Muon Chemistry and Kinetics in Deuterium

When a negative muon enters pure deuterium gas, it loses its kinetic energy in subsequent collisions with the molecules of the gas. The initial kinetic energy of the muon beam ranges from 5 - 20 MeV such that it is enough to penetrate the target and slow down to attain energies of the range of 10-20 eV (this is well within the range of the ionization potential of deuterium). This energy range is suitable for the muon to undergo atomic capture. Thus, it stops in the gas to dissociate a D_2 molecule (with binding energy of 15.5 eV) into two atoms and gets captured atomically with one atom in a high atomic orbital (in the range of 15 -20) [55] [56]. In the procedure of capture it tends to lose its polarization due to spin flip in arbitrary directions [55]. This occurs as the it cascades to the ground 1s state (n=1) and forms a muonic deuterium (μd) atom. Finally about 20% of the polarization is retained for the negative muon beam and thus it can be considered to be in an almost depolarized state [61], [57].
This resulting $d\mu$ atom has two-hyperfine spin states and is in a bound state for about a microsecond [49]. After this it undergoes any one of the following processes:

1. The muon decays ejecting a decay (or Michel) electron via 1.1 process.
2. The muon gets captured by the deuterium to a muonic deuterium atom μd in two-hyperfine states i.e. a doublet or a quartet state.
3. A nuclear capture occurs on the deuteron from either of the two-hyperfine states via 1.3 reaction.
4. The muon catalyzes a fusion by forming $d\mu d$ molecules that undergo fusion either by forming helium along with fusion neutrons or by forming a triton and a proton. The reactions for these are in Sec. 2.2.2.

These processes will be discussed in detail in the subsections below. This entire muon chemistry is depicted in Fig. 2.2.

2.2.1 Hyperfine States

After the deuteron atomically captures the muon it forms two-spin states with the deuteron, namely

1. muon spin aligned with deuteron which gives rise to a quartet state with a total spin 3/2 i.e. $d\mu(\uparrow\uparrow)$
2. muon spin antialigned with deuteron which gives rise to a doublet state with a total spin 1/2 i.e. $d\mu(\uparrow\downarrow)$

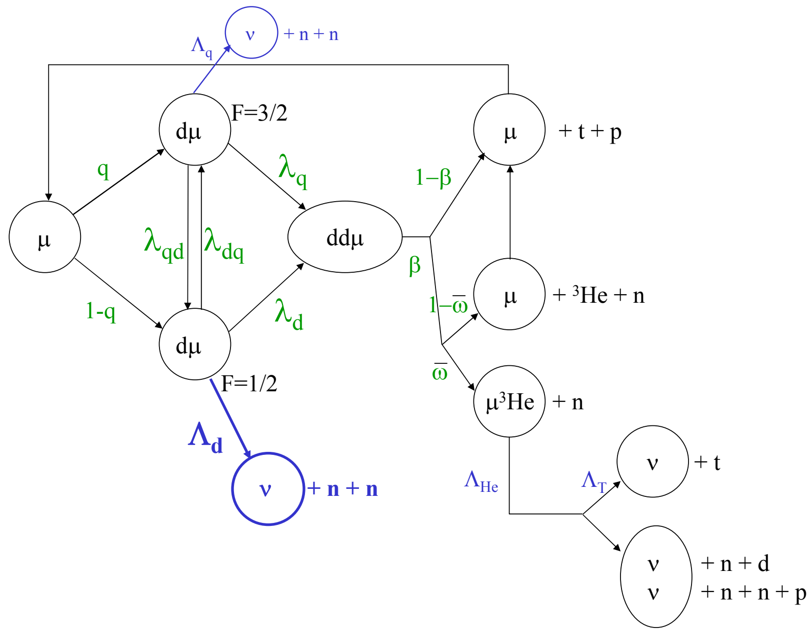

Figure 2.2: Muon Kinetics in deuterium. The branching ratio of muon sticking to He (called sticking probability) is 0.1206 which is negligible and so these side branches can be ignored. Image credit: [1].

The energy difference between these two states is 0.0485 eV. The ratio of the initial population of the hyperfine states of the $d\mu$ atom is proportional to the ratio of the number of spins states in each hyperfine state. The quartet state has four spin states namely m_s = -3/2, -1/2, 1/2, 3/2 whereas the doublet state has two-spin states i.e. m_s = -1/2, 1/2. So the initial population of the quartet to doublet hyperfine states is in the ratio of 2 : 1.

The V-A nature of weak interaction (vector - pseudovector) favours the capture of the the $d\mu$ atom from its doublet state. The capture rate from quartet state is 12 s^{-1} (denoted by Λ_q) as against the rate from doublet state that is 386 s^{-1} (denoted by Λ_d). These numbers are theoretically calculated and are taken from [1]. Thus, one major goal of this experiment was to increase the doublet population as much as possible. This is very well accomplished at a density of 5% liquid hydrogen and a temperature of 30 K [50]. In Fig. 2.2 the initial fraction of quartet state is shown by q = 2/3, and so the fraction of doublet state is given by 1-q = 1/3. Thus, the incoming muon initially gets captured forming a muonic deuterium in either of the two states. The population of muonic deuterium in the quartet state decays to muonic deuterium in the doublet state. This hyperfine transition rate is denoted by λ_{qd} in Fig. 2.2. It varies by a small amount from 35 - 40 μs^{-1} over a wide range of temperature from 30 to 300 K [77]. Its variation with temperature is shown in Fig. 2.3. It is more sensitive states of the the μd atom can be found by solving the sets of coupled differential equations given below:

$$\begin{aligned}
\frac{dn_{1/2}}{dt} &= -(\lambda_\mu + \phi\lambda_d)n_{1/2} + \phi\lambda_{qd}n_{3/2} + \frac{1}{3}\lambda_f(1-s)\,n_{dd} \\
\frac{dn_{3/2}}{dt} &= -(\lambda_\mu + \phi\lambda_{qd} + \phi\lambda_q)n_{3/2} + \frac{2}{3}\lambda_f(1-s)\,n_{dd} \\
\frac{dn_{dd}}{dt} &= \phi\lambda_d n_{1/2} + \phi\lambda_q n_{3/2} - (\lambda_\mu + \lambda_f)n_{dd}
\end{aligned} \tag{2.6}$$

where s is half the sticking probability of He i.e. 0.0603 (effective sticking probability of He), λ_f is the fusion rate, and ϕ is the density of the target.

2.3 Analytical Solution for Population of States

The analytical solutions of the population of μd atoms in quartet and doublet states are necessary to extract useful kinetic parameters, that would in turn interpret and compare the experimentally obtained fit results. The populations of the doublet ($n_{1/2}$) and quartet ($n_{3/2}$) states of the the μd atom can be found by solving the sets of coupled differential Eqs.(2.6). The symbols and their values are listed in table 2.2. These sets of coupled differential equations have been solved in here with details in the appendix 8.3.

We first study the effect of switching off recycling of muons in the process. Next we include recycling to understand its effect on the fusion time distribution.

Without muon recycling:

To begin with we deal with a simple case of no muon recycling and see how the system behaves by plotting the solutions for doublet and quartet states. If the sticking probability of He is 1 then, the muon would always stick to the He atom which would eliminate the possibility of muon recycling. Thus, the last term in Eq.(2.6) becomes zero as s = 1. Also the third set of equation is zero as the number of $d\mu d$ molecules formed is negligibly small. Thus Eq.(2.6) now reduces to

$$\begin{aligned}
\frac{dn_{3/2}}{dt} &= -(\lambda_\mu + \phi\lambda_{qd} + \phi\lambda_q)n_{3/2} \\
\frac{dn_{1/2}}{dt} &= -(\lambda_\mu + \phi\lambda_d)n_{1/2} + \phi\lambda_{qd}n_{3/2}
\end{aligned} \tag{2.7}$$

The eigenstate matrix of the above system of equations can be expressed as,

$$\begin{pmatrix} -\phi\lambda_q - \phi\lambda_{qd} - \lambda_\mu & 0 \\ \phi\lambda_{qd} & -\phi\lambda_d - \lambda_\mu \end{pmatrix}$$

The eigenvalues which represent the rate of each state λ_1 and λ_2 respectively are

$$\lambda_1 = \phi\lambda_d + \lambda_\mu \tag{2.8}$$

$$\lambda_2 = \phi\lambda_q + \phi\lambda_{qd} + \lambda_\mu \tag{2.9}$$

And the eigenvectors are, $\left\{\{0,1\},\left\{-\frac{-\lambda_d+\lambda_q+\lambda_{\mathrm{qd}}}{\lambda_{\mathrm{qd}}},1\right\}\right\}$

For simplicity we used the following substitution,

$$Y = -\frac{-\lambda_d + \lambda_q + \lambda_{\mathrm{qd}}}{\lambda_{\mathrm{qd}}} \tag{2.10}$$

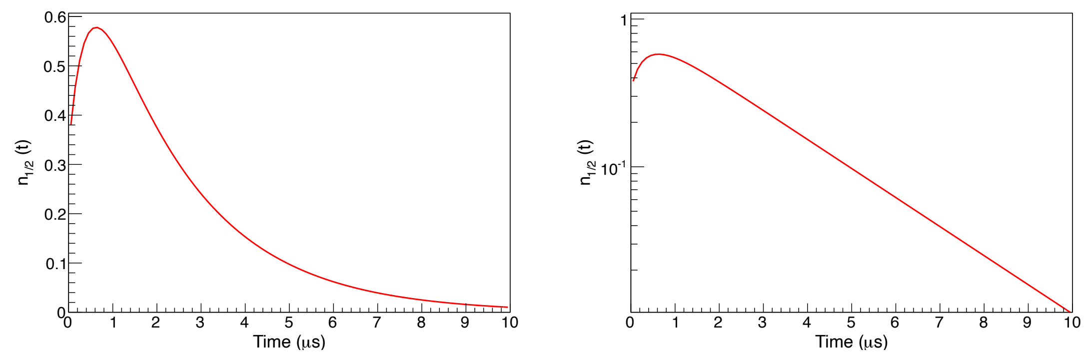

Figure 2.6: Plots showing the doublet state populations in linear and log scale with no recycling. This shows the initial population $n_{1/2}\ (0) = \frac{1}{3}$.

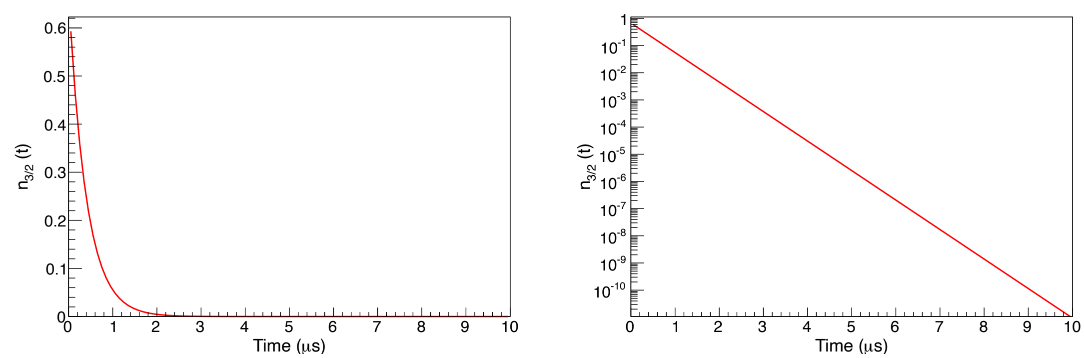

Figure 2.7: Plots showing the quartet state populations in linear and log scale with no recycling. This shows the initial population $n_{3/2}\ (0) = \frac{2}{3}$.

The initial population is proportional to the total spin states of each state and so $n_{1/2}$ and $n_{3/2}$ is in the ratio of $1/3$: $2/3$ initially. The final solution of Eq.(2.7) depicting the population of each state is given by, (refer appendix 8.3 for detailed steps)

$$n_{1/2} = \frac{(Y-2)}{3Y}e^{-\lambda_1 t} + \frac{2}{3Y}e^{-\lambda_2 t} \tag{2.11}$$

$$n_{3/2} = \frac{2}{3}e^{-\lambda_2 t} \tag{2.12}$$

The fusion distribution in general is the linear combination of the population of two states $n_{1/2}$ and $n_{3/2}$ and is thus given by,

$$n(t) = \lambda_d \beta_d n_{1/2} + \lambda_q \beta_q n_{3/2} \tag{2.13}$$

where β_d and β_q are the probabilities of the formation of fusion neutrons and He from the doublet state and quartet state respectively. λ_d and λ_q are the dμd molecular formation rates from the doublet state and quartet state of the μd atom respectively.

The experimentally obtained fusion neutron time distribution should thus be fitted with a two lifetime fit function. This can be compared with the above Eq.(2.13) and is given by a theoretical fit function as,

$$n(t) = A_1 e^{-\lambda_1 t} + A_2 e^{-\lambda_2 t} \tag{2.14}$$

A_1 and A_2 are the amplitudes corresponding to the two rates.
Simplifying Eq.(2.13) we get the ratio of the amplitudes as,

$$\frac{A_1}{A_2} = \frac{\lambda_d \beta_d (2 - Y)}{2(\lambda_d \beta_d + Y \lambda_q \beta_q)} \tag{2.15}$$

From Eq.(2.12) it is clear that the quartet state shows no recycling term and has just one rate. The ratio of the two amplitudes of the states is found to be, $A_1 : A_2 \approx 53 : 1$ using values of the constants and rates from table 2.2 [1]. It is also known that $\beta_d \approx \beta_q$ [83].

Including recycling of muons:

It is known that fusion of dμd molecule takes place almost instantly in 1 ns time [85] and so the population of dμd molecules attain equilibrium very fast. In equilibrium the formation of dμd molecules is equal to its disappearance. Hence the rate dn_{dd}/dt is negligibly small and can be taken to be zero for practical purposes. Substituting $dn_{dd}/dt = 0$ in Eq.(2.6) we get,

$$\begin{aligned} \phi \lambda_d n_{1/2} + \phi \lambda_q n_{3/2} &= (\lambda_\mu + \lambda_f) n_{dd} \\ n_{dd} &= \frac{\phi \lambda_d n_{1/2} + \phi \lambda_q n_{3/2}}{(\lambda_\mu + \lambda_f)} \end{aligned} \tag{2.16}$$

Substituting the expression of n_{dd} in Eq.(2.6) and simplifying we get the eigenstate matrix of this set of differential equations as,

$$\begin{pmatrix} -\phi\lambda_d - \lambda_\mu + \frac{(1-s)\phi\lambda_d\lambda_f}{3(\lambda_f+\lambda_\mu)} & \phi\lambda_{\mathrm{qd}} + \frac{(1-\ s)\phi\lambda_f\lambda_q}{3(\lambda_f+\lambda_\mu)} \\ +\frac{2(1-s)\phi\lambda_d\lambda_f}{3(\lambda_f+\lambda_\mu)} & -\phi\lambda_q - \phi\lambda_{\mathrm{qd}} - \lambda_\mu + \frac{2(1-s)\phi\lambda_f\lambda_q}{3(\lambda_f+\lambda_\mu)} \end{pmatrix}$$

In the term $\frac{(1-s)\phi\lambda_d\lambda_f}{3(\lambda_f+\lambda_\mu)}$ λ_d is very small for our experimental conditions 2.2 and $\lambda_f \gg \lambda_\mu$ and so we ignore this term. Thus the matrix reduces to

$$\begin{pmatrix} -\phi\lambda_d - \lambda_\mu + \frac{1}{3}(1-s)\phi\lambda_d & \phi\lambda_{\text{qd}} + \frac{1}{3}(1-s)\phi\lambda_q \\ \frac{2}{3}(1-s)\phi\lambda_d & -\phi\lambda_q + \frac{2}{3}(1-s)\phi\lambda_q - \phi\lambda_{\text{qd}} - \lambda_\mu \end{pmatrix}$$

The eigenvalues of the matrix give us the two lifetimes. If $\phi\lambda_d$ is ignored compared to the other terms we get approximate expression for the short lifetime as,

$$\lambda_1 \approx \frac{1}{3}(1+2s)\phi\lambda_q + \phi\lambda_{\text{qd}} + \lambda_\mu \tag{2.17}$$

Similarly the long lifetime is found to be

$$\lambda_2 \approx \lambda_\mu + \frac{1}{3}(2+s)\phi\lambda_d \tag{2.18}$$

Parameter	**Value at 30 K**	**Reference**
Fusion rate λ_f	1000 μs^{-1}	[85]
dμd formation rate from doublet state λ_d	0.053(3) μs^{-1}	[86]
dμd formation rate from quartet state λ_q	3.98(5) μs^{-1}	[86]
Density of the target ϕ	0.05 of LH$_2$	[86]
Hyperfine transition rate from quartet to doublet λ_{qd}	37.0(4) μs^{-1}	[86]
Muon decay rate λ_μ	0.455170(45) μs^{-1}	[7]

Table 2.2: The table shows the values of physical parameters used in the calculation of the muon chemistry kinetic parameters.

Thus, the populations of each of these states in terms of our defined symbols is given by the equations

$$n_{1/2} = \frac{X_1(2X_2-1)}{3(X_2-X_1)}e^{-\lambda_1 t} + \frac{X_2(1-2X_1)}{3(X_2-X_1)}e^{-\lambda_2 t} \tag{2.19}$$

$$n_{3/2} = \frac{(2X_2-1)}{3(X_2-X_1)}e^{-\lambda_1 t} + \frac{(1-2X_1)}{3(X_2-X_1)}e^{-\lambda_2 t} \tag{2.20}$$

where,

$$X_1 = \frac{\phi\lambda_q + \phi\lambda_{qd} - \frac{2}{3}(1-s)\phi\lambda_q - X}{\frac{4}{3}(1-s)\phi\lambda_d} \tag{2.21}$$

$$X_2 = \frac{\phi\lambda_q + \phi\lambda_{qd} - \frac{2}{3}(1-s)\phi\lambda_q + X}{\frac{4}{3}(1-s)\phi\lambda_d} \tag{2.22}$$

and

$$X = \sqrt{(\phi\lambda_q + \phi\lambda_{qd})^2 + \frac{4}{3}(1-s)\phi^2\lambda_q\left(1+\lambda_{qd}\right)} \tag{2.23}$$

where we again assumed $\lambda_{qd} \gg \lambda_d$ and thus ignored all terms involving λ_d compared to λ_{qd} in the numerator.

The time distribution for the ratio of the quartet population to that of the total population $\frac{n_{3/2}(t)}{n_{1/2}(t) + n_{3/2}(t)}$ is shown in the Fig. 2.8 in dotted blue line. The solid red line shows time distribution of the quartet population. Again from Eqs. 2.13 and

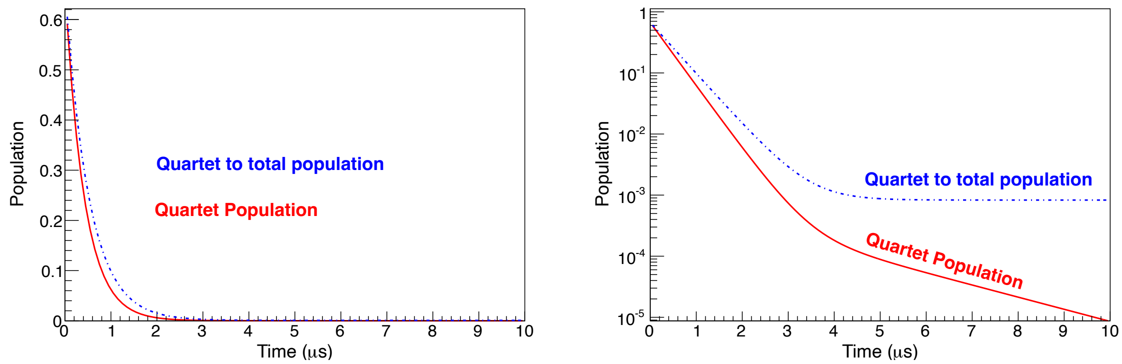

Figure 2.8: Plots showing the time distribution of the quartet state $n_{3/2}(t)$ (in solid red) and the quartet state to total population $\frac{n_{3/2}(t)}{n_{1/2}(t) + n_{3/2}(t)}$ (in dashed blue) in linear and log scale.

2.14, we get the ratio of the amplitudes as,

$$\frac{A_1}{A_2} = \frac{(\lambda_d X_1 + \lambda_q)(2X_2 - 1)}{(\lambda_d X_2 + \lambda_q)(1 - 2X_1)} \tag{2.24}$$

Substituting in the values of the parameters from table 2.2 and solving for X_1, X_2, we evaluate the ratio of $A_1 : A_2$ from Eq.(2.24) to be 46.56. This is slightly smaller than the ratio obtained with no recycling (≈ 53). In the subsection below we study the effect of changing the input parameters used in finding the amplitude ratio $A_1 : A_2$.

2.3.1 Effect of Changing Various Parameters on $A_1 : A_2$

A plot of the amplitude ratio vs. λ_{qd} is shown in the top plot of Fig. 2.9. This reveals that there is a small dependence of this ratio with λ_{qd}.

This can be accounted due to the fact that at the cryogenic temperatures of MuSun, almost all μd atoms in the quartet state quickly transit to the doublet state (as discussed in the muon chemistry section of experimental design), irrespective of the value of λ_{qd}.

A plot of the amplitude ratio vs. λ_d is shown in the middle plot of Fig. 2.9. The population of μd atoms in the quartet state would increase as λ_d increases. Thus, the amplitude ratio decreases. But the effect is very prominent in this case. A slight rise in λ_d decreases the value of the amplitude ratio. This could be attributed due to the fact that the moculelar formation occurs predominantly from quartet stste.

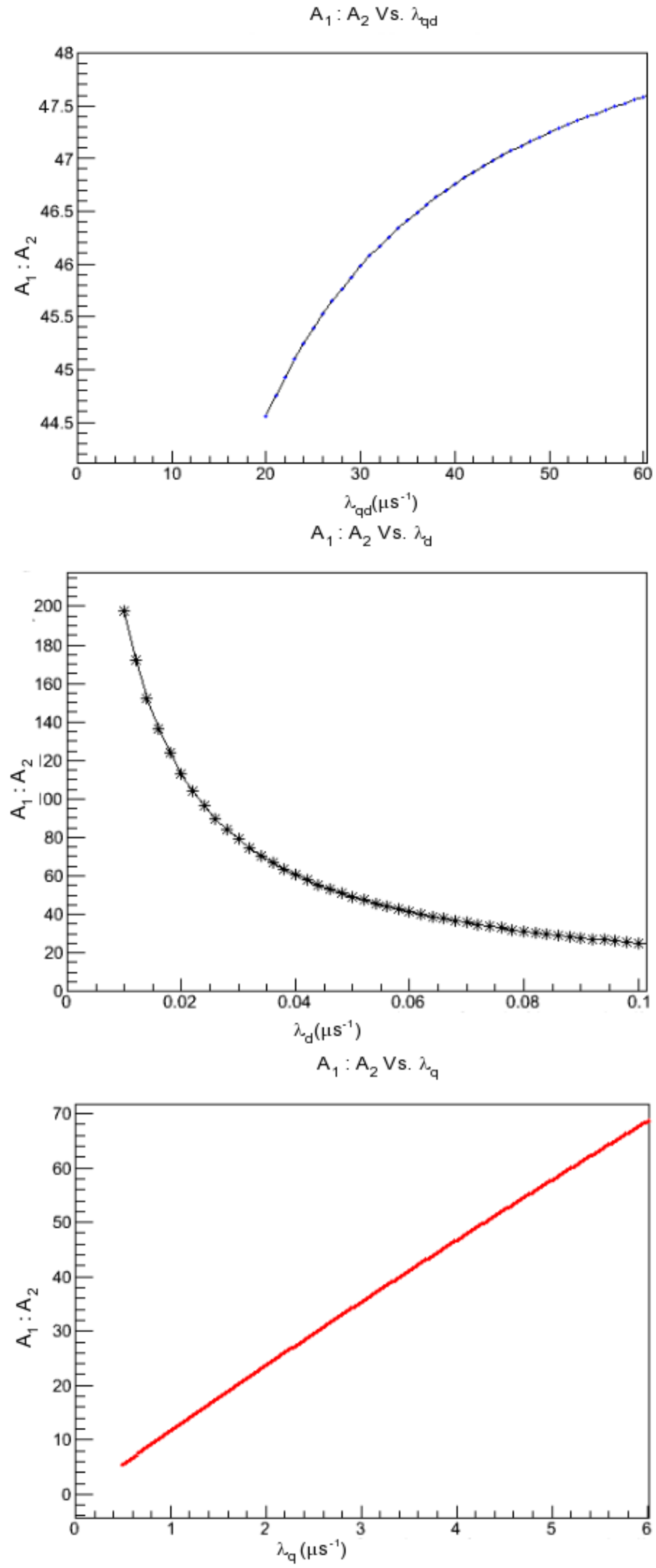

Figure 2.9: Plots showing the variation of the effect of changes in rates λ_{qd}, λ_d and λ_q with the amplitude ratio $A_1 : A_2$

A.2 Derivation of the Analytical solutions of population of states

For simplicity we assumed the coefficients of $n_{1/2}$ and $n_{3/2}$ in the above set of Eqs. (2.6) as a, b, c and d. Thus the problem now reduces down to solving the eigen values and eigen vectors of the matrix representation of the equation in terms of a, b, c and d as shown below

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The eigenvalues of this matrix is given by,

$$\left\{\frac{1}{2}\left(a+d-\sqrt{a^2+4bc-2ad+d^2}\right),\frac{1}{2}\left(a+d+\sqrt{a^2+4bc-2ad+d^2}\right)\right\} \quad (3)$$

The eigen vectors of this matrix in general are given by

$$\left\{-\frac{-a+d+\sqrt{a^2+4bc-2ad+d^2}}{2c},1\right\},\left\{-\frac{-a+d-\sqrt{a^2+4bc-2ad+d^2}}{2c},1\right\} \quad (4)$$

The above general equations will be used throughout to find eigenvalues and eigenvectors for different cases.

$$a = -\phi\lambda_{qd} - \phi\lambda_q - \lambda_\mu$$

$$b = 0$$

$$c = \phi\lambda_{qd}$$

$$d = -\phi\lambda_d - \lambda_\mu$$

The general solution to the set of coupled differential equation represented by Eq.(2.7) is given by,

$$\begin{pmatrix} n_{3/2} \\ n_{1/2} \end{pmatrix} = a_1 e^{\lambda_1 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + a_2 e^{\lambda_2 t}\begin{pmatrix} Y \\ 1 \end{pmatrix}$$

Next we plug in the initial conditions to find a_1 and a_2. Initially at t = 0, the population of doublet and quartet states is proportional to 1/3 and 2/3 respectively,

owing to their relative possible spin states, as explained previously. This gives us,
$\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} Ya_2 \\ a_1 + a_2 \end{pmatrix}$

From the above we can readily see that,

$a_1 = \frac{(Y-2)}{3Y}$ and $a_2 = \frac{2}{3Y}$

Including recycling: We find that the eigenstate matrix is given by,
$\begin{pmatrix} -\phi\lambda_d - \lambda_\mu + \frac{1}{3}(1-s)\phi\lambda_d & \phi\lambda_{\text{qd}} + \frac{1}{3}(1-s)\phi\lambda_q \\ \frac{2}{3}(1-s)\phi\lambda_d & -\phi\lambda_q + \frac{2}{3}(1-s)\phi\lambda_q - \phi\lambda_{\text{qd}} - \lambda_\mu \end{pmatrix}$

Again for simplicity the matrix elements were replaced by a, b c, and d given by the following equations:

$$a = -\phi\lambda_d - \lambda_\mu + \frac{1}{3}(1-s)\phi\lambda_d \tag{5}$$

$$b = \phi\lambda_{\text{qd}} + \frac{1}{3}(1-s)\phi\lambda_q \tag{6}$$

$$c = \frac{2}{3}(1-s)\phi\lambda_d \tag{7}$$

$$d = -\phi\lambda_q + \frac{2}{3}(1-s)\phi\lambda_q - \phi\lambda_{\text{qd}} - \lambda_\mu \tag{8}$$

Thus, from Eq.(3) we have,

$$\lambda_1 = \frac{1}{2}\left(a + d + \sqrt{a^2 + 4bc - 2ad + d^2}\right) \tag{9}$$

$$\lambda_2 = \frac{1}{2}\left(a + d - \sqrt{a^2 + 4bc - 2ad + d^2}\right) \tag{10}$$

Here again we assumed $\phi\lambda_q$ can be ignored compared to $\phi\lambda_{\text{qd}}$ and λ_μ which sets c = 0 and so we have,

$$\lambda_1 = d \approx -\frac{1}{3}(1+2s)\phi\lambda_q - \phi\lambda_{\text{qd}} - \lambda_\mu \tag{11}$$

Similarly the slow lifetime is found to be

$$\lambda_2 = a \approx -\lambda_\mu - \frac{1}{3}(2+s)\phi\lambda_d \tag{12}$$

The complete solution (without any approximation) gives,

$$\lambda_1 = \frac{-1}{6}\left((2+s)\phi\lambda_d + \phi\left((1+2s)\lambda_q + 3\lambda_{qd}\right) + 6\lambda_\mu + \right.$$
$$\sqrt{\left(\phi^2\left((2+s)^2\lambda_d^2 + \left(\lambda_q + 2s\lambda_q + 3\lambda_{qd}\right)^2 + 2\lambda_d\left((2+s(-13+2s))\lambda_q + 3(2-5s)\lambda_{qd}\right)\right)\right)} \tag{13}$$

Similarly the second eigen value of the matrix is found to be

$$\lambda_2 = \frac{-1}{6}\left((2+s)\phi\lambda_d + \phi\left((1+2s)\lambda_q + 3\lambda_{qd}\right) + 6\lambda_\mu - \right.$$
$$\sqrt{\left(\phi^2\left((2+s)^2\lambda_d^2 + \left(\lambda_q + 2s\lambda_q + 3\lambda_{qd}\right)^2 + 2\lambda_d\left((2+s(-13+2s))\lambda_q + 3(2-5s)\lambda_{qd}\right)\right)\right)} \tag{14}$$

Plugging in the values of a, b, c and d from the above equations without ignoring anything the approximate results for the eigen vectors in matrix form are as follows.
$\begin{pmatrix} X_1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} X_2 \\ 1 \end{pmatrix}$
where X_1 and X_2 are the first elements of the two eigen vectors given by the equations below:

$$\begin{aligned} X_1 &= \frac{a-d-X}{2c} \\ &= \frac{\phi\lambda_q + \phi\lambda_{qd} - \frac{2}{3}(1-s)\phi\lambda_q - X}{\frac{4}{3}(1-s)\phi\lambda_d} \end{aligned} \tag{15}$$

$$\begin{aligned} X_2 &= \frac{a-d+X}{2c} \\ &= \frac{\phi\lambda_q + \phi\lambda_{qd} - \frac{2}{3}(1-s)\phi\lambda_q + X}{\frac{4}{3}(1-s)\phi\lambda_d} \end{aligned} \tag{16}$$

where we again assumed $\lambda_{qd} \gg \lambda_d$ and thus ignored all terms involving λ_d compared to λ_{qd} in the numerator. X is a variable designated for the long expression shown below

$$\begin{aligned} X &= \sqrt{(a-d)^2 + 4bc} \\ &= \sqrt{(\phi\lambda_q + \phi\lambda_{qd})^2 - \frac{4}{3}(1-s)\phi^2\lambda_q\left(1+\lambda_{qd}\right)} \end{aligned} \tag{17}$$

The symbols X_1, X_2 and X have been used just for convenience and simplicity and have no other significance

Initial conditions:

The the solution to the set of coupled differential equation represented by Eq.(2.6) is given by,

$$\begin{pmatrix} n_{1/2} \\ n_{3/2} \end{pmatrix} = a_1 e^{-\lambda_1 t}\begin{pmatrix} X_1 \\ 1 \end{pmatrix} + a_2 e^{-\lambda_2 t}\begin{pmatrix} X_2 \\ 1 \end{pmatrix}$$

where a_1 and a_2 are the amplitudes of each state determined from the initial conditions. Initially at t = 0, the population of doublet and quartet states is proportional to 1/3 and 2/3 respectively, owing to their relative possible spin states, as explained previously.

Thus, for t = 0 we have,

$$\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} a_1 X_1 \\ a_1 \end{pmatrix} + \begin{pmatrix} a_2 X_2 \\ a_2 \end{pmatrix}$$

From the above we can readily see that,

$a_1 = \frac{(2X_2-1)}{3(X_2-X_1)}$ and $a_2 = \frac{(1-2X_1)}{3(X_2-X_1)}$

which gives, $\begin{pmatrix} n_{1/2} \\ n_{3/2} \end{pmatrix} = \frac{(2X_2-1)}{3(X_2-X_1)} e^{-\lambda_1 t} \begin{pmatrix} X_1 \\ 1 \end{pmatrix} + \frac{(1-2X_1)}{3(X_2-X_1)} e^{-\lambda_2 t} \begin{pmatrix} X_2 \\ 1 \end{pmatrix}$

Substituting the values of λ_1 and λ_2 from Eqs. 2.17 and 2.18 respectively and then solving the above matrix equation we get the simplified matrix equation as,

$$\begin{pmatrix} n_{1/2} \\ n_{3/2} \end{pmatrix} = \begin{pmatrix} \frac{X_1(2X_2-1)}{3(X_2-X_1)} e^{\lambda_1 t} + \frac{X_2(1-2X_1)}{3(X_2-X_1)} e^{\lambda_2 t} \\ \frac{(2X_2-1)}{3(X_2-X_1)} e^{\lambda_1 t} + \frac{(1-2X_1)}{3(X_2-X_1)} e^{\lambda_2 t} \end{pmatrix}$$

The above matrix gives the complete solution for the population of states from where we find the amplitudes A_1 and A_2 for the fusion neutron time spectrum.