18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 16. – **Thu, Oct 18, 2007**

Handouts: PS6 solutions, PS7.

Double integrals.

Recall integral in 1-variable calculus: $\int_a^b f(x)\,dx$ = area below graph $y = f(x)$ over $[a, b]$.

Now: double integral $\iint_R f(x, y)\,dA$ = volume below graph $z = f(x, y)$ over plane region R.

Cut R into small pieces $\Delta A \Rightarrow$ the volume is approximately $\sum f(x_i, y_i)\,\Delta A_i$. Limit as $\Delta A \to 0$ gives $\iint_R f(x, y)\,dA$. (picture shown)

How to compute the integral? By taking slices: $S(x)$ = area of the slice by a plane parallel to yz-plane (picture shown): then

$$\text{volume} = \int_{x_{min}}^{x_{max}} S(x)\,dx, \quad \text{and for given } x,\ S(x) = \int f(x, y)\,dy.$$

In the inner integral, x is a fixed parameter, y is the integration variable. We get an *iterated integral.*

Example 1: $z = 1 - x^2 - y^2$, region $0 \le x \le 1,\ 0 \le y \le 1$ (picture shown):

$$\int_0^1 \int_0^1 (1 - x^2 - y^2)\,dy\,dx.$$

(note: $dA = dy\,dx$, limit of $\Delta A = \Delta y\,\Delta x$ for small rectangles).

How to evaluate:

1) inner integral (x is constant): $\displaystyle\int_0^1 (1 - x^2 - y^2)\,dy = \left[(1 - x^2)y - \frac{1}{3}y^3\right]_0^1 = (1 - x^2) - \frac{1}{3} = \frac{2}{3} - x^2.$

2) outer integral: $\displaystyle\int_0^1 (\frac{2}{3} - x^2)\,dx = \left[\frac{2}{3}x - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$

Example 2: same function over the quarter disc $R : x^2 + y^2 < 1$ in the first quadrant.

How to find the bounds of integration? Fix x constant: what is a slice parallel to y-axis? bounds for y = from $y = 0$ to $y = \sqrt{1 - x^2}$ in the inner integral. For the outer integral: first slice is $x = 0$, last slice is $x = 1$. So we get:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2)\,dy\,dx.$$

(note the inner bounds depend on the outer variable x; the outer bounds are constants!)

Inner: $\displaystyle\left[(1 - x^2)y - y^3/3\right]_0^{\sqrt{1-x^2}} = \frac{2}{3}(1 - x^2)^{3/2}.$

Outer: $\displaystyle\int_0^1 \frac{2}{3}(1 - x^2)^{3/2}\,dx = \cdots = \frac{\pi}{8}.$

($\ldots$ = trig. substitution $x = \sin\theta$, $dx = \cos\theta\,d\theta$, $(1 - x^2)^{3/2} = \cos^3\theta$. Then use double angle formulas... complicated! I carried out part of the calculation to show how it would be done but then stopped before the end to save time; students may be confused about what happened exactly.)

Exchanging order of integration.

$\int_0^1 \int_0^2 dx\,dy = \int_0^2 \int_0^1 dy\,dx$, since region is a rectangle (shown). In general, more complicated!

Example 3: $\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y}\,dy\,dx$: inner integral has no formula. To exchange order:

1) draw the region (here: $x < y < \sqrt{x}$ for $0 \le x \le 1$ – picture drawn on blackboard).

2) figure out bounds in other direction: fixing a value of y, what are the bounds for x? here: left border is $x = y^2$, right is $x = y$; first slice is $y = 0$, last slice is $y = 1$, so we get

$$\int_0^1 \int_{y^2}^{y} \frac{e^y}{y}\,dx\,dy = \int_0^1 \frac{e^y}{y}(y - y^2)\,dy = \int_0^1 e^y - ye^y\,dy = [-ye^y + 2e^y]_0^1 = e - 2.$$

(the last integration can be done either by parts, or by starting from the guess $-ye^y$ and adjusting;).

18.02 Lecture 17. – **Fri, Oct 19, 2007**

Integration in polar coordinates. ($x = r\cos\theta$, $y = r\sin\theta$): useful if either integrand or region have a simpler expression in polar coordinates.

Area element: $\Delta A \simeq (r\Delta\theta)\,\Delta r$ (picture drawn of a small element with sides Δr and $r\Delta\theta$). Taking $\Delta\theta, \Delta r \to 0$, we get $dA = r\,dr\,d\theta$.

Example (same as last time): $\iint_{x^2+y^2\le 1,\ x\ge 0,\ y\ge 0} (1 - x^2 - y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^1 (1 - r^2)\,r\,dr\,d\theta.$

Inner: $\left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 = \frac{1}{4}$. Outer: $\int_0^{\pi/2} \frac{1}{4}\,d\theta = \frac{\pi}{2}\frac{1}{4} = \frac{\pi}{8}$.

In general: when setting up $\iint f\,r\,dr\,d\theta$, find bounds as usual: given a fixed θ, find initial and final values of r (sweep region by rays).

Applications.

1) The area of the region R is $\iint_R 1\,dA$. Also, the total mass of a planar object with density $\delta = \lim_{\Delta A = 0} \Delta m/\Delta A$ (mass per unit area, $\delta = \delta(x, y)$ – if uniform material, constant) is given by:

$$M = \iint_R \delta\,dA.$$

2) recall the average value of f over R is $\bar{f} = \frac{1}{Area}\iint_R f\,dA$. The *center of mass*, or *centroid*, of a plate with density δ is given by weighted average

$$\bar{x} = \frac{1}{mass}\iint_R x\,\delta\,dA, \qquad \bar{y} = \frac{1}{mass}\iint_R y\,\delta\,dA$$

3) **moment of inertia:** physical equivalent of mass for rotational motion. (mass = how hard it is to impart translation motion; moment of inertia about some axis = same for rotation motion around that axis)

Idea: kinetic energy for a single mass m at distance r rotating at angular speed $\omega = d\theta/dt$ (so velocity $v = r\omega$) is $\frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$; $I_0 = mr^2$ is the moment of inertia.

For a solid with density δ, $I_0 = \iint_R r^2\delta\,dA$ (moment of inertia / origin). (the rotational energy is $\frac{1}{2}I_0\omega^2$).

Moment of inertia about an axis: $I = \iint_R (\text{distance to axis})^2 \delta \, dA$. E.g. about x-axis, distance is $|y|$, so

$$I_x = \iint_R y^2 \delta \, dA.$$

Examples: 1) disk of radius a around its center ($\delta = 1$):

$$I_0 = \int_0^{2\pi} \int_0^a r^2 \, r \, dr \, d\theta = 2\pi \left[\frac{r^4}{4} \right]_0^a = \frac{\pi a^4}{2}.$$

2) same disk, about a point on the circumference?

Setup: place origin at point so integrand is easier; diameter along x-axis; then polar equation of circle is $r = 2a\cos\theta$ (explained on a picture). Thus

$$I_0 = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} r^2 \, r \, dr \, d\theta = \dots = \frac{3}{2}\pi a^4.$$

18.02 Lecture 18. – **Tue, Oct 23, 2007**

Change of variables.

Example 1: area of ellipse with semiaxes a and b: setting $u = x/a$, $v = y/b$,

$$\iint_{(x/a)^2+(y/b)^2<1} dx\,dy = \iint_{u^2+v^2<1} ab\,du\,dv = ab \iint_{u^2+v^2<1} du\,dv = \pi ab.$$

(substitution works here as in 1-variable calculus: $du = \frac{1}{a}\,dx$, $dv = \frac{1}{b}\,dy$, so $du\,dv = \frac{1}{ab}dx\,dy$.

In general, must find out the scale factor (ratio between $du\,dv$ and $dx\,dy$)?

Example 2: say we set $u = 3x - 2y$, $v = x + y$ to simplify either integrand or bounds of integration. What is the relation between $dA = dx\,dy$ and $dA' = du\,dv$? (area elements in xy- and uv-planes).

Answer: consider a small rectangle of area $\Delta A = \Delta x \Delta y$, it becomes in uv-coordinates a parallelogram of area $\Delta A'$. Here the answer is independent of which rectangle we take, so we can take e.g. the unit square in xy-coordinates.

In the uv-plane, $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, so this becomes a parallelogram with sides given by vectors $\langle 3, 1\rangle$ and $\langle -2, 1\rangle$ (picture drawn), and area = det $= \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} = 5 \left(= \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \right)$.

For any rectangle $\Delta A' = 5\Delta A$, in the limit $dA' = 5dA$, i.e. $du\,dv = 5dx\,dy$. So $\iint \ldots dx\,dy = \iint \ldots \frac{1}{5} du\,dv$.

General case: approximation formula $\Delta u \approx u_x \Delta x + u_y \Delta y$, $\Delta v \approx v_x \Delta x + v_y \Delta y$, i.e.

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

A small xy-rectangle is approx. a parallelogram in uv-coords, but scale factor depends on x and y now. By the same argument as before, the scale factor is the determinant.

Definition: the Jacobian is $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$. Then $du\,dv = |J|\,dx\,dy$.

(absolute value because area is the absolute value of the determinant).

Example 1: polar coordinates $x = r\cos\theta$, $y = r\sin\theta$:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

So $dx\,dy = r\,dr\,d\theta$, as seen before.

Example 2: compute $\int_0^1 \int_0^1 x^2 y\,dx\,dy$ by changing to $u = x$, $v = xy$ (usually motivation is to simplify either integrand or region; here neither happens, but we just illustrate the general method).

1) Area element: Jacobian is $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x$, so $du\,dv = x\,dx\,dy$, i.e. $dx\,dy = \frac{1}{x} du\,dv$.

2) Express integrand in terms of u, v: $x^2 y\,dx\,dy = x^2 y\,\frac{1}{x}\,du\,dv = xy\,du\,dv = v\,du\,dv$.

3) Find bounds (picture drawn): if we integrate $du\,dv$, then first we keep $v = xy$ constant, slice looks like portion of hyperbola (picture shown), parametrized by $u = x$. The bounds are: at the top boundary $y = 1$, so $v/u = 1$, i.e. $u = v$; at the right boundary, $x = 1$, so $u = 1$. So the inner

integral is $\int_v^1$. The first slice is $v = 0$, the last is $v = 1$; so we get

$$\int_0^1 \int_v^1 v\,du\,dv.$$

Besides the picture in xy coordinates (a square sliced by hyperbolas), I also drew a picture in uv coordinates (a triangle), which some students may find is an easier way of getting the bounds for u and v.

18.02 Lecture 19. – **Thu, Oct 25, 2007**

Handouts: PS7 solutions; PS8.

Vector fields.

$\vec{F} = M\hat{\boldsymbol{\imath}} + N\hat{\boldsymbol{\jmath}}$, where $M = M(x, y)$, $N = N(x, y)$: at each point in the plane we have a vector $\vec{F}$ which depends on x, y.

Examples: velocity fields, e.g. wind flow (shown: chart of winds over Pacific ocean); force fields, e.g. gravitational field.

Examples drawn on blackboard: (1) $\vec{F} = 2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ (constant vector field); (2) $\vec{F} = x\hat{\boldsymbol{\imath}}$; (3) $\vec{F} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ (radially outwards); (4) $\vec{F} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$ (explained using that $\langle -y, x\rangle$ is $\langle x, y\rangle$ rotated 90° counterclockwise).

Work and line integrals.

$W =$ (force).(distance) $= \vec{F} \cdot \Delta\vec{r}$ for a small motion $\Delta\vec{r}$. Total work is obtained by summing these along a trajectory C: get a "line integral"

$$W = \int_C \vec{F} \cdot d\vec{r} \;\left(= \lim_{\Delta\vec{r}\to 0} \sum_i \vec{F} \cdot \Delta\vec{r}_i \right).$$

To evaluate the line integral, we observe C is parametrized by time, and give meaning to the notation $\int_C \vec{F} \cdot d\vec{r}$ by

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} (\vec{F} \cdot \frac{d\vec{r}}{dt})\,dt.$$

Example: $\vec{F} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$, C is given by $x = t$, $y = t^2$, $0 \le t \le 1$ (portion of parabola $y = x^2$ from (0,0) to (1,1)). Then we substitute expressions in terms of t everywhere:

$$\vec{F} = \langle -y, x\rangle = \langle -t^2, t\rangle, \quad \frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}\rangle = \langle 1, 2t\rangle,$$

so $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt = \int_0^1 \langle -t^2, t\rangle \cdot \langle 1, 2t\rangle\,dt = \int_0^1 t^2\,dt = \frac{1}{3}$. (in the end things always reduce to a one-variable integral.)

In fact, the definition of the line integral does not involve the parametrization: so the result is the same no matter which parametrization we choose. For example we could choose to parametrize the parabola by $x = \sin\theta$, $y = \sin^2\theta$, $0 \le \theta \le \pi/2$. Then we'd get $\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \dots\, d\theta$, which would be equivalent to the previous one under the substitution $t = \sin\theta$ and would again be equal to $\frac{1}{3}$. In practice we always choose the simplest parametrization!

New notation for line integral: $\vec{F} = \langle M, N\rangle$, and $d\vec{r} = \langle dx, dy\rangle$ (this is in fact a differential: if we divide both sides by dt we get the component formula for the velocity $d\vec{r}/dt$). So the line integral

becomes
$$\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy.$$

The notation is dangerous: this is not a sum of integrals w.r.t. x and y, but really a line integral along C. To evaluate one must express everything in terms of the chosen parameter.

In the above example, we have $x = t$, $y = t^2$, so $dx = dt$, $dy = 2t\,dt$ by implicit differentiation; then
$$\int_C -y\,dx + x\,dy = \int_0^1 -t^2\,dt + t\,(2t)\,dt = \int_0^1 t^2\,dt = \frac{1}{3}$$
(same calculation as before, using different notation).

Geometric approach.

Recall velocity is $\dfrac{d\vec{r}}{dt} = \dfrac{ds}{dt}\,\hat{\boldsymbol{T}}$ (where $s=$arclength, $\hat{\boldsymbol{T}}=$unit tangent vector to trajectory).

So $d\vec{r} = \hat{\boldsymbol{T}}\,ds$, and $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{\boldsymbol{T}}\,ds$. Sometimes the calculation is easier this way!

Example: $C=$circle of radius a centered at origin, $\vec{F} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$, then $\vec{F}\cdot\hat{\boldsymbol{T}} = 0$ (picture drawn), so $\int_C \vec{F}\cdot\hat{\boldsymbol{T}}\,ds = \int 0\,ds = 0$.

Example: same C, $\vec{F} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$, then $\vec{F}\cdot\hat{\boldsymbol{T}} = |\vec{F}| = a$, so $\int_C \vec{F}\cdot\hat{\boldsymbol{T}}\,ds = \int a\,ds = a\,(2\pi a) = 2\pi a^2$; checked that we get the same answer if we compute using parametrization $x = a\cos\theta$, $y = a\sin\theta$.

18.02 Lecture 20. – **Fri, Oct 26, 2007**

Line integrals continued.

Recall: line integral of $\vec{F} = M\hat{\boldsymbol{\imath}} + N\hat{\boldsymbol{\jmath}}$ along a curve C: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy = \int_C \vec{F}\cdot\hat{\boldsymbol{T}}\,ds$.

Example: $\vec{F} = y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$, $\int_C \vec{F}\cdot d\vec{r}$ for $C = C_1 + C_2 + C_3$ enclosing sector of unit disk from 0 to $\pi/4$. (picture shown). Need to compute $\int_{C_i} y\,dx + x\,dy$ for each portion:

1) x-axis: $x = t$, $y = 0$, $dx = dt$, $dy = 0$, $0 \le t \le 1$, so $\int_{C_1} y\,dx + x\,dy = \int_0^1 0\,dt = 0$. Equivalently, geometrically: along x-axis, $y = 0$ so $\vec{F} = x\hat{\boldsymbol{\jmath}}$ while $\hat{\boldsymbol{T}} = \hat{\boldsymbol{\imath}}$ so $\int_{C_1} \vec{F}\cdot\hat{\boldsymbol{T}}\,ds = 0$.

2) C_2: $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$, $0 \le \theta \le \frac{\pi}{4}$. So
$$\int_{C_2} y\,dx + x\,dy = \int_0^{\pi/4} \sin\theta(-\sin\theta)d\theta + \cos\theta\cos\theta\,d\theta = \int_0^{\pi/4} \cos(2\theta)d\theta = \left[\frac{1}{2}\sin(2\theta)\right]_0^{\pi/4} = \frac{1}{2}.$$

3) C_3: line segment from $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ to $(0,0)$: could take $x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t$, $y=$same, $0 \le t \le 1$, ... but easier: C_3 backwards ("$-C_3$") is $y = x = t$, $0 \le t \le \frac{1}{\sqrt{2}}$. Work along $-C_3$ is opposite of work along C_3.
$$\int_{C_3} y\,dx + x\,dy = \int_{1/\sqrt{2}}^0 t\,dt + t\,dt = -\int_0^{1/\sqrt{2}} 2t\,dt = -[t^2]_0^{1/\sqrt{2}} = -\frac{1}{2}.$$

If $\vec{F}$ is a gradient field, $\vec{F} = \nabla f = f_x\hat{\boldsymbol{\imath}} + f_y\hat{\boldsymbol{\jmath}}$ (f is called "potential function"), then we can simplify evaluation of line integrals by using the fundamental theorem of calculus.

Fundamental theorem of calculus for line integrals:

$$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0) \text{ when } C \text{ runs from } P_0 \text{ to } P_1.$$

Equivalently with differentials: $\displaystyle\int_C f_x\,dx + f_y\,dy = \int_C df = f(P_1) - f(P_0)$. Proof:

$$\int_C \nabla f \cdot d\vec{r} = \int_{t_0}^{t_1} (f_x \frac{dx}{dt} + f_y \frac{dy}{dt})\,dt = \int_{t_0}^{t_1} \frac{d}{dt}(f(x(t), y(t))\,dt = [f(x(t), y(t))]_{t_0}^{t_1} = f(P_1) - f(P_0).$$

E.g., in the above example, if we set $f(x, y) = xy$ then $\nabla f = \langle y, x\rangle = \vec{F}$. So $\int_{C_i}$ can be calculated just by evaluating $f = xy$ at end points. Picture shown of C, vector field, and level curves.

Consequences: for a gradient field, we have:

• Path independence: if C_1, C_2 have same endpoints then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ (both equal to $f(P_1) - f(P_0)$ by the theorem). So the line integral $\int_C \nabla f \cdot d\vec{r}$ depends only on the end points, not on the actual trajectory.

• Conservativeness: if C is a closed loop then $\int_C \nabla f \cdot d\vec{r} = 0$ $(= f(P) - f(P))$.
(e.g. in above example, $\int_C = 0 + \frac{1}{2} - \frac{1}{2} = 0$.)

WARNING: this is only for gradient fields!

Example: $\vec{F} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$ is not a gradient field: as seen Thursday, along C = circle of radius a counterclockwise ($\vec{F} // \hat{\boldsymbol{T}}$), $\int_C \vec{F} \cdot d\vec{r} = 2\pi a^2$. Hence $\vec{F}$ is not conservative, and not a gradient field.

Physical interpretation.

If the force field $\vec{F}$ is the gradient of a potential f, then work of $\vec{F}$ = change in value of potential.

E.g.: 1) $\vec{F}$ = gravitational field, f = gravitational potential; 2) $\vec{F}$ = electrical field; f = electrical potential (voltage). (Actually physicists use the opposite sign convention, $\vec{F} = -\nabla f$).

Conservativeness means that energy comes from change in potential f, so no energy can be extracted from motion along a closed trajectory (conservativeness = conservation of energy: the change in kinetic energy equals the work of the force equals the change in potential energy).

We have four equivalent properties:

(1) $\vec{F}$ is conservative ($\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C)

(2) $\int F \cdot d\vec{r}$ is path independent (same work if same end points)

(3) $\vec{F}$ is a gradient field: $\vec{F} = \nabla f = f_x\hat{\boldsymbol{\imath}} + f_y\hat{\boldsymbol{\jmath}}$.

(4) $M\,dx + N\,dy$ is an exact differential ($= f_x\,dx + f_y\,dy = df$.)

((1) is equivalent to (2) by considering C_1, C_2 with same endpoints, $C = C_1 - C_2$ is a closed loop. (3) $\Rightarrow$ (2) is the FTC, $\Leftarrow$ will be key to finding potential function: if we have path independence then we can get $f(x, y)$ by computing $\int_{(0,0)}^{(x,y)} \vec{F} \cdot d\vec{r}$. (3) and (4) are reformulations of the same property).

18.02 Lecture 21. – **Tue, Oct 30, 2007**

Test for gradient fields.

Observe: if $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ is a gradient field then $N_x = M_y$. Indeed, if $\vec{F} = \nabla f$ then $M = f_x$, $N = f_y$, so $N_x = f_{yx} = f_{xy} = M_y$.

Claim: Conversely, if $\vec{F}$ is defined and differentiable at every point of the plane, and $N_x = M_y$, then $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ is a gradient field.

Example: $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$: $N_x = 1$, $M_y = -1$, so $\vec{F}$ is not a gradient field.

Example: for which value(s) of a is $\vec{F} = (4x^2 + axy)\hat{\imath} + (3y^2 + 4x^2)\hat{\jmath}$ a gradient field? Answer: $N_x = 8x$, $M_y = ax$, so $a = 8$.

Finding the potential: if above test says $\vec{F}$ is a gradient field, we have 2 methods to find the potential function f. Illustrated for the above example (taking $a = 8$):

Method 1: using line integrals (FTC backwards):

We know that if C starts at $(0,0)$ and ends at (x_1, y_1) then $f(x_1, y_1) - f(0,0) = \int_C \vec{F} \cdot d\vec{r}$. Here $f(0,0)$ is just an integration constant (if f is a potential then so is $f + c$). Can also choose the simplest curve C from $(0,0)$ to (x_1, y_1).

Simplest choice: take C = portion of x-axis from $(0,0)$ to $(x_1, 0)$, then vertical segment from $(x_1, 0)$ to (x_1, y_1) (picture drawn).

Then $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_{C_1 + C_2} (4x^2 + 8xy)\,dx + (3y^2 + 4x^2)\,dy$:

Over C_1, $0 \le x \le x_1$, $y = 0$, $dy = 0$: $\displaystyle\int_{C_1} = \int_0^{x_1} (4x^2 + 8x \cdot 0)\,dx = \left[\frac{4}{3}x^3\right]_0^{x_1} = \frac{4}{3}x_1^3.$

Over C_2, $0 \le y \le y_1$, $x = x_1$, $dx = 0$: $\displaystyle\int_{C_2} = \int_0^{y_1} (3y^2 + 4x_1^2)\,dy = \left[y^3 + 4x_1^2 y\right]_0^{y_1} = y_1^3 + 4x_1^2 y_1.$

So $f(x_1, y_1) = \dfrac{4}{3}x_1^3 + y_1^3 + 4x_1^2 y_1$ (+constant).

Method 2: using antiderivatives:

We want $f(x,y)$ such that (1) $f_x = 4x^2 + 8xy$, (2) $f_y = 3y^2 + 4x^2$.

Taking antiderivative of (1) w.r.t. x (treating y as a constant), we get $f(x,y) = \frac{4}{3}x^3 + 4x^2y +$ integration constant (independent of x). The integration constant still depends on y, call it $g(y)$.

So $f(x,y) = \frac{4}{3}x^3 + 4x^2 y + g(y)$. Take partial w.r.t. y, to get $f_y = 4x^2 + g'(y)$.

Comparing this with (2), we get $g'(y) = 3y^2$, so $g(y) = y^3 + c$.

Plugging into above formula for f, we finally get $f(x,y) = \frac{4}{3}x^3 + 4x^2y + y^3 + c$.

Curl.

Now we have: $N_x = M_y \Leftrightarrow^* \vec{F}$ is a gradient field $\Leftrightarrow \vec{F}$ is conservative: $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve.

(*): $\Rightarrow$ only holds if $\vec{F}$ is defined everywhere, or in a "simply-connected" region – see next week.

Failure of conservativeness is given by the *curl* of $\vec{F}$:

Definition: $\mathrm{curl}(\vec{F}) = N_x - M_y$.

Interpretation of curl: for a velocity field, curl = (twice) angular velocity of the rotation component of the motion.

(Ex: $\vec{F} = \langle a, b\rangle$ uniform translation, $\vec{F} = \langle x, y\rangle$ expanding motion have curl zero; whereas $\vec{F} = \langle -y, x\rangle$ rotation at unit angular velocity has curl $= 2$).

For a force field, curl $\vec{F} =$ torque exerted on a test mass, measures how $\vec{F}$ imparts rotation motion.

For translation motion: $\dfrac{\text{Force}}{\text{Mass}} = \text{acceleration} = \dfrac{d}{dt}(\text{velocity})$.

For rotation effects: $\dfrac{\text{Torque}}{\text{Moment of inertia}} = \text{angular acceleration} = \dfrac{d}{dt}(\text{angular velocity})$.

18.02 Lecture 22. – **Thu, Nov 1, 2007**

Handouts: PS8 solutions, PS9, practice exams 3A and 3B.

Green's theorem.

If C is a positively oriented closed curve enclosing a region R, then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\, dA \quad \text{which means} \quad \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA.$$

Example (reduce a complicated line integral to an easy $\iint$):
Let $C =$ unit circle centered at (2,0), counterclockwise. $R =$ unit disk at $(2, 0)$. Then

$$\oint_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy = \iint_R N_x - M_y\,dA = \iint_R (x + e^{-x}) - e^{-x}\,dA = \iint_R x\,dA.$$

This is equal to area $\cdot\, \bar{x} = \pi \cdot 2 = 2\pi$ (or by direct computation of the iterated integral). (Note: direct calculation of the line integral would probably involve setting $x = 2 + \cos\theta$, $y = \sin\theta$, but then calculations get really complicated.)

Application: proof of our criterion for gradient fields.

Theorem: if $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ is defined and continuously differentiable in the whole plane, then $N_x = M_y \Rightarrow \vec{F}$ is conservative ($\Leftrightarrow \vec{F}$ is a gradient field).

If $N_x = M_y$ then by Green, $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\,dA = \iint_R 0\,dA = 0$. So $\vec{F}$ is conservative.

Note: this only works if $\vec{F}$ and its curl are defined everywhere inside R. For the vector field on PS8 Problem 2, we can't do this if the region contains the origin – for example, the line integral along the unit circle is non-zero even though curl$(\vec{F})$ is zero wherever it's defined.

Proof of Green's theorem. 2 preliminary remarks:

1) the theorem splits into two identities, $\oint_C M\,dx = -\iint_R M_y\,dA$ and $\oint_C N\,dy = \iint_R N_x\,dA$.

2) additivity: if theorem is true for R_1 and R_2 then it's true for the union $R = R_1 \cup R_2$ (picture shown): $\oint_C = \oint_{C_1} + \oint_{C_2}$ (the line integrals along inner portions cancel out) and $\iint_R = \iint_{R_1} + \iint_{R_2}$.

Main step in the proof: prove $\oint_C M\,dx = -\iint_R M_y\,dA$ for "vertically simple" regions: $a < x < b$, $f_0(x) < y < f_1(x)$. (picture drawn). This involves calculations similar to PS5 Problem 3.

LHS: break C into four sides (C_1 lower, C_2 right vertical segment, C_3 upper, C_4 left vertical segment); $\int_{C_2} M\,dx = \int_{C_4} M\,dx = 0$ since $x =$ constant on C_2 and C_4. So

$$\oint_C = \int_{C_1} + \int_{C_3} = \int_a^b M(x, f_0(x))\,dx - \int_a^b M(x, f_1(x))\,dx$$

(using along C_1: parameter $a \le x \le b$, $y = f_0(x)$; along C_2, x from b to a, hence $-$ sign; $y = f_1(x)$).

RHS: $-\iint_R M_y\,dA = -\int_a^b\int_{f_0(x)}^{f_1(x)} M_y\,dy\,dx = -\int_a^b (M(x,f_1(x)) - M(x,f_0(x))\,dx$ (= LHS).

Finally observe: any region R can be subdivided into vertically simple pieces (picture shown); for each piece $\oint_{C_i} M\,dx = -\iint_{R_i} M_y\,dA$, so by additivity $\oint_C M\,dx = -\iint_R M_y\,dA$.

Similarly $\oint_C N\,dy = \iint_R N_x\,dA$ by subdividing into horizontally simple pieces. This completes the proof.

Example. The area of a region R can be evaluated using a line integral: for example, $\oint_C x\,dy = \iint_R 1dA = area(R)$.

This idea was used to build mechanical devices that measure area of arbitrary regions on a piece of paper: planimeters (photo of the actual object shown, and principle explained briefly: as one moves its arm along a closed curve, the planimeter calculates the line integral of a suitable vector field by means of an ingenious mechanism; at the end of the motion, one reads the area).

18.02 Lecture 23. – **Fri, Nov 2, 2007**

Flux. The flux of a vector field $\vec{F}$ across a plane curve C is $\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds$, where $\hat{\boldsymbol{n}}$ = normal vector to C, rotated 90° clockwise from $\hat{\boldsymbol{T}}$.

We now have two types of line integrals: work, $\int \vec{F}\cdot\hat{\boldsymbol{T}}\,ds$, sums $\vec{F}\cdot\hat{\boldsymbol{T}}$ = component of $\vec{F}$ in direction of C, along the curve C. Flux, $\int \vec{F}\cdot\hat{\boldsymbol{n}}\,ds$, sums $\vec{F}\cdot\hat{\boldsymbol{n}}$ = component of $\vec{F}$ perpendicular to C, along the curve.

If we break C into small pieces of length Δs, the flux is $\sum_i(\vec{F}\cdot\hat{\boldsymbol{n}})\,\Delta s_i$.

Physical interpretation: if $\vec{F}$ is a velocity field (e.g. flow of a fluid), flux measures how much matter passes through C per unit time.

Look at a small portion of C: locally $\vec{F}$ is constant, what passes through portion of C in unit time is contents of a parallelogram with sides Δs and $\vec{F}$ (picture shown with $\vec{F}$ horizontal, and portion of curve = diagonal line segment). The area of this parallelogram is $\Delta s\cdot$height $= \Delta s\,(\vec{F}\cdot\hat{\boldsymbol{n}})$. (picture shown rotated with portion of C horizontal, at base of parallelogram). Summing these contributions along all of C, we get that $\int(\vec{F}\cdot\hat{\boldsymbol{n}})\,ds$ is the total flow through C per unit time; counting positively what flows towards the right of C, negatively what flows towards the left of C, as seen from the point of view of a point travelling along C.

Example: C = circle of radius a counterclockwise, $\vec{F} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ (picture shown): along C, $\vec{F}//\hat{\boldsymbol{n}}$, and $|\vec{F}| = a$, so $\vec{F}\cdot\hat{\boldsymbol{n}} = a$. So

$$\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \int_C a\,ds = a\,\text{length}(C) = 2\pi a^2.$$

Meanwhile, the flux of $-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$ across C is zero (field tangent to C).

That was a geometric argument. What about the general situation when calculation of the line integral is required?

Observe: $d\vec{r} = \hat{\boldsymbol{T}}\,ds = \langle dx, dy\rangle$, and $\hat{\boldsymbol{n}}$ is $\hat{\boldsymbol{T}}$ rotated 90° clockwise; so $\hat{\boldsymbol{n}}\,ds = \langle dy, -dx\rangle$.

So, if $\vec{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}$ (using new letters to make things look different; of course we could call the components M and N), then

$$\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \int_C \langle P, Q\rangle\cdot\langle dy, -dx\rangle = \int_C -Q\,dx + P\,dy.$$

(or if $\vec{F} = \langle M, N\rangle$, $\int_C -N\,dx + M\,dy$).

So we can compute flux using the usual method, by expressing x, y, dx, dy in terms of a parameter variable and substituting (no example given).

Green's theorem for flux. If C encloses R counterclockwise, and $\vec{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}$, then

$$\oint_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \iint_R \operatorname{div}(\vec{F})\,dA, \quad \text{where} \quad \operatorname{div}(\vec{F}) = P_x + Q_y \quad \text{is the divergence of } \vec{F}.$$

Note: the counterclockwise orientation of C means that we count flux of $\vec{F}$ *out* of R through C.

Proof: $\oint_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \oint_C -Q\,dx + P\,dy$. Call $M = -Q$ and $N = P$, then apply usual Green's theorem $\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$ to get

$$\oint_C -Q\,dx + P\,dy = \iint_R (P_x - (-Q_y))\,dA = \iint_R \operatorname{div}(\vec{F})\,dA.$$

This proof by "renaming" the components is why we called the components P, Q instead of M, N. If we call $\vec{F} = \langle M, N\rangle$ the statement becomes $\oint_C -N\,dx + M\,dy = \iint_R (M_x + N_y)\,dA$.

Example: in the above example ($x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ across circle), $\operatorname{div}\vec{F} = 2$, so flux $= \iint_R 2\,dA = 2\,\text{area}(R) = 2\pi a^2$. If we translate C to a different position (not centered at origin) (picture shown) then direct calculation of flux is harder, but total flux is still $2\pi a^2$.

Physical interpretation: in an incompressible fluid flow, divergence measures source/sink density/rate, i.e. how much fluid is being added to the system per unit area and per unit time.

18.02 Lecture 24. – **Tue, Nov 6, 2007**

Simply connected regions. [slightly different from the actual notations used]

Recall Green's theorem: if C is a closed curve around R counterclockwise then line integrals can be expressed as double integrals:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}(\vec{F})\, dA, \qquad \oint_C \vec{F} \cdot \hat{\boldsymbol{n}}\, ds = \iint_R \text{div}(\vec{F})\, dA,$$

where $\text{curl}(M\hat{\boldsymbol{\imath}} + N\hat{\boldsymbol{\jmath}}) = N_x - M_y$, $\text{div}(P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}) = P_x + Q_y$.

For Green's theorem to hold, $\vec{F}$ must be defined on the *entire* region R enclosed by C.

Example: (same as in pset): $\vec{F} = \dfrac{-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}}{x^2 + y^2}$, C = unit circle counterclockwise, then $\text{curl}(\vec{F}) = \dfrac{\partial}{\partial x}(\dfrac{x}{x^2+y^2}) - \dfrac{\partial}{\partial y}(\dfrac{-y}{x^2+y^2}) = \cdots = 0$. So, if we look at both sides of Green's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = 2\pi \quad \text{(from pset)}, \qquad \iint_R \text{curl}\vec{F}\, dA = \iint_R 0\, dA = 0\,?$$

The problem is that R includes 0, where $\vec{F}$ is not defined.

Definition: a region R in the plane is simply connected if, given any closed curve in R, its interior region is entirely contained in R.

Examples shown.

So: Green's theorem applies safely when the domain in which $\vec{F}$ is defined and differentiable is simply connected: then we automatically know that, if $\vec{F}$ is defined on C, then it's also defined in the region bounded by C.

In the above example, can't apply Green to the unit circle, because the domain of definition of $\vec{F}$ is not simply connected. Still, we can apply Green's theorem to an annulus (picture shown of a curve C' = unit circle counterclockwise + segment along x-axis + small circle around origin clockwise + back to the unit circle allong the x-axis, enclosing an annulus R'). Then Green applies and says $\oint_{C'} \vec{F} \cdot d\vec{r} = \iint_{R'} 0\, dA = 0$; but line integral simplifies to $\oint_{C'} = \int_C - \int_{C_2}$, where C = unit circle, C_2 = small circle / origin; so line integral is actually the same on C and C_2 (or any other curve encircling the origin).

Review for Exam 3.

2 main objects: double integrals and line integrals. Must know how to set up and evaluate.

Double integrals: drawing picture of region, taking slices to set up the iterated integral.

Also in polar coordinates, with $dA = r\, dr\, d\theta$ (see e.g. Problem 2; not done)

Remember: mass, centroid, moment of inertia.

For evaluation, need to know: usual basic integrals (e.g. $\int \frac{dx}{x}$); integration by substitution (e.g. $\displaystyle\int_0^1 \frac{t\, dt}{\sqrt{1+t^2}} = \int_1^2 \frac{du}{2\sqrt{u}}$, setting $u = 1 + t^2$). Don't need to know: complicated trigonometric integrals (e.g. $\int cos^4\theta\, d\theta$), integration by parts.

Change of variables: recall method:

1) Jacobian: $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ c_x & v_y \end{vmatrix}$. Its absolute value gives ratio between $du\, dv$ and $dx\, dy$.

2) express integrand in terms of u, v.

3) set up bounds in uv-coordinates by drawing picture. The actual example on the test will be reasonably simple (constant bounds, or circle in uv-coords).

Line integrals: $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{\boldsymbol{T}}\,ds = \int_C M\,dx + N\,dy$. To evaluate, express both x, y in terms of a single parameter and substitute.

Special case: gradient fields. Recall: $\vec{F}$ is conservative $\Leftrightarrow \int \vec{F}\cdot d\vec{r}$ is path independent $\Leftrightarrow \vec{F}$ is the gradient of some potential $f \Leftrightarrow \operatorname{curl}\vec{F} = 0$ (i.e. $N_x = M_y$).

If this is the case, then we can look for a potential using one of the two methods (antiderivatives, or line integral); and we can then use the FTC to avoid calculating the line integral. (cf. Problem 3).

Flux: $\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds$ $(= \int_C -Q\,dx + P\,dy)$. Geometric interpretation.

Green's theorem (in both forms) (already written at beginning of lecture).

18.02 Lecture 25. – **Fri, Nov 9, 2007**

Handouts: Exam 3 solutions.

Triple integrals: $\displaystyle\iiint_R f\,dV$ (dV = volume element).

Example 1: region between paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$ (picture drawn), e.g. volume of this region: $\displaystyle\iiint_R 1\,dV = \int_?^? \int_?^? \int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx.$

To set up bounds, (1) for fixed (x, y) find bounds for z: here lower limit is $z = x^2+y^2$, upper limit is $z = 4-x^2-y^2$; (2) find the shadow of R onto the xy-plane, i.e. set of values of (x, y) above which region lies. Here: R is widest at intersection of paraboloids, which is in plane $z = 2$; general method: for which (x, y) is z on top surface $>$ z on bottom surface? Answer: when $4 - x^2 - y^2 > x^2 - y^2$, i.e. $x^2 + y^2 < 2$. So we integrate over a disk of radius $\sqrt{2}$ in the xy-plane. By usual method to set up double integrals, we finally get:

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx.$$

Evaluation would be easier if we used polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$: then

$$V = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} dz\,r\,dr\,d\theta.$$

(evaluation easy, not done).

Cylindrical coordinates. (r, θ, z), $x = r\cos\theta$, $y = r\sin\theta$. r measures distance from z-axis, θ measures angle from xz-plane (picture shown).

Cylinder of radius a centered on z-axis is $r = a$ (drawn); $\theta = 0$ is a vertical half-plane (not drawn).

Volume element: in rect. coords., $dV = dx\,dy\,dz$; in cylindrical coords., $dV = r\,dr\,d\theta\,dz$. In both cases this is justified by considering a small box with height Δz and base area ΔA, then volume is $\Delta V = \Delta A\,\Delta z$.

Applications: Mass: $M = \iiint_R \delta\,dV$.

Average value of f over R: $\bar{f} = \dfrac{1}{Vol}\iiint_R f\,dV$; weighted average: $\bar{f} = \dfrac{1}{Mass}\iiint_R f\,\delta\,dV$.

In particular, center of mass: $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x} = \dfrac{1}{Mass}\iiint_R x\,\delta\,dV$.

(Note: can sometimes avoid calculation using symmetry, e.g. in above example $\bar{x} = \bar{y} = 0$).

Moment of inertia around an axis: $I = \iiint_R (\text{distance from axis})^2\,\delta\,dV$.

About z-axis: $I_z = \iiint_R r^2\,\delta\,dV = \iiint_R (x^2+y^2)\,\delta\,dV$. (consistent with I_0 in 2D case)

Similarly, about x and y axes: $I_x = \iiint_R (y^2+z^2)\,\delta\,dV$, $I_y = \iiint_R (x^2+z^2)\,\delta\,dV$
(setting $z=0$, this is consistent with previous definitions of I_x and I_y for plane regions).

Example 2: moment of inertia I_z of solid cone between $z = ar$ and $z = b$ ($\delta = 1$) (picture drawn):

$$I_z = \iiint_R r^2\,dV = \int_0^b \int_0^{2\pi} \int_0^{z/a} r^2\,r\,dr\,d\theta\,dz \quad \left(= \frac{\pi b^5}{10a^4}\right).$$

(I explained how to find bounds in order $dr\,d\theta\,dz$: first we fix z, then slice for given z is the disk bounded by $r = z/a$; the first slice is $z = 0$, the last one is $z = b$).

Example 3: volume of region where $z > 1 - y$ and $x^2 + y^2 + z^2 < 1$? Pictures drawn: in space, slice by yz-plane, and projection to xy-plane.

The bottom surface is the plane $z = 1 - y$, the upper one is the sphere $z = \sqrt{1-x^2-y^2}$. So inner is $\displaystyle\int_{1-y}^{\sqrt{1-x^2-y^2}} dz$. The shadow on the xy-plane = points where $1 - y < \sqrt{1-x^2-y^2}$, i.e. squaring both sides, $(1-y)^2 < 1 - x^2 - y^2$ i.e. $x^2 < 2y - 2y^2$, i.e. $-\sqrt{2y-2y^2} < x < \sqrt{2y-2y^2}$. So we get:

$$\int_0^1 \int_{-\sqrt{2y-2y^2}}^{\sqrt{2y-2y^2}} \int_{1-y}^{\sqrt{1-x^2-y^2}} dz\,dx\,dy.$$

Bounds for y: either by observing that $x^2 < 2y - y^2$ has solutions iff $2y - y^2 > 0$, i.e. $0 < y < 1$, or by looking at picture where clearly leftmost point is on z-axis ($y = 0$) and rightmost point is at $y = 1$.