CHAPTER 2

Discrete-Time Signal Transforms

2.1 INTRODUCTION

Transforms are mathematical tools that allow representing signals and systems in domains other than time. In alternate domains, tasks involving digital signals and systems may be considerably easier to perform than equivalent tasks performed in discrete-time. As an example, let us recall that linear shift invariant (LSI) systems are identified in time by impulse response $h(n)$, and that the output signals for such systems are given by the convolution between the input signal and $h(n)$ as defined in Equation 1.36, that is:

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k). \qquad (2.1)$$

For real-life signals, calculation of Equation 2.1 is usually not an option. In the frequency domain, determination of the system response is much simpler because convolution is replaced by multiplication between the signals transforms. For practical problems, except perhaps in very special cases, specification, design, and analysis of digital systems are performed in domains other than time with the use of transforms. The existence of highly efficient algorithms implementing some transforms, which are examined in Chapter 4, has contributed to the use of transforms in most digital signal processing tasks, even for real-time applications.

In this chapter, we initially introduce the Fourier series that provides the basis for understanding transforms. We then focus on the Fourier and z transforms for discrete-time signals. Those are the two main transforms utilized in digital signal processing.

2.2 SIGNALS AS LINEAR COMBINATIONS OF COMPLEX EXPONENTIALS

Consider that the input to an LSI system is a complex exponential signal $x(n)=z^n$ with z a complex valued parameter given by $z=re^{j\omega}$. According to Equation 2.1, the resulting output signal $y(n)$ is:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)z^{(n-k)} = z^n \sum_{k=-\infty}^{\infty} h(k)z^{-k}$$

or

$$y(n) = z^n H(z) \tag{2.2}$$

where $H(z) = \sum_{k=-\infty}^{\infty} h(k)z^{-k}$.

Equation 2.2 means that the output of an LSI system with impulse response $h(n)$ to a complex exponential $x(n) = z^n$ is the input signal itself multiplied by a complex constant $H(z)$. As $H(z)$ is determined from $h(n)$, it characterizes (or identifies) the system. In Section 2.5, it is shown that $H(z)$, known as the system function, is the z transform of $h(n)$.

Now, let us consider that a signal is expressed as a linear combination of complex exponentials, that is, $x(n) = \Sigma_k a_k z_k^n$, with $z_k = r_k e^{j\omega_k}$. Applying Equation 2.2 and the superposition principle (Equation 1.24), it is easy to conclude that the output of an LSI system to this signal is:

$$y(n) = \sum_k b_k z_k^n. \tag{2.3}$$

Equation 2.3 shows that similarly to $x(n)$, $y(n)$ is also a linear combination of the set $\{z_k\}$ of complex exponentials. The new coefficients are $b_k = a_k H(z_k)$. Both signals are characterized by the respective sets of coefficients, $\{a_k\}$ for $x(n)$ and $\{b_k\}$ for $y(n)$. Thus, knowledge of $H(z)$ is all needed to determine the system output to an input signal expressed as a linear combination of complex exponentials.

The complex exponential $z_k = r_k e^{j\omega_k}$ is characterized by its frequency ω_k. Therefore, ω_k can be viewed as an independent variable for representing signals, as long as those signals can be expressed in the form of Equation 2.3. As a coefficient is assigned to each frequency value, coefficients become the dependent variable identifying each signal. That corresponds to a change of signal representation domain, from time to frequency, which constitutes the base for the theory of transforms described in this chapter.

2.3 FOURIER SERIES FOR PERIODIC SIGNALS

The Fourier series expresses a periodic signal as a linear combination of harmonically related complex exponentials, defined as:

$$\phi_k(n) = e^{j\frac{2\pi}{N}kn}, k = 0, \pm 1, \pm 2, \cdots. \quad (2.4)$$

In Chapter 1, we saw that there are only N distinct signals in the set defined by Equation 2.4, and that they are all periodic with period N. Therefore, any linear combination of those signals is also periodic with period N.

The Fourier series for a periodic signal $\hat{x}(n)$ with period N is given by the synthesis equation:

$$\hat{x}(n) = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn}, \quad n = 0, 1, ..., N-1. \quad (2.5)$$

Coefficients a_k are calculated by the analysis equation:

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} \hat{x}(n) e^{-j\frac{2\pi}{N}kn} \quad k = 0, 1, ..., N-1. \quad (2.6)$$

As both $\hat{x}(n)$ and a_k are periodic with period N, only N values of each need to be calculated. Due to periodicity, the sums in Equations 2.5 and 2.6 can be calculated over any interval of N consecutive values of a_k and $\hat{x}(n)$, respectively.

Example 2.1 *Using Euler's formula, the sinusoidal signal* $\hat{x}(n) = \sin(\frac{2\pi}{N}n)$ *is expressed as*:

$$\hat{x}(n) = \frac{1}{2j}e^{j\frac{2\pi}{N}n} - \frac{1}{2j}e^{-j\frac{2\pi}{N}n}.$$

This expression corresponds to the Fourier series of $\hat{x}(n)$*, as it is a linear combination of two complex exponentials. Comparing with Equation 2.5, we can conclude that* $a_1 = \frac{1}{2j}$*,* $a_{-1} = a_{N-1} = -\frac{1}{2j}$*, and* $a_k = 0$ *for all other values of* $k = 1, 2, ..., N-1$.

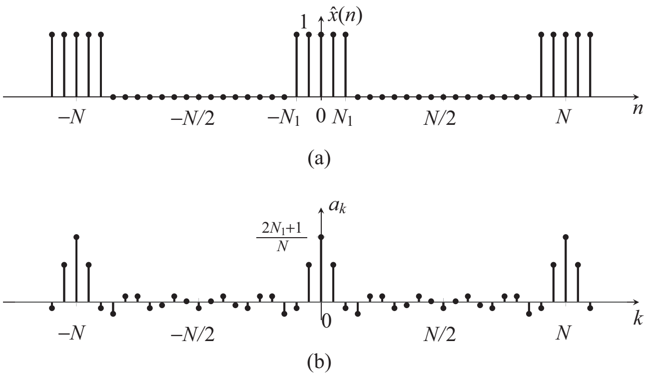

Figure 2.1. Example 2.2 for $N_1 = 2$ and $N = 20$: (a) periodic impulse train, (b) Fourier series coefficients.

Example 2.2 *Figure 2.1(a) shows the periodic impulse train, with period N. Utilizing Equation 2.6, the Fourier series coefficients for this signal can be obtained*:

$$a_k = \begin{cases} \dfrac{1}{N}\dfrac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(2\pi k/2N)}, & k \neq 0, \pm N, \pm 2N, \ldots \\ \\ \dfrac{2N_1 + 1}{N}, & k = 0, \pm N, \pm 2N, \ldots \end{cases}$$

The sequence a_k is periodic with period N, as shown in Figure 2.1(b).

2.4 THE FOURIER TRANSFORM

For a non-periodic signal $x(n)$, representation in the frequency domain is provided by the Fourier transform, defined as:

$$\mathscr{F}[x(n)] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}. \tag{2.7}$$

The inverse Fourier transform is the operation that recovers the signal in the time domain, that is,

$$x(n) = \mathscr{F}^{-1}[X(e^{j\omega})] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega. \tag{2.8}$$

The symbol $\mathscr{F}[.]$ represents the Fourier transform operator. Equations 2.7 and 2.8 are known as the analysis and synthesis equations of the Fourier transform, respectively. The relation between them is often depicted as:

$$x(n) \overset{\mathscr{F}}{\Longleftrightarrow} X(e^{j\omega}).$$

Because the integral is a linear operation, Equation 2.8 expresses the signal as a linear combination of complex exponentials. The difference to the Fourier series is that the exponentials are no longer constrained to take values on the discrete and finite harmonic set $\phi_k(n)$. As ω varies over a continuum of frequencies from $-\infty$ to $+\infty$, the Fourier transform can be used to represent any signal, periodic or not, for which Equation 2.8 converges. This convergence, and therefore, the existence of the Fourier transform is assured for any absolutely summable signal, that *is*, any signal that obeys $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$. A similarly sufficient condition is that $x(n)$ possesses finite energy, that is, $\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$.

Example 2.3 *Signal $x(n) = a^n u(n)$ with $|a| < 1$ is an absolutely summable decreasing sequence; therefore, its Fourier transform converges. From Equation 2.7*:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

which corresponds to the sum of the terms of infinite geometric series with ratio $q = ae^{-j\omega} < 1$. Therefore:

$$X(e^{j\omega}) = \frac{1}{1-q} = \frac{1}{1-ae^{-j\omega}}.$$

The magnitude and phase of $X(e^{j\omega})$ are plotted in Figure 2.2.

Table 2.1 shows the Fourier transforms pairs for some basic signals.

2.4.1 PROPERTIES OF THE FOURIER TRANSFORM

The main properties of discrete-time Fourier transform are presented next. Most of the demonstrations are left to the reader as an exercise.

Table 2.1. Basic Fourier transform pairs

Signal	Fourier transform
1	$2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)$
$\delta(n)$	1
$\delta(n - n_0)$	$e^{-j\omega n_0}$
$\sum_{k=-\infty}^{\infty} \delta(n - kN)$	$\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{N})$
$u(n)$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega - 2\pi k)$
$a^n u(n), \mid a \mid < 1$	$\frac{1}{1-ae^{-j\omega}}$
$(n+1)a^n u(n), \mid a \mid < 1$	$\frac{1}{(1-ae^{-j\omega})^2}$
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)$
$\cos(\omega_0 n)$	$\pi \sum_{l=-\infty}^{\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$
$\sin(\omega_0 n)$	$\frac{\pi}{j} \sum_{l=-\infty}^{\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$
$x(n) = \begin{cases} 1, & \mid n \mid \leq N_1 \\ 0, & \mid n \mid > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$
$\frac{\sin \omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \mid \omega \mid \leq \omega_c \\ 0, & \omega_c < \mid \omega \mid \leq \pi \end{cases}$

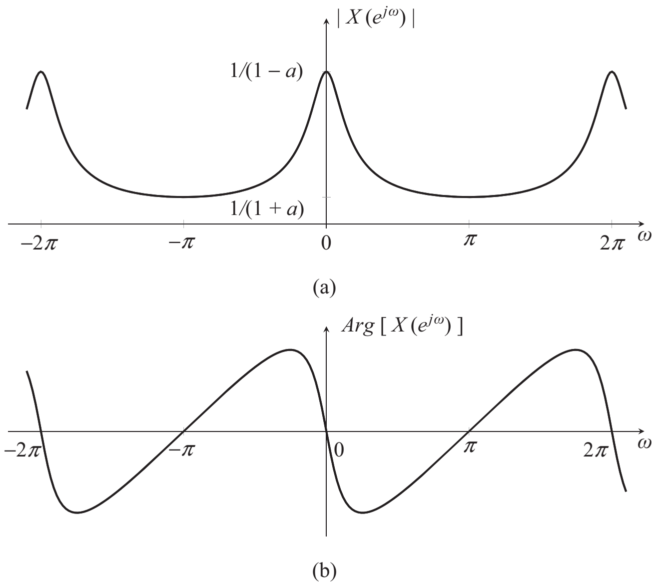

Figure 2.2. Example 2.3: (a) magnitude and (b) phase of $X(e^{j\omega})$.

Linearity

The Fourier transform is a linear operator, as both its direct and inverse forms are defined by linear operations sum and integral, respectively. This property implies that the Fourier transform obeys the superposition principle, That is, the transform of a linear combination of signals is the linear combination of the signals individual transforms. That is,

$$ax(n)+by(n)\overset{\mathcal{F}}{\Longleftrightarrow} aX(e^{j\omega})+bY(e^{j\omega})$$

where $X(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of $x(n)$ and $y(n)$, respectively, and a and b are constant coefficients. This equation holds for linear combinations of any number of signals.

Periodicity

The Fourier transform of a discrete-time signal is a continuous and periodic function of frequency ω, with period 2π. Thus:

$$X\left[e^{j(\omega+2\pi)}\right]=X\left(e^{j\omega}\right).$$

This property derives from the fact that $X(e^{j\omega})$ is a linear combination of complex exponentials $e^{-j\omega n}$. As those exponentials are all periodic with period 2π, so must be $X(e^{j\omega})$.

Time and Frequency Shifting

The effect of shifting a signal $x(n)$ by n_0 is to multiply the signal Fourier transform $X(e^{j\omega})$ by exponential $e^{j\omega n_0}$, that is,

$$x(n-n_0)\overset{\mathcal{F}}{\Longleftrightarrow} X(e^{j\omega})e^{-j\omega n_0}.$$

Because $\left|e^{-j\omega n_0}\right|=1$ and $Arg\left[e^{-j\omega n_0}\right]=-\omega n_0$, we have that:

$$|X(e^{j\omega})e^{-j\omega n_0}|=|X(e^{j\omega})|.|e^{-j\omega n_0}|=|X(e^{j\omega})|$$

$$Arg\left[X(e^{j\omega})e^{-j\omega n_0}\right]=Arg\left[X(e^{j\omega})\right]+Arg\left[e^{-j\omega n_0}\right]$$

$$=Arg\left[X(e^{j\omega})\right]-\omega n_0.$$

Therefore, the frequency domain effect of time shifting $x(n)$ is to add a phase that is a linear function of ω to its Fourier transform. The magnitude of $X(e^{j\omega})$ is not altered. This is in agreement with the fact that time shifting does not affect the shape of the signal, but only changes the signal position in time.

Frequency shifting the Fourier transform by ω_0 has the dual effect, that is, the signal is multiplied by an exponential in time:

$$e^{j\omega_0 n}x(n) \overset{\mathcal{F}}{\Longleftrightarrow} X[e^{j(\omega-\omega_0)}].$$

Conjugation and Conjugate Symmetry

For a signal $x(n)$ with Fourier transform $X(e^{j\omega})$, the conjugation property states that:

$$x^*(n) \overset{\mathcal{F}}{\Leftrightarrow} X^*(e^{-j\omega}),$$

where $x^*(n)$ is complex conjugate of $x(n)$.

If the signal $x(n)$ is real, it results that its Fourier transform presents conjugate symmetry, that is,

$$X(e^{j\omega}) = X^*(e^{-j\omega}).$$

Considering that $X(e^{j\omega}) = Re[X(e^{j\omega})] + jIm[X(e^{j\omega})]$, it can be shown that this property implies that $Re[X(e^{j\omega})]$ is an even function of ω and $Im[X(e^{j\omega})]$ is an odd function of ω. As $X(e^{j\omega})=|X(e^{j\omega})|e^{jArg[X(e^{j\omega})]}$, we arrive at:

$$|X(e^{j\omega})| = |X(e^{-j\omega})|$$

$$Arg[X(e^{j\omega})] = -Arg[X(e^{-j\omega})].$$

Thus, for any real signal $x(n)$, $|X(e^{j\omega})|$ is an even function of ω, while $Arg[X(e^{j\omega})]$ is an odd function of ω. Additionally, it can be shown that:

$$E[x(n)] \overset{\mathcal{F}}{\Longleftrightarrow} Re\left[X\left(e^{j\omega}\right)\right]$$

$$O[x(n)] \overset{\mathcal{F}}{\Longleftrightarrow} Im\left[X\left(e^{j\omega}\right)\right]$$

where $E[x(n)]$ and $O[x(n)]$ are the even and odd components of $x(n)$, respectively.

Convolution

Given two signals $h(n)$ and $x(n)$ with Fourier transforms $H(e^{j\omega})$ and $X(e^{j\omega})$, respectively, it can be demonstrated that:

$$y(n) = h(n) * x(n) \overset{\mathcal{F}}{\Longleftrightarrow} Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}),$$

where $h(n)*x(n)$ indicates the convolution between the two signals.

As shown in Chapter 1, the convolution expresses the input/output relationship for LSI systems. Thus, if $h(n)$ is the impulse response of an LSI system and $x(n)$ is the input signal, the output of the system can be obtained in the frequency domain by multiplying the respective Fourier transforms. The system frequency response is, therefore, given by $H(e^{j\omega}) = Y(e^{j\omega}) / X(e^{j\omega})$

Multiplication

This is the dual of the convolution property, asserting that multiplication of two signals in discrete-time corresponds to performing a periodic convolution of their Fourier transforms in frequency. Thus, consider signals $g(n)$ and $w(n)$ with Fourier transforms $G(e^{j\omega})$ and $W(e^{j\omega})$, respectively, and $z(n) = g(n).w(n)$ with Fourier transform $Z(e^{j\omega})$. The multiplication property states that:

$$Z\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{2\pi} G\left(e^{j\theta}\right)W\left(e^{j(\omega-\theta)}\right)d\theta. \tag{2.9}$$

Equation 2.9 expresses the periodic convolution between $G(e^{j\omega})$ and $W(e^{j\omega})$. As both functions are periodic with period 2π, so is $Z(e^{j\omega})$. The integral can be calculated over any frequency interval of length 2π.

Time Reversal

For $x(n)$ and its Fourier transform $X(e^{j\omega})$, we have:

$$x(-n) \overset{\mathcal{F}}{\Longleftrightarrow} X(e^{-j\omega}).$$

Differentiation in Frequency

For $x(n)$ and its Fourier transform $X(e^{j\omega})$, it can be shown that:

$$nx(n) \overset{\mathcal{F}}{\Longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.$$

Parseval's Relation

Parseval established the equivalence between the energy of a signal in time and in frequency as:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 \, d\omega.$$

Because of this relation, $|X(e^{j\omega})|^2$ is known as the *energy density spectrum* of signal $x(n)$.

The main properties of the Fourier transform are summarized in Table 2.2.

2.4.2 FOURIER TRANSFORM ANALYSIS OF LSI SYSTEMS

Difference equations with constant coefficients provide a convenient and much used way of representing LSI systems in the discrete-time domain. Those equations express the output of the system as a linear combination of past and present input values, as well as past output values. The general form of a constant coefficients difference equations is:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{2.10}$$

where $x(n)$ and $y(n)$ are respectively the input and output signals for the LSI system defined by coefficients a_k and b_k. In Chapter 5, it is shown how difference equations can be used to characterize FIR and IIR digital filters. From Equation 2.10, the output of the LSI system is expressed as:

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k)$$

where for simplicity we make $a_0 = 1$.

Applying the Fourier transform and its linearity and time shifting properties to both sides of Equation 2.10 yields

$$Y(e^{j\omega})\sum_{k=0}^{N} a_k e^{-jk\omega} = X(e^{j\omega})\sum_{k=0}^{M} b_k e^{-jk\omega}. \tag{2.11}$$

Table 2.2. Properties of the Fourier transform

Property	Signal	Fourier transform
	$x(n)$	$X(e^{j\omega})$ periodic with period 2π
	$y(n)$	$Y(e^{j\omega})$ periodic with period 2π
Linearity	$ax(n) + by(n)$	$aX(e^{j\omega}) + bY(e^{j\omega})$
Time shifting	$x(n - n_0)$	$X(e^{j\omega})e^{-j\omega n_0}$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*(n)$	$X^*(e^{-j\omega})$
Conjugate symmetry	$x(n)$ real	$X^*(e^{-j\omega})$
Convolution	$y(n) * x(n)$	$Y(e^{j\omega})X(e^{j\omega})$
Multiplication	$y(n)x(n)$	$\frac{1}{2\pi}\int_{2\pi} Y(e^{j\theta})X(e^{j(\omega-\theta)})d\theta$
Time reversal	$x(-n)$	$X(e^{-j\omega})$
Differentiation in frequency	$nx(n)$	$j\frac{dX(e^{j\omega})}{d\omega}$
Parseval's relation	$\sum_{-\infty}^{\infty} \mid x(n) \mid^2 = \frac{1}{2\pi}\int_{2\pi} \mid X(e^{j\omega}) \mid^2 d\omega$	

From this result, the convolution property provides the frequency response of the LSI system

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}. \tag{2.12}$$

Thus, $H(e^{j\omega})$ is expressed as the ratio between two polynomials on the variable $e^{-j\omega}$, one of degree M and the other of degree N. The next two examples show how this procedure can be utilized to analyze and determine the output of LSI systems represented by constant coefficients difference equations.

Example 2.4 *The constant coefficients difference equation*:

$$y(n) = \frac{1}{2}[x(n) + x(n-1)]$$

represents the system known as causal average filter. Applying the Fourier transform as previously described, we obtain:

$$Y(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X(e^{j\omega})e^{-j\omega}] = \frac{1}{2}X(e^{j\omega})[1 + e^{-j\omega}]$$

and

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{2}[1 + e^{-j\omega}].$$

Inverse transforming $H(e^{j\omega})$ *yields the system impulse response*:

$$h(n) = \frac{1}{2}[\delta(n) + \delta(n-1)].$$

The plot of the magnitude of $H(e^{j\omega})$ *is shown in Figure* 2.3. *Recalling that* $|Y(e^{j\omega})| = |X(e^{j\omega})||H(e^{j\omega})|$, *we can conclude from that graphic that low input frequencies* ($|\omega| \leq \frac{\pi}{2}$) *are not highly affected by the system, while high input frequencies* ($\frac{\pi}{2} < |\omega| \leq \pi$) *are greatly attenuated, characterizing this system as a lowpass filter*.

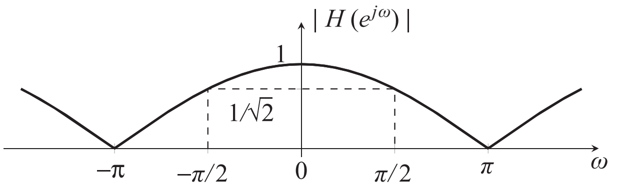

Figure 2.3. Example 2.4: Magnitude response of the average filter.

Let us now determine the output of this system to the input signal given by the sum of two real exponential sequences:

$$x(n) = -2\left(\frac{1}{3}\right)^n u(n) + 3\left(\frac{1}{2}\right)^n u(n).$$

From Table 2.1, we obtain $a^n u(n) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$, *for* $|a|<1$. *Applying that to* $x(n)$ *and adding the terms yields*

$$X(e^{j\omega}) = \frac{1}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}.$$

In the frequency domain, the output of the system is

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{2}\left[\frac{1+e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}\right],$$

or

$$Y(e^{j\omega}) = \frac{1}{2}[1+e^{-j\omega}]X(e^{j\omega}).$$

From the time shifting property, we obtain $y(n) = \frac{1}{2}[x(n) + x(n-1)]$, *which corresponds to*

$$y(n) = \frac{1}{2}\left[-2\left(\frac{1}{3}\right)^n + 3\left(\frac{1}{2}\right)^n\right]u(n) + \frac{1}{2}\left[-2\left(\frac{1}{3}\right)^{n-1} + 3\left(\frac{1}{2}\right)^{n-1}\right]u(n-1).$$

Example 2.5 *Find the output y(n) of the LSI system represented by the difference equation*:

$$y(n)+\frac{1}{10}y(n-1)-\frac{3}{10}y(n-2)=x(n)$$

for the input signal $x(n)=\left(\frac{1}{2}\right)^{n}u(n)$.
Fourier transforming both sides of the difference equation:

$$Y(e^{j\omega})\left[1+\frac{1}{10}e^{-j\omega}-\frac{3}{10}e^{-j2\omega}\right]=X(e^{j\omega}),$$

which leads to

$$H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{1}{1+\frac{1}{10}e^{-j\omega}-\frac{3}{10}e^{-j2\omega}}$$

or, factoring the denominator polynomial,

$$H(e^{j\omega})=\frac{1}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{3}{5}e^{-j\omega})}.$$

The Fourier transform of the input signal is

$$X(e^{j\omega})=\frac{1}{1-\frac{1}{2}e^{-j\omega}},$$

which yields

$$Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})=\frac{1}{(1-\frac{1}{2}e^{-j\omega})^{2}(1+\frac{3}{5}e^{-j\omega})}.$$

Expanding $Y(e^{j\omega})$ *in partial fractions we obtain*

$$Y(e^{j\omega})=\frac{\frac{30}{121}}{1-\frac{1}{2}e^{-j\omega}}+\frac{\frac{5}{11}}{(1-\frac{1}{2}e^{-j\omega})^{2}}+\frac{\frac{36}{121}}{1+\frac{3}{5}e^{-j\omega}}.$$

With the help of Table 2.1, y(n) is obtained

$$y(n)=\left[\frac{30}{121}\left(\frac{1}{2}\right)^n+\frac{5}{11}(n+1)\left(\frac{1}{2}\right)^n+\frac{36}{121}\left(-\frac{3}{5}\right)^n\right]u(n).$$

2.5 THE *Z* TRANSFORM

The z transform of a discrete-time signal is defined as

$$X(z)=\mathcal{Z}[x(n)]=\sum_{n=-\infty}^{\infty}x(n)z^{-n}. \tag{2.13}$$

The complex parameter $z = re^{j\omega}$ takes values on the z plane, a plane defined by the real and imaginary components of z, as represented in Figure 2.4.

Comparing Equation 2.13 with Equation 2.7, we can observe that

$$X(e^{j\omega})=\left[X(z)\right]_{z=e^{j\omega}}. \tag{2.14}$$

Equation 2.14 means that the Fourier transform corresponds to the z transform calculated for a particular set of points on the z plane. Those are the points with magnitude $r = 1$, that is, the points $z = e^{j\omega}$. Those values of z are located on the unit circle, a circle in the z plane with unit radius and centered at $z = 0$, shown in Figure 2.4.

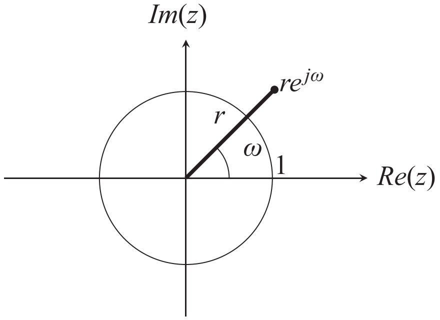

Figure 2.4. The z plane and the unit circle.

2.5.1 CONVERGENCE OF THE Z TRANSFORM

Allowing r to assume any value on the z plane, we have:

$$\left[X(z)\right]_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)(re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n)r^{-n}]e^{-j\omega n} = \mathscr{F}\left[x(n)r^{-n}\right]. \tag{2.15}$$

Thus, the z transform of $x(n)$ corresponds to the Fourier transform *of* $x(n)r^{-n}$. This result implies that $X(z)$ converges when $x(n)r^{-n}$ satisfies the convergence condition for the Fourier transform, that is,

$$\sum_{n=-\infty}^{\infty} |x(n)r^{-n}| = \sum_{n=-\infty}^{\infty} |x(n)||r^{-n}| < \infty. \tag{2.16}$$

Therefore, convergence (or existence) of $X(z)$ is assured for values of z in the region of the z plane where r satisfies Equation 2.16. That is the region where the Fourier transform of $x(n)r^{-n}$ converges, known as region of convergence, or ROC, of $X(z)$. The ROC is a ring-like region of the type $R_- < |z| = r < R_+$, as illustrated by the shaded area in Figure 2.5. R_- and R_+ are poles of $X(z)$. Depending on the signal, R_- can be zero and R_+ can be infinite, as shown in Examples 2.6 and 2.7.

We saw that the Fourier transform $X(e^{j\omega})$ of a signal is the z transform $X(z)$ calculated on the unity circle ($|z| = 1$) in the z plane, which implies

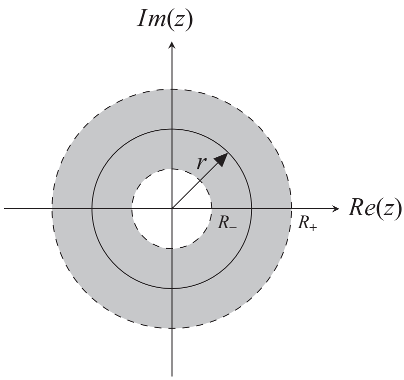

Figure 2.5. ROC representation: $R_- < |z| = r < R_+$.

that existence of $X(e^{j\omega})$ requires the unity circle to be inside the ROC of $X(z)$. Only signals that satisfy this requirement have a Fourier transform. Otherwise, when the ROC does not contain $|z| = 1$, the signal will have z transform but not Fourier transform. This conclusion shows that the z transform provides a more generic representation of signals that can be used in cases where the use of the Fourier transform is not possible.

2.5.2 POLE-ZERO PLOT AND ROC REPRESENTATION ON THE Z PLANE

For most signals of practical interest, the z transform is a rational function, that is, it may be expressed as a ratio of polynomials in z *or* z^{-1}. It is characterized by its zeros, the roots of the numerator polynomial, and its poles, the roots of the denominator polynomial. On the z plane, poles are represented by × and zeros are represented by ○, yielding what is known as a pole-zero plot. The ROC is bounded by poles, but it does not contain poles, as those are the values at which $X(z)$ goes to infinite. The ROC may contain zeros. Combined with the ROC, a pole-zero plot provides complete representation of the z transform, as shown in the following examples.

Example 2.6 *Let us determine the z transform of the signal*:

$$x(n) = a^n u(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

From Equation 2.13 we obtain:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

which is the sum of terms of a geometric series with ratio $a z^{-1}$. *This sum converges for* $|a z^{-1}| < 1$ *or* $|z|>|a|$. *For z in that region we have*

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

with ROC $|z| > |a|$, *one zero at* $z = 0$ *and one pole at* $z = a$, *as illustrated in Figure 2.6 where the shaded area corresponds to the ROC.*

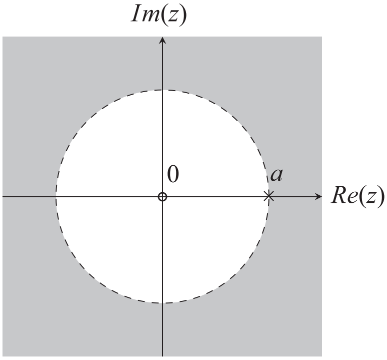

Figure 2.6. ROC and pole-zero plot for Example 2.6.

Example 2.7 *Now, let us determine the* z *transform of the signal*:

$$x(n) = -a^n u(-n-1) = \begin{cases} -a^n, & n \leq -1 \\ 0, & n > -1 \end{cases}$$

Again, from Equation 2.13:

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} (a^{-1}z)^n.$$

This geometric series converges for $|a^{-1}z| < 1$, *or* $|z|<|a|$, *which defines the ROC of* $X(z)$. *Thus, calculating for z in that region*:

$$X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} \tag{2.17}$$

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}.$$

The ROC $|z| < |a|$ *for this transform is illustrated by the shaded area in Figure 2.7, along with the pole-zero plot*: *one zero at* $z = 0$ *and one pole at* $z = a$.

Examples 2.6 and 2.7 show that the ROC is essential for identifying the z transform of a signal. The two signals analyzed in those examples

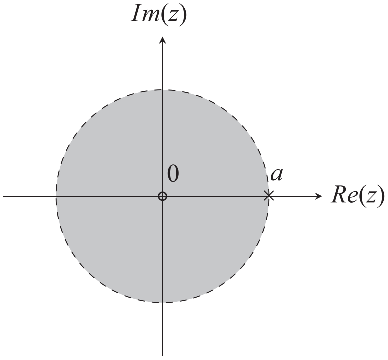

Figure 2.7. ROC and pole-zero plot for Example 2.7.

produce the same expression for $X(z)$ and the same pole-zero plot; distinction between them being provided only by the two different ROCs.

For causal signals, that is, signals which are zero for $n < 0$, $X(z)$ is a series that contains negative powers of z only. In those cases, it is convenient to express $X(z)$ as a ratio of polynomials function of z^{-1}, instead of polynomials in z. However, the poles and zeros are always the roots of the polynomials expressed as function of z.

2.5.3 SOME ROC FEATURES

The ROC does not contain poles, but is bounded by them. As we saw, the ROC consists of a ring in the z plane centered at the origin. The exact shape of the ROC depends on the time behavior of the signal, as described next.

1 Finite duration signals: Signals that are non-null only over a finite interval $N_1 \leq n \leq N_2$. The z transform for a finite duration $x(n)$ is a finite sum, that is:

$$X(z) = \sum_{n=N_1}^{N_2} x(n) z^{-n}. \tag{2.18}$$

Assuming $|x(n)|$ finite, each term in the sum (Equation 2.18) is finite, thus guaranteeing convergence of $X(z)$, except possibly for $z = 0$ and/or $z = \infty$. When $N_1 < 0$, the signal is at least partially on the negative half of the n axis, and therefore n takes some negative values in Equation 2.18. The corresponding terms in the sum are positive powers of n, which go to infinite when $z = \infty$. Thus $z = \infty$ is a pole, and thus does not belong in the ROC. If $N_2 \geq 0$, the signal is at least partially on the positive half of the n axis, and the corresponding terms in the sum are negative powers of n, which go to infinite when $z = 0$. Thus, for this case, a pole occurs at $z = 0$ and consequently, that point (the origin) is not in the ROC.

It is also possible that the signal is nonzero exclusively either for n positive or for n negative. As $N_1 < N_2$, this happens either when $N_1 \geq 0$ *or* when $N_2 < 0$, respectively. For $N_1 \geq 0$, the presence of only negative powers of z in Equation 2.18 implies that $z = \infty$ is not a pole and is included in the ROC. For $N_2 < 0$, the sum contains only positive powers of z, implying that $z = 0$ is not a pole and is contained in the ROC.

2 Right-sided signals: Signals in this category are null for n smaller than a certain value N_1. The z transform for a right-sided signal $x(n)$ is therefore:

$$X(z) = \sum_{n=N_1}^{\infty} x(n) z^{-n} \tag{2.19}$$

It can be shown that the ROC for Equation 2.19 is of the type $|z| > |a|$, where a is the pole of $X(z)$ with largest magnitude. As the sum in Equation 2.19 contains negative powers of z, it follows that the ROC of $X(z)$ for right-sided signals cannot contain $z = 0$. If N_1 is negative, some terms in the sum (Equation 2.19) are positive powers of z, and the ROC does not include $z = \infty$. If N_1 is positive ($N_1 \geq 0$), the sum contains negative powers of z only, and the ROC includes $z = \infty$.

Figure 2.6 shows the ROC of the right-sided signal in Example 2.6.

3 Left-sided signals: Those signals are zero for n greater that a certain value N_2. Thus, the z transform for a left-sided signal $X(n)$ is:

$$X(z) = \sum_{n=-\infty}^{N_2} x(n) z^{-n}. \tag{2.20}$$

It can be shown that the ROC of $X(z)$ for a left-sided signal is of the type $|z| < |b|$, where b is the pole of $X(z)$ with smallest magnitude. Thus, this ROC cannot include $z = \infty$. The value of N_2 determines whether $z = 0$ is in the ROC or not. For $N_2 < 0$, only positive powers of z are present in Equation 2.20 and $z = 0$ is in the ROC. Otherwise, for $N_2 \geq 0$, the presence of negative powers of z in the sum places poles at the origin, thus $z = 0$ is not contained in the ROC of $X(z)$. Figure 2.7 shows the ROC of the left-sided signal in Example 2.7.

4 Two-sided signals: Signals that take values for $-\infty < n < \infty$, are called two-sided. Those signals can be expressed as the sum of a right-sided plus a left-sided signal. The ROC for the z transform of the right-sided component is of the type $|z| > |a|$, while for the left-sided component that ROC is of the type $|z| < |b|$. Therefore, the ROC for the transform of the two-sided signal is $|a| < |z| < |b|$, the intersection of the two component ROCs (assuming $|a| < |b|$ so that there is an intersection). That ROC may, in fact, be larger than the intersection, but it must contain the intersection.

Table 2.3 shows the z transform for some signals commonly found in practice.

Example 2.8 *Determine the z transform of the two-sided signal*:

$$x(n) = \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$$

We have that $x(n) = x_1(n) + x_2(n)$, *where* $x_1(n) = \left(\frac{1}{3}\right)^n u(n)$ *is right-sided and* $x_2(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$ *is left-sided. From Examples 2.6 and 2.7, or from Table 2.3, we have*:

$$X_1(z) = \mathcal{Z}[x_1(n)] = \frac{1}{1-\frac{1}{3}z^{-1}}, ROC\ |z| > \frac{1}{3}$$

$$X_2(z) = \mathcal{Z}[x_2(n)] = \frac{1}{1-\frac{1}{2}z^{-1}}, ROC\ |z| < \frac{1}{2}$$

The two ROCs intersect at $\frac{1}{3} < |z| < \frac{1}{2}$. *The z transform is a linear operation, thus superposition applies yielding*

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}}$$

Table 2.3. Some commonly found signals and their z transforms

Signal	z Transform	ROC
$\delta(n)$	1	$\forall z$
$\delta(n-n_0)$	z^{-n_0}	$\forall z$ except $z=0$ (if $n_0>0$) or $z=\infty$ (if $n_0<0$)
$u(n)$	$\frac{1}{1-z^{-1}}$	$\mid z\mid>1$
$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\mid z\mid<1$
$u(n+n_0)-u(n-n_0+1)$	$\frac{z^{n_0}-z^{-(n_0+1)}}{1-z^{-1}}$	$\forall z$
$\alpha^n u(n)$	$\frac{1}{1-\alpha z^{-1}}$	$\mid z\mid>\mid\alpha\mid$
$\alpha^{\|n\|}$, $\alpha>0$	$\frac{z^{-1}\left(1-\alpha^2\right)}{(z^{-1}-\alpha)(z-\alpha)}$	$\alpha<\|z\|<\frac{1}{\alpha}$
$-\alpha^n u(-n-1)$	$\frac{1}{1-\alpha z^{-1}}$	$\mid z\mid<\mid\alpha\mid$
$n\alpha^n u(n)$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\mid z\mid>\mid\alpha\mid$
$-n\alpha^n u(-n-1)$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\mid z\mid<\mid\alpha\mid$
$cos(\omega_0 n)u(n)$	$\frac{1-cos(\omega_0)z^{-1}}{1-2cos(\omega_0)z^{-1}+z^{-2}}$	$\mid z\mid>1$
$\sin(\omega_0 n)u(n)$	$\frac{\sin(\omega_0)z^{-1}}{1-2cos(\omega_0)z^{-1}+z^{-2}}$	$\mid z\mid>1$
$\alpha^n cos(\omega_0 n)u(n)$	$\frac{1-\alpha cos(\omega_0)z^{-1}}{1-2\alpha cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}$	$\mid z\mid>\alpha$
$\alpha^n \sin(\omega_0 n)u(n)$	$\frac{\alpha\sin(\omega_0)z^{-1}}{1-2\alpha cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}$	$\mid z\mid>\alpha$

$$= \frac{2+\frac{5}{6}z^{-2}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}, \ ROC\ \frac{1}{3}<|z|<\frac{1}{2}.$$

Figure 2.8 shows the ROCs and pole-zero plots for the signals in this example. The ROC of X(z) is bounded by the poles $z_1 = \frac{1}{2}$ *and* $z_2 = \frac{1}{3}$. *The zeros of X(z) are imaginary located at* $z = \pm j\sqrt{\frac{5}{12}}$.

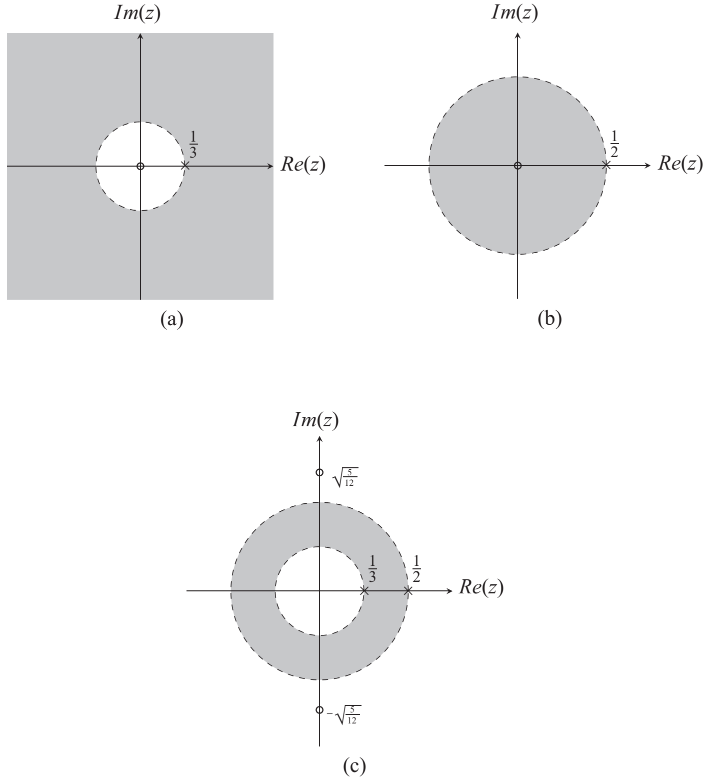

Figure 2.8. ROCs for Example 2.8: (a) $X_1(z)$:$|z|>1/3$, (b) $X_2(z)$:$|z| < 1/2$, (c) $X(z)$:$1/3<|z|<1/2$.

Example 2.9 *Determine the signal $x(n)$ with z transform*:

$$X(z)=\frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}-\frac{2}{9}z^{-2}},ROC|z|>\frac{2}{3}$$

Factoring the denominator of $X(z)$ yields:

$$X(z)=\frac{1-\frac{1}{2}z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)},ROC|z|>\frac{2}{3}$$

$X(z)$ has poles at $z=-\frac{1}{3}$ and $z=\frac{2}{3}$ and a zero at $z=\frac{1}{2}$. Expanding $X(z)$ in partial fractions, we obtain

$$X(z)=\frac{\frac{5}{6}}{1+\frac{1}{3}z^{-1}}+\frac{\frac{1}{6}}{1-\frac{2}{3}z^{-1}},ROC|z|>\frac{2}{3}$$

Thus, we can express $X(z)$ as:

$$X(z)=X_1(z)+X_2(z),ROC|z|>\frac{2}{3},$$

where

$$X_1(z)=\frac{\frac{5}{6}}{1+\frac{1}{3}z^{-1}},ROC\ |z|>\frac{1}{3},$$

$$X_2(z)=\frac{\frac{1}{6}}{1-\frac{2}{3}z^{-1}},ROC|z|>\frac{2}{3}.$$

From Table 2.3, we obtain the respective signals $x_1(n)$ and $x_2(n)$:

$$x_1(n)=\frac{5}{6}\left(-\frac{1}{3}\right)^n u(n)\overset{\mathcal{Z}}{\Leftrightarrow}X_1(z)=\frac{\frac{5}{6}}{1+\frac{1}{3}z^{-1}},ROC\ |z|>\frac{1}{3},$$

$$x_2(n)=\frac{1}{6}\left(\frac{2}{3}\right)^n u(n)\overset{\mathcal{Z}}{\Leftrightarrow} X_2(z)=\frac{\frac{1}{6}}{1-\frac{2}{3}z^{-1}}, ROC\,|\,z\,|>\frac{2}{3}.$$

Signal $x(n)$ *is the sum of* $x_1(n)$ *and* $x_2(n)$:

$$x(n)=x_1(n)+x_2(n)=\frac{5}{6}\left(-\frac{1}{3}\right)^n u(n)+\frac{1}{6}\left(\frac{2}{3}\right)^n u(n). \tag{2.21}$$

Example 2.10 *Let us examine how the same pole-zero pattern yields distinct ROCs, each corresponding to a different time signal. Consider again* $X(z)$ *of the previous example*:

$$X(z)=\frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}-\frac{2}{9}z^{-2}}=\frac{\frac{5}{6}}{1+\frac{1}{3}z^{-1}}+\frac{\frac{1}{6}}{1-\frac{2}{3}z^{-1}}.$$

Three ROCs are possible for this pole pattern:

- $|z|>\frac{2}{3}$. *This is the case examined in Example 2.9 for which* $x(n)$ *is the right-sided signal in Equation 2.21.*
- $|z|<\frac{1}{3}$. *In this case, we must have*:

$$X_1(z)=\frac{\frac{5}{6}}{1+\frac{1}{3}z^{-1}}, ROC\,|\,z\,|<\frac{1}{3},$$

$$X_2(z)=\frac{\frac{1}{6}}{1-\frac{2}{3}z^{-1}}, ROC\,|\,z\,|<\frac{2}{3},$$

such that the intersection of the two ROCs is $|\,z\,|<\frac{1}{3}$. *The preceding ROCs imply that all signals are left-sided*:

$$x_1(n)=-\frac{5}{6}\left(-\frac{1}{3}\right)^n u(-n-1),$$

$$x_2(n) = -\frac{1}{6}\left(\frac{2}{3}\right)^n u(-n-1),$$

and

$$x(n) = x_1(n) + x_2(n) = -\frac{5}{6}\left(-\frac{1}{3}\right)^n u(-n-1) - \frac{1}{6}\left(\frac{2}{3}\right)^n u(-n-1).$$

- $\frac{1}{3} < |z| < \frac{2}{3}$. *This ROC corresponds to the intersection of* $|z| > \frac{1}{3}$ *and* $|z| < \frac{2}{3}$, *thus*

$$X_1(z) = \frac{\frac{5}{6}}{1 + \frac{1}{3}z^{-1}}, ROC|z| > \frac{1}{3},$$

$$X_2(z) = \frac{\frac{1}{6}}{1 - \frac{2}{3}z^{-1}}, ROC|z| < \frac{2}{3}.$$

In this case, $x_1(n)$ *is right-sided and* $x_2(n)$ *left-sided*:

$$x_1(n) = \frac{5}{6}\left(-\frac{1}{3}\right)^n u(n),$$

$$x_2(n) = -\frac{1}{6}\left(\frac{2}{3}\right)^n u(-n-1).$$

$x(n)$ *is, therefore, a two-sided signal given by*:

$$x(n) = x_1(n) + x_2(n) = \frac{5}{6}\left(-\frac{1}{3}\right)^n u(n) - \frac{1}{6}\left(\frac{2}{3}\right)^n u(-n-1).$$

2.5.4 PROPERTIES OF THE Z TRANSFORM

Table 2.4 summarizes the main z transform properties. Some of those properties are similar to the ones described for the Fourier transform.

Table 2.4. Main z transform properties

Property	Signal	z transform	ROC
	$x(n)$ $y(n)$	$X(z)$ $Y(z)$	R_x R_y
Linearity	$ax(n) + by(n)$	$aX(z) + bY(z)$	$R \subset (R_x \cap R_y)$
Time shifting	$x(n-m)$	$z^{-m}X(z)$	R_x, $z = 0$ and/or $z = \infty$ may be excluded or added
Multiplication by exponential	${z_0}^n x(n)$	$X(z/z_0)$	$\|z_0\|R_x$
Time reversal	$x(-n)$	$X\left(\frac{1}{z}\right)$	$(R_x)^{-1}$
Convolution	$x(n) * y(n)$	$X(z).Y(z)$	$R \subset (R_x \cap R_y)$
Differentiation in z	$nx(n)$	$-z\frac{dX(z)}{dz}$	R_x
Summation	$\sum_{i=0}^{n} x(i)$	$\frac{1}{1-z^{-1}}X(z)$	$R \subset (R_x \cap \|z\| > 1)$
Initial value theorem	If $x(n) = 0$ for $n < 0 \Rightarrow x(0) = \lim_{z\to\infty} X(z)$		

However, when dealing with the z transform, we must examine how the operations associated with each property affect the pole-zero pattern, and consequently, affect the ROC.

When adding or multiplying z transforms, pole-zero cancelation may occur. As an example, consider:

$$X_1(z) = \frac{1-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)}, ROC_1 \, |z| > \frac{1}{3}$$

$$X_2(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{9}z^{-1}\right)}, ROC_2 \, |z| > \frac{1}{2}.$$

Multiplying the two transforms

$$X(z) = X_1(z)X_2(z) = \frac{1}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{2}{9}z^{-1}\right)}, ROC \, |z| > \frac{1}{3}.$$

The intersection of ROC_1 and ROC_2 is $|z| > \frac{1}{2}$. However, as a result of the multiplication, the zero of $X_1(z)$ *at* $z = \frac{1}{2}$ cancels with the pole of $X_2(z)$ also at $z = \frac{1}{2}$. Thus, the resulting ROC is bounded by the remaining pole of *X(z)* with largest magnitude, that is, the pole at $z = \frac{1}{3}$. Observe that the ROC $|z| > \frac{1}{3}$ contains the intersection of the two original ROCs. It will be equal to the intersection when no pole-zero cancelation occurs.

Although not as often as with multiplication, pole-zero cancelation may also occur when adding two or more z transforms. When two signals are convolved in time, the z transform of the result is the product of the signals transforms. The resulting ROC contains the intersection of the original ROCs.

Another example of how the ROC is affected by operating on the signal is provided by the multiplication by exponential property in Table 2.4, that is,

$$z_0{}^n x(n) \overset{\mathcal{Z}}{\Leftrightarrow} X(z/z_0), \; ROC \; |z_0| R_x$$

If $X(z)$ has a pole at $z = a$, it implies that $X(z/z_0)$ has a pole at $z = az_0$. The same applies for all poles and zeros. Due to this scaling of the poles, the ROC will also be scaled by $|z_0|$. Depending on whether $|z_0|$ is greater or smaller than unity the ROC will be expanded or contracted.

2.5.5 ANALYSIS OF LSI SYSTEMS WITH THE Z TRANSFORM

Consider an LSI system with impulse response $h(n)$, input signal $x(n)$, and

$$x(n) \overset{\mathcal{Z}}{\Leftrightarrow} X(z), ROC\ R_x,$$

$$h(n) \overset{\mathcal{Z}}{\Leftrightarrow} H(z), ROC\ R_h.$$

The output for such system is given in time by $y(n) = x(n) * h(n)$ or in the z domain by

$$Y(z) = X(z)H(z), ROC\ R_y \subseteq (R_x \cap R_h) \tag{2.22}$$

$H(z)$ is the system transfer function.

Some of the main properties of an LSI system can be obtained analyzing $H(z)$ and its ROC. For a causal system $h(n) = 0$ for $n < 0$, that is, $h(n)$ is a right-sided signal; thus, the ROC of $H(z)$ for a causal system is $|z| > |a|$, where a is the largest magnitude pole. For a stable LSI system, the ROC of $H(z)$ must contain the unity circle, as $h(n)$ is absolutely summable, and therefore its Fourier transform exists. Therefore, for a causal and stable LSI system, all the poles of $H(z)$ must be inside the unity circle in the z plane, such that both aforementioned conditions are satisfied.

Consider an LSI system represented by the constant coefficients difference equation:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k).$$

Applying the z transform to both sides of this equation, yields

$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{k=0}^{M} b_k z^{-k},$$

from which the system transfer function $H(z)$ can be obtained

$$H(z) = \frac{Y(z)}{x(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{2.23}$$

The ROC of $H(z)$ can be obtained from the poles of Equation 2.23. For a causal and stable system, all poles must lie inside the unity circle and the ROC will be $|z| > |a|$, where a is the largest magnitude pole. Thus, in order to have a system that is simultaneously causal and stable, the a_k coefficients in Equation 2.23 have to be chosen such that all poles of $H(z)$ have magnitude less than unity. The resulting $h(n)$ will be a right-sided absolutely summable sequence.

Example 2.11 *An LSI system is described by the difference equation*:

$$y(n) - 3.5y(n-1) + 1.5y(n-2) = x(n).$$

Let us determine the system transfer function H(z) and impulse response h(n) *for each of the following cases*:

(a) *causal system;*
(b) *stable system;*
(c) *non-causal system.*

First, we apply the z transform to both sides of Equation 2.24:

$$Y(z) - 3.5z^{-1}Y(z) + 1.5z^{-2}Y(z) = X(z)$$

or

$$Y(z)(1 - 3.5z^{-1} + 1.5z^{-2}) = X(z).$$

and

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 3.5z^{-1} + 1.5z^{-2}}.$$

Extracting the poles and expanding in partial fractions, we obtain:

$$H(z) = \frac{6/5}{1 - 3z^{-1}} - \frac{1/5}{1 - \frac{1}{2}z^{-1}}.$$

Thus, H(z) has poles at $z = 1/2$ *and* $z = 3$. *The expression for H*(z) *is the same regardless of the chosen ROC. Now, we analyze each of the proposed cases*:

(a) *For a causal system, the ROC must be* $|z|>3$, *so the impulse response is the right-sided signal*:

$$h(n)=\left[\frac{6}{5}(3)^n-\frac{1}{5}\left(\frac{1}{2}\right)^n\right]u(n).$$

(b) *For a stable system, the ROC must be* $\frac{1}{2}<|z|<3$ *in order to contain the unity circle. The impulse response is the two-sided signal:*

$$h(n)=\frac{6}{5}(3)^n u(n)+\frac{1}{5}\left(\frac{1}{2}\right)^n u(-n-1).$$

(c) *A non-causal system has a left-sided impulse response* ($h(n)\neq 0$, $n<0$), *which requires the ROC to be* $|z|<\frac{1}{2}$ *and*

$$h(n)=\left[\frac{1}{5}\left(\frac{1}{2}\right)^n-\frac{6}{5}(3)^n\right]u(-n-1).$$

2.6 CHAPTER OVERVIEW

Transforms are tools that enable us to represent signals and systems in domains other than time. These alternative representations are usually required in most practical problems. In this chapter, the two main signal transforms utilized in DSP have been analyzed: the Fourier transform and the z transform. Both are based on the idea of representing a signal as a linear combination of complex exponentials of the type z^n, where $z = re^{j\omega}$.

We start the chapter by the Fourier series that expresses periodic signals as a weighted sum of harmonically related exponentials. Next, we introduce the Fourier transform, obtained by making $|z| = r = 1$, and thus $z = e^{j\omega}$. It allows to represent signals, non-periodic as well as periodic, in the domain where the independent variable is frequency ω, thus called frequency domain. Fourier transform convergence, transform calculation for basic signals, and main properties are presented and exemplified. We finish by showing how to perform Fourier analysis of LSI systems represented by constant coefficients difference equations. In the frequency domain, an LSI system is represented by its frequency response $H(e^{j\omega})$, which is the Fourier transform of the impulse response $h(n)$ and can be obtained from the constant coefficients difference equations.

For the z transform, no restriction is placed on $|z| = r$, and the transform is defined on a complex plane called the z plane, where that variable takes values. Therefore, the z transform is more general and can be used to represent signals that do not possess Fourier transform. We analyze the relationship between the Fourier and the z transform and introduce the ROC, which is the complex plane region where the z transform converges. We show that the ROC is bounded by the poles of the transform and depends on the signal type, whether right-sided, left-sided, two-sided, or of finite duration. The main z transform properties are also analyzed in this chapter, as well as how the ROC is affected by operations on the signal. Finally, we examine the application of the z transform to LSI systems analysis and define the transfer function $H(z)$ of an LSI system, the z transform of the impulse response $h(n)$. We describe how $H(z)$ can be obtained from the constant coefficients difference equations representing the system and how its ROC is defined by the poles and by the system properties.