18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 24. – **Tue, Nov 6, 2007**

Simply connected regions. [slightly different from the actual notations used]

Recall Green's theorem: if C is a closed curve around R counterclockwise then line integrals can be expressed as double integrals:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \operatorname{curl}(\vec{F})\, dA, \qquad \oint_C \vec{F} \cdot \hat{\boldsymbol{n}}\, ds = \iint_R \operatorname{div}(\vec{F})\, dA,$$

where $\operatorname{curl}(M\hat{\boldsymbol{\imath}} + N\hat{\boldsymbol{\jmath}}) = N_x - M_y$, $\operatorname{div}(P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}) = P_x + Q_y$.

For Green's theorem to hold, $\vec{F}$ must be defined on the *entire* region R enclosed by C.

Example: (same as in pset): $\vec{F} = \dfrac{-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}}{x^2+y^2}$, C = unit circle counterclockwise, then $\operatorname{curl}(\vec{F}) = \dfrac{\partial}{\partial x}(\dfrac{x}{x^2+y^2}) - \dfrac{\partial}{\partial y}(\dfrac{-y}{x^2+y^2}) = \cdots = 0$. So, if we look at both sides of Green's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = 2\pi \quad \text{(from pset)}, \qquad \iint_R \operatorname{curl}\vec{F}\, dA = \iint_R 0\, dA = 0\,?$$

The problem is that R includes 0, where $\vec{F}$ is not defined.

Definition: a region R in the plane is simply connected if, given any closed curve in R, its interior region is entirely contained in R.

Examples shown.

So: Green's theorem applies safely when the domain in which $\vec{F}$ is defined and differentiable is simply connected: then we automatically know that, if $\vec{F}$ is defined on C, then it's also defined in the region bounded by C.

In the above example, can't apply Green to the unit circle, because the domain of definition of $\vec{F}$ is not simply connected. Still, we can apply Green's theorem to an annulus (picture shown of a curve $C' =$ unit circle counterclockwise + segment along x-axis + small circle around origin clockwise + back to the unit circle allong the x-axis, enclosing an annulus R'). Then Green applies and says $\oint_{C'} \vec{F} \cdot d\vec{r} = \iint_{R'} 0\, dA = 0$; but line integral simplifies to $\oint_{C'} = \int_C - \int_{C_2}$, where C = unit circle, C_2 = small circle / origin; so line integral is actually the same on C and C_2 (or any other curve encircling the origin).

Review for Exam 3.

2 main objects: double integrals and line integrals. Must know how to set up and evaluate.

Double integrals: drawing picture of region, taking slices to set up the iterated integral.

Also in polar coordinates, with $dA = r\, dr\, d\theta$ (see e.g. Problem 2; not done)

Remember: mass, centroid, moment of inertia.

For evaluation, need to know: usual basic integrals (e.g. $\int \frac{dx}{x}$); integration by substitution (e.g. $\displaystyle\int_0^1 \frac{t\, dt}{\sqrt{1+t^2}} = \int_1^2 \frac{du}{2\sqrt{u}}$, setting $u = 1 + t^2$). Don't need to know: complicated trigonometric integrals (e.g. $\int cos^4\theta\, d\theta$), integration by parts.

Change of variables: recall method:

1) Jacobian: $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ c_x & v_y \end{vmatrix}$. Its absolute value gives ratio between $du\, dv$ and $dx\, dy$.

2) express integrand in terms of u, v.

3) set up bounds in uv-coordinates by drawing picture. The actual example on the test will be reasonably simple (constant bounds, or circle in uv-coords).

Line integrals: $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{\boldsymbol{T}}\, ds = \int_C M\, dx + N\, dy$. To evaluate, express both x, y in terms of a single parameter and substitute.

Special case: gradient fields. Recall: $\vec{F}$ is conservative $\Leftrightarrow \int \vec{F} \cdot d\vec{r}$ is path independent $\Leftrightarrow \vec{F}$ is the gradient of some potential $f \Leftrightarrow \operatorname{curl} \vec{F} = 0$ (i.e. $N_x = M_y$).

If this is the case, then we can look for a potential using one of the two methods (antiderivatives, or line integral); and we can then use the FTC to avoid calculating the line integral. (cf. Problem 3).

Flux: $\int_C \vec{F} \cdot \hat{\boldsymbol{n}}\, ds$ $(= \int_C -Q\, dx + P\, dy)$. Geometric interpretation.

Green's theorem (in both forms) (already written at beginning of lecture).

18.02 Lecture 25. – **Fri, Nov 9, 2007**

Handouts: Exam 3 solutions.

Triple integrals: $\displaystyle\iiint_R f\, dV$ (dV = volume element).

Example 1: region between paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$ (picture drawn), e.g. volume of this region: $\displaystyle\iiint_R 1\, dV = \int_?^? \int_?^? \int_{x^2+y^2}^{4-x^2-y^2} dz\, dy\, dx.$

To set up bounds, (1) for fixed (x, y) find bounds for z: here lower limit is $z = x^2+y^2$, upper limit is $z = 4-x^2-y^2$; (2) find the shadow of R onto the xy-plane, i.e. set of values of (x, y) above which region lies. Here: R is widest at intersection of paraboloids, which is in plane $z = 2$; general method: for which (x, y) is z on top surface $>$ z on bottom surface? Answer: when $4 - x^2 - y^2 > x^2 - y^2$, i.e. $x^2 + y^2 < 2$. So we integrate over a disk of radius $\sqrt{2}$ in the xy-plane. By usual method to set up double integrals, we finally get:

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\, dy\, dx.$$

Evaluation would be easier if we used polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$: then

$$V = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} dz\, r\, dr\, d\theta.$$

(evaluation easy, not done).

Cylindrical coordinates. (r, θ, z), $x = r\cos\theta$, $y = r\sin\theta$. r measures distance from z-axis, θ measures angle from xz-plane (picture shown).

Cylinder of radius a centered on z-axis is $r = a$ (drawn); $\theta = 0$ is a vertical half-plane (not drawn).

Volume element: in rect. coords., $dV = dx\, dy\, dz$; in cylindrical coords., $dV = r\, dr\, d\theta\, dz$. In both cases this is justified by considering a small box with height Δz and base area ΔA, then volume is $\Delta V = \Delta A\, \Delta z$.

Applications: Mass: $M = \iiint_R \delta\, dV$.

Average value of f over R: $\bar{f} = \dfrac{1}{Vol} \iiint_R f\, dV$; weighted average: $\bar{f} = \dfrac{1}{Mass} \iiint_R f\, \delta\, dV$.

In particular, center of mass: $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x} = \dfrac{1}{Mass} \iiint_R x\, \delta\, dV$.

(Note: can sometimes avoid calculation using symmetry, e.g. in above example $\bar{x} = \bar{y} = 0$).

Moment of inertia around an axis: $I = \iiint_R (\text{distance from axis})^2\, \delta\, dV$.

About z-axis: $I_z = \iiint_R r^2\, \delta\, dV = \iiint_R (x^2 + y^2)\, \delta\, dV$. (consistent with I_0 in 2D case)

Similarly, about x and y axes: $I_x = \iiint_R (y^2 + z^2)\, \delta\, dV$, $I_y = \iiint_R (x^2 + z^2)\, \delta\, dV$ (setting $z = 0$, this is consistent with previous definitions of I_x and I_y for plane regions).

Example 2: moment of inertia I_z of solid cone between $z = ar$ and $z = b$ ($\delta = 1$) (picture drawn):

$$I_z = \iiint_R r^2\, dV = \int_0^b \int_0^{2\pi} \int_0^{z/a} r^2\, r\, dr\, d\theta\, dz \quad \left(= \frac{\pi b^5}{10a^4} \right).$$

(I explained how to find bounds in order $dr\, d\theta\, dz$: first we fix z, then slice for given z is the disk bounded by $r = z/a$; the first slice is $z = 0$, the last one is $z = b$).

Example 3: volume of region where $z > 1 - y$ and $x^2 + y^2 + z^2 < 1$? Pictures drawn: in space, slice by yz-plane, and projection to xy-plane.

The bottom surface is the plane $z = 1 - y$, the upper one is the sphere $z = \sqrt{1 - x^2 - y^2}$. So inner is $\displaystyle\int_{1-y}^{\sqrt{1-x^2-y^2}} dz$. The shadow on the xy-plane = points where $1 - y < \sqrt{1 - x^2 - y^2}$, i.e. squaring both sides, $(1 - y)^2 < 1 - x^2 - y^2$ i.e. $x^2 < 2y - 2y^2$, i.e. $-\sqrt{2y - 2y^2} < x < \sqrt{2y - 2y^2}$. So we get:

$$\int_0^1 \int_{-\sqrt{2y-2y^2}}^{\sqrt{2y-2y^2}} \int_{1-y}^{\sqrt{1-x^2-y^2}} dz\, dx\, dy.$$

Bounds for y: either by observing that $x^2 < 2y - y^2$ has solutions iff $2y - y^2 > 0$, i.e. $0 < y < 1$, or by looking at picture where clearly leftmost point is on z-axis ($y = 0$) and rightmost point is at $y = 1$.

18.02 Lecture 26. – **Tue, Nov 13, 2007**

Spherical coordinates (ρ, ϕ, θ).

ρ = rho = distance to origin. $\phi = \varphi$ = phi = angle down from z-axis. θ = same as in cylindrical coordinates. Diagram drawn in space, and picture of 2D slice by vertical plane with z, r coordinates.

Formulas to remember: $z = \rho\cos\phi$, $r = \rho\sin\phi$ (so $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$).

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$. The equation $\rho = a$ defines the sphere of radius a centered at 0.

On the surface of the sphere, ϕ is similar to *latitude*, except it's 0 at the north pole, $\pi/2$ on the equator, π at the south pole. θ is similar to *longitude*.

$\phi = \pi/4$ is a cone (asked using flash cards) ($z = r = \sqrt{x^2+y^2}$). $\phi = \pi/2$ is the xy-plane.

Volume element: $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

To understand this formula, first study surface area on sphere of radius a: picture shown of a "rectangle" corresponding to $\Delta\phi$, $\Delta\theta$, with sides = portion of circle of radius a, of length $a\Delta\phi$, and portion of circle of radius $r = a\sin\phi$, of length $r\Delta\theta = a\sin\phi\Delta\theta$. So $\Delta S \approx a^2\sin\phi\,\Delta\phi\Delta\theta$, which gives the surface element $dS = a^2 \sin\phi\, d\phi d\theta$.

The volume element follows: for a small "box", $\Delta V = \Delta S\,\Delta\rho$, so $dV = d\rho\, dS = \rho^2\sin\phi\, d\rho d\phi d\theta$.

Example: recall the complicated example at end of Friday's lecture (region sliced by a plane inside unit sphere). After rotating coordinate system, the question becomes: volume of the portion of unit sphere above the plane $z = 1/\sqrt{2}$? (picture drawn). This can be set up in cylindrical (left as exercise) or spherical coordinates.

For fixed ϕ, θ we are slicing our region by rays straight out of the origin; ρ ranges from its value on the plane $z = 1/\sqrt{2}$ to its value on the sphere $\rho = 1$. Spherical coordinate equation of the plane: $z = \rho\cos\phi = 1/\sqrt{2}$, so $\rho = \sec\phi/\sqrt{2}$. The volume is:

$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\frac{1}{\sqrt{2}}\sec\phi}^{1} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$

(Bound for ϕ explained by looking at a slice by vertical plane $\theta =$constant: the edge of the region is at $z = r = \frac{1}{\sqrt{2}}$).

Evaluation: not done. Final answer: $\dfrac{2\pi}{3} - \dfrac{5\pi}{6\sqrt{2}}$.

Application to gravitation.

Gravitational force exerted on mass m at origin by a mass ΔM at (x,y,z) (picture shown) is given by $|\vec{F}| = \dfrac{G\,\Delta M\, m}{\rho^2}$, $dir(\vec{F}) = \dfrac{\langle x,y,z\rangle}{\rho}$, i.e. $\vec{F} = \dfrac{G\,\Delta M\, m}{\rho^3}\langle x,y,z\rangle$. ($G$ = gravitational constant).

If instead of a point mass we have a solid with density δ, then we must integrate contributions to gravitational attraction from small pieces $\Delta M = \delta\,\Delta V$. So

$$\vec{F} = \iiint_R \frac{Gm\,\langle x,y,z\rangle}{\rho^3}\,\delta\, dV, \quad \text{i.e. } z\text{-component is} \quad F_z = Gm\iiint_R \frac{z}{\rho^3}\delta\, dV, \ \dots$$

If we can set up to use symmetry, then F_z can be computed nicely using spherical coordinates.

General setup: place the mass m at the origin (so integrand is as above), and place the solid so that the z-axis is an axis of symmetry. Then $\vec{F} = \langle 0, 0, F_z\rangle$ by symmetry, and we have only one component to compute. Then

$$F_z = Gm \iiint_R \frac{z}{\rho^3}\,\delta\,dV = Gm \iiint_R \frac{\rho\cos\phi}{\rho^3}\,\delta\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = Gm \iiint_R \delta\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta.$$

Example: Newton's theorem: the gravitational attraction of a spherical planet with uniform density δ is the same as that of the equivalent point mass at its center.

[[Setup: the sphere has radius a and is centered on the positive z-axis, tangent to xy-plane at the origin; the test mass is m at the origin. Then

$$F_z = Gm \iiint_R \frac{z}{\rho^3}\,\delta\,dV = Gm \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2a\cos\phi} \delta\,\cos\phi\,\sin\phi\,d\rho\,d\phi\,d\theta = \cdots = \frac{4}{3}Gm\delta\,\pi a = \frac{GMm}{a^2}$$

where M = mass of the planet $= \frac{4}{3}\pi a^3\delta$. (The bounds for ρ and ϕ need to be explained carefully, by drawing a diagram of a vertical slice with z and r coordinate axes, and the inscribed right triangle with vertices the two poles of the sphere + a point on its surface, the hypothenuse is the diameter $2a$ and we get $\rho = 2a\cos\phi$ for the spherical coordinate equation of the sphere).]]

18.02 Lecture 27. – **Thu, Nov 15, 2007**

Handouts: PS10 solutions, PS11

Vector fields in space.

At every point in space, $\vec{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\boldsymbol{k}}$, where P, Q, R are functions of x, y, z.

Examples: force fields (gravitational force $\vec{F} = -c\langle x, y, z\rangle/\rho^3$; electric field $\mathbf{E}$, magnetic field $\mathbf{B}$); velocity fields (fluid flow, $\boldsymbol{v} = \boldsymbol{v}(x, y, z)$); gradient fields (e.g. temperature and pressure gradients).

Flux.

Recall: in 2D, flux of a vector field $\vec{F}$ across a curve $C = \int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds$.

In 3D, flux of a vector field is a *double* integral: flux through a *surface*, not a curve!

$\vec{F}$ vector field, S surface, $\hat{\boldsymbol{n}}$ unit normal vector: Flux $= \iint \vec{F}\cdot\hat{\boldsymbol{n}}\,dS$.

Notation: $d\vec{S} = \hat{\boldsymbol{n}}\,dS$. (We'll see that $d\vec{S}$ is often easier to compute than $\hat{\boldsymbol{n}}$ and dS).

Remark: there are 2 choices for $\hat{\boldsymbol{n}}$ (choose which way is counted positively!)

Geometric interpretation of flux:

As in 2D, if $\vec{F}$ = velocity of a fluid flow, then flux = flow per unit time across S.

Cut S into small pieces, then over each small piece: what passes through ΔS in unit time is the contents of a parallelepiped with base ΔS and third side given by $\vec{F}$.

Volume of box = base × height $= (\vec{F}\cdot\hat{\boldsymbol{n}})\,\Delta S$.

• Examples:

1) $\vec{F} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ through sphere of radius a centered at 0.

$\hat{\boldsymbol{n}} = \frac{1}{a}\langle x, y, z\rangle$ (other choice: $-\frac{1}{a}\langle x, y, z\rangle$; traditionally choose $\hat{\boldsymbol{n}}$ pointing out).

$\vec{F}\cdot\hat{\boldsymbol{n}} = \langle x, y, z\rangle\cdot\hat{\boldsymbol{n}} = \frac{1}{a}(x^2+y^2+z^2) = a$, so $\iint_S \vec{F}\cdot\hat{\boldsymbol{n}}dS = \iint_S a\,dS = a\,(4\pi a^2)$.

2) Same sphere, $\vec{H} = z\hat{\boldsymbol{k}}$: $\vec{H}\cdot\hat{\boldsymbol{n}} = \frac{z^2}{a}$.

$$\iint_S \vec{H}\cdot d\vec{S} = \iint_S \frac{z^2}{a}\,dS = \int_0^{2\pi}\int_0^{\pi} \frac{a^2\cos^2\phi}{a}\,a^2\sin\phi\,d\phi d\theta = 2\pi a^3 \int_0^{\pi}\cos^2\phi\sin\phi\,d\phi = \frac{4}{3}\pi a^3.$$

Setup. Sometimes we have an easy geometric argument, but in general we must compute the surface integral. The setup requires the use of two parameters to describe the surface, and $\vec{F}\cdot\hat{\boldsymbol{n}}\,dS$ must be expressed in terms of them. How to do this depends on the type of surface. For now, formulas to remember:

0) plane $z = a$ parallel to xy-plane: $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{k}}$, $dS = dx\,dy$. (similarly for planes // xz or yz-plane).

1) sphere of radius a centered at origin: use ϕ, θ (substitute $\rho = a$ for evaluation); $\hat{\boldsymbol{n}} = \frac{1}{a}\langle x, y, z\rangle$, $dS = a^2\sin\phi\,d\phi\,d\theta$.

2) cylinder of radius a centered on z-axis: use z, θ (substitute $r = a$ for evaluation): $\hat{\boldsymbol{n}}$ is radially out in horizontal directions away from z-axis, i.e. $\hat{\boldsymbol{n}} = \frac{1}{a}\langle x, y, 0\rangle$; and $dS = a\,dz\,d\theta$ (explained by drawing a picture of a “rectangular” piece of cylinder, $\Delta S = (\Delta z)\,(a\Delta\theta)$).

3) graph $z = f(x, y)$: use x, y (substitute $z = f(x, y)$). We'll see on Friday that $\hat{\boldsymbol{n}}$ and dS separately are complicated, but $\hat{\boldsymbol{n}}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy$.

18.02 Lecture 28. – **Fri, Nov 16, 2007**

Last time, we defined the flux of $\vec{F}$ through surface S as $\iint \vec{F}\cdot\hat{\boldsymbol{n}}\,dS$, and saw how to set up in various cases. Continue with more:

Flux through a graph. If S is the graph of some function $z = f(x, y)$ over a region R of xy-plane: use x and y as variables. Contribution of a small piece of S to flux integral?

Consider portion of S lying above a small rectangle $\Delta x\,\Delta y$ in xy-plane. In linear approximation it is a parallelogram. (picture shown)

The vertices are $(x, y, f(x, y))$; $(x + \Delta x, y, f(x + \Delta x, y))$; $(x, y + \Delta y, f(x, y + \Delta y))$; etc. Linear approximation: $f(x + \Delta x, y) \simeq f(x, y) + \Delta x\,f_x(x, y)$, and $f(x, y + \Delta y) \simeq f(x, y) + \Delta y\,f_y(x, y)$.

So the sides of the parallelogram are $\langle \Delta x, 0, \Delta x\,f_x\rangle$ and $\langle 0, \Delta y, \Delta y\,f_y\rangle$, and

$$\Delta\vec{S} = (\Delta x\,\langle 1, 0, f_x\rangle) \times (\Delta y\,\langle 0, 1, f_y\rangle) = \Delta x\Delta y \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1\rangle\Delta x\Delta y.$$

So $d\vec{S} = \pm\langle -f_x, -f_y, 1\rangle dx\,dy$.

(From this we can get $\hat{\boldsymbol{n}} = \mathrm{dir}(d\vec{S}) = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}}$ and $dS = |d\vec{S}| = \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$. The conversion factor $\sqrt{\cdots}$ between dS and dA relates area on S to area of projection in xy-plane.)

• Example: flux of $\vec{F} = z\hat{\boldsymbol{k}}$ through S = portion of paraboloid $z = x^2 + y^2$ above unit disk, oriented with normal pointing up (and into the paraboloid): geometrically flux should be > 0 (asked using flashcards). We have $\hat{\boldsymbol{n}}\,dS = \langle -2x, -2y, 1\rangle\,dx\,dy$, and

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S z\,dx\,dy = \iint_S (x^2 + y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 r^2\,r\,dr\,d\theta = \pi/2.$$

Parametric surfaces. If we can describe S by parametric equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ (i.e. $\vec{r} = \vec{r}(u, v)$), then we can set up flux integrals using variables u, v. To find $d\vec{S}$, consider a small portion of surface corresponding to changes Δu and Δv in parameters, it's a parallelogram with sides $\vec{r}(u+\Delta u, v) - \vec{r}(u,v) \approx (\partial\vec{r}/\partial u)\,\Delta u$ and $(\partial\vec{r}/\partial v)\,\Delta v$, so

$$\Delta\vec{S} = \pm\left(\frac{\partial\vec{r}}{\partial u}\Delta u\right)\times\left(\frac{\partial\vec{r}}{\partial v}\Delta v\right), \qquad d\vec{S} = \pm\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)du\,dv.$$

(This generalizes all formulas previously seen; but won't be needed on exam).

Implicit surfaces: If we have an implicitly defined surface $g(x,y,z)=0$, then we have a (non-unit) normal vector $\mathbf{N} = \nabla g$. (similarly for a slanted plane, from equation $ax+by+cz=d$ we get $\mathbf{N} = \langle a,b,c\rangle$).

Unit normal $\hat{\boldsymbol{n}} = \pm\mathbf{N}/|\mathbf{N}|$; surface element $\Delta S = ?$ Look at projection to xy-plane: $\Delta A = \Delta S\cos\alpha = (\mathbf{N}\cdot\hat{\boldsymbol{k}}/|\mathbf{N}|)\,\Delta S$ (where α = angle between slanted surface element and horizontal: projection shrinks one direction by factor $\cos\alpha = (\mathbf{N}\cdot\hat{\boldsymbol{k}})/|\mathbf{N}|$, preserves the other).

Hence $dS = \dfrac{|\mathbf{N}|}{\mathbf{N}\cdot\hat{\boldsymbol{k}}}\,dA$, and $\hat{\boldsymbol{n}}\,dS = \dfrac{|\mathbf{N}|\hat{\boldsymbol{n}}}{\mathbf{N}\cdot\hat{\boldsymbol{k}}}\,dx\,dy = \pm\dfrac{\mathbf{N}}{\mathbf{N}\cdot\hat{\boldsymbol{k}}}\,dx\,dy$.

(In fact the first formula should be $dS = \dfrac{|\mathbf{N}|}{|\mathbf{N}\cdot\hat{\boldsymbol{k}}|}\,dA$, I forgot the absolute value).

Note: if S is vertical then the denominator is zero, can't project to xy-plane any more (but one could project e.g. to the xz-plane).

Example: if S is a graph, $g(x,y,z) = z - f(x,y) = 0$, then $\mathbf{N} = \langle g_x, g_y, g_z\rangle = \langle -f_x, -f_y, 1\rangle$, $\mathbf{N}\cdot\hat{\boldsymbol{k}} = 1$, so we recover the formula $d\vec{S} = \langle -f_x, -f_y, 1\rangle dx\,dy$ seen before.

Divergence theorem. ("Gauss-Green theorem") – 3D analogue of Green theorem for flux.

If S is a closed surface bounding a region D, with normal pointing outwards, and $\vec{F}$ vector field defined and differentiable over all of D, then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\vec{F}\,dV, \quad \text{where} \quad \operatorname{div}(P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\boldsymbol{k}}) = P_x + Q_y + R_z.$$

Example: flux of $\vec{F} = z\hat{\boldsymbol{k}}$ out of sphere of radius a (seen Thursday): $\operatorname{div}\vec{F} = 0+0+1 = 1$, so $\iint_S \vec{F}\cdot d\vec{S} = 3\,\mathrm{vol}(D) = 4\pi a^3/3$.

Physical interpretation (mentioned very quickly and verbally only): $\operatorname{div}\vec{F}$ = source rate = flux generated per unit volume. So the divergence theorem says: the flux outwards through S (net amount leaving D per unit time) is equal to the total amount of sources in D.

18.02 Lecture 29. – **Tue, Nov 20, 2007**

Recall statement of divergence theorem: $\iint_S \mathbf{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\mathbf{F}\,dV$.

Del operator. $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle$ (symbolic notation!)

$\nabla f = \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z\rangle =$ gradient.

$\nabla\cdot\mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle \cdot \langle P, Q, R\rangle = P_x + Q_y + R_z =$ divergence.

Physical interpretation. $\operatorname{div}\mathbf{F}$ = source rate = flux generated per unit volume. Imagine an incompressible fluid flow (i.e. a given mass occupies a fixed volume) with velocity $\mathbf{F}$, then $\iiint_D \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS =$ flux through S is the net amount leaving D per unit time = total amount of sources (minus sinks) in D.

Proof of divergence theorem. To show $\iint_S \langle P,Q,R\rangle\cdot d\vec{S} = \iiint_D (P_x+Q_y+R_z)\,dV$, we can separate into sum over components and just show $\iint_S R\hat{\boldsymbol{k}}\cdot d\vec{S} = \iiint_D R_z\,dV$ & same for P and Q.

If the region D is vertically simple, i.e. top and bottom surfaces are graphs, $z_1(x,y) \le z \le z_2(x,y)$, with (x,y) in some region U of xy-plane: r.h.s. is

$$\iiint_D R_z\,dV = \iint_U \Big(\int_{z_1(x,y)}^{z_2(x,y)} R_z\,dz\Big)\,dx\,dy = \iint_U (R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\,dx\,dy.$$

Flux through top: $d\vec{S} = \langle -\partial z_2/\partial x, -\partial z_2/\partial y, 1\rangle dx\,dy$, so $\iint_{\text{top}} R\hat{\boldsymbol{k}}\cdot d\vec{S} = \iint R(x,y,z_2(x,y))\,dx\,dy$.

Bottom: $d\vec{S} = -\langle -\partial z_1/\partial x, -\partial z_1/\partial y, 1\rangle dx\,dy$, so $\iint_{\text{bottom}} R\hat{\boldsymbol{k}}\cdot d\vec{S} = \iint -R(x,y,z_1(x,y))\,dx\,dy$.

Sides: sides are vertical, $\hat{\mathbf{n}}$ is horizontal, $\mathbf{F}$ is vertical so flux = 0.

Given any region D, decompose it into vertically simple pieces (illustrated for a donut). Then $\iiint_D$ = sum of pieces (clear), and $\iint_S$ = sum of pieces since the internal boundaries cancel each other.

Diffusion equation: governs motion of smoke in (immobile) air (dye in solution, ...)

The equation is: $\dfrac{\partial u}{\partial t} = k\nabla^2 u = k\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right)$,
where $u(x,y,z,t)$ = concentration of smoke; we'll also introduce $\mathbf{F}$ = flow of the smoke. It's also the heat equation (u = temperature).

Equation uses two inputs:

1) Physics: $\mathbf{F} = -k\nabla u$ (flow goes from highest to lowest concentration, faster if concentration changes more abruptly).

2) Flux and quantity of smoke are related: if D bounded by closed surface S, then $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\frac{d}{dt}\iiint_D u\,dV$. (flux out of D = - variation of total amount of smoke inside D)

By differentiation under integral sign, the r.h.s. is $-\iiint_D \frac{\partial}{\partial t}u\,dV$ (This can be explained in terms of integral as a sum of $u(x_i,y_i,z_i,t)\Delta V_i$ and derivative of sum is sum of derivatives) and by divergence theorem the l.h.s. is $\iiint_D \operatorname{div}\mathbf{F}\,dV$. Dividing by volume of D, we get

$$-\frac{1}{vol(D)}\iiint_D \frac{\partial u}{\partial t}\,dV = \frac{1}{vol(D)}\iiint_D \operatorname{div}\mathbf{F}\,dV.$$

Same average values over any region; taking limit as D shrinks to a point, get $\partial u/\partial t = -\operatorname{div}\mathbf{F}$.

Combining, we get the diffusion equation: $\partial u/\partial t = -\operatorname{div}\mathbf{F} = +k\operatorname{div}(\nabla u) = k\nabla^2 u$.

18.02 Lecture 30. – **Tue, Nov 27, 2007**

Handouts: practice exams 4A and 4B.

Clarification from end of last lecture: we derived the diffusion equation from 2 inputs. u = concentration, $\mathbf{F}$ = flow, satisfy:

1) from physics: $\mathbf{F} = -k\nabla u$,

2) from divergence theorem: $\partial u/\partial t = -\operatorname{div}\mathbf{F}$.

Combining, we get the diffusion equation: $\partial u/\partial t = -\operatorname{div}\mathbf{F} = +k\operatorname{div}(\nabla u) = k\nabla^2 u$.

Line integrals in space.

Force field $\mathbf{F} = \langle P, Q, R\rangle$, curve C in space, $d\vec{r} = \langle dx, dy, dz\rangle$

$\Rightarrow$ Work $= \int_C \mathbf{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$.

Example: $\mathbf{F} = \langle yz, xz, xy\rangle$. C: $x = t^3$, $y = t^2$, $z = t$. $0 \le t \le 1$. Then $dx = 3t^2dt$, $dy = 2tdt$, $dz = dt$ and substitute:

$$\int_C \mathbf{F}\cdot d\vec{r} = \int_C yzdx + xzdy + xydz = \int_0^1 6t^5dt = 1$$

(In general, express (x, y, z) in terms of a *single* parameter: 1 degree of freedom)

Same $\mathbf{F}$, curve C' = segments from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$. In the xy-plane, $z = 0 \implies \mathbf{F} = xy\hat{\boldsymbol{k}}$, so $\mathbf{F}\cdot d\vec{r} = 0$, no work. For the last segment, $x = y = 1$, $dx = dy = 0$, so $\mathbf{F} = \langle z, z, 1\rangle$ and $d\vec{r} = \langle 0, 0, dz\rangle$. We get $\int_0^1 1\,dz = 1$.

Both give the same answer because $\mathbf{F}$ is conservative, in fact $\mathbf{F} = \nabla(xyz)$.

Recall the fundamental theorem of calculus for line integrals:

$$\int_{P_0}^{P_1} \nabla f\cdot d\vec{r} = f(P_1) - f(P_0).$$

Gradient fields.

$\mathbf{F} = \langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$?

Then $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$, so $P_y = Q_x$, $P_z = R_x$, $Q_z = R_y$.

Theorem: $\mathbf{F}$ is a gradient field if and only if these equalities hold (assuming defined in whole space or simply connected region)

Example: for which a, b is $axy\hat{\boldsymbol{\imath}} + (x^2 + z^3)\hat{\boldsymbol{\jmath}} + (byz^2 - 4z^3)\hat{\boldsymbol{k}}$ a gradient field?

$P_y = ax = 2x = Q_x$ so $a = 2$; $P_z = 0 = 0 = R_x$; $Q_z = 3z^2 = bz^2 = R_y$ so $b = 3$.

Systematic method to find a potential: (carried out on above example)

$f_x = 2xy$, $f_y = x^2 + z^3$, $f_z = 3yz^2 - 4z^3$:

$f_x = 2xy \Rightarrow f(x, y, z) = x^2y + g(y, z)$.

$f_y = x^2 + g_y = x^2 + z^3 \Rightarrow g_y = z^3 \Rightarrow g(y, z) = yz^3 + h(z)$, and $f = x^2y + yz^3 + h(z)$.

$f_z = 3yz^2 + h'(z) = 3yz^2 - 4z^3 \Rightarrow h'(z) = -4z^3 \Rightarrow h(z) = -z^4 + c$, and $f = x^2y + yz^3 - z^4 + c$.

Other method: $f(x_1, y_1, z_1) = f(0,0,0) + \int_{P_0}^{P_1} \mathbf{F}\cdot d\vec{r}$ (use a curve that gives an easy computation, e.g. 3 segments parallel to axes).

Curl: encodes by how much $\mathbf{F}$ fails to be conservative.

$\operatorname{curl}\langle P, Q, R\rangle = (R_y - Q_z)\hat{\boldsymbol{\imath}} + (P_z - R_x)\hat{\boldsymbol{\jmath}} + (Q_x - P_y)\hat{\boldsymbol{k}}$.

How to remember the formula? Use the del operator $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$.

Recall from last week that $\nabla \cdot \mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z = \operatorname{div} \mathbf{F}$.

Now we have: $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \operatorname{curl} \mathbf{F}$.

Interpretation of curl for velocity fields: curl measures *angular velocity.*

Example: rotation around z-axis at constant angular velocity ω (trajectories are circles centered on z-axis): $\boldsymbol{v} = \langle -\omega y, \omega x, 0 \rangle$.

Then $\nabla \times \boldsymbol{v} = ... = 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + (\omega + \omega)\hat{\boldsymbol{k}} = 2\omega\hat{\boldsymbol{k}}$. So length of curl = twice angular velocity, and direction = axis of rotation.

A general motion can be complicated, but decomposes into various effects.

• curl measures the *rotation* component of a complex motion.

18.02 Lecture 31. – **Thu, Nov 29, 2007**

Handouts: PS11 solutions, PS12.

Stokes' theorem is the 3D analogue of Green's theorem for work (in the same sense as the divergence theorem is the 3D analogue of Green for flux).

Recall $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$.

Stokes' theorem: if C is a *closed curve*, and S *any* surface bounded by C, then

$$\oint_C \mathbf{F} \cdot d\vec{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS.$$

Orientation: compatibility of an orientation of C with an orientation of S (changing orientation changes sign on both sides of Stokes).

Rule: if I walk along C in positive direction, with S to my left, then $\hat{\mathbf{n}}$ is pointing up. (Various examples shown.)

Another formulation (right-hand rule): if thumb points along C (1-D object), index finger towards S (2-D object), then middle finger points along $\hat{\mathbf{n}}$ (3-D object).

More examples shown.

Example: Stokes vs. Green. If S is a portion of xy-plane bounded by a curve C counterclockwise, then $\oint_C \mathbf{F} \cdot d\vec{r} = \int_C P\, dx + Q\, dy$, by Green this is equal to $\iint_S (Q_x - P_y)\, dx\, dy = \iint_S \nabla \times \mathbf{F} \cdot \hat{\boldsymbol{k}}\, dx\, dy = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$, so Green and Stokes say the same thing in this example.

Remark. In Stokes' theorem we are free to choose any surface S bounded by C! (e.g. if C = circle, S could be a disk, a hemisphere, a cone, ...)

"Proof" of Stokes.

1) if C and S are in the xy-plane then the statement follows from Green.

2) if C and S are in an arbitrary plane: this also reduces to Green in the given plane. Green/Stokes works in any plane because of *geometric invariance* of work, curl and flux under rotations of space. They can be defined in purely geometric terms so as not to depend on the coordinate system (x, y, z); equivalently, we can choose coordinates (u, v, w) adapted to the given plane, and work

with those coordinates, the expressions of work, curl, flux will be the familiar ones replacing x, y, z with u, v, w.

3) in general, we can decompose S into small pieces, each piece is nearly flat (slanted plane); on each piece we have approximately work = flux by Green's theorem. When adding pieces, the line integrals over the inner boundaries cancel each other and we get the line integral over C; the flux integrals add up to flux through S.

Example: verify Stokes for $\mathbf{F} = z\hat{\imath} + x\hat{\jmath} + y\hat{\boldsymbol{k}}$, C = unit circle in xy-plane (counterclockwise), S = piece of paraboloid $z = 1 - x^2 - y^2$.

Direct calculation: $x = \cos\theta$, $y = \sin\theta$, $z = 0$, so $\oint_C \mathbf{F}\cdot d\vec{r} = \int_C z\,dx + x\,dy + y\,dz = \oint_C x\,dy = \int_0^{2\pi} \cos^2\theta\,d\theta = \pi$.

By Stokes: $\operatorname{curl}\mathbf{F} = \langle 1, 1, 1\rangle$, and $\hat{\mathbf{n}}\,dS = \langle -f_x, -f_y, 1\rangle dx\,dy = \langle 2x, 2y, 1\rangle\,dx\,dy$.

$$\iint_S (\nabla\times\mathbf{F})\cdot d\vec{S} = \iint \langle (2x + 2y + 1)\,dx\,dy = \iint 1\,dx\,dy = \text{area(disk)} = \pi.$$

($\iint x\,dx\,dy = 0$ by symmetry and similarly for y).

18.02 Lecture 32. – Fri, Nov 30, 2007

Stokes and path independence.

Definition: a region is simply connected if every closed loop C inside it bounds some surface S inside it.

Example: the complement of the z-axis is not simply connected (shown by considering a loop encircling the z-axis); the complement of the origin is simply connected.

Topology: uses these considerations to classify for example surfaces in space: e.g., the mathematical proof that a sphere and a torus are "different" surfaces is that the sphere is simply connected, the torus isn't (in fact it has two "independent" loops that don't bound).

Recall: if $\mathbf{F}$ is a gradient field then $\operatorname{curl}(\mathbf{F}) = 0$.

Conversely, Theorem: if $\nabla\times\mathbf{F} = 0$ in a *simply connected* region then $\mathbf{F}$ is conservative (so $\int \mathbf{F}\cdot d\vec{r}$ is path-independent and we can find a potential).

Proof: Assume R simply connected, $\nabla\times\mathbf{F} = 0$, and consider two curves C_1 and C_2 with same end points. Then $C = C_1 - C_2$ is a closed curve so bounds some S; $\int_{C_1}\mathbf{F}\cdot d\vec{r} - \int_{C_2}\mathbf{F}\cdot d\vec{r} = \oint_C \mathbf{F}\cdot d\vec{r} = \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = 0$.

Orientability. We can apply Stokes' theorem to any surface S bounded by C... provided that it has a well-defined normal vector! Counterexample shown: the Möbius strip. It's a one-sided surface, so we can't compute flux through it (no possible consistent choice of orientation of $\hat{\mathbf{n}}$). Instead, if we want to apply Stokes to the boundary curve C, we must find a two-sided surface with boundary C. (pictures shown).

Stokes and surface independence.

In Stokes we can choose any S bounded by C: so if a same C bounds two surfaces S_1, S_2, then $\oint_C \mathbf{F}\cdot d\vec{r} = \iint_{S_1} \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_2} \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$? Can we prove directly that the two flux integrals are equal?

Answer: change orientation of S_2, then $S = S_1 - S_2$ is a closed surface with $\hat{\mathbf{n}}$ outwards; so we can apply the divergence theorem: $\iint_S \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_D \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV$. But $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$, always. (Checked by calculating in terms of components of $\mathbf{F}$; also, symbolically: $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, much like $u \cdot (u \times v) = 0$ for genuine vectors).

Review for Exam 4.

We've seen three types of integrals, with different ways of evaluating:

1) $\iiint f\, dV$ in rect., cyl., spherical coordinates (I re-explained the general setup and the formulas for dV); applications: center of mass, moment of inertia, gravitational attraction.

2) surface integrals, flux. Setting up $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$, by knowing formulas for $\hat{\mathbf{n}}\, dS$.

We have seen: planes parallel to coordinate planes (e.g. yz-plane: $\hat{\mathbf{n}} = \pm\hat{\boldsymbol{\imath}}$, $dS = dy\, dz$); spheres and cylinders ($\hat{\mathbf{n}}$ = straight out/in from center or axis; $dS = a\, dz\, d\theta$ for cylinders, $a^2 \sin\phi\, d\phi d\theta$ for spheres); if we can express $z = f(x, y)$, $\hat{\mathbf{n}}\, dS = \pm\langle -f_x, -f_y, 1\rangle dx\, dy$ (recall $\langle \dots \rangle$ is not $\hat{\mathbf{n}}$ and $dx\, dy$ is not dS); if S has a given normal vector $\vec{N}$ (e.g. if S is given by $g(x, y, z) = 0$), $\hat{\mathbf{n}}\, dS = \pm\vec{N}/(\vec{N} \cdot \hat{\boldsymbol{k}})\, dx\, dy$.

3) line integrals $\int_C \mathbf{F} \cdot d\vec{r}$ ($=\int_C P\, dx + Q\, dy + R\, dz$), evaluate by parameterizing C and expressing in terms of a single variable.

While these various types of integrals are completely different in terms of interpretation and method of evaluation, we've seen some theorems that establish bridges between them:

a) ($\iint$ vs $\iiint$) divergence theorem: S closed surface, $\hat{\mathbf{n}}$ outwards, then $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_D (\operatorname{div} \mathbf{F})\, dV$.

b) ($\int$ vs $\iint$) Stokes' theorem: C closed curve bounding S compatibly oriented, then $\int_C \mathbf{F} \cdot d\vec{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$.

Both sides of these theorems are integrals of the types discussed above, and are evaluated by the usual methods! (even if the integrand happens to be a div or a curl).

In fact, another conceptually similar bridge exists between no integral at all and line integral: the fundamental theorem of calculus, $f(P_1) - f(P_0) = \int_C \nabla f \cdot d\vec{r}$.

One more topic: given $\mathbf{F}$ with $\operatorname{curl} \mathbf{F} = 0$, finding a potential function.

18.02 Lecture 33. – Thu, Dec 6, 2007

Handouts: PS12 solutions; exam 4 solutions; review sheet and practice final.

Applications of div and curl to physics.

Recall: curl of velocity field = 2· angular velocity vector (of the rotation component of motion).

E.g., for uniform rotation about z-axis, $\boldsymbol{v} = \omega(-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}})$, and $\nabla \times \boldsymbol{v} = 2\omega\hat{\boldsymbol{k}}$.

Curl singles out the rotation component of motion (while div singles out the stretching component).

Interpretation of curl for force fields.

If we have a solid in a force field (or rather an acceleration field!) $\mathbf{F}$ such that the force exerted on Δm at (x, y, z) is $\mathbf{F}(x, y, z)\,\Delta m$: recall the *torque* of the force about the origin is defined as $\tau = \vec{r} \times \mathbf{F}$ (for the entire solid, take $\iiint \dots \,\delta\, dV$), and measures how $\mathbf{F}$ imparts rotation motion.

For translation motion: $\dfrac{\text{Force}}{\text{Mass}} = \text{acceleration} = \dfrac{d}{dt}(\text{velocity})$.

For rotation effects: $\dfrac{\text{Torque}}{\text{Moment of inertia}} = \text{angular acceleration} = \dfrac{d}{dt}(\text{angular velocity})$.

Hence: $\text{curl}(\dfrac{\text{Force}}{\text{Mass}}) = 2\,\dfrac{\text{Torque}}{\text{Moment of inertia}}$.

Consequence: if $\mathbf{F}$ derives from a potential, then $\nabla \times \mathbf{F} = \nabla \times (\nabla f) = 0$, so $\mathbf{F}$ does not induce any rotation motion. E.g., gravitational attraction by itself does not affect Earth's rotation. (not strictly true: actually Earth is deformable; similarly, friction and tidal effects due to Earth's gravitational attraction explain why the Moon's rotation and revolution around Earth are synchronous).

Div and curl of electrical field. – part of Maxwell's equations for electromagnetic fields.

1) Gauss-Coulomb law: $\nabla \cdot \vec{E} = \dfrac{\rho}{\epsilon_0}$ (ρ = charge density, ϵ_0 = physical constant).

By divergence theorem, can reformulate as: $\displaystyle\iint_S \vec{E} \cdot \hat{\mathbf{n}}\, dS = \iiint_D \nabla \cdot \vec{E}\, dV = \frac{Q}{\epsilon_0}$, where Q = total charge inside the closed surface S.

This formula tells how charges influence the electric field; e.g., it governs the relation between voltage between the two plates of a capacitor and its electric charge.

2) Faraday's law: $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ ($\vec{B}$= magnetic field).

So in presence of a varying magnetic field, $\vec{E}$ is no longer conservative: if we have a closed loop of wire, we get a non-zero voltage ("induction" effect). By Stokes, $\displaystyle\oint_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt}\iint_S \vec{B} \cdot \hat{\mathbf{n}}\, dS$.

This principle is used e.g. in transformers in power adapters: AC runs through a wire looped around a cylinder, which creates an alternating magnetic field; the flux of this magnetic field through another output wire loop creates an output voltage between its ends.

There are two more Maxwell equations, governing div and curl of $\vec{B}$: $\nabla \cdot \vec{B} = 0$, and $\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \dfrac{\partial \vec{E}}{\partial t}$ (where $\vec{J}$ = current density).