} \right>$$

we can find of gradient of $f(x,y)$ at $(-1,1)$

By substituting $x=-1$ and $y=1$

$$\nabla f(-1,1) = \left< \frac{-1}{\sqrt{(-1)^2+(1)^4}}, \frac{2(1^3)}{\sqrt{(-1)^2+(1)^4}} \right>$$

$$\nabla f(-1,1) = \left< -\frac{\sqrt{2}}{2}, \sqrt{2} \right>$$

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(a) maximum rate of change of the given function at $(-1,1)$ is the magnitude of the gradient of $f(x,y)$ at $(-1,1)$

at (-1.1)

$$\nabla f(-1.1) = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + (\sqrt{2}^2)}$$

$$= \frac{\sqrt{10}}{2}$$

$$\boxed{\nabla f(-1.1) = 1.581}$$

(b) Direction in which maximum rate of change of given function occurs at $(-1.1) = \left\langle -\frac{\sqrt{2}}{2}, \sqrt{2} \right\rangle$

$$\boxed{= \left\langle -\frac{\sqrt{2}}{2}, \sqrt{2} \right\rangle}$$

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(6) find the local extrema of the given function

$f(x,y) = 4 + x^3 + y^3 - 3xy$

$\frac{\partial f}{\partial x} = 0$

$3x^2 - 3y = 0$

$3(x^2 - y) = 0$

$x^2 = y \rightarrow (1)$

$\frac{\partial f}{\partial y} = 0$

$3y^2 - 3x = 0$

$3(y^2 - x) = 0$

$x = y^2 \rightarrow 2$

consider (1)

$x^2 = y$

substitute (2) in (1)

$y = x^2$

$y = (y^2)^2$

$= y^4 - y = 0$

$y(y^3 - 1) = 0$

$y = 0 \quad y = 1$

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for $y=0$ sub in eq ①

$$x^2 = y$$
$$x^2 = 0$$
$$x = 0 \Rightarrow x, y = (0,0)$$

for $y=1$ sub in eq ②

$$y^2 = x$$
$$x = 1$$
$$(x.y) = (1.1)$$

Critical points are $(0,0)$ & $(1,1)$

find the second derivate

$$\frac{df}{dx} = 3x^2 - 3y \qquad \frac{df}{dy} = 3y^2 - 3x$$

$$\frac{df^2}{dx^2} = 6x \qquad \frac{df^2}{dy^2} = 6y$$

$$\frac{df}{dxy} = -3 \qquad \frac{df}{dxy} = -3$$

Local exterma $\left(\frac{df^2}{dx^2}\right)\left(\frac{d^2f}{dy^2}\right) - \left(\frac{df}{dxy}\right)^2$

$$= (6x)(6y) - (-3)^2$$
$$= 36xy - 9$$

$\therefore$ Local exterma is $36xy - 9$

(7) for the function $f(x,y) = x^2+y^2$, use the iterative optimization method to calculate the first four steps of the gradient descent for the function $f(x,y)$ with $\langle x_0, y_0\rangle = \langle 1,1\rangle$, $\alpha = 0.1$ and precision $= 0.000001$ or 10^(-6)

sol: $f(x,y) = x^2+y^2$

The iterative optimization method, gradient descent for $f(x,y)$ is given by

$$x^{k+1} = x^k + \alpha d^k$$

where the descent direction d^k is given by the negative of the gradient of $f(x,y)$ and $x^{k+1} = (x^{k+1}, y^{k+1})$

$$d^k = -\nabla f(x^k, y^k) = -\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}_{/(x^k,y^k)}$$

$$= \begin{pmatrix} -2x \\ -2y \end{pmatrix}_{/(x^k,y^k)}$$

Now given that, $(x^0, y^0) = (1,1)$

$\alpha = 0.1$

$$x' = x^0 + \alpha d^0$$

$= (1,1) + 0.1\,(-2\times 1, -2\times 1)$ $\because [d^0 = -\nabla f(x^0,y^0) = (-2.1, -2.1)$

$= (1,1) - \left(\frac{2}{10}, \frac{2}{10}\right)$

$= \left(\frac{4}{5}, \frac{4}{5}\right) = (0.8, 0.8)$

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$$x^2 = x^1 + \alpha d^1$$
$$= (0.8, 0.8) + 0.1\,(-2\times 0.8, -2\times 0.8)$$
$$= (0.8, 0.8) - (0.16, 0.16)$$
$$= (0.64, 0.64)$$

$$x^3 = x^2 + \alpha d^2$$

$$= (0.64, 0.64) + 0.1\,(-2\times 0.64, -2\times 0.64)$$
$$= (0.64, 0.64) - (0.128, 0.128)$$
$$= (0.512, 0.512)$$

$$x^4 = x^3 + \alpha d^3 = (0.512, 0.512) + 0.1(-2\times 0.512, -2\times 0.512)$$
$$= (0.512, 0.512) - (0.1024, 0.1024)$$
$$= (0.4096, 0.4096)$$

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