18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 1. – Thu, Sept 6, 2007

Handouts: syllabus; PS1; flashcards.

Goal of multivariable calculus: tools to handle problems with several parameters – functions of several variables.

Vectors. A vector (notation: $\vec{A}$) has a direction, and a length ($|\vec{A}|$). It is represented by a directed line segment. In a coordinate system it's expressed by components: in space, $\vec{A} = \langle a_1, a_2, a_3 \rangle = a_1 \hat{\imath} + a_2 \hat{\jmath} + a_3 \hat{k}$. (Recall in space x-axis points to the lower-left, y to the right, z up).

Scalar multiplication

Formula for length? Showed picture of $\langle 3, 2, 1 \rangle$ and used flashcards to ask for its length. Most students got the right answer ($\sqrt{14}$).

You can explain why $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ by reducing to the Pythagorean theorem in the plane (Draw a picture, showing $\vec{A}$ and its projection to the xy-plane, then derived $|\vec{A}|$ from length of projection + Pythagorean theorem).

Vector addition: $\vec{A} + \vec{B}$ by head-to-tail addition: Draw a picture in a parallelogram (showed how the diagonals are $\vec{A} + \vec{B}$ and $\vec{B} - \vec{A}$); addition works componentwise, and it is true that $\vec{A} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$ on the displayed example.

Dot product.

Definition: $\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$ (a scalar, not a vector).

Theorem: geometrically, $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$.

Explained the theorem as follows: first, $\vec{A} \cdot \vec{A} = |\vec{A}|^2 \cos 0 = |\vec{A}|^2$ is consistent with the definition. Next, consider a triangle with sides $\vec{A}$, $\vec{B}$, $\vec{C} = \vec{A} - \vec{B}$. Then the law of cosines gives $|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos\theta$, while we get

$$|\vec{C}|^2 = \vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A} \cdot \vec{B}.$$

Hence the theorem is a vector formulation of the law of cosines.

Applications. 1) computing lengths and angles: $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$.

Example: triangle in space with vertices $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, 2)$, find angle at P:

$$\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{\langle -1, 1, 0 \rangle \cdot \langle -1, 0, 2 \rangle}{\sqrt{2}\sqrt{5}} = \frac{1}{\sqrt{10}}, \qquad \theta \approx 71.5^\circ.$$

Note the sign of dot product: positive if angle less than 90°, negative if angle more than 90°, zero if perpendicular.

2) detecting orthogonality.

Example: what is the set of points where $x + 2y + 3z = 0$? (possible answers: empty set, a point, a line, a plane, a sphere, none of the above, I don't know).

Answer: plane; can see "by hand", but more geometrically use dot product: call $\vec{A} = \langle 1, 2, 3 \rangle$, $P = (x, y, z)$, then $\vec{A} \cdot \overrightarrow{OP} = x + 2y + 3z = 0 \Leftrightarrow |\vec{A}||\overrightarrow{OP}| \cos\theta = 0 \Leftrightarrow \theta = \pi/2 \Leftrightarrow \vec{A} \perp \overrightarrow{OP}$. So we get the plane through O with normal vector $\vec{A}$.

18.02 Lecture 2. – Fri, Sept 7, 2007

We've seen two applications of dot product: finding lengths/angles, and detecting orthogonality. A third one: finding components of a vector. If $\hat{\boldsymbol{u}}$ is a unit vector, $\vec{A}\cdot\hat{\boldsymbol{u}} = |\vec{A}|\cos\theta$ is the component of $\vec{A}$ along the direction of $\hat{\boldsymbol{u}}$. E.g., $\vec{A}\cdot\hat{\boldsymbol{\imath}} =$ component of $\vec{A}$ along x-axis.

Example: pendulum making an angle with vertical, force = weight of pendulum $\vec{F}$ pointing downwards: then the physically important quantities are the components of $\vec{F}$ along tangential direction (causes pendulum's motion), and along normal direction (causes string tension).

Area. E.g. of a polygon in plane: break into triangles. Area of triangle $= \frac{1}{2}$ base $\times$ height $= \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta$ (= 1/2 area of parallelogram). Could get $\sin\theta$ using dot product to compute $\cos\theta$ and $\sin^2+\cos^2 = 1$, but it gives an ugly formula. Instead, reduce to complementary angle $\theta' = \pi/2-\theta$ by considering $\vec{A}' = \vec{A}$ rotated 90° counterclockwise (drew a picture). Then area of parallelogram $= |\vec{A}||\vec{B}|\sin\theta = |\vec{A}'||\vec{B}|\cos\theta' = \vec{A}'\cdot\vec{B}$.

Q: if $\vec{A} = \langle a_1, a_2\rangle$, then what is $\vec{A}'$? (showed picture, used flashcards). Answer: $\vec{A}' = \langle -a_2, a_1\rangle$. (explained on picture). So area of parallelogram is $\langle b_1, b_2\rangle\cdot\langle -a_2, a_1\rangle = a_1b_2 - a_2b_1$.

Determinant. Definition: $\det(\vec{A},\vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

Geometrically: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm$ area of parallelogram.

The sign of 2D determinant has to do with whether $\vec{B}$ is counterclockwise or clockwise from $\vec{A}$, without details.

Determinant in space: $\det(\vec{A},\vec{B},\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$.

Geometrically: $\det(\vec{A},\vec{B},\vec{C}) = \pm$ volume of parallelepiped. Referred to the notes for more about determinants.

Cross-product. (only for 2 vectors in space); gives a vector, not a scalar (unlike dot-product).

Definition: $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\boldsymbol{\imath}}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{\boldsymbol{k}}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$.

(the 3x3 determinant is a *symbolic* notation, the actual formula is the expansion).

Geometrically: $|\vec{A}\times\vec{B}|$ = area of space parallelogram with sides $\vec{A}$, $\vec{B}$; direction = normal to the plane containing $\vec{A}$ and $\vec{B}$.

How to decide between the two perpendicular directions = right-hand rule. 1) extend right hand in direction of $\vec{A}$; 2) curl fingers towards direction of $\vec{B}$; 3) thumb points in same direction as $\vec{A}\times\vec{B}$.

Flashcard Question: $\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}} = ?$ (answer: $\hat{\boldsymbol{k}}$, checked both by geometric description and by calculation).

Triple product: volume of parallelepiped = area(base) $\cdot$ height $= |\vec{B}\times\vec{C}|\,(\vec{A}\cdot\hat{n})$, where $\hat{n} = \vec{B}\times\vec{C}/|\vec{B}\times\vec{C}|$. So volume $= \vec{A}\cdot(\vec{B}\times\vec{C}) = \det(\vec{A},\vec{B},\vec{C})$. The latter identity can also be checked directly using components.

18.02 Lecture 3. – Tue, Sept 11, 2007

Remark: $\boldsymbol{A}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{A}$, $\boldsymbol{A}\times\boldsymbol{A} = 0$.

Application of cross product: equation of plane through P_1, P_2, P_3: $P = (x, y, z)$ is in the plane iff $\det(\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = 0$, or equivalently, $\overrightarrow{P_1P}\cdot\boldsymbol{N} = 0$, where $\boldsymbol{N}$ is the normal vector $\boldsymbol{N} = \overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}$. I explained this geometrically, and showed how we get the same equation both ways.

Matrices. Often quantities are related by linear transformations; e.g. changing coordinate systems, from $P = (x_1, x_2, x_3)$ to something more adapted to the problem, with new coordinates (u_1, u_2, u_3). For example

$$\begin{cases} u_1 = 2x_1 + 3x_2 + 3x_3 \\ u_2 = 2x_1 + 4x_2 + 5x_3 \\ u_3 = x_1 + x_2 + 2x_3 \end{cases}$$

Rewrite using matrix product: $\begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, i.e. $AX = U$.

Entries in the matrix product = dot product between rows of A and columns of X. (here we multiply a 3x3 matrix by a column vector = 3x1 matrix).

More generally, matrix multiplication AB:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot & 0 \\ \cdot & 3 \\ \cdot & 0 \\ \cdot & 2 \end{bmatrix} = \begin{bmatrix} \cdot & 14 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

(Also explained one can set up A to the left, B to the top, then each entry of AB = dot product between row to its left and column above it).

Note: for this to make sense, width of A must equal height of B.

What AB means: BX = apply transformation B to vector X, so $(AB)X = A(BX)$ = apply first B then A. (so matrix multiplication is like composing transformations, but from right to left!)

(Remark: matrix product is not commutative, AB is in general not the same as BA – one of the two need not even make sense if sizes not compatible).

Identity matrix: identity transformation $IX = X$. $\qquad I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example: $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ = plane rotation by 90 degrees counterclockwise.

$R\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}$, $R\hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\imath}}$, $R^2 = -I$.

Inverse matrix. Inverse of a matrix A (necessarily square) is a matrix $M = A^{-1}$ such that $AM = MA = I_n$.

A^{-1} corresponds to the reciprocal linear relation.

E.g., solution to linear system $AX = U$: can solve for X as function of U by $X = A^{-1}U$.

Cofactor method to find A^{-1} (efficient for small matrices; for large matrices computer software uses other algorithms): $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$ ($\mathrm{adj}(A)$ = "adjoint matrix").

Illustration on example: starting from $A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}$

1) matrix of minors (= determinants formed by deleting one row and one column from A): $\begin{bmatrix} 3 & -1 & -2 \\ 3 & 1 & -1 \\ 3 & 4 & 2 \end{bmatrix}$ (e.g. top-left is $\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 3$).

2) cofactors = flip signs according to checkerboard diagram $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$: get $\begin{bmatrix} 3 & +1 & -2 \\ -3 & 1 & +1 \\ 3 & -4 & 2 \end{bmatrix}$

3) transpose = exchange rows / columns (read horizontally, write vertically) get the adjoint matrix $M^T = adj(A) = \begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$

4) divide by $\det(A)$ (here = 3): get $A^{-1} = \dfrac{1}{3}\begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$.

18.02 Lecture 4. – Thu, Sept 13, 2007

Handouts: PS1 solutions; PS2.

Equations of planes. Recall an equation of the form $ax + by + cz = d$ defines a plane.

1) plane through origin with normal vector $\boldsymbol{N} = \langle 1, 5, 10\rangle$: $P = (x, y, z)$ is in the plane $\Leftrightarrow$ $\boldsymbol{N} \cdot \overrightarrow{OP} = 0 \Leftrightarrow \langle 1, 5, 10\rangle \cdot \langle x, y, z\rangle = x + 5y + 10z = 0$. Coefficients of the equation are the components of the normal vector.

2) plane through $P_0 = (2, 1, -1)$ with same normal vector $\boldsymbol{N} = \langle 1, 5, 10\rangle$: parallel to the previous one! P is in the plane $\Leftrightarrow \boldsymbol{N} \cdot \overrightarrow{P_0P} = 0 \Leftrightarrow (x-2) + 5(y-1) + 10(z+1) = 0$, or $x + 5y + 10z = -3$. Again coefficients of equation = components of normal vector.

(Note: the equation multiplied by a constant still defines the same plane).

So, to find the equation of a plane, we really need to look for the normal vector $\boldsymbol{N}$; we can e.g. find it by cross-product of 2 vectors that are in the plane.

Flashcard question: the vector $\boldsymbol{v} = \langle 1, 2, -1\rangle$ and the plane $x + y + 3z = 5$ are 1) parallel, 2) perpendicular, 3) neither?

(A perpendicular vector would be proportional to the coefficients, i.e. to $\langle 1, 1, 3\rangle$; let's test if it's in the plane: equivalent to being $\perp \boldsymbol{N}$. We have $\boldsymbol{v} \cdot \boldsymbol{N} = 1 + 2 - 3 = 0$ so $\boldsymbol{v}$ is parallel to the plane.)

Interpretation of 3x3 systems. A 3x3 system asks for the intersection of 3 planes. Two planes intersect in a line, and usually the third plane intersects it in a single point (picture shown). The unique solution to $AX = B$ is given by $X = A^{-1}B$.

Exception: if the 3rd plane is parallel to the line of intersection of the first two? What can happen? (asked on flashcards for possibilities).

If the line $\mathcal{P}_1 \cap \mathcal{P}_2$ is contained in $\mathcal{P}_3$ there are infinitely many solutions (the line); if it is parallel to $\mathcal{P}_3$ there are no solutions. (could also get a plane of solutions if all three equations are the same)

These special cases correspond to systems with $\det(A) = 0$. Then we can't invert A to solve the system: recall $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$. Theorem: A is invertible $\Leftrightarrow \det A \neq 0$.

Homogeneous systems: $AX = 0$. Then all 3 planes pass through the origin, so there is the obvious ("trivial") solution $X = 0$. If $\det A \neq 0$ then this solution is unique: $X = A^{-1}\,0 = 0$. Otherwise, if $\det A = 0$ there are infinitely many solutions (forming a line or a plane).

Note: $\det A = 0$ means $\det(\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3) = 0$, where $\boldsymbol{N}_i$ are the normals to the planes $\mathcal{P}_i$. This means the parallelepiped formed by the $\boldsymbol{N}_i$ has no area, i.e. they are coplanar (showed picture of 3 planes intersecting in a line, and their coplanar normals). The line of solutions is then perpendicular to the plane containing $\boldsymbol{N}_i$. For example we can get a vector along the line of intersection by taking a cross-product $\boldsymbol{N}_1 \times \boldsymbol{N}_2$.

General systems: $AX = B$: compared to $AX = 0$, all the planes are shifted to parallel positions from their initial ones. If $\det A \neq 0$ then unique solution is $X = A^{-1}B$. If $\det A = 0$, either there are infinitely many solutions or there are no solutions.

(We don't have tools to decide whether it's infinitely many or none, although elimination will let us find out).

18.02 Lecture 5. – Fri, Sept 14, 2007

Lines. We've seen a line as intersection of 2 planes. Other representation = parametric equation = as trajectory of a moving point.

E.g. line through $Q_0 = (-1, 2, 2)$, $Q_1 = (1, 3, -1)$: moving point $Q(t)$ starts at Q_0 at $t = 0$, moves at constant speed along line, reaches Q_1 at $t = 1$: its "velocity" is $\vec{v} = \overrightarrow{Q_0Q_1}$; $\overrightarrow{Q_0Q(t)} = t\overrightarrow{Q_0Q_1}$. On example: $\langle x+1, y-2, z-2\rangle = t\langle 2, 1, -3\rangle$, i.e.

$$\begin{cases} x(t) = -1 + 2t, \\ y(t) = 2 + t, \\ z(t) = 2 - 3t \end{cases}$$

Lines and planes. Understand where lines and planes intersect.

Flashcard question: relative positions of Q_0, Q_1 with respect to plane $x + 2y + 4z = 7$? (same side, opposite sides, one is in the plane, can't tell).

(A sizeable number of students erroneously answered that one is in the plane.)

Answer: plug coordinates into equation of plane: at Q_0, $x+2y+4z = 11 > 7$; at Q_1, $x+2y+4z = 3 < 7$; so opposite sides.

Intersection of line Q_0Q_1 with plane? When does the moving point $Q(t)$ lie in the plane? Check: at $Q(t)$, $x + 2y + 4z = (-1 + 2t) + 2(2 + t) + 4(2 - 3t) = 11 - 8t$, so condition is $11 - 8t = 7$, or $t = 1/2$. Intersection point: $Q(t = \frac{1}{2}) = (0,\ 5/2,\ 1/2)$.

(What would happen if the line was parallel to the plane, or inside it. Answer: when plugging the coordinates of $Q(t)$ into the plane equation we'd get a constant, equal to 7 if the line is contained in the plane – so all values of t are solutions – or to something else if the line is parallel to the plane – so there are no solutions.)

General parametric curves.

Example: cycloid: wheel rolling on floor, motion of a point P on the rim. (Drew picture, then showed an applet illustrating the motion and plotting the cycloid).

Position of P? Choice of parameter: e.g., θ, the angle the wheel has turned since initial position. Distance wheel has travelled is equal to arclength on circumference of the circle $= a\theta$.

Setup: x-axis = floor, initial position of P = origin; introduce A = point of contact of wheel on floor, B = center of wheel. Decompose $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}$.

$\overrightarrow{OA} = \langle a\theta, 0\rangle$; $\overrightarrow{AB} = \langle 0, a\rangle$. Length of $\overrightarrow{BP}$ is a, and direction is θ from the $(-y)$-axis, so $\overrightarrow{BP} = \langle -a\sin\theta, -a\cos\theta\rangle$. Hence the *position vector* is $\overrightarrow{OP} = \langle a\theta - a\sin\theta, a - a\cos\theta\rangle$.

Q: What happens near bottom point? (flashcards: corner point with finite slopes on left and right; looped curve; smooth graph with horizontal tangent; vertical tangent (cusp)).

Answer: use Taylor approximation: for $t \to 0$, $f(t) \approx f(0) + tf'(0) + \frac{1}{2}t^2 f''(0) + \frac{1}{6}t^3 f'''(0) + \ldots$. This gives $\sin\theta \approx \theta - \theta^3/6$ and $\cos\theta \approx 1 - \theta^2/2$. So $x(\theta) \simeq \theta^3/6$, $y(\theta) \simeq \theta^2/2$ Hence for $\theta \to 0$, $y/x \simeq (\frac{1}{2}\theta^2)/(\frac{1}{6}\theta^3) = 3/\theta \to \infty$: vertical tangent.

18.02 Lecture 6. – Tue, Sept 18, 2007

Handouts: Practice exams 1A and 1B.

Velocity and acceleration. Last time: position vector $\vec{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}$ $[+z(t)\hat{\boldsymbol{k}}]$.

E.g., cycloid: $\vec{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$.

Velocity vector: $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \langle \dfrac{dx}{dt}, \dfrac{dy}{dt}\rangle$. E.g., cycloid: $\vec{v}(t) = \langle 1 - \cos t, \sin t\rangle$. (at $t = 0$, $\vec{v} = \vec{0}$: translation and rotation motions cancel out, while at $t = \pi$ they add up and $\vec{v} = \langle 2, 0\rangle$).

Speed (scalar): $|\vec{v}|$. E.g., cycloid: $|\vec{v}| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$. (smallest at $t = 0, 2\pi, ...$, largest at $t = \pi$).

Acceleration: $\vec{a}(t) = \dfrac{d\vec{v}}{dt}$. E.g., cycloid: $\vec{a}(t) = \langle \sin t, \cos t\rangle$ (at $t = 0$ $\vec{a} = \langle 0, 1\rangle$ is vertical).

Remark: the speed is $|\frac{d\vec{r}}{dt}|$, which is NOT the same as $\frac{d|\vec{r}|}{dt}$!

Arclength, unit tangent vector. s = distance travelled along trajectory. $\dfrac{ds}{dt}$ = speed = $|\vec{v}|$. Can recover length of trajectory by integrating ds/dt, but this is not always easy... e.g. the length of an arch of cycloid is $\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$ (can't do).

Unit tangent vector to trajectory: $\hat{T} = \dfrac{\vec{v}}{|\vec{v}|}$. We have: $\dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{ds}\dfrac{ds}{dt} = \hat{T}\dfrac{ds}{dt} = \hat{T}|\vec{v}|$.

In interval Δt: $\Delta\vec{r} \approx \hat{T}\,\Delta s$, dividing both sides by Δt and taking the limit $\Delta t \to 0$ gives us the above identity.

Kepler's 2nd law. (illustration of efficiency of vector methods) Kepler 1609, laws of planetary motion: the motion of planets is in a plane, and area is swept out by the line from the sun to the planet at a constant rate. Newton (about 70 years later) explained this using laws of gravitational attraction.

Kepler's law in vector form: area swept out in Δt is area $\approx \frac{1}{2}|\vec{r} \times \Delta\vec{r}| \approx \frac{1}{2}|\vec{r} \times \vec{v}|\,\Delta t$
So $\frac{d}{dt}(\text{area}) = \frac{1}{2}|\vec{r} \times \vec{v}|$ is constant.

Also, $\vec{r} \times \vec{v}$ is perpendicular to plane of motion, so $\text{dir}(\vec{r} \times \vec{v})$ = constant. Hence, Kepler's 2nd law says: $\vec{r} \times \vec{v}$ = constant.

The usual product rule can be used to differentiate vector functions: $\frac{d}{dt}(\vec{a} \cdot \vec{b})$, $\frac{d}{dt}(\vec{a} \times \vec{b})$, being careful about non-commutativity of cross-product.

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = \vec{r} \times \vec{a}.$$

So Kepler's law $\Leftrightarrow$ $\vec{r} \times \vec{v}$ = constant $\Leftrightarrow$ $\vec{r} \times \vec{a} = 0$ $\Leftrightarrow$ $\vec{a} // \vec{r}$ $\Leftrightarrow$ the force $\vec{F}$ is central.

(so Kepler's law really means the force is directed $//\vec{r}$; it also applies to other central forces – e.g. electric charges.)

18.02 Lecture 7. – Thu, Sept 20, 2007

Handouts: PS2 solutions, PS3.

Review. Material on the test = everything seen in lecture. The exam is similar to the practice exams, or very slightly harder. The main topics are (Problem numbers refer to Practice 1A):

1) vectors, dot product. $\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta = \sum a_i b_i$. Finding angles. (e.g. Problem 1.)

2) cross-product, area of space triangles $\frac{1}{2}|\boldsymbol{A} \times \boldsymbol{B}|$; equations of planes (coefficients of equation = components of normal vector) (e.g. Problem 5.)

3) matrices, inverse matrix, linear systems (e.g. Problem 3.)

4) finding parametric equations by decomposing position vector as a sum; velocity, acceleration; differentiating vector identities (e.g. Problems 2,4,6).