Interactive Many-Particles Collision Simulator

(Guided by Prof. Girish S. Setlur)

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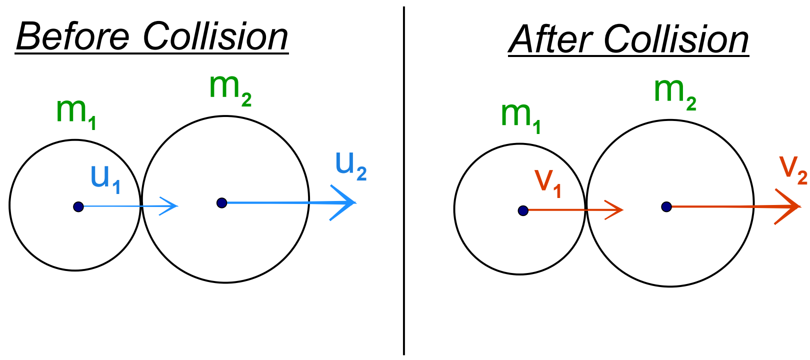
Problem Statement :

- Imagine a flat smooth horizontal circular table with a circular wall on the circumference.
- Imagine many small pellets can slide without friction on the table.
- These pellets are all of equal mass and can collide elastically with each other and with the boundary wall.
- Write a code in some language that gives the velocity and position of each pellet after time t if at t=0 the positions and velocities are known.

First, let us consider the most simple case of :

$1-\mathrm{D}$ (**heads on**) Elastic Collision



· **Conservation of Linear Momentum :** $$m_1 \cdot u_1 + m_2 \cdot u_2 = m_1 \cdot v_1 + m_2 \cdot v_2 \quad \ldots (1)$$

· **Conservation of Energy :** $$\frac{m_1}{2} \cdot u_1^2 + \frac{m_2}{2} \cdot u_2^2 = \frac{m_1}{2} \cdot v_1^2 + \frac{m_2}{2} \cdot v_2^2 \quad \ldots (2)$$

From (2) and (1), we get : $$u_1 - u_2 = -v_1 + v_2 \quad \ldots (3)$$

From (3) and (1), we get : $$\begin{pmatrix} m_1 & m_2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\frac{1}{m_1 - (-m_2)} \begin{pmatrix} 1 & -m_2 \\ 1 & m_1 \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

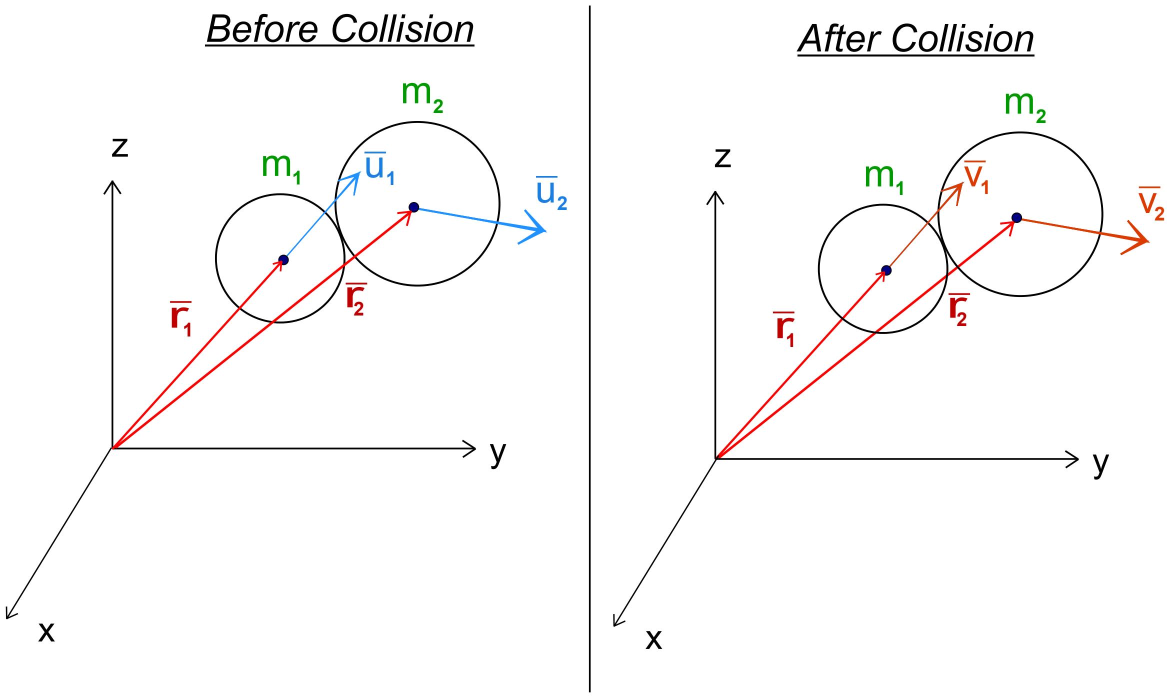
$$\frac{1}{m_1 + m_2} \begin{pmatrix} m_1 - m_2 & 2\, m_2 \\ 2\, m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \ldots (4)$$

$$\Longrightarrow \boxed{v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\cdot u_1 + \left(\frac{2\,m_2}{m_1 + m_2}\right)\cdot u_2}$$

$$\boxed{v_2 = \left(\frac{2\,m_1}{m_1 + m_2}\right)\cdot u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)\cdot u_2}$$

Now, let us move on to collisions in $2/3-D$ (*i.e* $-\vec{u}_1$ *and* $\vec{u}_2$ *are not necessarily colinear*):

$2/3-D$ Elastic Collision (*between hard pellets / spheres*)



Consider the following coordinate system, where :

$$position(\text{body 1}) = \vec{r}_1$$
$$position(\text{body 2}) = \vec{r}_2$$

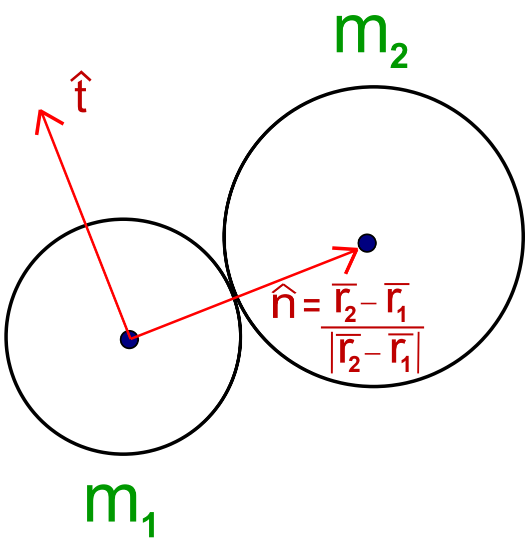
Let the initial and final velocities be:

$$initial\ velocity(\text{body 1}) = \vec{u}_1$$
$$initial\ velocity(\text{body 2}) = \vec{u}_2$$
$$final\ velocity(\text{body 1}) = \vec{v}_1$$
$$final\ velocity(\text{body 2}) = \vec{v}_2$$

Also, consider the directions:

$$normal\ direction\ at\ the\ point\ of\ contact \rightarrow \hat{n} = \frac{\vec{r}_2 - \vec{r}_1}{\left|\vec{r}_2 - \vec{r}_1\right|} \quad \ldots (5)$$

$$tangential\ direction\ at\ the\ point\ of\ contact \rightarrow \hat{t}$$



Firstly, we will show that :

"any change in the velocity of these bodies will only be in the normal direction"

· **Conservation of Linear Momentum :**

$$m_1 \vec{u}_1 + m_2 \vec{u}_2 = m_1 \vec{v}_1 + m_2 \vec{v}_2 \quad \ldots (6)$$

$$\left[m_1 (\vec{v}_1 - \vec{u}_1) = -m_2 (\vec{v}_2 - \vec{u}_2) \right]$$

· **Conservation of Angular Momentum :**

$$\vec{r}_1 \times (m_1 \vec{u}_1) + \vec{r}_2 \times (m_2 \vec{u}_2) = \vec{r}_1 \times (m_1 \vec{v}_1) + \vec{r}_2 \times (m_2 \vec{v}_2) \quad \ldots (7)$$

$$\left[m_1 \vec{r}_1 \times (\vec{v}_1 - \vec{u}_1) = -m_2 \vec{r}_2 \times (\vec{v}_2 - \vec{u}_2) \right]$$

From (6) and (7), we get :

$$(\vec{r}_2 - \vec{r}_1) \times (\vec{v}_1 - \vec{u}_1) = \vec{0} \quad \text{and} \quad (\vec{r}_2 - \vec{r}_1) \times (\vec{v}_2 - \vec{u}_2) = \vec{0}$$

From (5), we know that $(\vec{r}_2 - \vec{r}_1) \propto \hat{n}$

$$\Longrightarrow \begin{pmatrix} \vec{v}_1 = \vec{u}_1 + \lambda_1 \hat{n} \\ \\ \vec{v}_2 = \vec{u}_2 + \lambda_2 \hat{n} \end{pmatrix} \quad \ldots (8) \quad (\text{where } \lambda_1 \text{ and } \lambda_2 \text{ are some scalars})$$

Hence,

"any change in the velocity of these bodies will only be in the normal direction"

This means, the velocity vectors can be expressed as a linear combination of normal and tangential components.

$$\vec{u}_1 = \vec{u}_{1,n} + \vec{u}_{1,t}$$

$$= (\vec{u}_1 \cdot \hat{n})\, \hat{n} + \left[\vec{u}_1 - (\vec{u}_1 \cdot \hat{n})\, \hat{n} \right]$$

$$\vec{u}_2 = \vec{u}_{2,n} + \vec{u}_{2,t}$$

$$= (\vec{u}_2 \cdot \hat{n})\hat{n} + \left[\vec{u}_2 - (\vec{u}_2 \cdot \hat{n})\hat{n}\right]$$

From 8, we know that $\underline{\vec{v}_{1,t} = \vec{u}_{1,t}}$ *and* $\underline{\vec{v}_{2,t} = \vec{u}_{2,t}}$ ...9

Now, because $\vec{u}_n$ & $\vec{u}_t$ *are orthogonal to each other, final velocities can simply be found considering the collision to be a* $1-D$ *elastic collision in normal direction.*

i.e − using 4, we get:

$$\left(\begin{array}{l} \vec{v}_{1,n} = \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right) \cdot \vec{u}_{1,n} + \left(\dfrac{2\,m_2}{m_1 + m_2}\right) \cdot \vec{u}_{2,n} \\ \vec{v}_{2,n} = \left(\dfrac{2\,m_1}{m_1 + m_2}\right) \cdot \vec{u}_{1,n} + \left(\dfrac{m_2 - m_1}{m_1 + m_2}\right) \cdot \vec{u}_{2,n} \end{array} \right) \quad ...10$$

Using 9 and 10, we get :

$$\begin{aligned} \vec{v}_1 &= \vec{v}_{1,t} + \vec{v}_{1,n} \\ &= \vec{u}_{1,t} + \left[\left(\frac{m_1 - m_2}{m_1 + m_2}\right) \cdot \vec{u}_{1,n} + \left(\frac{2\,m_2}{m_1 + m_2}\right) \cdot \vec{u}_{2,n}\right] \\ &= \left[\vec{u}_1 - (\vec{u}_1 \cdot \hat{n})\hat{n}\right] + \left[\left(\frac{m_1 - m_2}{m_1 + m_2}\right) \cdot \left[(\vec{u}_1 \cdot \hat{n})\hat{n}\right] + \left(\frac{2\,m_2}{m_1 + m_2}\right) \cdot \left[(\vec{u}_2 \cdot \hat{n})\hat{n}\right]\right] \\ &= \vec{u}_1 + \left(\frac{2\,m_2}{m_1 + m_2}\right)(\vec{u}_2 - \vec{u}_1) \cdot \hat{n}\ \hat{n} \end{aligned}$$

$$\boxed{\vec{v}_1 = \vec{u}_1 + \left(\frac{2\,m_2}{m_1 + m_2}\right)(\vec{u}_2 - \vec{u}_1) \cdot \hat{n}\ \hat{n}} \quad ...11$$

Similarly,

$$\boxed{\vec{v}_2 = \vec{u}_2 - \left(\frac{2\,m_1}{m_1 + m_2}\right)(\vec{u}_2 - \vec{u}_1) \cdot \hat{n}\ \hat{n}} \quad ...12$$

$$\text{, where } \hat{n} = \frac{\vec{r}_2 - \vec{r}_1}{\left|\vec{r}_2 - \vec{r}_1\right|}$$

So, if we know the <u>*initial position and velocities*</u> *of the bodies before collision* $(\vec{r}_1, \vec{r}_2, \vec{u}_1, \vec{u}_2)$, *using 5, 11, and 12, one can easily find out the* <u>*velocities after collision*</u> $(\vec{v}_1, \vec{v}_2)$.

Lastly, let's consider the <u>*collision with wall*</u>.

At the point of contact, again a normal and tangential vector can be found.

Informally,

one could simply assume $(m_1 = m\ (<\infty);\ u_1 = u;\ v_1 = v)$ *and* $\left(m_2 \to \infty\ ;\ \|\vec{u}_2\| = 0\right)$

in (11) *and arrive at the formula* :

$$\vec{v} = \vec{u} + (2)(-\vec{u})\cdot\hat{n}\ \hat{n}$$

$$\boxed{\vec{v} = \vec{u} - 2(\vec{u}\cdot\hat{n})\,\hat{n}} \qquad \ldots (13)$$

$\downarrow$

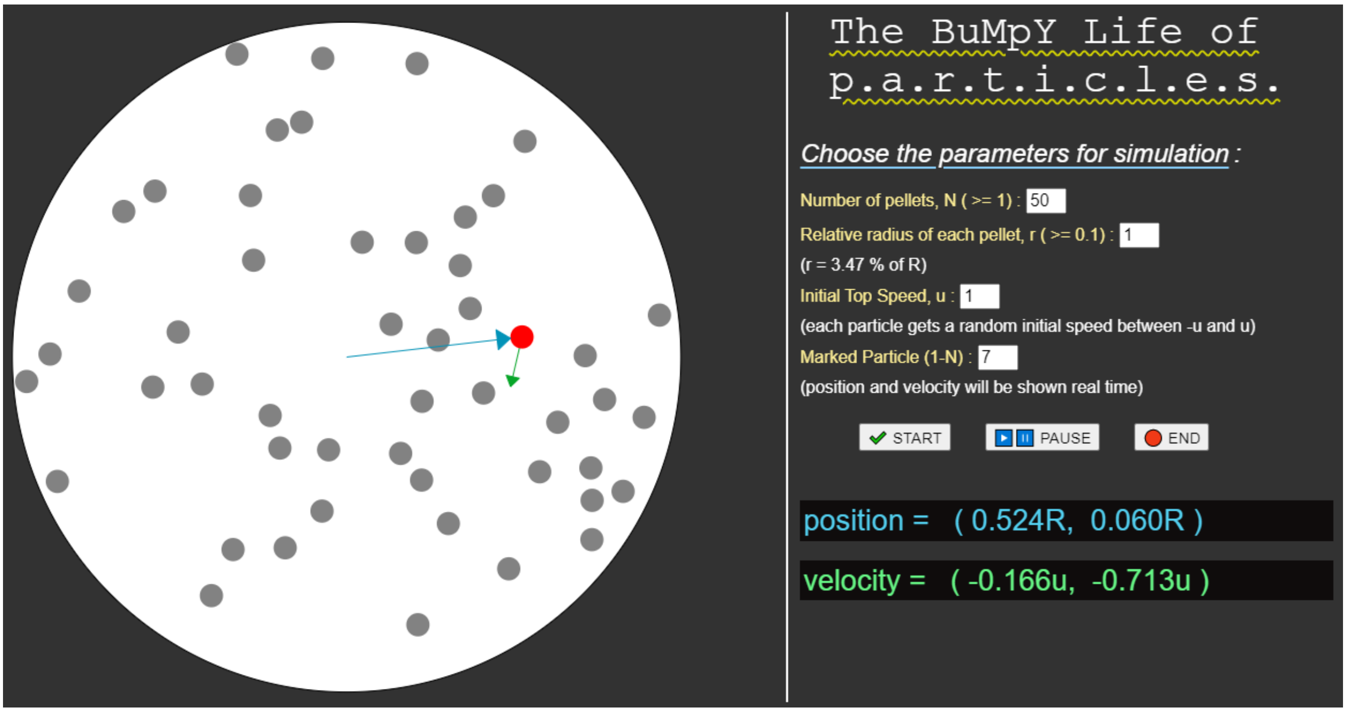
Assuming, circular / spherical wall, $(\hat{n} = \hat{r})$

This simply is again a reflection of conservation of angular momentum, linear momentum, and total energy of the system.

Now that we are equipped with *equations* (5), (11), (12), (13), the only thing that remains to be figurred out is <u>*collision-detection*</u>; and to check if at any instant of time, some bodies are colliding is to "simply check all possible-pairs of particles" ${}^{N}C_2\ \left[\approx O\left(N^2\right)\ \textit{operation}\right]$, where $N \to \#\ \textit{bodies}$.

Using these principles, here is a simple simulation, that has been programmed in Javascript and deployed on the following address :

https://nandwani-rohit.github.io/many-particles/



- Click on START to start the simulation.
- PAUSE to pause/resume the simulation at any moment
- Change **Marked Particle** input number from 1 to any number to *see it's respective position and velocities real-time.*
- To change the number of pellets (N), or radius (r) or initial top speed (u), change the values as you see fit, and then end and start the simulation, again by clicking on respective buttons.

⟹ So, basically once a simulation starts, one can only change Marked Particle input value to see whichever pellet they wish to see.

OR

Pause and move at one's own luxury from particle to particle seeing which one's going where in it's life. For other values, one needs to end the simulation and start again with those update values plugged in.

★ **References** :

- Wikipedia Article on Elastic Collision
- 2-Dimensional Elastic Collisions without Trigonometry

The End