UNIT-II - SET THEORY:

A set is any well-defined collection or class of distinct object.

Eg; 1) $\{1, 2, 3\} \rightarrow$ set of elements.

2) Rivers in India $\rightarrow$ set of objects.

ELEMENTS OF A SET:-

The objects in a set are called elements or numbers. The elements in the set must be distinct.

Eg: 1) $x \in A$

2) $x \notin A$.

SET DISCRIPTION:-

There are two methods in it,

a) ROSTER METHOD/ TABULAR / ENUMERATION METHOD:-

In this methods elements of a set are described by elimination within the braces. i.e, the elements of a set by listing the elements inside ~~the~~ of brackets.

Eg; $V = \{a, e, i, o, u\}$.

$V = \{1, 2, 3, 4, \ldots\}$ the set of all natural no's.

b) SET BUILDER/ SELECTION METHOD:-

The elements of a set are represented by stating a property (qualitative, quantitative, or both) that uniquely chategorised them.

Eg; $\{x \in N \mid 0 < x < 11, x \text{ is odd}\}$.

TYPES OF SETS:-

1) FINITE & INFINITE SET:-

A set is finite if it contains finite number of different elements.

Eg; $A = \{1, 2, 3, 4, 5\}$.

A set which contains infinite no. of. elements is known as infinite set.

Eg; $W = \{0, 1, 2, 3, \ldots\}$

$N = \{1, 2, 3, \ldots\}$.

2) SINGLETON SET / UNIT SET:-

A set which contains only one element is called a singleton set.

Eg; $A = \{5\}$.

3) NULL SET / EMPTY / VOID:-

A set which contains no element is called as null set.

It is denoted by ϕ.

Eg; $A = \{\ \}$

4) EQUALITY OF SETS:-

Two sets A & B are said to be equal if every element of A is an element of B and also every element of B is an element of A. The equality of 2 sets A & B is denoted by $A=B$. Symbolically, $A=B$ iff $x \in A$, this is known as axiom of extention or identity.

5) IV EQUALENT SET:- Two sets A & B are said to be equivalent if they have the same cardinality i.e, $n(A) = n(B)$.

If the elements of one set can be put into one to one correspondence with the elements of another sets then the two sets are called equalent. It is denoted by $\sim$ or $\equiv$

Generally, two sets are equivalent to each other if the no. of. elements in both the sets is equal.

SUBSET:-

Let A & B be two sets. The set A is a subset of B iff every element of A is an element of B. In other words, the set A is a subset of B if,

$x \in A \Rightarrow x \in B$.

$B = \{1, 2, 3, 4, 5\}$

$A = \{2, 3, 4\}$.

$A \subseteq B$.

*) $\subseteq$ → Subset | contained.

*) If A contained B then B is called superset of A.

PROPERTIES OF SUBSET:-

*) If the set A is a subset of B, then B is called superset.

*) If the set A is a subset of B, and B is a subset of A.

$A \subseteq B$

$B \subseteq A$

$\boxed{A = B.}$

i.e, then the sets A & B are said to be equal.

*) If the set A is a subset of B and B is a subset of c.

$A \subseteq B$

$B \subseteq C.$

Then $A \subseteq C$.

i.e, A is a subset of c.

PROPER SUBSET:-

The set A is a proper subset of the set B or A is properly contained in B iff,

*) every element of subset A is also an element of B.

i.e, $A \subseteq B$.

*) there is atleast one element in the set B that is not in A i.e, $A \neq B$.

For eg; If $A = \{1,3,5\}$ then $B = \{1,5\}$ A is propersubset of A. Suppose $C = \{1,3,5\}$ is a subset of A, but it is not a proper subset of A since $C = A$.

FAMILY OF SETS:-

If the element of the set are set themself then, such a set is called family of sets. The word collection and class are also used for a set of sets.

Eg; $B = \{a, b\}$.

$\{\{a\}, \{b\}, \{ab\}, \phi\}$.

POWER SET:-

If S is any set then the family of all the subset of S is called the powerset of S and it is denoted by P(S).

Eg; $S = \{a, b, c\}$.

$P(S) = \{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

The no. of members of a set is often written as $|S|$, so when S has n member we can write :

$$|P(S)| = 2^n.$$

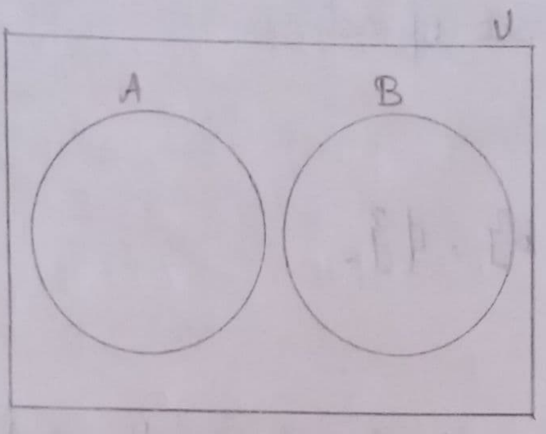
Eg; $S = \{1, 2, 3, 4, 5\}$.

$|P(S)| = 2^n = 2^5 = 32$.

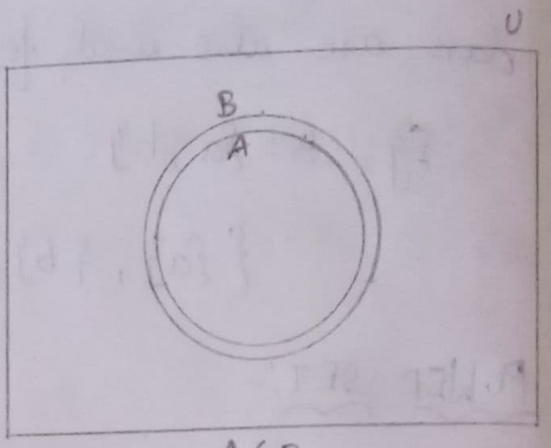
UNIVERSAL SET:-

In application of set theory all the sets under discussion are assumed to be the subset of the fixed large set is called the universal set or universal of discourse. It is denoted by U or E.

VENN - EULER DIAGRAM:-



A, B are disjoint.



$A \subseteq B$

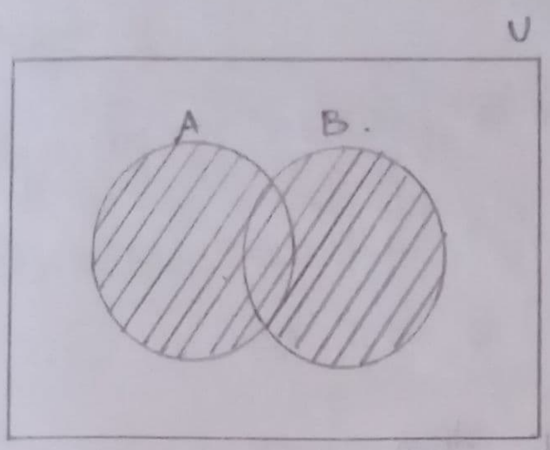
SET OPERATIONS AND ITS PROPERTIES:-

1) Union of sets
2) Intersection of sets
3) Disjoint sets
4) Difference of sets
5) complement of sets.

1) UNION OF SETS:-

Let A & B the two non-empty set, the union of A & B is the set of all elements which all either in A or B and in both A & B and it is denoted by $A \cup B$.

$$A \cup B = \{x ; x \in A \text{ (or) } x \in B\}.$$

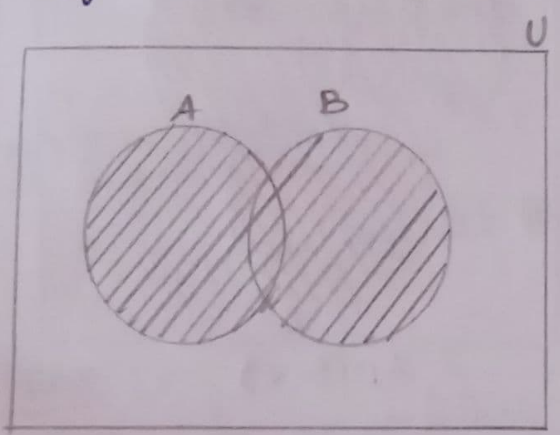


$A \cup B$.

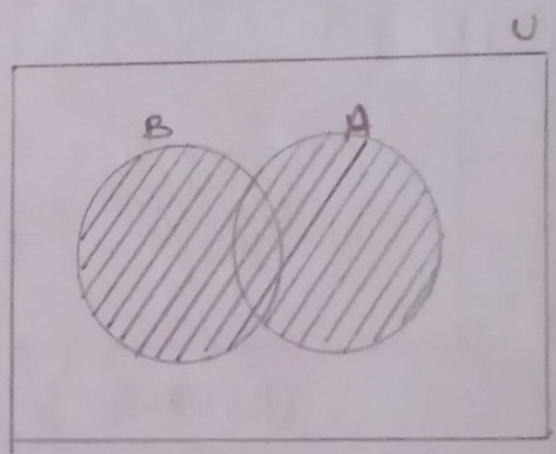
PROPERTIES:-

1) COMMUTATIVE:-

If A & B are any 2 sets then $A \cup B = B \cup A$.



$A \cup B$



$B \cup A$

PROOF:-

Let x arbitary element

$x \in A \cup B \iff x \in A$ or $x \in B$.

$\iff x \in B$ or $x \in A$.

$\iff x \in B \cup A$.

$A \cup B \subseteq B \cup A \rightarrow$ 1.

$y \in B \cup A \iff y \in B$ or $y \in A$

$\iff y \in A$ or $y \in B$

$\iff A \cup B$ or $y \in A \cup B$.

$B \cup A \subseteq A \cup B \rightarrow$ ②

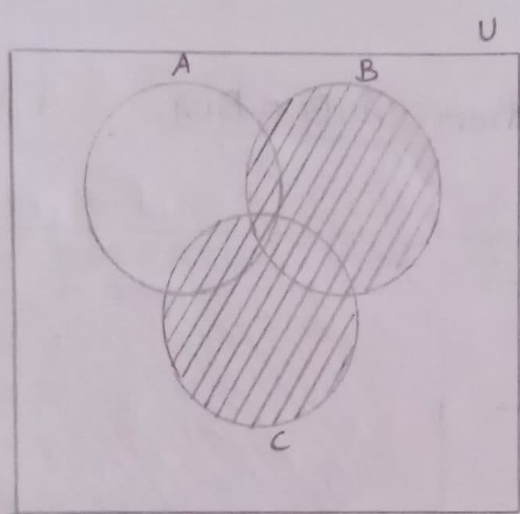
from ① & ②

$\boxed{A \cup B = B \cup A.}$

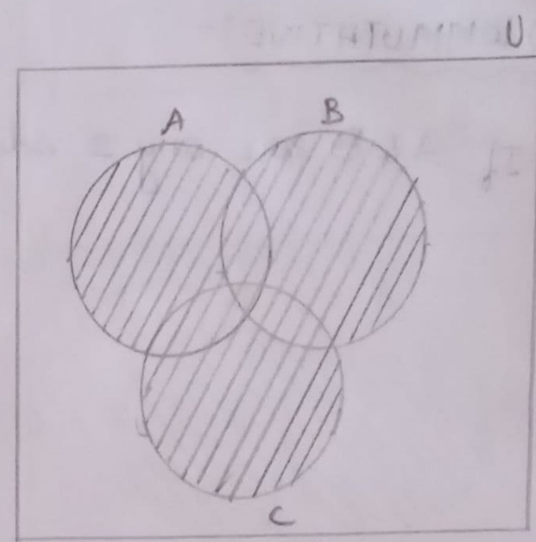
2) ASSOCIATIVE PROPERTY:-

A, B & C are any 3 sets then.

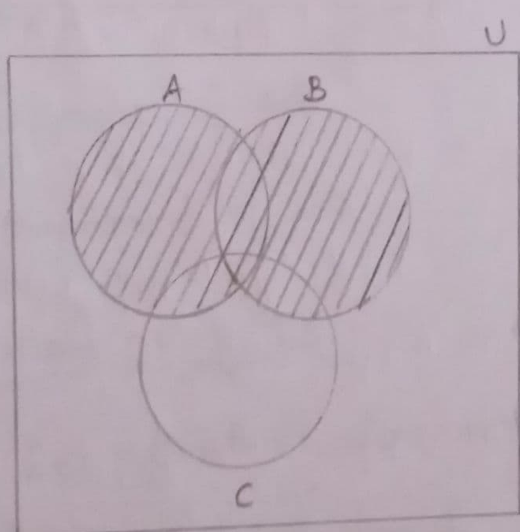
$\boxed{A \cup (B \cup C) = (A \cup B) \cup C}$



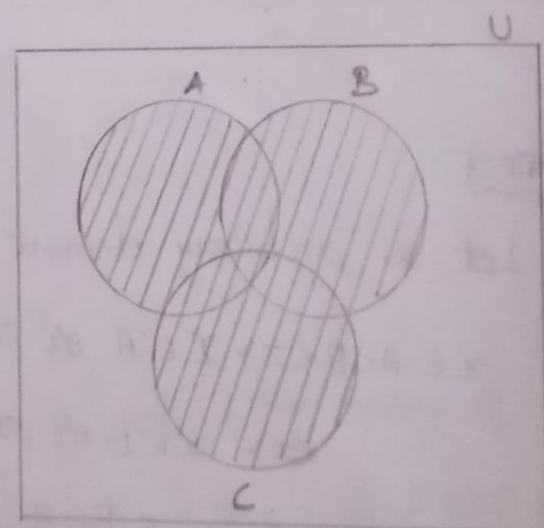
$(B \cup C)$



$A \cup (B \cup C)$ $\rightarrow$ ①



$(A \cup B)$



$(A \cup B) \cup C$ $\rightarrow$ ②

① = ② $\Rightarrow A \cup (B \cup C) = (A \cup B) \cup C.$

PROOF:-

Let x be an arbitary element of $A\cup(B\cup C)$

$x \in A\cup(B\cup C) \Leftrightarrow x \in A$ or $x \in (B\cup C)$

$\Leftrightarrow x\in A$ or $(x \in B$ (or) $x \in C)$

see the right hand side ie, $(A\cup B)\cup C$ so, changing the brackets.

$\Leftrightarrow (x\in A$ or $x\in B)$ or $x \in C$.

$\Leftrightarrow x \in (A\cup B)$ or $x\in C$.

$\Leftrightarrow x \in (A\cup B)\cup C$.

$\Leftrightarrow (A\cup B)\cup C$.

$\therefore A\cup(B\cup C) = (A\cup B)\cup C$.

3) IDEMPOTENT:-

If A is any set then $A\cup A = A$.

PROOF:-

If x be an arbitary element of $A\cup A$.

$x \in A\cup A = x\in A$ or $x\in A$.

$= x\in A$.

$= A$.

$\therefore A\cup A = A$.

4) If A & B are any 2 set then $A \subseteq A\cup B$ & $B\subseteq A\cup B$.

5) If A is any set then $A\cup\phi = A$.

6) If A is any subset of the universal set 'U' then $A\cup U = U$ ($A\cup U = U$)

7) If $A\subseteq B$, then $A\cup B = B$ and $B\subseteq A$ then $A\cup B = A$

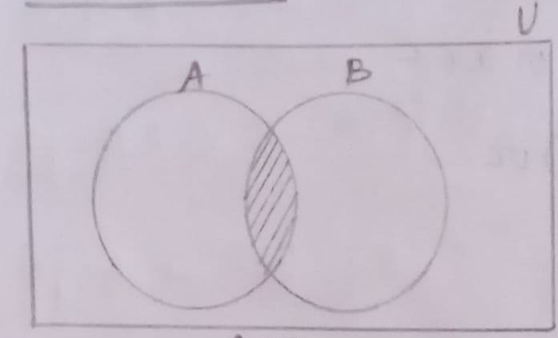
2) INTERSECTION OF SETS:-

Let A and B be two non empty set then intersection of A and B is the set of all elements which are in both A & B.

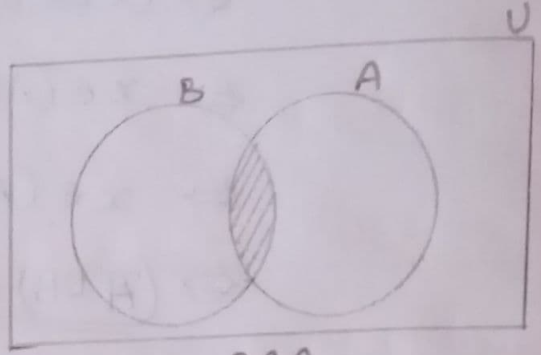
PROPERTIES:-

1) COMMUTATIVE:-

$A \cap B = B \cap A.$



$A \cap B$



$B \cap A$

Let x be an arbitary element of $A \cap B$.

$x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$.

$\Leftrightarrow x \in B$ and $x \in A$.

$\Leftrightarrow x \in B \cap A$.

$A \cap B \subseteq B \cap A \rightarrow (1)$.

Let y be an arbitary element of $B \cap A$.

$y \in B \cap A \Leftrightarrow y \in B$ and $y \in A$

$\Leftrightarrow y \in A$ and $y \in B$.

$\Leftrightarrow y \in A \cap B$.

$B \cap A \subseteq A \cap B \longrightarrow (2)$

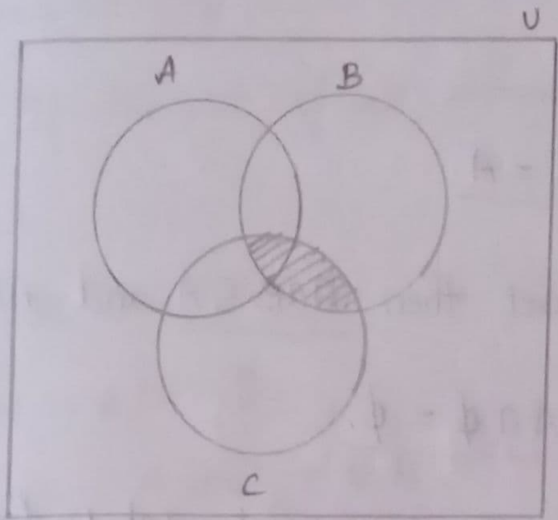
from (1) and (2)

$\boxed{A \cap B = B \cap A}$.

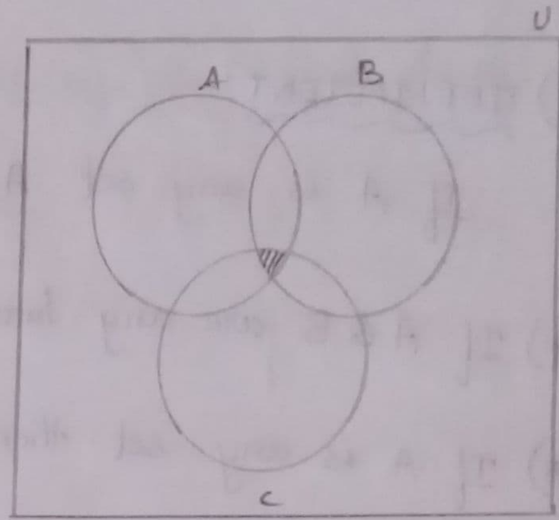
2) ASSOCIATIVE :-

A, B & C are any 3 sets then

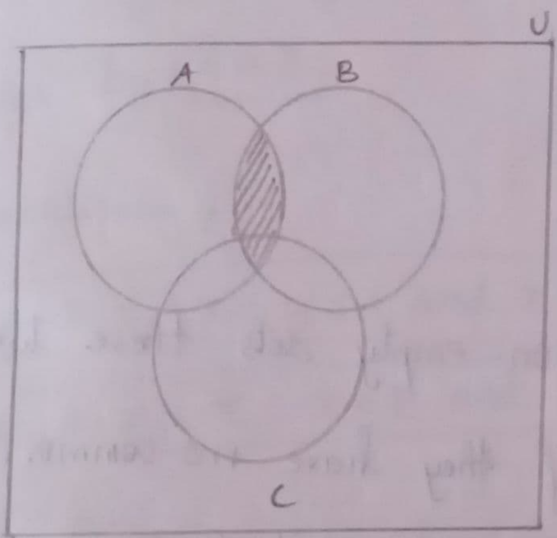
$A \cap (B \cap C) = (A \cap B) \cap C.$



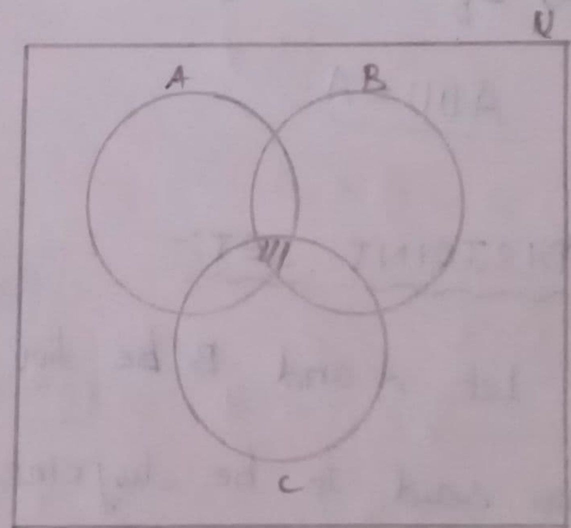
$B \cap C$



$A \cap (B \cap C)$ → ①



$A \cap B$



$(A \cap B) \cap C$ → ②

① = ② $\Rightarrow A \cap (B \cap C) = (A \cap B) \cap C.$

PROOF:

Let x be an arbitary element of $A \cap (B \cap C)$.

$x \in A \cap (B \cap C) \Leftrightarrow x \in A$ and $x \in (B \cap C)$;

$\Leftrightarrow x \in A$ and $(x \in B$ and $x \in C)$

$\Leftrightarrow \underbrace{(x \in A \text{ and } x \in B)}$ and $x \in C$.

$\Leftrightarrow x \in (A \cap B)$ and $x \in C$.

$\Leftrightarrow x \in (A \cap B) \cap C.$

$\Leftrightarrow (A \cap B) \cap C.$

$\therefore \boxed{A \cap (B \cap C) = (A \cap B) \cap C}.$

3) IDEMPOTENT :-

If A is any set $A \cap A = A$.

4) If A & B are any two set then $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

5) If A is any set then $A \cap \phi = \phi$.

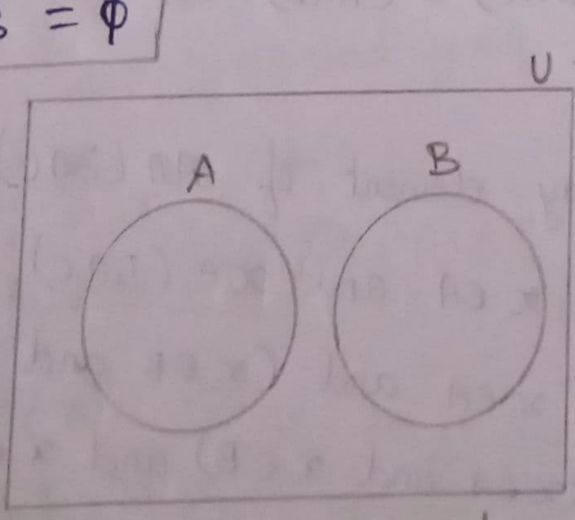
6) If A is any subset of the universal set U then

$A \cup U = A$.

3) DISJOINT SET :-

Let A and B be two non-empty sets these two sets are said to be disjoint if they have no common elements i.e, the intersection is null set symbolically.

$\boxed{A \cap B = \phi}$



Disjoint set.

(5) 4) DIFFERENCE OF TWO SETS:-

If A & B are any two sets then the difference of A & B is denoted by A−B or A\B.

If the set of all element belongs to A but do not belongs to B.

$x \in A, x \notin B.$

$x \in A - B.$

$x \in A$ and $x \notin B.$

The difference of A and B is read as A\B or A−B is denoted by $A \sim B$.

Symbolically,

$$\boxed{A-B = \{x ; x \in A \text{ and } x \notin B\}}$$

$$\boxed{B-A = \{x ; x \in B \text{ and } x \notin A\}}$$

Eg; $A = \{0,2,4,9\}$ $B = \{0,3,6,8,9\}$

$A - B = \{0,2,4,9\} - \{0,3,6,8,9\}.$

$= \{2,4\}$

$B - A = \{0,3,6,8,9\} - \{0,2,4,9\}$

$= \{3,6,8\}.$

PROPERTIES :-

1) $A - A = \phi$

Proof :-

Let x belongs to $A - A$

i.e, $x \in A - A$.

$\Leftrightarrow x \in A$ and $x \notin A$.

$\therefore$ there is no element satisfying both of these condition

hence there is no element belongs to $A - A$.

$\therefore A - A = \phi$

2) $A - \phi = A$.

proof:

$x \in A - \phi$

$\Leftrightarrow x \in A$ and $x \notin \phi$.

$\therefore A - \phi = A$.

3) $A - B \subseteq A$.

proof :

Let $x \in A - B$.

$x \in A$ & $x \notin B$.

$x \in A$

$\therefore A - B \subseteq A$.

4) $A-B$, $A \cap B$ and $B-A$ are mutually disjoint.

5) $(A-B) \cap B = \phi$.

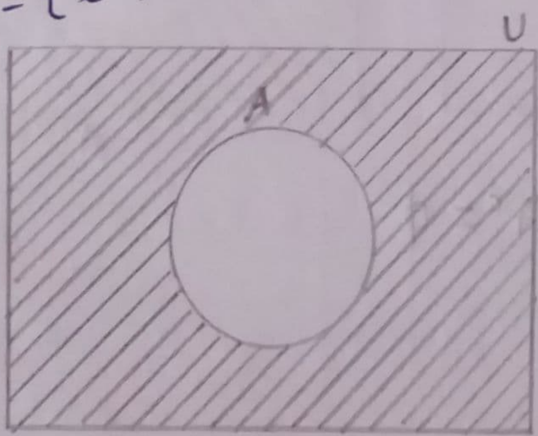
6) $(A-B) \cup A = A$.

5) COMPLEMENT OF A SET:-

Let A be any set the complement of A is the set of all elements that belongs to the universal set but do not belongs to A. If U is an universal set the complement of belongs to A. If U is an universal set the complement of 'A' is the set $U-A$ is denoted by $\bar{A}$ or A' or A^c.

$\therefore A^c = U - A$

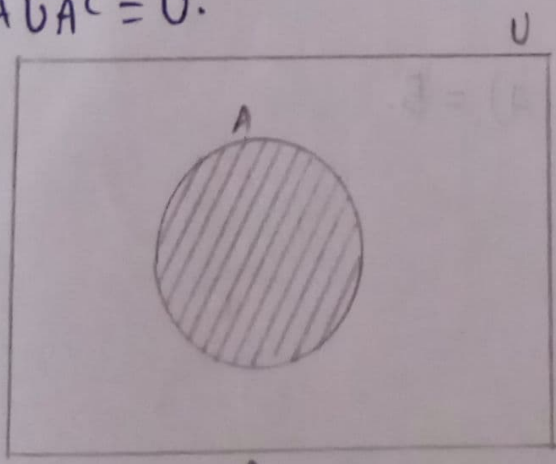
$A^c = \{x ; x \in U \text{ and } x \notin A\}$.



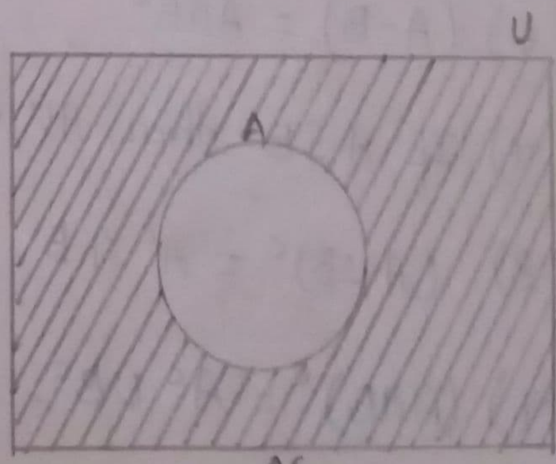
A^c

PROPERTIES:-

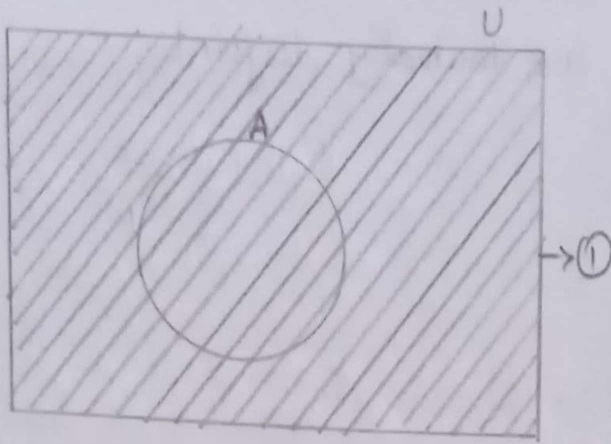
1) $A \cup A^c = U$.



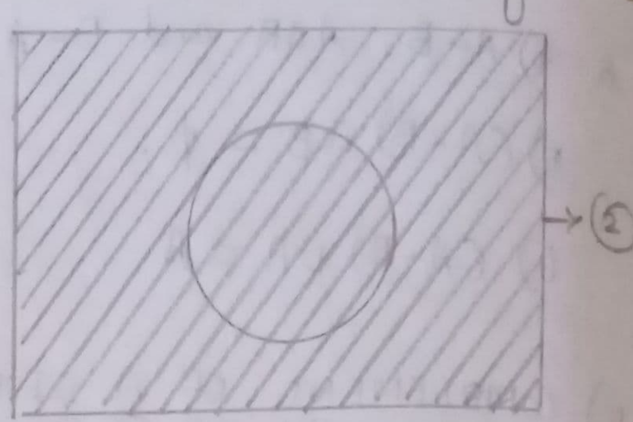
A



A^c



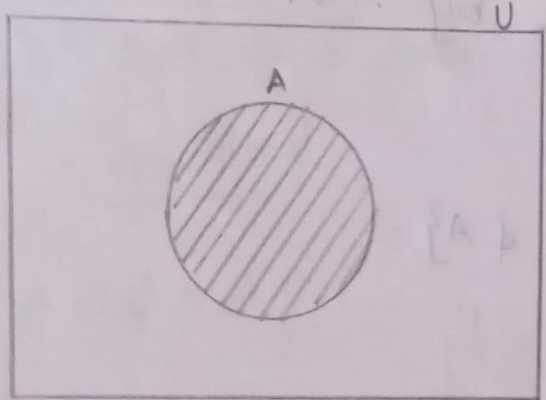
$A \cup A^c$



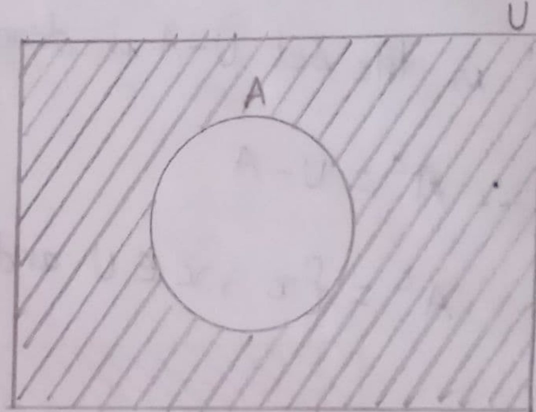
U

Here ① = ②.

2) $A \cap A^c = \phi$



A



A^c

$\therefore A \cap A^c = \phi$

3) $U^c = \phi$

4) $\phi^c = U$

5) $(A^c)^c = A$

6) $(A-B) = A \cap B^c$

7) If $A \subseteq B$ then $A \cup (B-A) = B$.

8) $(A \cup B)^c = A^c \cap B^c$

9) $(A \cap B)^c = A^c \cup B^c$.

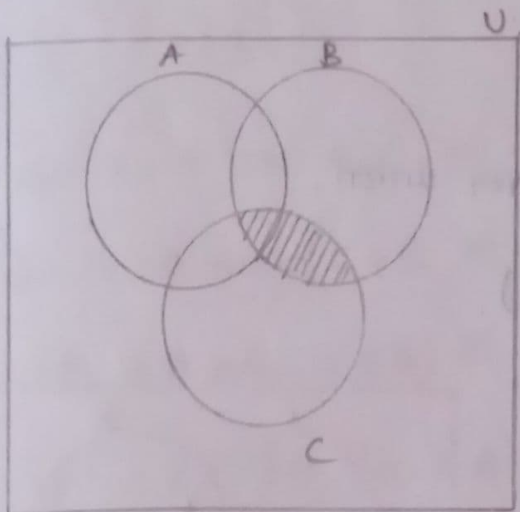
DISTRIBUTIVE LAWS:-

1) Union of distributive over intersection:-

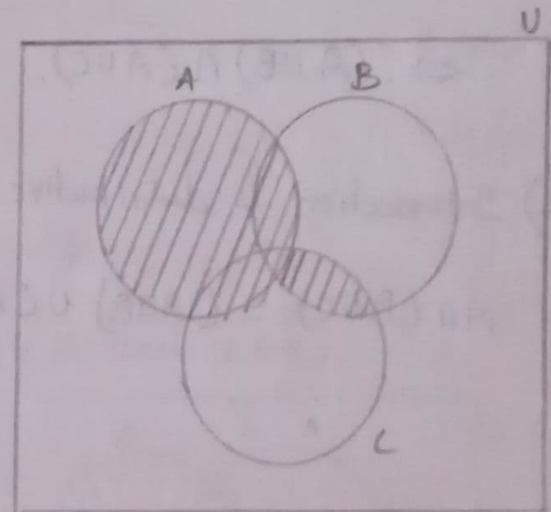
If A, B & C are any 3 sets then,

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

PROOF:-

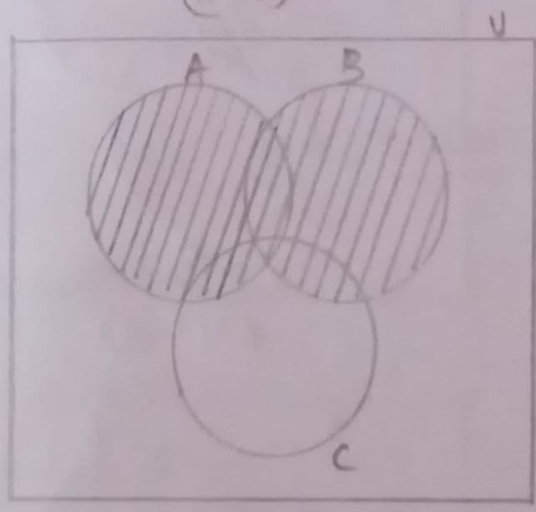


$(B \cap C)$

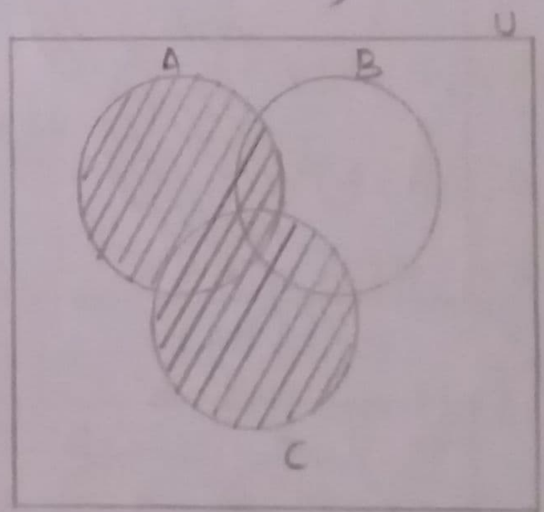


→ ①

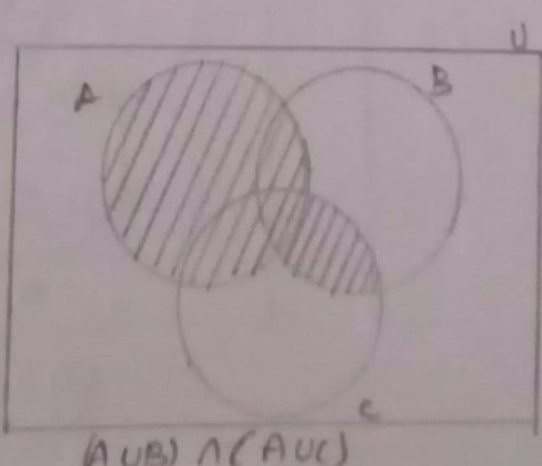
$A \cup (B \cap C)$



$A \cup B$



$A \cup C$



→ ②

$\therefore$ ① = ②

$(A \cup B) \cap (A \cup C)$

Proof:

Let $x \in A \cup (B \cap C)$

$\Leftrightarrow x \in A$ or $x \in (B \cap C)$.

$\Leftrightarrow x \in A$ or $(x \in B$ and $x \in C)$.

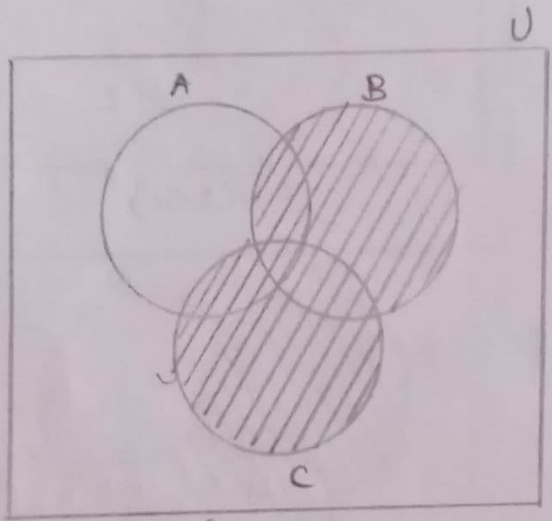
$\Leftrightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Leftrightarrow (x \in A \cup B) \cap (x \in A \cup C)$.

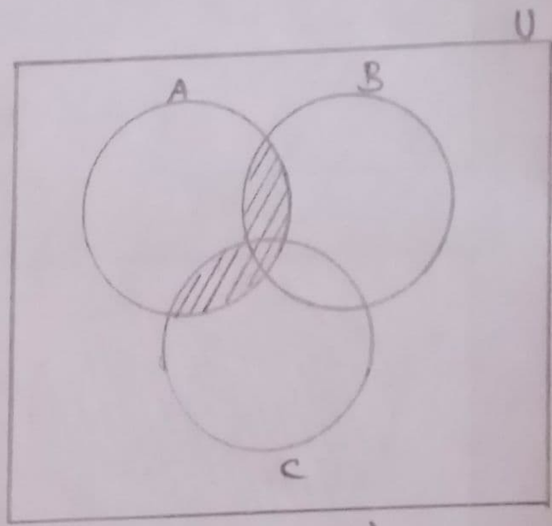
$\Leftrightarrow (A \cup B) \cap (A \cup C)$.

2) Intersection is distributive over union.

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

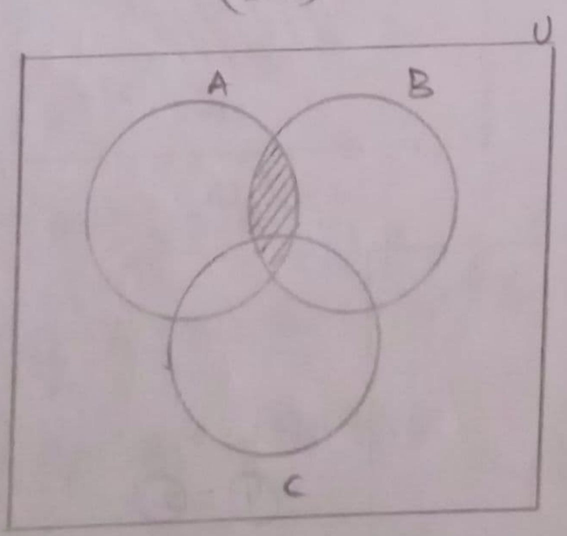


$(B \cup C)$

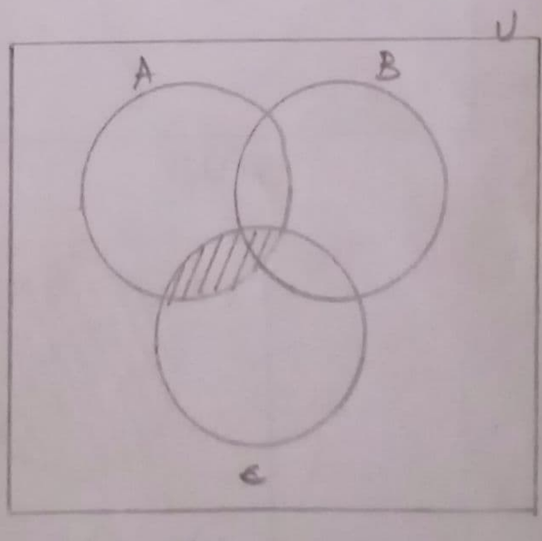


$\to$ ①

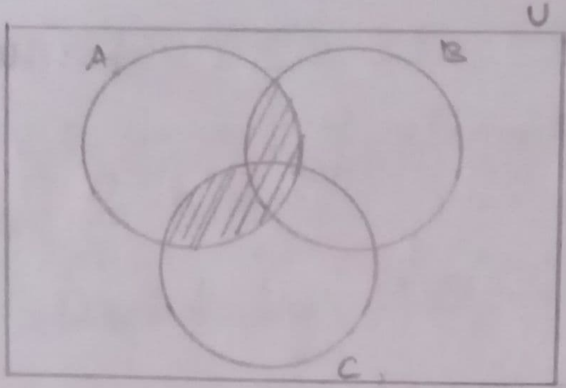
$A \cap (B \cup C)$



$(A \cap B)$



$(A \cap C)$



$\rightarrow$ ②

$(A \cap B) \cup (A \cap C)$

$\therefore$ ① = ②

Proof:-

Let $x \in A \cap (B \cup C)$

$\Leftrightarrow x \in A$ and $x \in (B \cup C)$

$\Leftrightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$.

$\Leftrightarrow (x \in A \cap B) \cup (x \in A \cap C)$.

$(A \cap B) \cup (A \cap C)$.

SOME BASIC SET IDENTITIES:-

1) Idempotent law:-

$A \cup A = A$

$A \cap A = A$.

2) Associative laws:-

$(A \cup B) \cup C = A \cup (B \cup C)$.

$(A \cap B) \cap C = A \cap (B \cap C)$.

3) Commutative laws:

E : Universal set.

$A \cup B = B \cup A$

$A \cap B = B \cap A$.

4) Distributive law.

4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

5) $A \cup \phi = A$

$A \cap E = A$.

6) $A \cup E = E$

$A \cap \phi = \phi$.

7) $A \cup \sim A = E$

$A \cap \sim A = \phi$.

8) Absorption laws:

$A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$.

9) DeMorgan's laws:

9) $\sim (A \cup B) = \sim A \cap \sim B$.

$\sim (A \cap B) = \sim A \cup \sim B$.

10) $\sim \phi = E$

$\sim E = \phi$

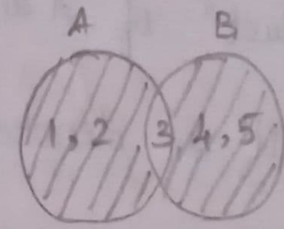
11) $\sim (\sim A) = A$.

SYMMETRIC DIFFERENCE OF TWO SETS:-

*) Let A & B is set of elements which are either in A or in B but not in both.

*) It is denoted by $A \oplus B$ or $A \Delta B$.

*) It can be said as,



$$\boxed{\begin{aligned} A \oplus B &= (A \cup B) - (A \cap B). \\ A \oplus B &= (A - B) \cup (B - A). \end{aligned}}$$

Eg; $A = \{1,2,3\}$, $B = \{3,4,5\}$ find $A \oplus B$.

$$A \oplus B = (A \cup B) - (A \cap B).$$

$$= \{1,2,3,4,5\} - \{3\}$$

$$A \oplus B = \{1,2,4,5\}.$$

PROPERTIES:-

1) $A \oplus B = B \oplus A$ → Commutative.

2) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ → Associative.

3) $A \oplus A = \phi$

4) $A \oplus (A \cap B) = \phi$

5) $(A \oplus B) \cup (A \cap B) = A \cup B$.

6) $A \oplus B = (A - B) \cup (B - A)$.

7) $A \oplus B = \phi \Leftrightarrow A = B$

CARTESIAN PRODUCT:-

*) Let A and B are 2 sets, then the set of all order pairs ordered pair (a,b) where $a \in A$ and $b \in B$ is called the "cartesian product" of A and B.

*) It is denoted by $A \times B$.

*) $A \times B$ is defined as : $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$.

*) $B \times A$ is defined as : $B \times A = \{(b,a) \mid b \in B \text{ and } a \in A\}$.

Note : $A \times B \neq B \times A$

*) If set A has 'm' elements and set B has 'n' elements, then $A \times B$ has $m \times n$ elements.

A = 3 element, B = 3 element
$A \times B = 3 \times 3 \Rightarrow 9$ ordered pair

Eg; $A = \{1,2,3\}$ $B = \{4,5,6\}$ find out $A \times B$, $B \times A$. check whether both are equivalent or not ?

$$A \times B = \{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}.$$

$$B \times A = \{(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(6,1),(6,2),(6,3)\}.$$

$\therefore A \times B \neq B \times A$.

Eg 2: $A = \phi$, $B = \{1, 2, 3\}$ what are $A \times B$ and $B \times A$?

Sol $A \times B = \phi = B \times A$.

Consider the expressions $(A \times B) \times C$ and $A \times (B \times C)$. From the definition it follows that,

$$(A \times B) \times C = \{((a,b),c) \mid ((a,b) \in A \times B \wedge (c \in C)\}.$$

$$= \{(a,b,c) \mid (a \in A) \wedge (b \in B) \wedge (c \in C)\}.$$

$$A \times (B \times C) = \{(a,(b,c)) \mid (a \in A) \wedge ((b,c) \in B \times C)\}.$$

$$\therefore \boxed{(A \times B) \times C \neq A \times (B \times C)}$$

PROPERTIES OF CARTESIAN PRODUCT:-

$A \times (B \cup C) = (A \times B) \cup (A \times C)$.

$A \times (B \cap C) = (A \times B) \cap (A \times C)$.

PROOF:-

$$A \times (B \cup C) = \{(x,y) \mid (x \in A) \wedge (y \in B \cup C)\}$$

$$= \{(x,y) \mid (x \in A) \wedge ((y \in B) \vee (y \in C))\}.$$

$$= \{(x,y) \mid ((x \in A) \wedge (y \in B)) \vee ((x \in A) \wedge (y \in C))\}.$$

$$= \underline{(A \times B) \cup (A \times C)}.$$

$A \times (B \cap C) = \{(x,y) \mid (x \in A) \wedge (y \in B \cap C)\}$

$= \{(x,y) \mid (x \in A) \wedge ((y \in B) \wedge (y \in C))\}$

$= \{(x,y) \mid ((x \in A) \wedge (y \in B)) \wedge ((x \in A) \wedge (y \in C))\}$

$= (A \times B) \wedge (A \times C)$.

*) The cartesian product $A \times A$ is also written as A^2, and similary $A \times A \times A$ as A^3, and so on.

RELATIONS / BINARY RELATION:-

*) A relation R b/w two sets A & B is a subset of Cartesian products $A \times B$. A, B -2 sets ind, athinu $A \times B$ kandupidichu athinu oru subset eduthu kazhinjal aanu relation aavunnathu.

*) R is a set of ordered pairs (a, b) such that $a \in A$ and $b \in B$.

*) $\boxed{R \subseteq (A \times B) = \{(x,y) : x \in A \text{ and } y \in B\}}$.

Relation is a subset of $A \times B$, where we have ordered pairs x & y, x is a $\in A$ & $y \in B$.

*) Eg; $A = \{1,2,6,9\}$ $B = \{1,3,5,7\}$ & ~~R~~

$R \subseteq (A \times B) = \{(x,y) : x < y\}$.

Sol $A = \{1, 2, 6, 9\}$, $B = \{1, 3, 5, 7\}$.

$A \times B = \{(1,1), (1,3), (1,5), (1,7), (2,1), (2,3), (2,5), (2,7), (6,1), (6,3), (6,5), (6,7), (9,1), (9,3), (9,5), (9,7)\}$.

Here the relation ~~A×B~~ should have $x < y$.

$R = \{(1,3), (1,5), (1,7), (2,3), (2,5), (2,7), (6,7)\}$.

PROPERTIES OF RELATION:-

1) Reflexive
2) Symmetric
3) Transitive
4) Antisymmetric.

1) REFLEXIVE RELATION:-

A relation R in set A is said to be reflexive relation if $(x, x) \in R$ for all $x \in A$.

$xRx \ \forall x \in A$.

if We saw same elements in ordered pair then we can say that this relation is reflexive.

Eg; $A = \{2, 3, 4\}$.

$A \times A = \{2, 3, 4\} \times \{2, 3, 4\}$

$= \{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4), (4,2), (4,3), (4,4)\}$.

∴ The relation is reflexive.

2) SYMMETRIC RELATION:-

*) A relation R in set A is said to be symmetric if for every $(x,y) \in R$ there exist an $(y,x) \in R$.

*) $xRy \Rightarrow yRx$. if x, y exists then y, x should also exists.

*) Eg; $A = \{2,3,4\}$

$$A \times A = \{2,3,4\} \times \{2,3,4\}$$

$$= \{(2,2), \underbrace{(2,3)}_{1}, \underbrace{(2,4)}_{2}, \underbrace{(3,2)}_{1}, (3,3), \underbrace{(3,4)}_{3}, \underbrace{(4,2)}_{2}, \underbrace{(4,3)}_{3}, (4,4)\}.$$

∴ The relation is said to be symmetric.

3) TRANSITIVE RELATION:-

*) Let A be a set in which the relation R defined.

*) R is said to be transitive, if for every $(x,y) \in R$ and $(y,z) \in R$ there exist an $(x,z) \in R$.

*) $(2,3)$ & $(3,4) \Rightarrow (2,4)$.

*) $xRy \wedge yRz \Rightarrow xRz$.

Eg; $R = \{(3,4), (2,3), (3,2), (4,1), (3,3)\}$.

Transitive relation $R = \{(3,4), (2,3), (3,2), (4,1), (3,3), (3,1), (2,4), (2,2)\}$

$\overset{x\,y}{(3,4)}$ and & $\overset{y\,z}{(4,1)}$ ind but $\overset{x\,z}{(3,1)}$ illa so added

(2,3) ind & (3,4) ind but (2,4) illa so added.

(2,3) ind & (3,2) ind but (2,2) illa so added.

(3,2) ind & (2,3) ind and (3,3) ind.

(4,1) ind but (1,4) illa so leave it

$\overset{x\,y}{(3,3)}$ ind & $\overset{y\,z}{(3,4)}$ ind and we have (3,4) so don't need to repeat

(3,3) ind & (3,2) ind and we have (3,2) so don't need to repeat.

So, this relation is transitive relation.

4) ANTISYMMETRIC RELATION:-

*) A relation R on a set A is said to be antisymmetric relation, if there is no pair of distinct or dissimilar elements of A, each of which is related by R to be other.

*) Whenever (x,y) is in relation R, then (y,x) is not. If (y,x) is in R then $x=y$. $\overset{x\,y}{(2,2)}$, & (y,x) is also (2,2) so, $x=y$.

*) Defn: if $(x,y) \in R$ and $(y,x) \in R$ then $\underline{x=y}$.

*) $\underline{xRy \wedge yRx \Rightarrow x=y}$.

*) Eg; Antisymmetric relation $R = \{(2,3), (3,4), (3,3)\}$.

Q) Give an example of a relation which is neither reflexive nor irreflexive.

Sol A relation R is neither reflexive nor irreflexive, if there exist 2 different elements $a \in A$ and $b \in A$ such that $(a,a) \in R$ and $(b,b) \notin R$.

Eg; Let $A = \{1,2\}$, $R = \{(1,1)\}$.

Then,

R is not reflexive, because $(2,2) \notin R$.

R is not irreflexive, because $(1,1) \in R$.

Q). Give an eg of a relation which is both symmetric and antisymmetric.

Sol $R = \{(a,b) \mid a = b\}$ it says that a & b are equal.

It is symmetric and antisymmetric.

If $(a,b) \in R$ then $(b,a) \in R$ (if $a=b$), since $(a,b) \in R$ and $(b,a) \in R$ if and only if $a=b$, then it is antisymmetric.

Q) If relation R and S are both reflexive, S.T $R \cup S$ and $R \cap S$ are also reflexive?

Sol. A relation R on the set A is reflexive if $(a,a) \in R$ for every element $a \in A$.

Union: $A \cup B$: All elements that are either in A or in B.

R and S are reflexive relation on a set A.

proof: Let $a \in A$.

since R is reflexive :

$(a,a) \in R$.

since S is reflexive :

$(a,a) \in S$.

then union $R \cup S$ contains all elements in either R or S.

$(a,a) \in R \cup S$.

R and S are reflexive relation on a set A.

Let $a \in A$.

since R is reflexive:

$(a,a) \in R$.

since S is reflexive:

$(a,a) \in S$.

the ~~union~~ $R \cap S$ contains all elements in both R and S.

$(a,a) \in R \cap S$.

Q) If relation R and S are reflexive, symmetric, and transitive, show that $R \cap S$ is also reflexive, symmetric and transitive.

Sol. Equivalence relation means reflexive, symmetric & transitive. Let an element $a \in A$. Since R and S are equivalence relations they are reflexive. ∴

$\therefore (a,a) \in R$ and $(a,a) \in S$.

So, $(a,a) \in R$ and $(a,a) \in S$.

So, $(a,a) \in R \cap S$.

$\therefore R \cap S$ is reflexive.

Let $(a,b) \in R \cap S$.

$(a,b) \in R$ and $(a,b) \in S$.

$(b,a) \in R$ and $(b,a) \in S$.

$(b,a) \in R \cap S$.

$\therefore R \cap S$ is symmetric.

Let $(a,b), (b,c) \in R \cap S$.

$(a,b), (b,c) \in R \Rightarrow (a,c) \in R$.

$(a,b), (b,c) \in S \Rightarrow (a,c) \in S$.

$(a,c) \in R \cap S$.

$\therefore R \cap S$ is transitive.

RELATION MATRICES:-

A relation R from a finite set X to a finite set Y can ~~be~~ also be represented by a matrix called the relation matrix of R.

Let $X = \{x_1, x_2, \ldots x_m\}$, $Y = \{y_1, y_2, \ldots y_n\}$ and R be a relation from x to Y. The relation matrix of R can be obtained by first constructing a table whose columns are preceded by a column consisting of successive elements of X and whose rows are headed by a row consisting of the successive elements of Y.

If $x_i R y_i$, then we enter a 1 in the i^{th} row and j^{th} column. If $x_k \not R x_l$, then we enter a zero in the kth ~~to~~ row and lth column. As a special case, consider $m=3$, $n=2$, and R given by:-

$$R = \{(x_1, y_1), (x_2, y_1), (x_3, y_2), (x_2, y_2)\}.$$

Table representation.

	y_1	y_2
x_1	1	0
x_2	1	1
x_3	0	1

If we assume that the elements of X and Y appear in a certain order, then the relation R can be represented by a matrix whose elements are 1s and 0s. this matrix can be defined in the following manner.

$$r_{ij} = \begin{cases} 1 & \text{if } x_i R y_j \\ 0 & \text{if } x_i \not R y_j \end{cases}$$

the matrix representation of the relation

$R = \{ (x_1, y_1), (x_2, y_1), (x_3, y_2), (x_2, y_2) \}$ is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

*) Properties:-

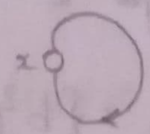
1) If a relation is reflexive, then all the diagonal entries must be 1.

2) If a relation is symmetric, then the relation matrix is symmetric.

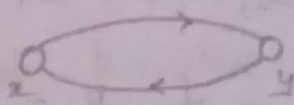
3) If a relation is antisymmetric, then its matrix is such that $r_{ij} = 1$, then $r_{ji} = 0$ for $i \neq j$

GRAPHS OF RELATIONS:-

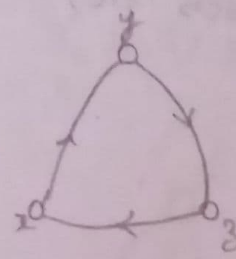
$x \, R \, y$



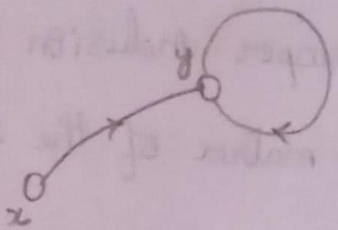
$x \, R \, x$



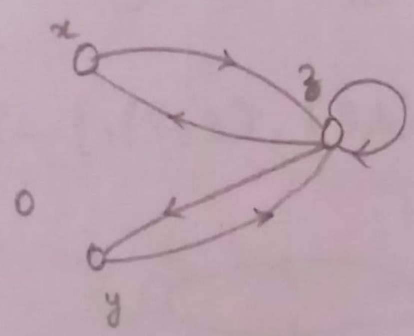
$x R y \wedge y R x$



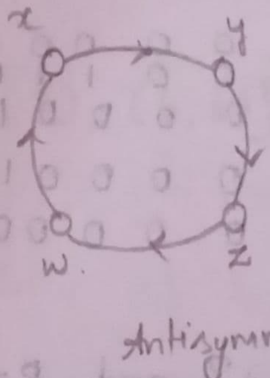
$x R y \wedge y R z \wedge z R x$

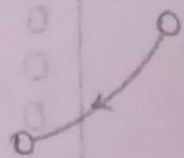


$x R y \wedge y R y$



Symmetric



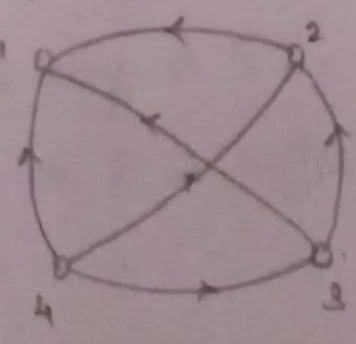


Antisymmetric.

Q) Let $X = \{1, 2, 3, 4\}$ and $R = \{(x, y) \mid x > y\}$. Draw the graph of R and also give its matrix.

Sol: The graph and the corresponding relation matrix for the relation

$$R = \{(4,1), (4,2), (4,3), (3,1), (3,2), (2,1)\}$$



$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Q) Let $A = \{a, b, c\}$ and denote the subsets of A by $B_0, \ldots B_7$ according to the convention. Thus

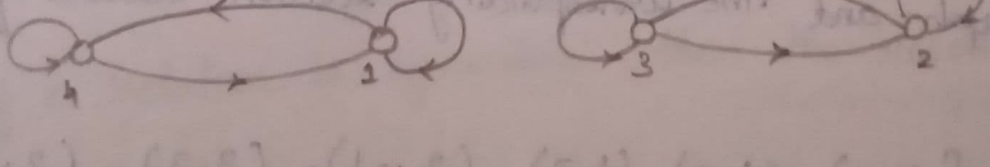
$B_0 = \phi$, $B_1 = \{c\}$, $B_2 = \{b\}$, $B_3 = \{b, c\}$, $B_4 = \{a\}$, $B_5 = \{a, c\}$, $B_6 = \{a, b\}$ and $B_7 = \{a, b, c\}$. If R is the relation of proper inclusion on the subsets $B_0, \ldots B_7$, then give the matrix of the relation.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Q) Let $X = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,4), (4,1), (4,4), (2,2), (2,3), (3,2), (3,3)\}$.

Write the matrix of R and sketch its graph.

Sol.

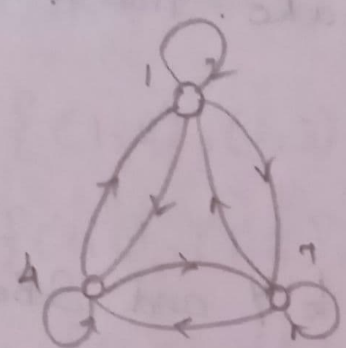


$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

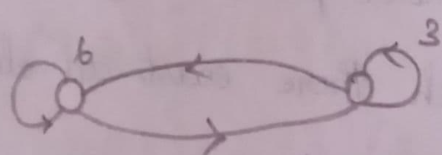
Q) Let $X = \{1, 2, \ldots 7\}$ and $R = \{(x, y) \mid x - y$ is divisible by $3\}$. Show that R is an equivalence relation. Draw the graph of R.

Sol Equivalence relation is, reflexive, symmetric, transitive.

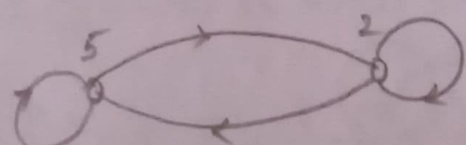
$$R = \{(1,1), (1,4), (4,1), (1,7), (7,1), (7,7), (7,4), (4,7), (4,4)\}.$$



$$R = \{(3,3), (3,6), (6,3), (6,6)\}$$



$$R = \{(2,2), (2,5), (5,2), (5,5)\}$$



Hence it is equivalence.

Another method:-

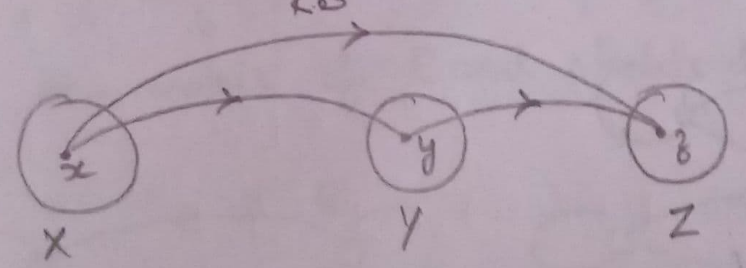
1) For any $a \in X$, $a - a$ is divisible by 3; hence aRa, ~~or~~ R is reflexive

2) For any $a, b \in X$, if $a-b$ is divisible by 3, then $b-a$ is also divisible by 3, that is, $aRb \Rightarrow bRa$. Thus R is symmetric.

3) For $a, b, c \in X$, if aRb and bRc then both $a-b$ and $b-c$ are divisible by 3, so that $a-c = (a-b)+(b-c)$ is also divisible by 3, and hence aRc. Thus R is transitive.

COMPOSITION OF RELATIONS :-

Let R be a relation from X to Y and S be a relation from Y to Z. Then the composite relation of R and S is the relation consisting of order pairs (x, z), where $x \in X$, $z \in Z$ and for which there exist an element $y \in Y$ such that $(x, y) \in R$ and $(y, z) \in S$.



where

$R : X \to Y$ $\quad x \in X$

$S : Y \to Z$ $\quad y \in Y$

$RoS : X \to Z$ $\quad z \in Z$.

Eg: If $A = \{1,2,3,4\}$ and R and S are two relations on set A defined by

$R = \{(1,2),(1,3),(2,4),(4,4)\}$

$S = \{(1,1),(1,2),(1,3),(1,4),(2,3),(2,4)\}$.

Find out RoS, SoR, RoR, SoS, $Ro(SoR)$, $(RoS)oR$, $(RoR)oR$, $(SoS)oS$.

Sol. $R = \{(1,2),(1,3),(2,4),(4,4)\}$

$S = \{(1,1),(1,2),(1,3),(1,4),(2,3),(2,4)\}$

1st ordered pair of R

$RoS = \{(1,3),(1,4)\}$ $(1,2) \in R$

ordered pair starts with 2 in S! $(2,3) \in S$ So, $(1,2),(2,3) \Rightarrow (1,3)$

$SoR = \{(1,2),(1,3),(1,4),$
$(2,4),\}$.

$(1,2) \in R$
$(2,4) \in R$
$(1,2),(2,4) \Rightarrow (1,4)$

$RoR = \{(1,2),(1,3),(2,4),(4,4)\} \circ \{(1,2),(1,3),(2,4),(4,4)\}$.

$= \{(1,4),(2,4),(4,4)\}$

$SoS = \{(1,1),(1,2),(1,3),(1,4),(2,3),(2,4)\} \circ \{(1,1),(1,2),$
$(1,3),(1,4),(2,3),(2,4)\}$

$= \{(1,1),(1,2),(1,3),(1,4)\}$ ~~(1,3)~~,

$Ro(SoR) =$

$SoR = \{(1,2),(1,3),(1,4),(2,4)\}$

$\ddagger$

$R \circ (S \circ R) = \{(1,2),(1,3),(2,4),(4,4)\} \circ \{(1,2),(1,3),$
$(1,4),(2,4)\}.$

$= \{(1,4)\}$

$\overbrace{(R \circ S)}^{R \circ S} \circ R = \{(1,3),(1,4)\} \circ \{(1,2),(1,3),(2,4),(4,4)\}.$

$= \{(1,4)\}.$

$\because \boxed{R \circ (S \circ R) = (R \circ S) \circ R}. \Rightarrow R \circ (S \circ R) = \{(1,4)\} \Rightarrow$ Equal
$(R \circ S) \circ R = \{(1,4)\}$

$(R \circ R) \circ R = \{(1,4),(2,4),(4,4)\} \circ \{(1,2),(1,3),(2,4),(4,4)\}$

$= \{(1,4),(2,4),(4,4)\}$

$(S \circ S) \circ S = \{(1,1),(1,2),(1,3),(1,4)\} \circ \{(1,1),(1,2),$
$(1,3),(1,4),(2,3),(2,4)\}$

$= \{(1,1),(1,2),(1,3),(1,4)\} \cancel{(1,3),(1,4)\}}$

Q) Let R and S be two relations on a set of positive integer I:

$R = \{(x,2x) \mid x \in I\} \quad S = \{(x,7x) \mid x \in I\}.$

Find $R \circ S$, $R \circ R$, $R \circ R \circ R$, and $R \circ S \circ R$.

Sol. $R \circ S = \{(x,2x) \mid x \in I\} \circ \{x,7x) \mid x \in I\}$

$= \{(x,14x) \mid x \in I\}$

$= S \circ R.$

$R \circ R = \{(x,2x) \mid x \in I\} \circ \{(x,2x) \mid x \in I\}.$

$= \{(x,4x) \mid x \in I\}.$

$$R \circ R \circ R = \{(x,2x) \mid x \in I\} \circ \{(x,2x) \mid x \in I\} \circ \{(x,2x) \mid x \in I\}$$
$$= \{(x,8x) \mid x \in I\}.$$

$$R \circ S \circ R = \{(x,2x) \mid x \in I\} \circ \{(x,7x) \mid x \in I\} \circ \{(x,2x) \mid x \in I\}.$$
$$= \{(x,28x) \mid x \in I\}.$$

Q) Let $R = \{(1,2),(3,4),(2,2)\}$, $S = \{(4,2),(2,5),(3,1),(1,3)\}$, 1) Find $R \circ S$, $S \circ R$, $R \circ (S \circ R)$, $(R \circ S) \circ R$, $R \circ R$, $S \circ S$ and $R \circ R \circ R$. 2) obtain relation matrices for $R \circ S$ and $S \circ R$.

Sol

$R \circ S = \{(1,5),(3,2),(2,5)\}$

$S \circ R = \{(4,2),(3,2),(1,4)\}$.

Not $\boxed{R \circ S \neq S \circ R}$

$(R \circ S) \circ R = \{(3,2)\}$.

$R \circ (S \circ R) = \{(3,2)\} = (R \circ S) \circ R$.

$R \circ R = \{(1,2),(2,2)\}$.

$S \circ S = \{(4,5),(3,3),(1,1)\}$.

$R \circ R \circ R = \{(1,2),(2,2)\}$.

~~Rela~~

Relation Matrices.

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & & & & & \\ 2 & & & & & \\ 3 & & & & & \\ 4 & & & & & \\ 5 & & & & & \end{array}$$

$$\underbrace{\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \quad 5 \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\end{array}}_{M_R} \circ \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{M_S} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{M_{ROS}}$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{M_S} \circ \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{M_R} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{M_{SOR}}$$

EQUIVALENCE & PARTIAL ORDER RELATIONS:-

*) A relation R on a set A is said to be an equivalence relation iff R is reflexive, Symmetric and transitive.

*) Let $A = \{1, 2, 3, 4\}$.

$R = \{(1,1), (1,2), (2,2), (2,1), (3,3), (4,4)\}$.

PARTIAL ORDER RELATION:-

A relation R is said to be partial order relation if it is reflexive, transitive and anti symmetric.

$Z^+ \Rightarrow$ Set of +ve integers

Eg; $xRy : x \leq y,\ x, y \in Z^+$

Proof:

Reflexive :

$(x, x) \in R \Rightarrow x \leq x.$

Hence the Relation is reflexive.

transitive:

We have to prove $(x, y) \in R$ & $(y, z) \in R \Rightarrow (x, z) \in R.$

$(x, y) \in R \Rightarrow x \leq y. \rightarrow ①$

$(y, z) \in R \Rightarrow y \leq z. \rightarrow ②$

from ① & ②

$x \leq y \leq z \Rightarrow x \leq z \Rightarrow (x, z) \in R.$

Hence the relation is transitive.

Antisymmetric:-

We have to prove $(x, y) \in R$ & $(y, x) \in R \Rightarrow x = y.$

$(x, y) \in R \Rightarrow x \leq y \rightarrow ①$

$(y, x) \in R \Rightarrow y \leq x \rightarrow ②$

x yine kaalum cheruthu &
y xine kaalum cheruthu. Ohikilum
less than property hold cheyyila.
same number angotum ingotum
less aayirikan pattila. Oru no
less aanu engil adutha number greater aayirikanam. So, less than work

aavila but we have =, so = work cheyyum so $x=y$ &
$y=x$

From ① & 2.

$x=y$

Hence the relation is antisymmetric.

So, the relation $xRy : x \leq y, x,y \in Z^+$ is reflexive, transitive and antisymmetric. Hence it is partial order relation.

FREQUENTLY USED PARTIAL ORDER RELATION:-

1) Less than or equal to, greater than equal to.

2) Inclusion

3) Divides and integral multiple.

4) Lexicographic ordering.

1) Less than or equal to, greater than equal to :-

Let R be the set of real numbers. The relation "less than or equal to", or $\leq$, is a partial ordering on R. The converse of this relation, "greater than or equal to", or $\geq$ is also a partial ordering on R. Associated relations are < and > respectively.

2) Inclusion :

Let $p(A) = 2^A = X$ be the powerset of A, that is

X is the set of subsets of A. The relation of inclusion ($\subseteq$) on X is a partial ordering. Associated with the relation $\subseteq$ is a relation called proper inclusion ($\subset$) which is irreflexive, antisymmetric, and transitive.

As a special case, we let $A = \{a, b, c\}$. Then,

$X = p(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, A\}$.

3) Divide and integral multiple:- $|$ $\leq$ and $\geq$

If a and b are +ve integers, then we say "a divisible b", we say that $a|b$, iff there is an integer c such that $ac = b$.

Alternatively, we say that "b is an integral multiple of a". The relation "divides" is a partial order relation. Let X be the set of +ve integers.

The relations "divides" and "integral multiple of" are partial ordering on X, and each is the converse of the other.

Eg: $X = \{2, 3, 6, 8\}$ and let $\leq$ be the relation "divides" on X. Then.

$\leq = \{(2,2), (3,3), (6,6), (8,8), (2,8), (2,6), (3,6)\}$.

$\geq = \{(2,2), (3,3), (6,6), (8,8), (8,2), (6,2), (6,3)\}$.

4) Lexicographic ordering:-

Let R be the set of real numbers and let $P = R \times R$. The relation $\geq$ on R is assumed to be the usual relation of "greater than or equal to". For any two ordered pairs (x_1, y_1) and (x_2, y_2) in P, we define the total ordering relation S as follows,

$$(x_1, y_1) \, S \, (x_2, y_2) \Leftrightarrow (x_1 > x_2) \vee (x_1 = x_2) \wedge (y_1 \geq y_2)$$

It is clear that if $(x_1, y_1) \not S (x_2, y_2)$, then we must have $(x_2, y_2) \, S \, (x_1, y_1)$ so that S is a total ordering on P. The partial ordering S is called the lexicographic ordering.