19. CONVOLUTION

Topics:

Objectives:

19.1 INTRODUCTION

• Can be used to find normal input reponses for linear systems

• It is most useful for finding an output response for a system given an arbitrary input function

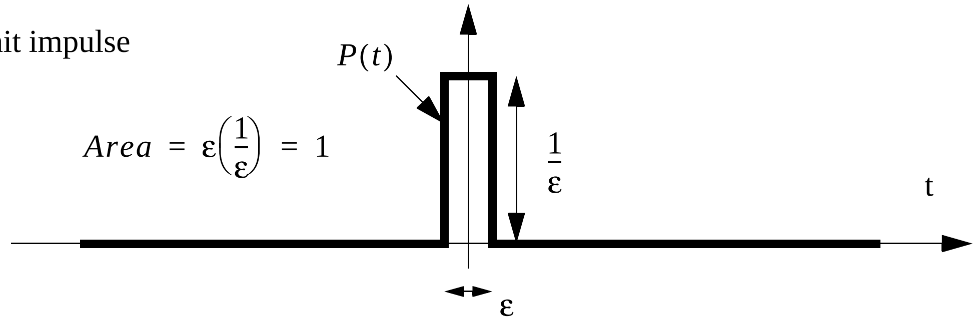
• It is also the basis for other methods that come later for system analysis.

19.2 UNIT IMPULSE FUNCTIONS

• A unit impulse function

Unit impulse

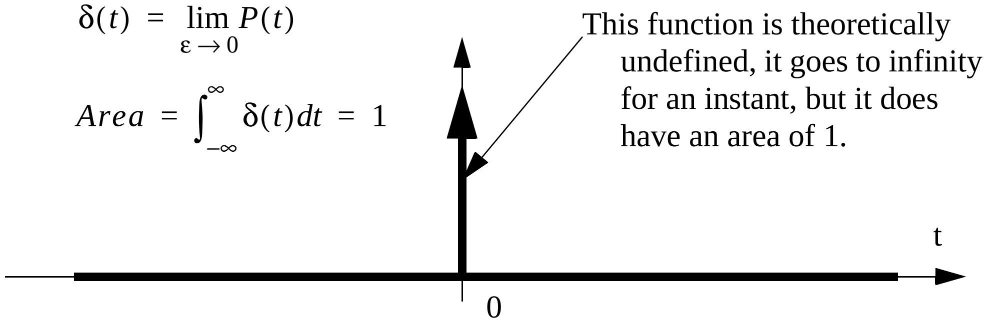
$$Area = \varepsilon\left(\frac{1}{\varepsilon}\right) = 1$$



Dirac delta function

$$\delta(t) = \lim_{\varepsilon \to 0} P(t)$$

$$Area = \int_{-\infty}^{\infty} \delta(t)dt = 1$$



• For a unit step function,

$$u(t<0) = 0$$

$$u(t \geq 0) = 1$$

$$\frac{d}{dt}u(t) = \delta(t)$$

• If we look at an input signal (force here) we can break it into very small segments in time. As the time becomes small we can approximate it with a set of impulses.

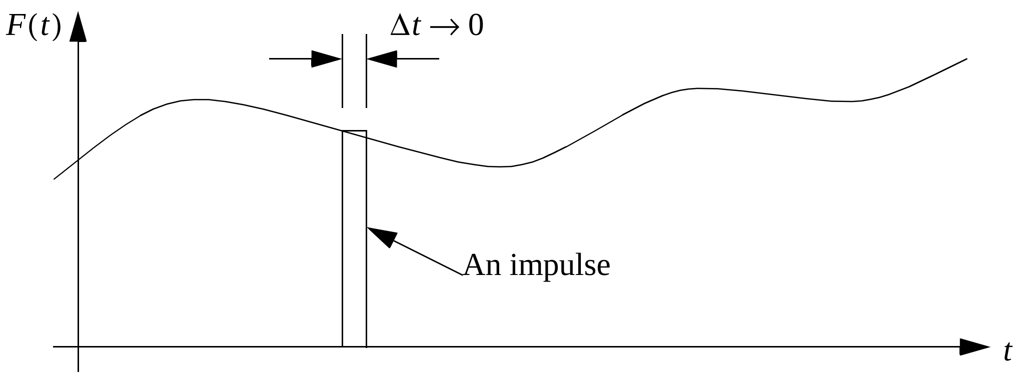


Figure 19.1 An impulse as a brief duration (instant) pulse

19.3 IMPULSE RESPONSE

• If we put an impulse into a system the output will be an impulse response.

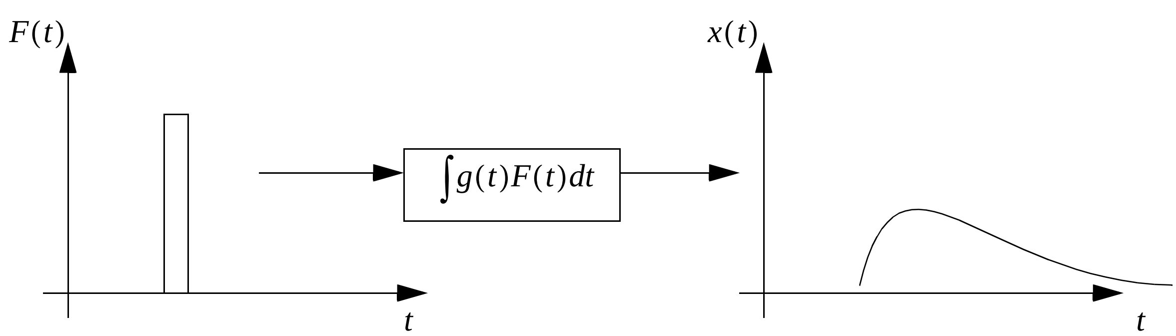
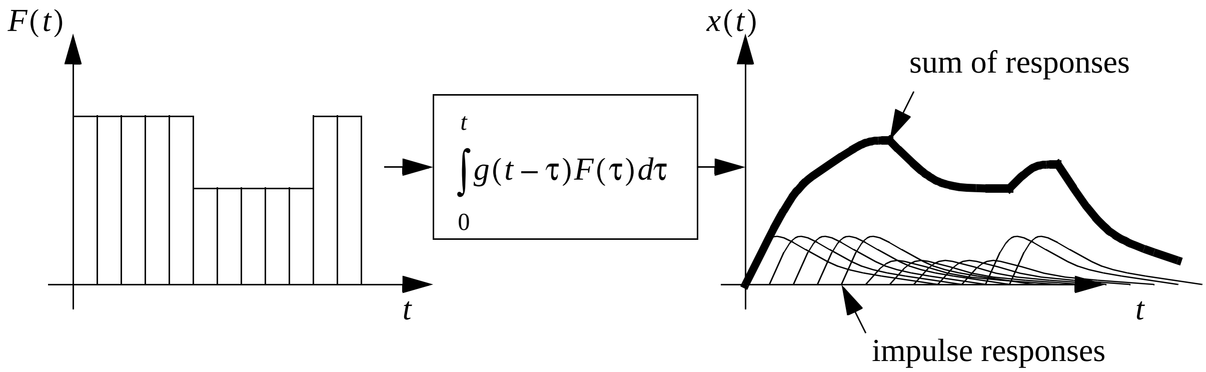


Figure 19.2 Response of the system to a single pulse

• If we add all of the impulse responses together we will get a total system response. This operation is called convolution.



The convolution integral

$$c(t) = \int_0^t g(t-\tau)r(\tau)d\tau$$

Figure 19.3 A set of pulses for a system gives summed responses to give the output

• Consider the unit impulse of a system with the given differential equation. Note: This method is only valid for trivial differential equations with only one homogeneous term. The preferred method is shown later.

The following first order differential equation has an input 'F'. The input can be replaced with a unit impulse function.

$$\dot{x} + 0.5x = 2F$$
$$\dot{X} + 0.5X = 2\delta(t)$$

The homogeneous solution can be found.

$$\dot{X} + 0.5X = 0 \qquad \text{guess,} \qquad X_h = e^{At} \qquad \dot{X}_h = Ae^{At}$$
$$A + 0.5 = 0$$
$$X_h = Ce^{-0.5t}$$

The particular solution is found.

$$\dot{X} + 0.5X = 2\delta(t) \qquad \text{guess,} \qquad X_p = A \qquad \dot{X}_p = 0$$
$$0 + 0.5A = 2(0)$$
$$X_p = A = 0$$

The initial condition caused by the impulse function found, assuming a zero initial condition.

$$\left(\frac{1}{dt}\right)X_0 + 0.5(0) = 2\left(\frac{1}{dt}\right)$$
$$X_0 = 2$$

The initial condition caused by the impulse function found, assuming a zero initial condition.

$$X(t) = Ce^{-0.5t}$$
$$X(0) = 2 = Ce^{0}$$
$$X(t) = 2e^{-0.5t}$$

• Note that the derivation of the unit impulse function assumed zero initial conditions, so the process of convolution must also assume systems start at rest and undeflected.

19.4 CONVOLUTION

• The following example shows the use of the convolution integral to find a number of responses.

The unit impulse response to a step input can be calculated using the convolution integral.

$$x(t) = \int_0^t X(t-\tau)F(\tau)d\tau = \int_0^t (2e^{-0.5(t-\tau)})(1)d\tau = 2e^{-0.5t}\int_0^t e^{0.5\tau}d\tau = 2e^{-0.5t}\frac{e^{0.5\tau}}{0.5}\Big|_0^t$$

$$x(t) = 2e^{-0.5t}\left(\frac{e^{0.5t}}{0.5} - \frac{e^{0.5(0)}}{0.5}\right) = 4e^{-0.5t}(e^{0.5t} - 1) = 4(1 - e^{-0.5t})$$

The unit impulse response to a unit ramp input can be calculated using the convolution integral.

$$x(t) = \int_0^t X(t-\tau)F(\tau)d\tau = \int_0^t 2e^{-0.5(t-\tau)}\tau d\tau = 2e^{-0.5t}\int_0^t e^{0.5\tau}\tau d\tau$$

$x(t) =$ FINISH THE INTEGRAL

The unit impulse response to a sinusoidal input can be calculated using the convolution integral.

$$x(t) = \int_0^t X(t-\tau)F(\tau)d\tau = \int_0^t 2e^{-0.5(t-\tau)}\sin(\tau)d\tau = 2e^{-0.5t}\int_0^t e^{0.5\tau}\sin(\tau)d\tau$$

$x(t) =$ FINISH THE INTEGRAL

19.5 NUMERICAL CONVOLUTION

• The convolution integral can also be solved numerically. This is particularly useful for systems with arbitrary inputs.

$$x(t) = h\sum_{i=0}^{n-1} X(t-(i+0.5)h)F((i+0.5)h)$$

where,

$$n = \frac{t}{h} = \text{number of steps}$$

$$h = \text{step size (s)}$$

$$X(t) = \text{the unit impulse response function}$$

$$F(t) = \text{the input function}$$

• This can be applied to the previous example for a unit step input to find the system position at 10 seconds, with a 2 second time step.

$$x(t) = h \sum_{i=0}^{n-1} 2e^{-0.5(t-(i+0.5)h)} F((i+0.5)h)$$

$$n = \frac{10}{2} = 5$$

$$x(10) = 2 \sum_{i=0}^{5-1} 2e^{-0.5(10-2(i+0.5))}(1)$$

$$x(10) = 4 \sum_{i=0}^{4} e^{-4.5+i}$$

$$x(10) = 4(e^{-4.5+0} + e^{-4.5+1} + e^{-4.5+2} + e^{-4.5+3} + e^{-4.5+4})$$

$$x(10) = 4(e^{-4.5} + e^{-3.5} + e^{-2.5} + e^{-1.5} + e^{-0.5})$$

$$x(10) = 4(0.01111 + 0.03020 + 0.08208 + 0.22313 + 0.6065)$$

$$x(10) = 3.81208$$

This value can be compared to the exact value calculated below. The accuracy of the numerical value would increase substantially if the step size were decreased.

$$x(10) = 4(1 - e^{-0.5(10)}) = 4(1 - 0.006738) = 3.973048$$

• A Scilab program to perform the previous calculation numerically is shown below.

```
// A numerical convolution example

function foo=X(t) // The impulse response function
	foo = 2 * exp(-0.5*t);
endfunction

function foo=F(t) // The input function
	foo = 1; // a step function
endfunction

// define the variables

function foo = convolution(t, h) // The integration function
	n = t / h;
	foo = 0;
	for i=0:n-1
				foo = foo + h * X(t - (i+0.5)*h) * F((i+0.5)*h);
	end
endfunction

h = 1; // the time step for the integration
t = 10; // the time point to calculate
printf("The estimated value x(%fs) = %f \n", t, convolution(t, h));
x_calc = 4*(1-exp(-0.5*t));
printf("The actual value is x(%fs) = %f \n", t, x_calc);
```

19.6 LAPLACE IMPULSE FUNCTIONS

• The convolution integral can be difficult to deal with because of the time shift. But, the Laplace transform for the convolution integral turns it into a simple multiplication.

$$c(t) = \int_0^t g(t-\tau)r(\tau)d\tau$$

$$C(s) = G(s)R(s)$$

Figure 19.4 The convolution integral in the Laplace s-domain

19.7 SUMMARY

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19.8 PRACTICE PROBLEMS

19.9 PRACTICE PROBLEM SOLUTIONS

19.10 ASSIGNMENT PROBLEMS

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