CHAPTER 1

# Discrete-Time Signals and Systems

## 1.1 INTRODUCTION

A discrete-time signal is a sequence of values usually representing the behavior of a physical phenomenon. In electrical engineering problems, those values are samples of a continuous-time-varying electrical signal taken at uniform rate, called sampling rate or sampling frequency. The inverse of the sampling rate is called sampling interval or sampling period. Figure 1.1 shows the graphical representations of a continuous-time signal and its discrete-time uniformly sampled version. For the latter, time is normalized by the sampling period becoming the index $n$. This can be stated as:

$$s_d(n) = s(nT) \tag{1.1}$$

where $T$ is the sampling period.

In practice, continuous-time or analog signals are electrical events (voltage or current) representing the behavior of some physical phenomenon such as speech or temperature as a function of time. Devices known as transducers are utilized to convert physical variations (of pressure or temperature, for example) into voltage or electrical current changes, thus creating an electrical signal. To be digitally processed, electrical signals have to be sampled, time discretized, quantized, and encoded, thus becoming a digital signal.

Therefore, a digital signal is a discrete-time signal, as represented in Figure 1.1(b), for which the amplitude is quantized, that is, it is constrained to assume values in a finite set. Quantization is usually accomplished by rounding or truncating the amplitude sample to the nearest value in the discrete set. It is always present in digital signal processing, as samples

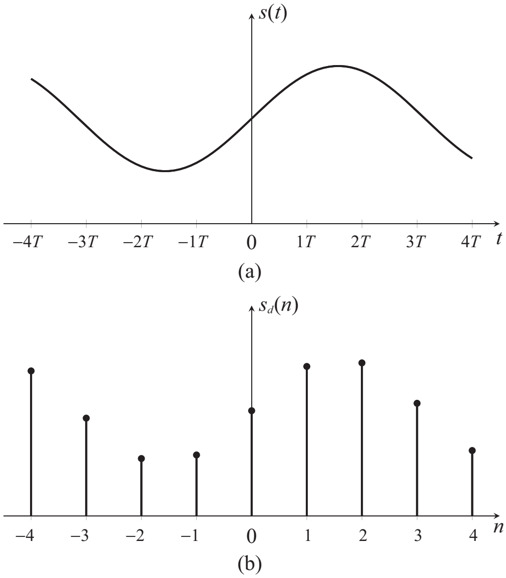


**Figure 1.1.** (a) Continuous-time signal $s(t)$, (b) Discrete-time signal $s_d(n) = s(nT)$.

must be stored in finite length registers. Chapter 3 analyzes the sampling and quantization processes.

In this chapter, we present the main properties of discrete-time signals, demonstrate how those signals are affected by operations on the independent variable, and introduce some signals that are important in digital signal processing.

## 1.2 PROPERTIES OF DISCRETE-TIME SIGNALS

Most properties of continuous-time (analog) signals are also common to discrete-time (digital) signals. In this section, we review some relevant properties to digital signal processing.

### 1.2.1 PERIODICITY

A discrete-time signal is periodic if there exists an integer $N$ such that:

$$x(n) = x(n+N) \tag{1.2}$$

for any value of $n$. The integer $N$ is called the period of the signal.

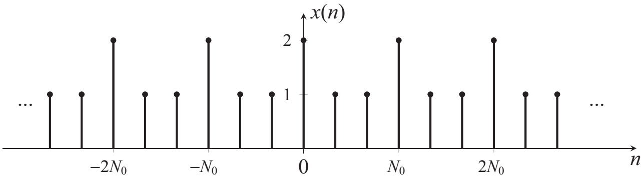


**Figure 1.2.** Discrete-time periodic signal (segment) with period $N_0$.

Equation 1.2 holds for any integer multiple of $N$. Figure 1.2 shows a periodic signal where $N_0$ is the smallest value that period $N$ can assume, called fundamental period. Thus, this signal is periodic for any period $N = kN_0$, $k$ integer. Equation 1.2, thus, generalizes to:

$$x(n) = x(n + kN_0) \quad \forall\ n, k \in Z. \tag{1.3}$$

### 1.2.2 POWER AND ENERGY

The energy of a discrete-time signal $x(n)$ is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 . \tag{1.4}$$

The average power of a discrete-time signal is defined as:

$$P = \lim_{N\to\infty} \frac{1}{N} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} |x(n)|^2 . \tag{1.5}$$

If $x(n)$ has finite energy ($E < \infty$), then $P = 0$, and the signal is called an energy signal. If $E$ is infinite and $P$ is finite and non-null, then $x(n)$ is known as a power signal.

All finite duration signals with finite amplitude are energy signals. All periodic signals are power signals, but not all power signals are necessarily periodic.

### 1.2.3 EVEN AND ODD SIGNALS

An even signal is symmetric with respect to the vertical axis. Therefore, for an even signal, we have:

$$x(n) = x(-n).$$

If a signal is antisymmetric with respect to the vertical axis, that is, if:

$$x(n) = -x(-n)$$

it is called an odd signal.

Any signal $x(n)$ can be expressed as the sum of an even component $x_e(n)$ and an odd component $x_o(n)$ such that:

$$x(n) = x_e(n) + x_o(n).$$

The even and odd components of the signal can be obtained as:

$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right] \tag{1.6}$$

and

$$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]. \tag{1.7}$$

**Example 1.1** *Consider the following discrete-time signal*:

$$x(n) = \begin{cases} 1, 0 \le n \le 4 \\ 0, \textit{otherwise}. \end{cases}$$

*The even and odd components of this signal can be obtained, respectively, from Equations 1.6 and 1.7, yielding*:

$$x_e(n) = \begin{cases} \frac{1}{2}, 1 \le |n| \le 4 \\ 1, \ n = 0 \\ 0, \ \textit{otherwise} \end{cases}$$

and

$$x_o(n) = \begin{cases} \frac{1}{2}, & 1 \le n \le 4 \\ -\frac{1}{2}, & -4 \le n \le -1 \\ 0, & \textit{otherwise}. \end{cases}$$

*Signals* $x(n)$, $x_e(n)$ *and* $x_o(n)$ *for this example are shown in Figure 1.3.*

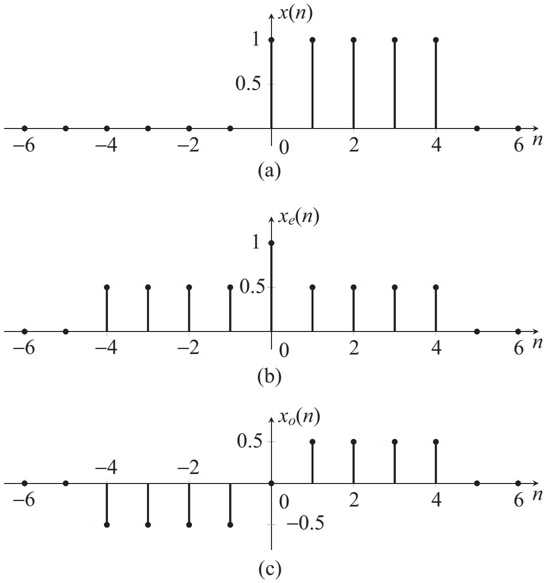


**Figure 1.3.** (a) Signal $x(n)$, (b) $x_e(n)$, even component of $x(n)$, (c) $x_o(n)$, odd component of $x(n)$.

### 1.2.4 OPERATIONS ON THE INDEPENDENT VARIABLE

Discrete-time signals are functions of the discrete variable $n$. Therefore, all operations defined for functions, such as sum and multiplication, are also valid for signals. More specific to our interest is how signals are affected by operations performed on the independent variable, which we examine next.

#### 1.2.4.1 Time Shifting

Time shifting of a signal is accomplished by replacing the independent variable $n$ by $n - n_0$. If $n_0$ is a positive integer, the signal is delayed; otherwise, (negative $n_0$) the signal is advanced. Delaying means that the signal is right-shifted, whereas advancing implies a left shift.

**Example 1.2.** *Consider* $v(n) = \frac{2}{5}n^2 + \frac{1}{5}n$. *Shifting* $v(n)$ *by* $n_0 = +2$ *means to delay it by 2*:

$$g(n) = v(n - n_0) = v(n-2)$$

$$g(n) = \frac{1}{5}[2(n-2)^2 + (n-2)] = \frac{1}{5}(2n^2 - 7n + 6)$$

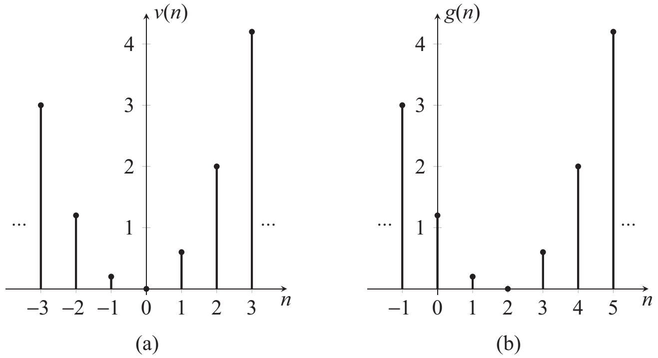


**Figure 1.4.** Time shifting: (a) original signal $v(n)$, (b) delayed signal $g(n) = v(n–2)$.

*A segment of the delayed signal* $g(n) = v(n-2)$ *is illustrated Figure 1.4.*

### *1.2.4.2 Time Reversal*

Time reversing a signal consists in changing the sign of the independent variable $n$. Thus, the time-reversed version of a discrete-time signal $x(n)$ is $x(-n)$. This reversion has the effect of reflecting the signal about the vertical axis.

**Example 1.3.** *Let us time-revert the signal* $g(n) = \frac{1}{5}(2n^2 - 7n + 6)$ *from Example 1.2*:

$$g(-n) = \frac{1}{5}[2(-n)^2 + -7(-n) + 6]$$

$$g(-n) = \frac{1}{5}(2n^2 + 7n + 6)$$

*Both* $g(n)$ *and* $g(-n)$ *are illustrated in Figure 1.5.*

### *1.2.4.3 Time Scaling*

Time scaling by an integer factor $\alpha$ is achieved by multiplying or dividing the independent variable $n$ by $\alpha$, with $|\alpha| > 1$. Multiplication implies in time compression, whereas division implies in time expansion. Therefore,

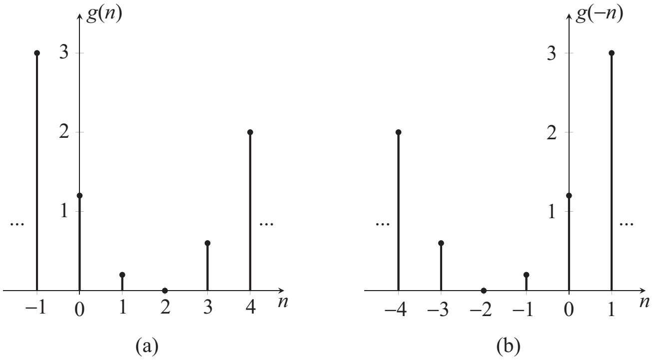


**Figure 1.5.** Time reversal: (a) original signal $g(n)$, (b) reversed signal $g(-n)$.

$x(\alpha n)$ and $x(n/\alpha)$ are, respectively, time-compressed and time-expanded versions of $x(n)$.

For discrete-time, compression corresponds to sampling the signal at rate α, meaning that, for each α samples, one is preserved and the others are discarded. Inversely, expansion in discrete-time means that $\alpha - 1$ samples are inserted between two original ones. The values of the inserted samples must be estimated by some interpolation technique.

**Example 1.4.** *Consider the discrete-time sinusoidal signal* $x(n) = \sin(\frac{\pi}{6})n$, (*more on sinusoidal signals in Section 1.4*). *Compressing this signal by a factor* $\alpha = 2$, *we obtain the signal* $g(n)$ *given by*:

$$g(n) = x(2n) = \sin\left(\frac{2\pi n}{6}\right) = \sin\left(\frac{\pi n}{3}\right).$$

*Expansion of* $x(n)$ *by factor 2 corresponds to signal* $v(n)$:

$$v(n) = \begin{cases} x(n/2) = \sin\left(\frac{\pi}{6}\cdot\frac{n}{2}\right) = \sin\left(\frac{\pi n}{12}\right), & \text{n even} \\ 0, & \text{n odd} \end{cases}$$

*Figure 1.6 shows signals* $x(n)$, $g(n)$, *and* $v(n)$ *for* $0 \le n \le 12$. *One full cycle of* $x(n)$ *fits in that interval, as shown in Figure 1.6*(*a*). *Due to time compression, two cycles of* $g(n)$ *fit in the same interval. However, it can be seen in Figure 1.6*(*b*) *that some samples of* $x(n)$ *are lost to produce* $g(n)$. *In this example expansion is accomplished by inserting zero between each two original values of* $x(n)$. *As a result, only half cycle of* $v(n)$ *fits in the*

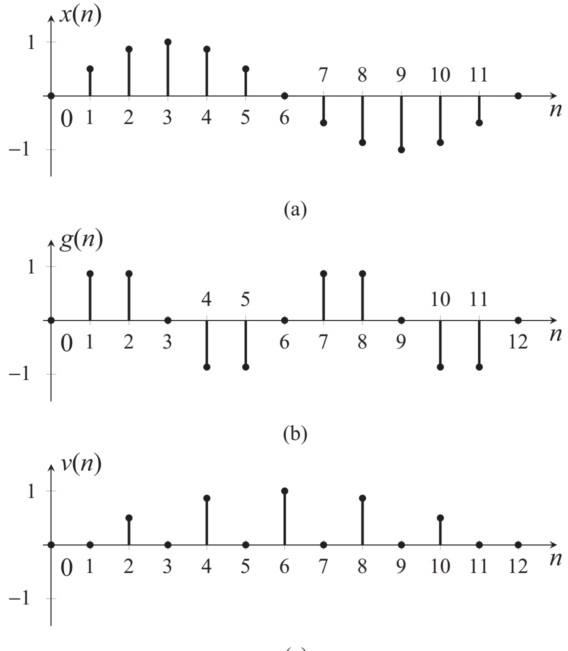


**Figure 1.6.** Time scaling: (a) original signal $x(n) = \sin\left(\frac{\pi n}{6}\right)$, (b) compressed signal $g(n)$, (c) expanded signal $v(n)$.

*interval, as shown in Figure 1.6(c). Other techniques can be used to fill the gaps between samples, such as replication of the previous value or linear interpolation.*

## 1.3 THE UNIT STEP AND UNIT IMPULSE SIGNALS

The discrete-time unit step signal is called $u(n)$ and is defined as:

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \tag{1.8}$$

as shown in Figure 1.7(a).

The discrete-time unit impulse signal called $\delta(n)$ is shown in Figure 1.7(b) and is defined as:

$$\delta(n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{1.9}$$

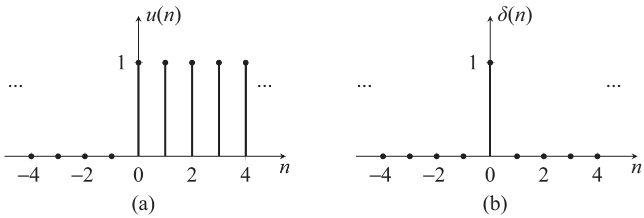


**Figure 1.7.** (a) Unit step signal $u(n)$, (b) unit impulse signal $\delta(n)$.

Signals $\delta(n)$ and $u(n)$ are related by the following equation:

$$u(n) = \sum_{k=-\infty}^{n} \delta(k). \tag{1.10}$$

Signals $\delta(n)$ and $u(n)$ play a very important role in the study of digital signals and systems. These signals can be used to express other discrete-time signals, as shown in the following examples.

**Example 1.5.** *Consider the following signal*:

$$x(n) = \begin{cases} \left(\frac{1}{2}\right)^{n-2}, & n \geq 2 \\ 0, & n < 2 \end{cases}$$

*shown in Figure 1.8. An alternative expression for this signal can be obtained with a shifted unit step signal as*:

$$x(n) = \left(\frac{1}{2}\right)^{n-2} u(n-2).$$

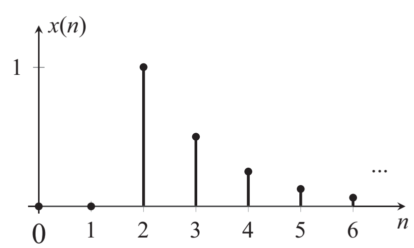


**Figure 1.8.** Signal $x(n)$ for Example 1.5.

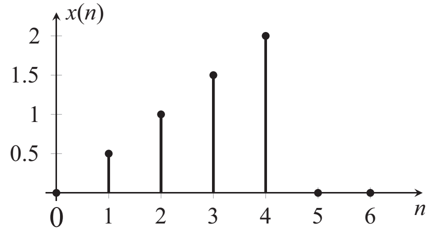


**Figure 1.9.** Signal $x(n)$ for Example 1.6.

**Example 1.6.** *Consider the finite duration signal shown in Figure 1.9 and given by*:

$$x(n) = \begin{cases} \dfrac{n}{2}, & 0 \le n \le 4 \\ 0, & otherwise. \end{cases}$$

*Signal* $x(n)$ *can be expressed using two unit steps, one of which shifted to* $n = 5$

$$x(n) = \frac{n}{2}\left[u(n) - u(n-5)\right].$$

*Alternatively, this signal can be expressed in terms of a sum of shifted weighted impulses*:

$$x(n) = 0\delta(n) + 0.5\delta(n-1) + \delta(n-2) + 1.5\delta(n-3) + 2\delta(n-4).$$

## 1.4 COMPLEX EXPONENTIAL AND SINUSOIDAL SIGNALS

A discrete-time complex exponential signal is defined as:

$$x(n) = z^n = r^n e^{j\omega n} \tag{1.11}$$

where $z$ is a complex parameter given by $z = re^{j\omega}$. When $\omega = 0$, the imaginary component of $z$ is null, and the complex exponential becomes a real exponential signal $x(n) = r^n$. In this case, for $|r| > 1$, we have an increasing exponential, whereas for $|r| < 1$, $x(n)$ is a decreasing exponential.

Of particular interest is the case $|r|=1$ that implies $z=e^{j\omega_0}$. From Euler's formula, we have:

$$x(n)=e^{j\omega_0 n}=\cos(\omega_0 n)+j\sin(\omega_0 n) \tag{1.12}$$

where

$$\cos(\omega_0 n)=\frac{1}{2}\left(e^{j\omega_0 n}+e^{-j\omega_0 n}\right), \tag{1.13}$$

and

$$\sin(\omega_0 n)=\frac{1}{2j}\left(e^{j\omega_0 n}-e^{-j\omega_0 n}\right). \tag{1.14}$$

Equations 1.13 and 1.14 define, respectively, the discrete-time cosine and sine signals, jointly known as sinusoidal signals. A general expression for this class of signals is given by:

$$x(n)=\cos\left(\omega_0 n+\phi\right), \tag{1.15}$$

where $\phi$ is the phase of the sinusoidal signal. For $\phi=0$, we have $x(n)=\cos(\omega_0 n)$, whereas $\phi=\frac{\pi}{2}$ implies that $x(n)=\sin(\omega_0 n)$.

Figure 1.10 shows one period ( $N=12$ ) of $x(n)=\sin\left(\frac{\pi}{6}n\right)$.

### 1.4.1 PERIODICITY OF DISCRETE-TIME EXPONENTIALS

Assuming the existence of a period $N$ such that $x(n)=x(n+N)$ (Section 1.2.1) for the complex exponential defined in Equation 1.12, we have: $x(n+N)=e^{j\omega_0(n+N)}=e^{j\omega_0 n}\cdot e^{j\omega_0 N}=x(n)$

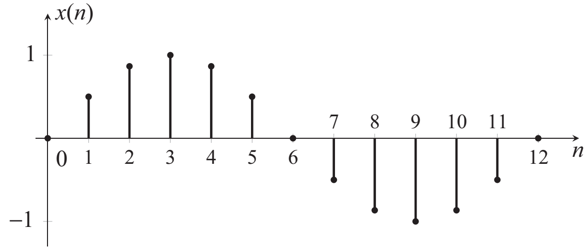


**Figure 1.10.** One period of sinusoidal signal $x(n)=\sin\left(\frac{\pi}{6}n\right)$.

which leads to:

$$e^{j\omega_0 n} \cdot e^{j\omega_0 N} = e^{j\omega_0 n}. \tag{1.16}$$

Equation 1.16 implies that, if $x(n)$ is periodic, then $\omega_0 N$ must be such that $e^{j\omega_0 N} = 1$. This condition is satisfied for $\omega_0 N = 2\pi m$, where $m$ is an integer, that is, for values of $\omega_0 N$ that are integer multiples of $2\pi$. The fundamental period is, thus, given by:

$$\frac{N}{m} = \frac{2\pi}{\omega_0} \tag{1.17}$$

or

$$\frac{\omega_0}{2\pi} = \frac{m}{N}. \tag{1.18}$$

The fundamental frequency defined as:

$$\omega_0 = \frac{2\pi}{N} m \; rad \tag{1.19}$$

is inversely proportional to the fundamental period. As $N$ is dimensionless $\omega_0$ is expressed in radian, the dimension of 2π.

Equation 1.18 states that the discrete-time complex exponential $e^{j\omega_0 n}$ is periodic if and only if $\frac{\omega_0}{2\pi}$ is a rational number, that is, a number that can be expressed as the ratio between two integers $\left(\frac{m}{N}\right)$. Due to the equivalence between the exponential and sinusoidal signals, expressed in Equations 1.12, 1.13, and 1.14, the condition of Equation 1.18 also applies to the latter class of signals. That periodicity condition is characteristic of discrete-time signals, as continuous-time complex exponentials, as well as cosines and sines are always periodic.

**Example 1.7.** *Figure 1.11 shows a segment of signal* $x(n) = \cos\left(\frac{2}{3}n\right)$. *Despite being a cosine, for this signal,* $\omega_0 = \frac{2}{3}$ and $\frac{\omega_0}{2\pi} = \frac{1}{3\pi}$, *which is not a rational number. Signal* $x(n)$, *therefore, is non-periodic.*

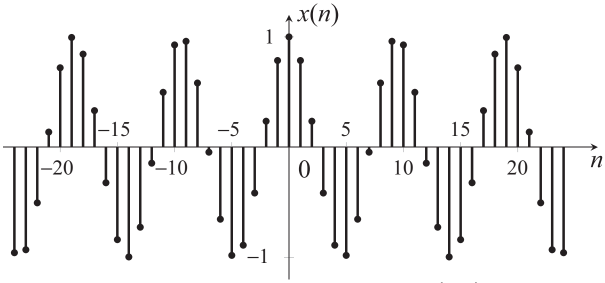


**Figure 1.11.** Non-periodic cosine $x(n) = \cos\left(\frac{2}{3}n\right)$ for Example 1.7.

### 1.4.2 HARMONICALLY RELATED EXPONENTIALS

A set of discrete-time complex exponentials is harmonic if all signals in the set share a common period $N$. This implies that those signals' frequencies are integer multiples of $\frac{2\pi}{N}$. Thus, the set of harmonically related exponentials is defined as:

$$\phi_k(n) = e^{j\frac{2\pi}{N}kn}, k = 0, \pm 1, \pm 2, \cdots. \tag{1.20}$$

Due to the periodicity of the discrete complex exponential, we have

$$\phi_{k+N}(n) = e^{j(k+N)\frac{2\pi}{N}n} = e^{jk\frac{2\pi n}{N}} e^{j2\pi n} = \phi_k(n). \tag{1.21}$$

Equation 1.21 shows that the signals are periodic with respect to the variable $k$ with period $N$. This periodicity implies that there are only $N$ distinct harmonic signals in the set defined by Equation 1.20; these signals are $\phi_0(n)=1$, $\phi_1(n) = e^{j\frac{2\pi n}{N}}$, $\phi_2(n) = e^{j\frac{4\pi n}{N}}, \cdots, \phi_{N-1}(n) = e^{-j\frac{2\pi n}{N}}$. Any value of $k$ outside the interval $[0, N-1]$ leads to one of the signals aforementioned, that is, $\phi_N(n) = \phi_0(n), \phi_{N+1}(n) = \phi_1(n), \phi_{N+2}(n) = \phi_2(n)$, and so on. Thus, the set of harmonically related discrete-time exponentials is finite of size $N$. This is distinct from the continuous-time case where all harmonically related exponentials of the form $e^{j\frac{2\pi}{T}kt}$ ($k$ is an integer) are unique, and therefore, the set of signals $\phi_k(t)$ is infinite.

Harmonically related exponentials form the base for representing periodic discrete-time signals by Fourier series, as described in Chapter 2.

## 1.5 DISCRETE-TIME SYSTEMS

A system is a transformation that modifies signals to adapt them to an intended application. This operation can be expressed as:

$$y(n) = T\left[x(n)\right]. \tag{1.22}$$

Figure 1.12 shows the graphical representation of a discrete-time system. Signals $x(n)$ and $y(n)$ are the input and output signal, respectively. Transformation $T[\cdot]$ can be implemented as a hardware device that processes signal samples in real time or as a software-implemented operation.

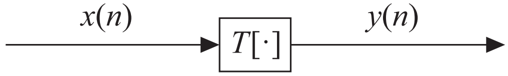


**Figure 1.12.** Representation of a discrete-time system.

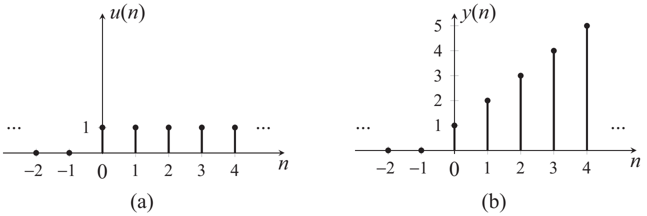


**Figure 1.13.** Accumulator of Example 1.8: (a) input signal $x(n) = u(n)$, (b) output signal $y(n)$.

**Example 1.8.** *The accumulator is a system that adds up past and present values of* $x(n)$ *to produce* $y(n)$. *Its characteristic transformation is given by*:

$$y(n) = T\left[x(n)\right] = \sum_{k=-\infty}^{n} x(k). \tag{1.23}$$

*Figure 1.13 shows the output signal* $y(n)$ *produced by the accumulator of Equation 1.23 for an unit step signal applied at the input.*

### *1.5.1 PROPERTIES OF DISCRETE-TIME SYSTEMS*

The most relevant properties of discrete-time systems are described next.

#### *1.5.1.1 Linearity*

A linear system simultaneously obeys the homogeneity and additivity properties, that is:

$$\text{Homogeneity}: \quad T\left[ax(n)\right] = aT\left[x(n)\right]$$

$$\text{Additivity}: \quad T\left[x_1(n) + x_2(n)\right] = T\left[x_1(n)\right] + T\left[x_2(n)\right].$$

The combination of these two properties is known as the superposition property or superposition principle, which is the condition for linearity, expressed as:

$$\text{Linearity}: T\left[ax_1(n) + bx_2(n)\right] = aT\left[x_1(n)\right] + bT\left[x_2(n)\right]. \quad (1.24)$$

Thus, the superposition principle states that, if the input signal $x(n)$ to a linear system can be decomposed into a weighted sum (or linear combination) of components $x_1(n)$ and $x_2(n)$, the output of the system is the weighted sum of the outputs produced individually by each component. The sum weights (or coefficients of the linear combination) at the output are the same as in the input.

Equation 1.24 holds for any number of signals combined at the input:

$$T\left[\sum_{i=-\infty}^{\infty} b_i x_i(n)\right] = \sum_{i=-\infty}^{\infty} b_i T\left[x_i(n)\right] = \sum_{i=-\infty}^{\infty} b_i y_i(n) \quad (1.25)$$

where

$$y_i(n) = T\left[x_i(n)\right] \quad (1.26)$$

and $b_i$ are the linear combination coefficients.

Signals $y_i(n) = T\left[x_i(n)\right]$, $i = 0, \pm 1, \pm 2, \pm 3, \ldots$ in Equation 1.26 are the outputs produced by the system in response to the $x_i(n)$ components of input signal $x(n)$, respectively. Thus, Equations 1.25 and 1.26 imply that the output of a linear system to any linear combination of input signals is the linear combination of the outputs produced by each of the input components individually. The set of coefficients $b_i$ at the input is preserved at the output.

**Example 1.9.** *Consider the system characterized by*:

$$y(n) = T[x(n)] = 3x(n) + 5. \quad (1.27)$$

*To determine whether the system aforementioned is linear, it suffices to verify whether Equation 1.27 satisfies the superposition principle. Let us assume that $x_1(n)$ and $x_2(n)$ are two input signals and their respective outputs are $y_1(n) = 3x_1(n) + 5$ and $y_2(n) = 3x_2(n) + 5$. A linear combination of the two output signals is*:

$$ay_1(n) + by_2(n) = a[3x_1(n) + 5] + b[3x_2(n) + 5]$$

$$= 3[ax_1(n) + bx_2(n)] + 5(a+b) \tag{1.28}$$

*where a and b are constants.*

*Let the signal* $x_3(n) = ax_1(n) + bx_2(n)$ *, with the same constants a and b, be also input to the system to produce output* $y_3(n) = 3x_3(n) + 5$*. We then have*:

$$y_3(n) = 3x_3(n) + 5 = 3[ax_1(n) + bx_2(n)] + 5. \tag{1.29}$$

*Equations 1.28 and 1.29 are distinct, and therefore, the system is non-linear. Observe that the non-linearity of this system is due to the term* $5$ *added to* $3x(n)$ *in Equation 1.27. If this term was null, the system would became* $y(n) = 3x(n)$ *, which is a linear system.*

**Example 1.10.** *To determine the linearity of the accumulator defined in Equation 1.23, we follow the same procedure as in the previous example, that is*:

$$y_1(n) = \sum_{k=-\infty}^{n} x_1(k)$$

$$y_2(n) = \sum_{k=-\infty}^{n} x_2(k) \tag{1.30}$$

*where* $x_1(n)$ *and* $x_2(n)$ *are two input signals. Therefore, for constant coefficients a and b*:

$$ay_1(n) + by_2(n) = a\sum_{k=-\infty}^{n} x_1(k) + b\sum_{k=-\infty}^{n} x_2(k)$$

$$= \sum_{k=-\infty}^{n} \left[ax_1(k) + bx_2(k)\right]. \tag{1.31}$$

*For input signal* $x_3(n) = ax_1(n) + bx_2(n)$ *, the output* $y_3(n)$ *is given by*:

$$y_3(n) = \sum_{k=-\infty}^{n} x_3(k) = \sum_{k=-\infty}^{n} \left[ax_1(k) + bx_2(k)\right]. \tag{1.32}$$

*Equations 1.31 and 1.32 are equal, which means that the accumulator is a linear system.*

### *1.5.1.2 Shift Invariance*

Assuming that $x(n)$ and $y(n)$ are the input and output signals, respectively, for a shift invariant system:

$$T\left[x\left(n-n_0\right)\right]=y\left(n-n_0\right). \tag{1.33}$$

The characteristic transformation $T[\cdot]$ of a shift invariant system is not a function of the independent variable $n$; consequently, the system does not change with time. Thus, the output signal $y(n)$ does not depend on the particular value of $n$ at which $x(n)$ is applied to the system input.

To determine whether a system is shift-invariant, it suffices to verify whether $T[\cdot]$ satisfies Equation 1.33. If it does not, then the system is shift-varying.

**Example 1.11.** *The accumulator of Equation 1.23 is a shift-invariant system. To verify this, consider its characteristic equation*:

$$T\left[x\left(n-n_0\right)\right]=\sum_{l=-\infty}^{n} x\left(l-n_0\right). \tag{1.34}$$

*Making* $k=l-n_0$ *and replacing in the preceding equation, we obtain*:

$$T\left[x\left(n-n_0\right)\right]=\sum_{k=-\infty}^{n-n_0} x(k)=y\left(n-n_0\right).$$

**Example 1.12.** *An example of a shift-varying system is given by*:

$$y(n)=n^2x(n).$$

*As the input signal is multiplied by* $n^2$, *the characteristic transformation is a function of n, which makes the system shift-varying. Verification is reasonably straightforward. For the shifted input, the system output is*:

$$T[x(n-n_0)]=n^2x(n-n_0)$$

*and the shifted output* (*from Equation 1.33*) *is*:

$$y(n-n_0)=(n-n_0)^2x(n-n_0).$$

*Because* $y(n-n_0)$ *is not equal to* $T[x(n-n_0)]$, *the system is shift-varying.*

### 1.5.1.3 Causality

Causality implies that the output of the system at a given instant $n$ only depends on past and/or present values of the input signal. Thus, for a causal system, there is a cause-and-effect relationship between the input and output signals. There must be a present or past event at the input for any event produced at the system output. Causal systems are also known as *non-anticipative* as the output does not depend on future input values. Any physical system is causal because future inputs cannot be processed in the real world.

The accumulator is an example of a causal system, as its output is the sum of all past inputs and present input value.

### 1.5.1.4 Stability

A system is stable if a bounded input signal always results in a bounded output signal. Thus, for a stable system, we have:

$$|x(n)| \leq B \Rightarrow |y(n)| \leq L \cdot B \tag{1.35}$$

for finite $L$ and $B$.

It is easy to see that the accumulator is an unstable system, as it performs a sum of infinite values.

### 1.5.1.5 Invertibility

A system is invertible when the input signal can be recovered from the output signal. Thus, for any invertible system $T[\cdot]$, there exists an inverse system $G[\cdot]$ such that $G[\cdot] = T^{-1}[\cdot]$. Invertibility implies that distinct inputs produce distinct outputs; otherwise, input recovery would not be possible.

**Example 1.13.** *The accumulator is an example of invertible system. Its inverse is the system given by*:

$$G[x(n)] = x(n) - x(n-1).$$

## 1.6 LINEAR SHIFT-INVARIANT SYSTEMS

A system that simultaneously obeys the properties of linearity and shift invariance is a linear shift-invariant (LSI) system. The relevance of this class of systems comes from the fact that a large number of digital signal processing systems can be modeled as an LSI system within some practical limits. This is very convenient because, for LSI systems, the relationship between the input and output signals is given by an operation called convolution and the system is identified by its response to an input impulse signal, called impulse response. These characteristics provide a practical procedure to determine the output of an LSI system for any input signal, in the discrete-time domain, as well as in the frequency domain through the use of transforms.

### *1.6.1 IMPULSE RESPONSE AND DISCRETE-TIME CONVOLUTION*

The impulse response of a discrete-time LSI system is called $h(n)$. It is defined as the output signal produced in response to an impulse applied at the system input. Thus, for an LSI system characterized by transformation *T,*

$$h(n) = T[\delta(n)].$$

It can be shown that the output $y(n)$ of an LSI system to any input signal $x(n)$ is the result of the discrete-time convolution between $x(n)$ and the impulse response $h(n)$. This operation is defined as:

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k). \tag{1.36}$$

Equation 1.36 allows to determine the output of an LSI system for any input signal, provided that $h(n)$ is known. Thus, an LSI system is characterized or identified by its impulse response $h(n)$ as illustrated in Figure 1.14.

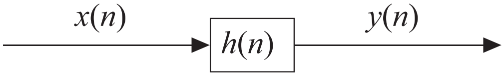


**Figure 1.14.** LSI system represented by its impulse response $h(n)$.

Computation of Equation 1.36 will not be examined here, as it is seldom performed in practice. Most often, in signal processing problems, time convolution is replaced by a frequency domain multiplication, as will be seen in Chapter 2. Derivation of the convolution sum (Equation 1.36), as well as details about its computation, can be found in signals and systems analysis textbooks.

The convolution is an operation that can be performed between any two signals. Thus, the convolution between signals $g(n)$ e $v(n)$ is expressed as:

$$g(n) * v(n) = \sum_{k=-\infty}^{\infty} g(k)v(n-k).$$

### 1.6.2 PROPERTIES OF LSI SYSTEMS

It is straightforward to show that convolution is a commutative operation:

$$g(n) * v(n) = v(n) * g(n) = \sum_{k=-\infty}^{\infty} v(k)g(n-k)$$

associative:

$$g(n) * \left[v(n) * z(n)\right] = \left[g(n) * v(n)\right] * z(n)$$

and distributive:

$$g(n) * \left[v(n) + z(n)\right] = \left[g(n) * v(n)\right] + \left[g(n) * z(n)\right].$$

The preceding properties have implications on the interconnections of LSI systems. Specifically, they imply that the order in which LSI systems are serially connected is irrelevant as illustrated in Figures 1.15 (a) and (b). The serial connection is also equivalent to a single system with $h(n)$ given by the convolution of the impulse responses of the connected systems, as in Figure 1.15 (c). A parallel connection of LSI systems is equivalent to a single system with $h(n)$ given by the sum of the impulse responses of the connected systems, as shown in Figure 1.16. Although illustrated for two systems, these equivalences are valid for any number of connected LSI systems.

In Section 1.5.1, we learned that a system is stable when a bounded input signal always produces a bounded output signal. For an LSI system,

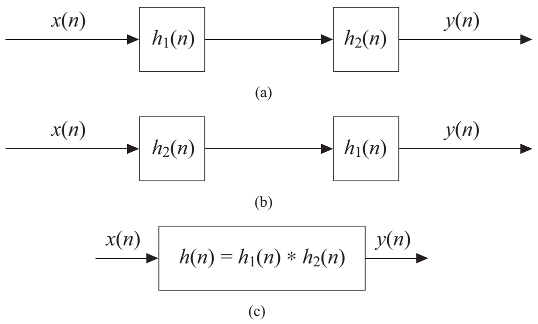


**Figure 1.15.** Equivalences among serial connections of LSI systems.

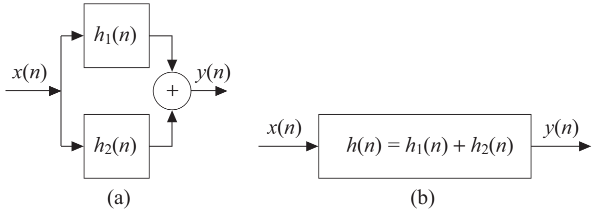


**Figure 1.16.** (a) Parallel connection of two LSI systems, (b) equivalent single system.

this property implies that the impulse response is absolutely summable, that is, $h(n)$ must obey:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty. \tag{1.37}$$

Thus, the impulse response of a stable LSI system is either a finite duration sequence or an infinite convergent sequence, both assuming finite values only.

**Example 1.14.** *The LSI system with impulse response* $h_1(n) = (1/2)^n u(n)$ *is stable, while the LSI system with impulse response* $h_2(n) = u(n)$ *is unstable. To verify for* $h_1(n)$, *we can calculate*:

$$\sum_{n=-\infty}^{\infty} |h_1(n)| = \sum_{n=0}^{\infty} (1/2)^n = 2 < \infty.$$

*Therefore,* $h_1(n)$ *is absolutely summable. For* $h_2(n)$, *we have*:

$$\sum_{n=-\infty}^{\infty} |h_2(n)| = \sum_{n=0}^{\infty} 1 = \infty. \tag{1.38}$$

*Thus, $h_2(n)$ is not absolutely summable.*

The impulse response of a causal LSI system must obey:

$$h(n) = 0, \quad n < 0. \tag{1.39}$$

This property derives from the fact that causal systems are non-anticipative. As the impulse signal is null for $n < 0$, no event occurs at the input of the LSI system before $n = 0$ to cause an output event. Thus, $h(n)$ must be 0 for $n < 0$.

## 1.7 CHAPTER OVERVIEW

In this chapter, we introduce the basic concepts related to discrete-time signals and systems. We start by defining discrete-time and digital signals and analyze the main properties presented by those signals. The unit impulse and unit step signals, both of which play a relevant role in the area of digital signal processing, are also defined, and some of their applications are exemplified. We also introduce the complex exponential and sinusoidal signals, showing the relationship between them, as well as how the concepts of period and frequency apply to these two types of discrete-time signals. We then introduce discrete-time systems and define their main properties.

The final section of this chapter is dedicated to the class of LSI systems used to model most systems of practical interest. The input and output for LSI systems are related through the convolution operation, which is defined and analyzed. We end the chapter by describing the properties of LSI systems and how they characterize the corresponding impulse responses.