Fernando Vallecillos Ruiz

[1] Let A be a set. Consider the poset (P(A);⊆). Prove that, for any B ⊆ P(A), the least upperbound of B is ∪B. Also, what is the greatest lowerbound of B? (Hint: Be careful of the case when B = ∅).

Any pair of elements in a powerset has a least upper bound and a greatest lower bound. The least upper bound is the union of the two elements.

- The union of two elements is its upper bound, for any two set B and C, B⊆B∪C.
- The union of two elements is the least upper bound for B and C because there is no set D smaller than B∪C that contains both B or C.

The greatest lower bound is the intersection of the two elements.

- The intersection of two elements is its lower bound, for any two set B and C, B⊆B∩C.
- The intersection of two elements is the greatest lower bound for B and C because there is no set D bigger than B∩C that contains both B and C.

［2］In the class, we have discussed that every complete lattice is a cpo. Does the converse hold? That is, is every cpo a complete lattice? If yes, briey explain why. If no, give an example of a cpo that is not a complete lattice.

If we assume a set: {∅,{0},{1}} ordered by inclusion. This is a cpo, but not a lattice since {0} and {1} have no common upper bound.

π init = {X ↦ ⊤;Y ↦ +; Z ↦ -																		
	L1			L2			L3			L4			L5			L6		
Steps	X	Y	Z	X	Y	Z	X	Y	Z	X	Y	Z	X	Y	Z	X	Y	Z
1	⊤	+	-	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
2	⊤	+	-	0	+	-	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
3	⊤	+	-	0	+	-	0	+	+	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
4	⊤	+	-	0	+	-	0	+	+	0	+	+	⊥	⊥	⊥	⊥	⊥	⊥
5	⊤	+	-	0	+	-	0	+	+	0	+	+	0	+	+	⊥	⊥	⊥
6	⊤	+	-	0	+	-	+	+	+	0	+	+	0	+	+	⊥	⊥	⊥
7	⊤	+	-	0	+	-	+	+	+	+	+	+	0	+	+	+	+	+
8	⊤	+	-	0	+	-	+	+	+	+	+	+	+	+	+	+	+	+
9	⊤	+	-	0	+	-	+	+	+	+	+	+	+	+	+	+	+	+

［4］We have seen that there are cases where the abstract interpretation described in the class does not give precise results (cf. slide 22 of the abstract interpretation lecture slides for an example).
Suggest possible improvements to the analysis method so that it would be able to give more precise results. (There is no fixed right answer to the question.)

Depending on what we want to express we could come up with different ideas. Usually, the more abstraction that you have, the more you can express.
In class we have seen abstractions as {0, +, -, T}.
We could include the possibility that a number is positive/negative or zero. Then, we could have:
➢ + : positive integer
➢ - : negative number
➢ $+_0$: positive integer or zero
➢ $-_0$: negative number or zero
We could also use another abstraction to express that something is not zero or is empty:
➢ ≠0 : not zero
➢ ⊥ : Ø

$$\dfrac{\vDash \mathrm{true} \Rightarrow 0 = 0 \qquad \dfrac{\dfrac{\vdash \{0 = 0\}\, X \coloneqq 0\, \{X = 0\} \qquad \vdash \{\mathrm{X} = 0\}\, \mathrm{Y} \coloneqq 0\, \{\mathrm{X} = \mathrm{Y}\}}{\vdash \{0 = 0\}\, X \coloneqq 0; Y \coloneqq 0\, \{X = Y\}} \qquad \dfrac{\dfrac{\mathrm{A}\ \ldots}{\vdash \{\mathrm{X} = \mathrm{Y} \wedge \mathrm{Z} > 0\} Y \coloneqq Y + W; X \coloneqq X + W; Z \coloneqq Z - 1 \{\mathrm{X} = \mathrm{Y}\}}}{\vdash \{\mathrm{X} = \mathrm{Y}\}\, \mathit{while}\ Z > 0\ \mathit{do}\ (Y \coloneqq Y + W; X \coloneqq X + W; Z \coloneqq Z - 1)\ \{X = Y \wedge 0 \geq \mathrm{Z}\}}}{\vdash \{0 = 0\} X \coloneqq 0; Y \coloneqq 0; \mathit{while}\ Z > 0\ \mathit{do}\ (Y \coloneqq Y + W; X \coloneqq X + W; Z \coloneqq Z - 1\ ) \{X = Y \wedge 0 \geq \mathrm{Z}\}} \qquad \vDash \mathrm{X} = \mathrm{Y} \Rightarrow \mathrm{X} = \mathrm{Y} \wedge 0 \geq \mathrm{Z}}{\vdash \{\mathrm{true}\} X \coloneqq 0; Y \coloneqq 0; \mathit{while}\ Z > 0\ \mathit{do}\ (Y \coloneqq Y + W; X \coloneqq X + W; Z \coloneqq Z - 1 \{X = Y\}}$$

$$A \ = \ \dfrac{\vDash \mathrm{X} = \mathrm{Y} \wedge \mathrm{Z} > 0 \Rightarrow \mathrm{X} + \mathrm{W} = \mathrm{Y} + \mathrm{W} \qquad \dfrac{\dfrac{\vdash \{\mathrm{X} + \mathrm{W} = \mathrm{Y} + \mathrm{W}\}\mathrm{Y} \coloneqq \mathrm{Y} + \mathrm{W}\{\mathrm{X} + \mathrm{W} = \mathrm{Y}\} \quad \vdash \{\mathrm{X} + \mathrm{W} = \mathrm{Y}\}\mathrm{X} \coloneqq \mathrm{X} + \mathrm{W}\{\mathrm{X} = \mathrm{Y}\}}{\vdash \{\mathrm{X} + \mathrm{W} = \mathrm{Y} + \mathrm{W}\} Y \coloneqq Y + W; X \coloneqq X + W\{X = Y\}} \qquad \vdash \{X = Y\}Z \coloneqq Z - 1\{X = Y\}}{\vdash \{\mathrm{X} + \mathrm{W} = \mathrm{Y} + \mathrm{W}\} Y \coloneqq Y + W; X \coloneqq X + W; Z \coloneqq Z - 1\{\mathrm{X} = \mathrm{Y}\}}}{\vdash \{\mathrm{X} = \mathrm{Y} \wedge \mathrm{Z} > 0\} Y \coloneqq Y + W; X \coloneqq X + W; Z \coloneqq Z - 1\{\mathrm{X} = \mathrm{Y}\}}$$

$$\dfrac{vc(X \coloneqq 0, Z \times Y = X) = (Z \times Y = 0, Z \leq 0) \quad \dfrac{\dfrac{vc(X \coloneqq 0) = (Z > 0, true) \quad vc(T \coloneqq Z) = (Z > 0, true)}{vc(X \coloneqq 0; T \coloneqq Z,) = (Z > 0, true)} \quad \dfrac{\dfrac{vc(X \coloneqq X + Y) = (, true) \quad vc(T \coloneqq T - 1) = (T - 1 \neq -1, true)}{vc(X \coloneqq X + Y; T \coloneqq T - 1, Z \times Y = X) = (T - 1 \neq -1, true)}}{vc(while[Z \times Y = X] T \neq 0\ do\ (X \coloneqq X + Y; T \coloneqq T - 1), Z \times Y = X \wedge T = 0) = (Z \times Y = X, (Z \times Y = X \wedge T \neq 0 \rightarrow T - 1 \neq -1) \wedge (Z \times Y = X \wedge \mathrm{T} = 0 \rightarrow Z \times Y = X \wedge \mathrm{Z} > 0 \wedge T = 0))}}{vc(X \coloneqq 0; T \coloneqq Z; while[Z \times Y = X] T \neq 0\ do\ (X \coloneqq X + Y; T \coloneqq T - 1), Z \times Y = X \wedge Z > 0 \wedge T = 0) = (Y = 0 \wedge Z \times Y = X, (Z \times Y = X \wedge T \neq 0 \rightarrow T - 1 \neq -1) \wedge (Z \times Y = X \wedge \mathrm{T} = 0 \rightarrow Z \times Y = X \wedge \mathrm{Z} > 0 \wedge T = 0))}}{vc\big(if\ Z \leq 0\ then\ X \coloneqq 0\ else\ \big(X \coloneqq 0; T \coloneqq Z; while[Z \times Y = X] T \neq 0\ do\ (X \coloneqq X + Y; T \coloneqq T - 1)\big), Z \times Y = X\big) = ((Z \leq 0 \rightarrow Z \times Y = 0) \wedge (Z > 0 \rightarrow Y = 0 \wedge Z \times Y = X), Z \leq 0 \wedge (Y = 0 \wedge Z \times Y = X, (Z \times Y = X \wedge T \neq 0 \rightarrow T - 1 \neq -1) \wedge (Z \times Y = X \wedge \mathrm{T} = 0 \rightarrow Z \times Y = X \wedge \mathrm{Z} > 0 \wedge T = 0)))}$$

We have the following inputs:

- C = $\big(if\ Z \leq 0\ then\ X \coloneqq 0\ else\ \big(X \coloneqq 0; T \coloneqq Z; while\ T \neq 0\ do\ (X \coloneqq X + Y; T \coloneqq T - 1)\big), Z \times Y = X\big)$
- $\phi_{\text{post}} = \mathrm{Z\ x\ Y = X}$
- Loop invariant = $Z \times Y = X$
- $\phi_{\text{pre}} = \mathrm{Z} \geq 0$

1. Annotating with the loop invariant, c becomes:
    a. C= $\big(if\ Z \leq 0\ then\ X \coloneqq 0\ else\ \big(X \coloneqq 0; T \coloneqq Z; while[Z \times Y = X] T \neq 0\ do\ (X \coloneqq X + Y; T \coloneqq T - 1)\big), Z \times Y = X\big)$
2. Computing wp(c, ϕ post) gives (ϕ, ψ) where
    a. $-\phi = Z \times Y = X$
    b. $-\psi =(Z \times Y = X \wedge T \neq 0 \rightarrow T - 1 \neq -1) \wedge (Z \times Y = X \wedge \mathrm{T} = 0 \rightarrow Z \times Y = X \wedge \mathrm{Z} > 0 \wedge T = 0))$
3. Check if ϕ_{pre} -> ϕ and ψ are valid.
    a. They are valid.

$$\dfrac{vc(X \coloneqq 0, (Z - T) \times Y = X) = ((Z - T) \times Y = 0, Z \leq 0) \quad \dfrac{\dfrac{vc(X \coloneqq 0) = (Z > 0, true) \quad vc(T \coloneqq Z) = (Z > 0, true)}{vc(X \coloneqq 0; T \coloneqq Z,) = (Z > 0, true)} \quad \dfrac{\dfrac{vc(X \coloneqq X + Y) = (, true) \quad vc(T \coloneqq T - 1) = (T - 1 \neq -1, true)}{vc(X \coloneqq X + Y; T \coloneqq T - 1, (Z - T) \times Y = X) = (T - 1 \neq -1, true)}}{vc(while[(Z - T) \times Y = X] T \neq 0\ do\ (X \coloneqq X + Y; T \coloneqq T - 1), (Z - T) \times Y = X \wedge T = 0) = ((Z - T) \times Y = X, ((Z - T) \times Y = X \wedge T \neq 0 \rightarrow T - 1 \neq -1) \wedge ((Z - T) \times Y = X \wedge \mathrm{T} = 0 \rightarrow (Z - T) \times Y = X \wedge \mathrm{Z} > 0 \wedge T = 0))}}{vc(X \coloneqq 0; T \coloneqq Z; while[(Z - T) \times Y = X] T \neq 0\ do\ (X \coloneqq X + Y; T \coloneqq T - 1), (Z - T) \times Y = X \wedge Z > 0 \wedge T = 0) = (Y = 0 \wedge (Z - T) \times Y = X, ((Z - T) \times Y = X \wedge T \neq 0 \rightarrow T - 1 \neq -1) \wedge ((Z - T) \times Y = X \wedge \mathrm{T} = 0 \rightarrow (Z - T) \times Y = X \wedge \mathrm{Z} > 0 \wedge T = 0))}}{vc\big(if\ Z \leq 0\ then\ X \coloneqq 0\ else\ \big(X \coloneqq 0; T \coloneqq Z; while[(Z - T) \times Y = X] T \neq 0\ do\ (X \coloneqq X + Y; T \coloneqq T - 1)\big), (Z - T) \times Y = X\big) = ((Z \leq 0 \rightarrow (Z - T) \times Y = 0) \wedge (Z > 0 \rightarrow Y = 0 \wedge (Z - T) \times Y = X), Z \leq 0 \wedge (Y = 0 \wedge (Z - T) \times Y = X, ((Z - T) \times Y = X \wedge T \neq 0 \rightarrow T - 1 \neq -1) \wedge ((Z - T) \times Y = X \wedge \mathrm{T} = 0 \rightarrow (Z - T) \times Y = X \wedge \mathrm{Z} > 0 \wedge T = 0)))}$$

- C = $\big(if\ Z \leq 0\ then\ X \coloneqq 0\ else\ \big(X \coloneqq 0; T \coloneqq Z; while\ T \neq 0\ do\ (X \coloneqq X + Y; T \coloneqq T - 1)\big), Z \times Y = X\big)$
- $\phi_{\text{post}} = \mathrm{Z\ x\ Y = X}$
- Loop invariant = $(Z - T) \times Y = X$
- $\phi_{\text{pre}} = \mathrm{Z} \geq 0$

1. Annotating with the loop invariant, c becomes:
    a. C= $\big(if\ Z \leq 0\ then\ X \coloneqq 0\ else\ \big(X \coloneqq 0; T \coloneqq Z; while[(Z - T) \times Y = X] T \neq 0\ do\ (X \coloneqq X + Y; T \coloneqq T - 1)\big), (Z - T) \times Y = X\big) = ((Z \leq 0 \rightarrow (Z - T) \times Y = 0) \wedge (Z > 0 \rightarrow Y = 0 \wedge (Z - T) \times Y = X), Z \leq 0 \wedge$ $(Y = 0 \wedge (Z - T) \times Y = X, ((Z - T) \times Y = X \wedge T \neq 0 \rightarrow T - 1 \neq -1) \wedge ((Z - T) \times Y = X \wedge \mathrm{T} = 0 \rightarrow (Z - T) \times Y = X \wedge \mathrm{Z} > 0 \wedge T = 0)))$
2. Computing wp(c, ϕ post) gives (ϕ, ψ) where
    a. $-\phi =(Z - T) \times Y$
    b. $-\psi =((Z - T) \times Y = X \wedge T \neq 0 \rightarrow T - 1 \neq -1) \wedge ((Z - T) \times Y = X \wedge \mathrm{T} = 0 \rightarrow (Z - T) \times Y = X \wedge \mathrm{Z} > 0 \wedge T = 0)$
3. Check if ϕ_{pre} -> ϕ and ψ are valid.
    a. They are not valid.