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Note

The Regularity of Some 1-Additive Sequences

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For positive integers $u < v$, the 1-*additive sequence* $s_1, s_2, s_3, \ldots$ *based on* u, v is defined so that $s_1 = u$, $s_2 = v$, and s_{n+2} is the least integer greater than s_{n+1} having a unique representation in the form $s_{n+2} = s_i + s_j$, where $1 \leqslant i < j \leqslant n+1$. These sequences were defined by Ulam [4] and studied by, among others, Queneau [3] and Finch [1, 2]. Many 1-additive sequences appear to have quite erratic behavior (for example, when $u = 1$ or when $u = 2$ and $v = 3$). However, Finch [2] conjectures that the 1-additive sequence based on 2, v (where $v > 3$ is odd) has only two even terms, and consequently is regular. A sequence $s_1, s_2, s_3, \ldots$ is said to be *regular* if the sequence $s_2 - s_1, s_3 - s_2, s_4 - s_3, \ldots$ of successive differences is eventually periodic. Concerning this conjecture, Finch states in [2]: "Substantial computer work ... supports [this conjecture], but rigorous proof seems impossible." In this note we prove this conjecture in the following theorem.

THEOREM 1. *If $v > 3$ is odd, then the 1-additive sequence $s_1, s_2, s_3, \ldots$ based on 2, v has precisely two even terms, namely $s_1 = 2$ and $s_k = 2v + 2$, where $k = (v+7)/2$.*

In the rest of this paper, we let $v > 3$ be an odd integer, and we let $s_1, s_2, s_3, \ldots$ be the 1-additive sequence based on 2, v. We also let $d = 2v + 2$, and we let $S = \{s_1, s_2, s_3, \ldots\}$.

Queneau [3] determines the first $2v + 5$ terms of the sequence $s_1, s_2, s_3, \ldots$. We need only the first $\frac{1}{2}(3v + 11)$ terms, from which it easily follows that 2 and d are the only even terms not greater than $8v + 8$.

LEMMA 2. *Consider the sequence $s_1, s_2, s_3, \ldots$.*

(1) *The first $\frac{1}{2}(3v+11)$ terms are* $2, v, v+2, v+4, \ldots, 2v-3, 2v-1, 2v+1, 2v+2, 2v+3, 2v+5, \ldots, 3v-4, 3v-2, 3v, 3v+4, 3v+8, \ldots, 5v-2, 5v+2, 5v+4, 5v+10$.

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(2) *There are no even terms other than* 2 *and* d *which are* $\leqslant 8v+8$.

Proof of Theorem. Assume it is false and let x be the least even integer in S which is greater than d.

LEMMA 3. *Let r be an odd integer such that $1 \leqslant r < x-2v$. Then there is an i such that $0 \leqslant i \leqslant v$ and $r+2i \in S$.*

Proof. Assume the lemma is false and let r be the least positive odd integer for which $r < x-2v$ and $\{r+2i: 0 \leqslant i \leqslant v\} \cap S = \phi$. Clearly, $r \geqslant 3$. Since $r+2v$, $r+2v-2 \notin S$, it must be that $r+2v-d = r-2 \notin S$. But then $\{r-2+2i: 0 \leqslant i \leqslant v\} \cap S = \phi$, contradicting the minimality of r. ∎

By Lemma 2(2), $x > 8v+8$. Let $a < b$ be (necessarily odd) elements of S such that $a+b=x$. Letting $r = x-3v$ in Lemma 3, we see that there is an i such that $0 \leqslant i \leqslant v$ and $x-3v+2i \in S$. Since, by Lemma 2(1) $\{3v-2j: 0 \leqslant j \leqslant v\} \subseteq S$ we must have that $a = 3v-2i$ and $b = x-3v+2i$, and then by the 1-additivity of S, $x-3v+2j \notin S$ whenever $0 \leqslant j \leqslant v$ and $j \neq i$. Thus, we know whether or not each of $x-v$, $x-v-2$, $x-v-4, \ldots, x-3v+2$, $x-3v$ is in S. We continue determining, in order, whether or not each of $x-3v-2$, $x-3v-4$, $x-3v-6, \ldots, x-5v+2$, $x-5v$, $x-5v-2$ is in S, noting that $x-3v+2j-d = x-5v+2j-2$ is in S iff precisely one of $x-3v+2j$ and $x-3v+2j-2$ is in S. We thus get the following:

(1) If $0 \leqslant i < v$ and $0 \leqslant j \leqslant v$, then $x-3v+2j-d \in S$ iff $j=i$ or $j=i+1$.

(2) If $i=v$ and $0 \leqslant j \leqslant v$, then $x-3v+2j-d \in S$ iff $j=0$ or $j=v$.

Case 1. $0 \leqslant i < v$. If $\{x-3v+2i-d, x-3v+2i-d+2\} \cap \{a, b, x/2\} = \phi$, then we can conclude from (1) that $3v-2i+d$, $3v-2i+d-2 \notin S$. This implies that $3v-2i \notin S$, contradicting Lemma 2(1). Thus we need to prove that $\{x-3v+2i-d,\ x-3v+2i-d+2\} \cap \{a, b, x/2\} = \phi$.

Clearly $x-3v+2i-d < x-3v+2i-d+2 < b$. If $x-3v+2i-d = a$ or $x-3v+2i-d+2 = a$, then we get that $8v+2-4i \geqslant x > 8v+8$, which is a contradiction.

If $x-3v+2i-d = x/2$, then $x = 10v-4i+4$, so it follows from (1) that $5v-2i+2$, $5v-2i+4 \in S$. By Lemma 2(1) this is impossible except when $i=0$, but then $x = (5v-6)+(5v+10)$ is another representation of x where, by Lemma 2(1), $5v-6$, $5v+10 \in S$.

If $x-3v+2i-d+2 = x/2$, then as in the previous paragraph, $5v-2i-2$, $5v-2i \in S$. But by Lemma 2(1) this is impossible for all i for which $0 \leqslant i < v$. This completes Case 1.

Case 2: $i=v$. Letting $j=0$ in (2) we see that $x-3v-d \in S$. From Lemma 2(1) we get $3v+d = 5v+2 \in S$ and therefore $x = (x-3v-d)+(5v+2)$ is a representation different from $x = a+b$ since $x > 6v+2$.

Therefore, it must be that $x-3v-d=5v+2$. Then letting $j=1$ in (2) yields $x-3v+2-d \notin S$, yet $x-3v+2-d=5v+4 \in S$ by Lemma 2(1), a contradiction. This completes Case 2, and we have established Theorem 1. ∎

Finch [1] notes that a 1-additive sequence with only finitely many even terms is regular. Let $N(v)$ be the period of the sequence s_2-s_1, s_3-s_2, $s_4-s_3, \ldots$ of successive differences, and let the *fundamental difference* be $D(v)=s_{N(v)+n}-s_n$ for large n. Let $d(v)$ be the density of $s_1, s_2, s_3, \ldots$; clearly $d(v)=N(v)/D(v)$. For integers $n \geqslant 2$, let $a_0^n, a_1^n, a_2^n, \ldots$ be the mod 2 sequence where $a_0^n=0$ and $a_1^n=a_2^n=\cdots=a_{n-1}^n=1$, and $a_{n+k}^n=a_{n+k-1}^n+a_k^n$ if $k \geqslant 0$. This sequence is periodic; let $p(n)$ be its period, and let $q(n)$ be the number of 1's among $a_1^n, a_2^n, \ldots, a_{p(n)}^n$. Note that $2k-2+v \in S$ iff $a_k^{v+1}=1$. It follows from Theorem 1 that for odd $v>3$, $D(v)=2p(v+1)$ and $N(v)=q(v+1)$. Thus we get the following bounds.

COROLLARY 4. *Let $v>3$ be odd. Then $N(v) \leqslant 2^v$, $D(v) \leqslant 2^{v+2}-2$, and $d(v) \geqslant 1/2(v+1)$.*

The bounds for $N(v)$ and $D(v)$ are attained when $v=5$. Improvements can be made to the lower bound for $d(v)$.

In certain special cases we can get exact values for $N(v)$ and $D(v)$. The following proves another conjecture of Finch [2].

THEOREM 5. *If $m \geqslant 3$, then $N(2^m-1)=3^m-1$ and $D(2^m-1)=2(4^m-1)$.*

Proof. It is easily checked that $p(2)=3$ and $q(2)=2$. To prove the theorem it suffices to prove the recurrences $p(2^{r+1})=4p(2^r)+3$ and $q(2^{r+1})=3q(2^r)+2$ for $r \geqslant 1$. This is easily done, and left to the reader to do, once the following observations are made. Let $m=2^r$ and $n=2^{r+1}$. If $k=in+j$, where $0 \leqslant i, j<n$, then

$$a_k^n=\begin{cases} a_{im+j}^m & \text{if } i, j<m, \\ a_{im+j-m}^m & \text{if } i<m \leqslant j \text{ and } (i, j) \neq (0, m), \\ a_{(i-m)m+j}^m & \text{if } j<m \leqslant i \text{ and } (i, j) \neq (m, 0), \\ 0 & \text{if } m \leqslant i, j, \\ 1 & \text{if } (i, j)=(0, m) \text{ or } (i, j)=(m, 0). \end{cases}$$

In verifying the above it is helpful to note that

$$a_k^n=\begin{cases} 0 & \text{if } j=n-1, i \neq 0, \text{ or } i=n-1, j \neq 0, \\ 1 & \text{if } (i, j)=(0, n-1) \text{ or } (i, j)=(n-1, 0). \end{cases}$$

This completes the proof. ∎

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