4.4 Bloch Equation and Static-Field Solutions

The differential equations (4.11) and (4.14) for magnetization in the presence of a magnetic field and with relaxation terms can be combined into one vector equation,

$$\frac{d\vec{M}}{dt} = \gamma \vec{M} \times \vec{B}_{ext} + \frac{1}{T_1}(M_0 - M_z)\hat{z} - \frac{1}{T_2}\vec{M}_\perp \tag{4.21}$$

This empirical vector equation is referred to as the Bloch equation. The relaxation terms describe the return to equilibrium, but only for a field pointing along the z-axis. The quantum mechanical underpinnings of the Bloch equation are described in Chs. 5 and 6.

Let us solve the Bloch equation for the constant field case, $\vec{B}_{ext} = B_0\hat{z}$. A calculation of the components of the cross product in (4.21) produces the three component equations

$$\frac{dM_z}{dt} = \frac{M_0 - M_z}{T_1} \tag{4.22}$$

$$\frac{dM_x}{dt} = \omega_0 M_y - \frac{M_x}{T_2} \tag{4.23}$$

$$\frac{dM_y}{dt} = -\omega_0 M_x - \frac{M_y}{T_2} \tag{4.24}$$

where $\omega_0 \equiv \gamma B_0$. The first equation is the same as (4.11) whose solution is (4.12). For the last two equations, the relaxation terms can be easily eliminated by the change of variables, $M_x = m_x e^{-t/T_2}$ and $M_y = m_y e^{-t/T_2}$ (i.e., by the introduction of integrating factors). The resulting differential equations for m_x and m_y have exactly the form of the equations found, and solved, for μ_x and μ_y in Ch. 2.[8] In terms of the original variables, the complete set of solutions is therefore

$$M_x(t) = e^{-t/T_2}\left(M_x(0)\cos\omega_0 t + M_y(0)\sin\omega_0 t\right) \tag{4.25}$$

$$M_y(t) = e^{-t/T_2}\left(M_y(0)\cos\omega_0 t - M_x(0)\sin\omega_0 t\right) \tag{4.26}$$

$$M_z(t) = M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1}) \tag{4.27}$$

The equilibrium or steady-state solution can be found from the asymptotic limit $t \to \infty$ of (4.25)–(4.27). In that limit, all the exponentials vanish implying the steady-state solution

$$M_x(\infty) = M_y(\infty) = 0, \quad M_z(\infty) = M_0 \tag{4.28}$$

The general time-dependent solution for the transverse components, (4.25) and (4.26), is seen to have sinusoidal terms modified by a decay factor. The sinusoidal terms correspond to the precessional motion discussed in Ch. 2, and the damping factor comes from the transverse relaxation effect. The magnitude $|\vec{M}|$ is not fixed: The longitudinal component relaxes from its initial value to the equilibrium value M_0; the transverse component rotates clockwise and it decreases in magnitude. Recall that the transverse decay time T_2 is in general different from (smaller than) the longitudinal decay time T_1. An example of the resulting left-handed 'corkscrew' trajectory for an initial magnetization lying in the transverse plane is illustrated in Fig. 4.3.

[8]See, in particular, Prob. 2.6.