UNIT - IV

Theory of Games :-

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcomes.

The mathematical analysis of competitive problems is fundamentally based upon the 'minimax (maximin) criterion' of J. Von Neumann. This criterion implies the assumption of rationality from which it is argued that each player will act so as to maximize his minimum gain or minimize his maximum loss.

Game is defined as an activity between two or more persons involving activities by each person according to a set of rules, at the end of which each person receives some benifit or satisfaction or suffers loss.

Characteristics of Game Theory :-

(1) <u>Chance of Strategy</u> :- If in a game, activities are determined by skill, it is said to be a game of strategy, if they are determined by chance, it is a game of chance.

(2) <u>Number of Persons</u> :- A game is called an n-person game if the number of persons playing is n.

The person means an individual or a group aiming at a particular objective.

(3) Number of activities :- These may be finite or infinite.

(4) Number of alternatives available to each person :- It may also be finite or infinite. A finite game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is

(5) Information to the players about the past activities of other players :- is completely available, partly available or not available at all.

(6) Payoff :- A quantitative measure of satisfaction a person gets at the end of each play is called a payoff. It is a real valued function of variables in the game. Let v_i be the payoff to the player P_i, $1 \le i \le n$, in an n-person game. If $\sum_{i=1}^{n} v_i = 0$, then the game is said to be a zero-sum game.

Basic Terminologies :-

Game :- A competitive situation is called as a game if it has following properties:

(i) There are finite number of participants called players.

(ii) Each player has finite number of strategies available to him.

(iii) Every game results in an outcome.

Number of Players:- If a game involves only two players, then it is called a two-person game. However, if the number of players are more than two, the game is known as n-person game.

Sum of gains and losses:- If in a game the gains of one player are exactly the losses to another player, such that sum of gains and losses equal to zero, then the game is said to be a zero-sum game. Otherwise it is said to be non-zero sum game.

Strategy:- The strategy for a player is the list of all possible actions that he will take for every pay off the might arise. It is assumed that the rules governing the choices are known in advance to the players. The outcome resulting from a particular choice is also known to the players in advance and is expressed in terms of numerical values. Here, it is not necessary that players have a definite information about each other strategies.

Optimal strategy:- The particular strategy by which a player optimises his gains or losses without knowing the competitor's strategies is called optimal strategy.

Value of the game:- The expected outcome pay play when players follow their optimal strategies is called the value of the game.

Pure strategy:- It is a decision rule which is always used by the player to select the particular course of action. Thus, each player knows in advance of all the strategies out of which he always selects only one particular strategy, irrespective of the strategy others may choose and the objective of the players is to maximise gains or minimise losses.

Mixed strategy:- When both players are guessing as to which course of action is to be selected on a particular occasion with some fixed probability, it is a mixed strategic game. Thus, there is a probabilistic situation and objective of the players is to maximise expected gains or to minimise expected losses by making a solution among pure strategies with fixed probabilities.

Two person zero sum game:- A game with only two persons is said to be two-person zero-sum game if the gain of one player is equal to the loss of the other.

Pay off matrix:- The pay offs in terms of gains or losses, when players selected their particular ~~start~~ strategies, can be represented in the form of a matrix, called the pay off matrix.

Player B's strategies.

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{array}{c} \text{Player A's strategies} \end{array} \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Games with Saddle Point:-

Minimax and Maximin Principle:- Consider the pay matrix of a game which represents pay off of player A. Now, the objective of the study is to know how these players must elect their respective strategies so that they may optimise their pay off. Such a decision-making criterion is referred to as the minimax-maximin principle.

For player A minimum value in each row represents the least gain (pay off) to him if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that gives largest gain among the row minimum values. This choice of player A is called the maximin principle, and the corresponding gain is called the maximin value of the game denoted by v.

For player B (loser), the maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the pay off matrix by column minima. He will then select the strategy that gives minimum loss among the column maximum values. This choice of player B is called the minimax principle, and the corresponding loss is the minimax value of the game denoted by $\bar{v}$.

Saddle point:- A saddle point of a pay off matrix is that position in the pay off matrix where maximum of row minima coincides with the minimum of the column maxima. The saddle point need not be unique.

Value of the game:- The pay off of the saddle point is called the value of the game denoted by v.

Fair game:- A game is said to be fair if $v=0=\bar{v}$.

Strictly determinable game :- A game is said to be strictly determinable if $\underline{v} = v = \bar{v}$.

Procedure to determine saddle point :- .

1) Select the minimum ~~va~~ element in each row and enclose it in a rectangle. (□)
2) Select the maximum element in each column and enclose it in a circle (o).
3) Find out the element which is enclosed by the rectangle as well as the circle. Such element is the value of the game and that position is called as the saddle point.

Problem :- Solve the game whose payoff matrix is given by

Player B

		I	II	III	IV	V
Player A	I	-2	0	0	5	3
	II	3	2	1	2	2
	III	-4	-3	0	-2	6
	IV	5	3	-4	2	-6

Sol :- Select the row minimum and enclose it in a rectangle. Then, select the column maximum and enclose it in a circle.

Player B.

	I	II	III	IV	V
I	$\boxed{-2}$	0	0	5	3
II	3	2	$\boxed{1}$	2	2
III	$\boxed{-4}$	-3	0	-2	6
IV	5	3	-4	2	$\boxed{-6}$

Player A

It is clear that saddle point is (II, III) and the value of game $v=1$.

Player A uses his course of action II throughout.

Player B uses his course of action III throughout.

Problem:- Find the range of values of p & q which will render the entry (2,2) a saddle point for the game.

Player B.

Player A	2	4	5
	10	7	q
	4	p	6

Sol: First ignoring the values of p & q determine the maximin and minimax values of the pay off matrix.

Player B.

	B_1	B_2	B_3
A_1	$\boxed{2}$	4	5
A_2	10	$\boxed{7}$	q
A_3	$\boxed{4}$	p	6

Maximin value $\underline{v} = 7$ = minimax value.

This imposes the condition on p as $p \leq 7$ and on q as $q \geq 7$. Hence, the range of p and q will be $p \leq 7$, $q \leq 7$.

Games without Saddle Point Mixed Strategies:-

There are some games for which no saddle point exists. In such cases both the players must determine an optimal combination of strategies to find a saddle point. The optimal strategy combination for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called mixed strategies because they are probabilistic combination of available choices of strategy.

The value of game obtained by the use of mixed strategies represents least pay off which player A can expect to win and the least which player B can lose. The expected pay off to a player in a game with arbitrary pay off matrix $[a_{ij}]$ of order $m \times n$ is defined as

$$E(P, Q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$$

$$= P^T A Q.$$

where, $P = (p_1, p_2, \cdots, p_m)$ and $Q = (q_1, \cdots, q_n)$ denotes the mixed strategies for players A & B. Also, $p_1 + p_2 + \cdots + p_m = 1$ and $q_1 + q_2 + \cdots + q_n = 1$.

A particular strategy with particular probability a player chooses can also be interpreted as the relative frequency with which a strategy is chosen from the number of strategies of the game.

A mixed strategy game can be solved by different solution methods such as

(1) Algebraic method.

(2) Analytical or calculus method.

(3) matrix method.

(4) Graphical method

5. Linear programming method.

Dominance Property :- Some times reduce the size of a game's pay off matrix by eliminating a course of action which is so inferior to another as never to be used. Such a course of action is said to be dominated by the other. The concept of dominance is especially useful for the evaluation of two-person zero-sum games where a saddle point does not exist.

General Rule :-

1. If all the elements of a row, say k^{th}, are less than or equal to the corresponding elements of any other row, say r^{th}, then k^{th} row is dominated by the r^{th} row; ~~then kth row~~ is
2. If all the elements of a column, say k^{th} are greater than or equal to the corresponding elements of any other column, say r^{th}, then k^{th} column is dominated by r^{th} column.
3. Omit dominated rows or columns.
4. If some linear combination of some rows dominates i^{th} row, then i^{th} row will be deleted. Similar argument follows for columns.

Problem :- Reduce the size of the game whose matrix is given by:

Player B.

	I	II	III
I	-4	6	3
II	-3	-3	4
III	2	-3	4

Sol:-

	I	II	III
I	$\boxed{-4}$	6	3
II	$\boxed{-3}$	$\boxed{-3}$	4
III	2	$\boxed{-3}$	4

No saddle point exist. Consider Ist & IIIrd columns from the player B's point of view. Observe that each pay off in the IIIrd column is greater than the corresponding element in the 1st column regardless of player A's strategy. Evidently, the choice of IIIrd strategy by the player B will always result in the greater loss compared to that of selecting the ~~I~~ Ist strategy. Column III is inferior to I and is never to be used. Hence, deleting the IIIrd column which is dominated by I, the reduced size pay off matrix is:

$$
\begin{array}{cc|cc}
 & & \multicolumn{2}{c}{B} \\
 & & \mathrm{I} & \mathrm{II} \\
 & \mathrm{I} & -4 & 6 \\
A & \mathrm{II} & -3 & -3 \\
 & \mathrm{III} & 2 & -3
\end{array}
$$

Again, if the reduced matrix is looked at from player A's point of view, it is seen that the player A will never use the IInd strategy which is dominated by III. Hence, the size of the matrix can be reduced further by deleting the IInd row. Hence, the reduced matrix is

$$
\begin{array}{cc|cc}
 & & \multicolumn{2}{c}{B} \\
 & & \mathrm{I} & \mathrm{II} \\
A & \mathrm{I} & -4 & 6 \\
 & \mathrm{II} & 2 & -3
\end{array}
$$

Graphical Method (for $2 \times n$ or $m \times 2$ games):-

The graphical method is useful for the game where the pay off matrix is of the size $2 \times n$ or $m \times 2$. That is, the game with mixed strategies that has only two pure strategies for one of the players in the two person zero-sum game. optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. Hence, this method is useful in finding out which of the two strategies can be used.

Consider the $2 \times n$ pay off matrix of a game without a saddle point.

Player B.

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \end{array}$$

Let the mixed strategy for player A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$, where $p_1 + p_2 = 1$ and $p_1 \geq 0$, $p_2 \geq 0$.

Now, for each of the pure strategies available to B, expected pay off for player A would be as follows:

B's pure move	A's expected pay off $E(P)$
B_1	$E_1(P) = a_{11}p_1 + a_{21}p_2$
B_2	$E_2(P) = a_{12}p_1 + a_{22}p_2$
$\vdots$	$\vdots$
B_n.	$E_n(P) = a_{1n}p_1 + a_{2n}p_2$

The player B would like to choose that pure move B_j against S_A for which $E_j(P)$ is a minimum for $j = 1, \ldots, n$. Let us denote this minimum expected payoff for A by

$$v = \min\{E_j(P),\ j = 1, \ldots, n\}.$$

The objective of player A is select p_1 and hence p_2 in such a way that v is as large as possible. This may be done by the plotting straight lines.

$$E_j(P) = a_{1j}p_1 + a_{2j}p_2 = (a_{1j} - a_{2j})p_1 + a_{2j} \quad (j = 1, 2, \ldots, n)$$

as linear function of p_1.

The highest point on the lower boundary of these lines will give maximum expected pay off among the minimum expected pay offs on the lower boundary and the optimum value of probability p_1 and p_2.

Now, the two strategies of player B corresponding to those lines which pass through the maximum point can be determined.

The (mx2) games are also treated in the same way except that the upper boundary of the straight lines corresponding to B's expected pay off will give the maximum expected pay off (minimax value) and the optimum value of probability q_1 and q_2.

Problem:- Obtain the optimal strategies for both persons and the value of the game for zero-sum two-person game whose pay off matrix is as follows:

Player B

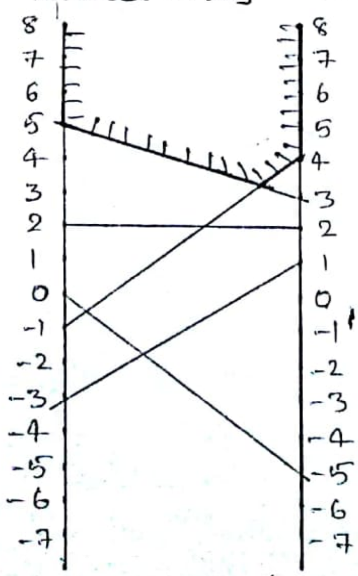
$$\text{Player A}\quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array}\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix} \end{array}$$

(9)

Sol:- Observe that the given problem does not possess any saddle point. So, let the player B play the mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_2 = 1 - q_1$ against player A. Then, B's expected pay offs against A's pure moves are given by:

A's pure move	B's expected pay off $E(q_1)$
A_1	$q_1 - 3q_2$
A_2	$3q_1 + 5q_2$
A_3	$-q_1 + 6q_2$
A_4	$4q_1 + q_2$
A_5	$2q_1 + 2q_2$
A_6	$-5q_1 + 0q_2$

The expected pay off equations are then plotted as functions of q_1 in the graph:



Since the player B wishes to minimise his maximum expected pay off, we consider the minimax point on the upper envelope of B's, expected payoff equations. Hence, the given pay off matrix of the game is reduced to

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player A} \begin{array}{c} A_1 \\ A_4 \end{array} & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

Let $S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & p_1 & 0 & p_2 \end{bmatrix}$ be the optimal strategy of player A.

Then, $$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{(1-4)}{(3+1)-(4+5)} = \frac{3}{5}$$

Hence, $p_2 = 1 - p_1 = 1 - \frac{3}{5} = \frac{2}{5}$.

Let $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ be the optimal strategy of player B.

Then $$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{(1-5)}{(3+1)-(4+5)} = \frac{4}{5}.$$

So, $q_2 = 1 - q_1 = 1 - \frac{4}{5} = \frac{1}{5}$

value of the game,

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$$= \frac{(3\times1) - (4\times5)}{(3+1) - (4+5)} = \frac{17}{5} \; //$$