PH415 : Homework 6

(Post_Midsem Homework 1)

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Problem Statement :

"Derive Continuity and Boltzmann equation in terms of convective derivatives."

To describe the dynamics of gases and fluids there are, in principle, two different ways:

→ the Eulerian point of view

and

→ the Lagrangian point of view

In the Eulerian point of view, the time differentiation of a given quantity $Q(\mathbf{r}, t)$ is taken with respect to a fixed point.

We denote Eulerian time derivations, with: $\dfrac{\partial}{\partial t}(\cdot)$

In the Lagrangian point of view, on the other hand, one takes the time differentiation associated with a certain fluid element, which is moving with the fluid velocity $\mathbf{v}$

The Lagrangian time differentiation, denoted by: $\dfrac{D}{Dt}(\cdot)$, is defined as

$$\frac{D}{Dt}(Q(\mathbf{r},t)) = \lim_{\delta t \to 0} \frac{Q(\mathbf{r}+\mathbf{v}\,\delta t,\ t+\delta t) - Q(\mathbf{r},t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\left(Q(\mathbf{r},t) + \mathbf{v}\,\delta t \cdot \nabla Q(\mathbf{r},t) + \delta t\, \frac{\partial Q(\mathbf{r},t)}{\partial t}\right) - Q(\mathbf{r},t)}{\delta t} \qquad \text{(Taylor Series Expansion)}$$

$$= \frac{\partial Q(\mathbf{r},t)}{\partial t} + \mathbf{v}\cdot\nabla Q(\mathbf{r},t)$$

$$\Longrightarrow \boxed{\frac{D}{Dt}(\cdot) \equiv \frac{\partial}{\partial t}(\cdot) + \mathbf{v}\cdot\nabla(\cdot)} \qquad ....(1)$$

where $\mathbf{v} = \mathbf{v}(\mathbf{r},t)$

$$\nabla = \nabla_{\mathbf{r}} = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial x}$$

The objective of this assignment is to convert the Continuity and Boltzmann equations from Eulerian Point of View to Lagrangian Point of View (i.e - using the convective derivatives $\dfrac{D}{Dt}(\cdot)$)

1.) **Continuity Equation**

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \nabla\cdot\mathbf{J}(\mathbf{r},t) = 0 \qquad ....(2)$$

where $\rho(\mathbf{r},t) \rightarrow \textit{Number Density}$

$\mathbf{J}(\boldsymbol{r},t) \rightarrow \textit{Current Density}$

$= \rho(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)$

From eq^n (1),

$$\frac{\partial}{\partial t}(\cdot) = \left(\frac{D}{Dt} - \mathbf{v}\cdot\nabla\right)(\cdot) \qquad ....(3)$$

$$\Longrightarrow \frac{\partial}{\partial t}\rho + \nabla\cdot\mathbf{J} = \frac{D}{Dt}\rho - (\mathbf{v}\cdot\nabla)\rho + \nabla\cdot(\rho\,\mathbf{v})$$

$$= \frac{D}{Dt}\rho - \cancel{(\mathbf{v}\cdot\nabla)\rho} + \left[\rho\nabla\cdot\mathbf{v} + \cancel{(\mathbf{v}\cdot\nabla)\rho}\right] \qquad \text{(chain rule)}$$

$$= \frac{D}{Dt}\rho + \rho\nabla\cdot\mathbf{v}$$

$$= 0$$

$$\Longrightarrow \boxed{\frac{D}{Dt}\rho(\mathbf{r},t) + \rho(\mathbf{r},t)\,\nabla\cdot\mathbf{v}(\mathbf{r},t) = 0} \qquad ....(4)$$

2.) **Boltzmann Equation**

$$\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t) = -\frac{1}{m}\rho(\mathbf{r},t)\,\nabla V_{\text{ext}}(\mathbf{r}) - \frac{1}{m}\rho(\mathbf{r},t)\,\nabla\int\rho(\mathbf{x},t)\,v(\mathbf{r}-\mathbf{x})\,d\mathbf{x}$$

$$-\,\mathbf{v}(\mathbf{r},t)\,\nabla\cdot(\rho(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)) - \rho(\mathbf{r},t)(\mathbf{v}(\mathbf{r},t)\cdot\nabla)\mathbf{v}(\mathbf{r},t) \qquad ....(5)$$

where:

$V_{\text{ext}}(\mathbf{r}) \rightarrow$ Potential Energy due to External Force

$\mathbf{J}(\mathbf{r},t) = \mathbf{J} \rightarrow$ Current Density

$\rho(\mathbf{r},t) = \rho \rightarrow$ Number Density

$v(\mathbf{r}-\mathbf{x}) \rightarrow$ Interaction Energy

(influence of "particle" at $\mathbf{x}$ on "particle" at $\mathbf{r}$)

$m \rightarrow$ mass of "each particle"

(this will ultimately be gone from the equation)

Taking 3rd and 4th terms on RHS to LHS in eq^n (5), and then multiplying both sides by $\frac{m}{\rho}$, we get:

$$\frac{m}{\rho}\frac{\partial}{\partial t}\mathbf{J} \;+\; \frac{m}{\rho}\,\mathbf{v}\,\nabla\cdot(\rho\,\mathbf{v}) \;+\; m(\mathbf{v}\cdot\nabla)\mathbf{v} \;=\; -\nabla V_{\text{ext}} \;-\; \nabla\int \rho(\mathbf{x},t)\; v(\mathbf{r}-\mathbf{x})\; d\mathbf{x}$$

$$\frac{m}{\rho}\frac{\partial}{\partial t}(\rho\mathbf{v}) \;+\; \frac{m}{\rho}\,\mathbf{v}\,\nabla\cdot\mathbf{J} \;+\; m(\mathbf{v}\cdot\nabla)\mathbf{v} \;=$$

$$\frac{m}{\rho}\left[\rho\frac{\partial\,\mathbf{v}}{\partial t} \;+\; \mathbf{v}\frac{\partial\rho}{\partial t}\right] \;+\; \frac{m}{\rho}\,\mathbf{v}\,\nabla\cdot\mathbf{J} \;+\; m(\mathbf{v}\cdot\nabla)\mathbf{v} \;= \qquad \text{(using Chain rule)}$$

$$\underbrace{m\left[\frac{\partial\,\mathbf{v}}{\partial t} \;+\; (\mathbf{v}\cdot\nabla)\mathbf{v}\right]} \;+\; \frac{m}{\rho}\,\mathbf{v}\;\underbrace{\left[\frac{\partial\rho}{\partial t} \;+\; \nabla\cdot\mathbf{J}\right]} \;=$$

$$m\left[\frac{D\,\mathbf{v}}{Dt}\right] \;+\; \frac{m}{\rho}\,\mathbf{v}\,[0] \;= \qquad \left(\text{using } eq^n\ (1) \text{ and } eq^n\ (2)\right)$$

$$\Longrightarrow \boxed{m\frac{D}{Dt}\mathbf{v}(\mathbf{r},t) \;=\; -\;\nabla V_{\text{eff}}(\mathbf{r},t)} \qquad \text{.... (6)}$$

where:

$$V_{\text{eff}}(\mathbf{r},t) \;:=\; V_{\text{ext}}(\mathbf{r}) \;+\; \int \rho(\mathbf{x},t)\; v(\mathbf{r}-\mathbf{x})\; d\mathbf{x} \qquad \text{.... (7)}$$

✶ ***References :***

(1) J. Schober. *The Small-Scale Dynamo: Amplification of Magnetic Fields in the Early Universe* (Chapter 2, Section 2 : Describing Fluids), *University of Heidelberg*. 2011

(2) G. Setlur. *Fluid description of a system of classical particles*. 2021

*** The End ***