1 Thin Wall Approximation

24th November 2023

1.1 Dynamics of Domain Walls in The Thin Wall Approximation

Preceding Julian's notes ('Dynamics of Domain Walls in the Thin Wall approximation'). I could not get a symmetric stress-energy tensor from equations (18) and (4) in said notes. I then calculated the determinant ($g^{(3)}$) for myself, and by using that expression (Eq. (1.3) and Eq. (1.4) below) the functional derivative $\delta g^{(3)}/\delta g_{\rho\sigma}$ becomes symmetric, hence $T^{\rho\sigma}$ is symmetric.

Covariant action. Consider symmetron potential, thin wall limit. Surface tension is

$$\sigma \simeq \int_{\phi_-}^{\phi_+} \mathrm{d}\phi\ \sqrt{2V_{\mathrm{eff}}(\phi) - 2V_{\mathrm{eff}}(\phi_\pm)}, \tag{1.1}$$

where $\phi_\pm = \phi(z \to \pm\infty)$. We write the covariant action as $S_{\mathrm{dw}} = -\sigma \int d^3\xi \sqrt{-g^{(3)}}$. The induced metric is

$$g^{(3)}_{AB} = g_{\mu\nu} \frac{\mathrm{d}x_{\mathrm{dw}}{}^{\mu}}{\mathrm{d}\xi^A} \frac{\mathrm{d}x_{\mathrm{dw}}{}^{\nu}}{\mathrm{d}\xi^B}; \quad A, B = 0, 1, 2, \tag{1.2}$$

where $x_{\mathrm{dw}}{}^{\mu}(\xi^A)$ is the embedding function. The determinant of the world volume metric is

$$g^{(3)} = \tilde{\varepsilon}_{ABC}\, g^{(3)}_{0A} g^{(3)}_{1B} g^{(3)}_{2C} = g_{\mu\nu} g_{\kappa\lambda} g_{\alpha\beta} \mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta}, \tag{1.3}$$

where $\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} = \tilde{\varepsilon}_{ABC} \Delta^{\mu}{}_{0} \Delta^{\nu}{}_{A} \Delta^{\kappa}{}_{1} \Delta^{\lambda}{}_{B} \Delta^{\alpha}{}_{2} \Delta^{\beta}{}_{C}$; $\Delta^{\mu}{}_{A} \equiv \mathrm{d}x_{\mathrm{dw}}{}^{\mu} \big/ \mathrm{d}\xi^A$. In particular,

$$\begin{aligned} \mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} = {} & \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{0}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{1}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{2} + \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{1}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{2}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{0} + \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{2}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{0}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{1} \\ & - \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{0}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{2}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{1} - \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{1}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{0}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{2} - \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{2}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{1}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{0}. \end{aligned} \tag{1.4}$$

Stress-energy tensor. We consider a planar wall lying in the xy-plane with a small perturbation in the z-direction. The stress-energy tensor is given by

$$T^{\rho\sigma} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g_{\rho\sigma}} = \frac{\sigma \delta(z - z_{\mathrm{dw}})}{\sqrt{-g}\sqrt{-g^{(3)}}} \frac{\delta g^{(3)}}{\delta g_{\rho\sigma}}. \tag{1.5}$$

We need the functional derivative of $g^{(3)}$ and the quantity $\sigma\delta(z - z_{\mathrm{dw}})$.

1.1.1 My Calculation

We vary $g^{(3)}$ w.r.t. $g_{\rho\sigma}$, ignoring $\mathcal{O}((\delta g_{\rho\sigma})^2)$-terms:

$$\begin{aligned} g^{(3)} + \delta g^{(3)} &= \left(g_{\mu\nu} + \delta g_{\mu\nu}\right)\left(g_{\kappa\lambda} + \delta g_{\kappa\lambda}\right)\left(g_{\alpha\beta} + \delta g_{\alpha\beta}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} \\ &= g^{(3)} + \left(\delta g_{\mu\nu} g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu}\delta g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu} g_{\kappa\lambda}\delta g_{\alpha\beta}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} \\ &= g^{(3)} + \left(\frac{\partial g_{\mu\nu}}{\partial g_{\rho\sigma}}\delta g_{\rho\sigma} g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu}\frac{\partial g_{\kappa\lambda}}{\partial g_{\rho\sigma}}\delta g_{\rho\sigma} g_{\alpha\beta} + g_{\mu\nu} g_{\kappa\lambda}\frac{\partial g_{\alpha\beta}}{\partial g_{\rho\sigma}}\delta g_{\rho\sigma}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} \\ &= g^{(3)} + \left(\delta^{\rho}{}_{\mu}\delta^{\sigma}{}_{\nu} g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu}\delta^{\rho}{}_{\kappa}\delta^{\sigma}{}_{\lambda} g_{\alpha\beta} + g_{\mu\nu} g_{\kappa\lambda}\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta}\cdot\delta g_{\rho\sigma} \\ &= g^{(3)} + \left(g_{\kappa\lambda} g_{\alpha\beta}\mathbf{Q}^{\rho\sigma\kappa\lambda\alpha\beta} + g_{\mu\nu} g_{\alpha\beta}\mathbf{Q}^{\mu\nu\rho\sigma\alpha\beta} + + g_{\mu\nu} g_{\kappa\lambda}\mathbf{Q}^{\mu\nu\kappa\lambda\rho\sigma}\right)\cdot\delta g_{\rho\sigma} \end{aligned} \tag{1.6}$$

Thus,

$$\frac{\delta g^{(3)}}{\delta g_{\rho\sigma}} = g_{\kappa\lambda} g_{\alpha\beta}\left\{\mathbf{Q}^{\rho\sigma\kappa\lambda\alpha\beta} + \mathbf{Q}^{\alpha\beta\rho\sigma\kappa\lambda} + \mathbf{Q}^{\kappa\lambda\alpha\beta\rho\sigma}\right\}. \tag{1.7}$$

Flat FRW universe. With $g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ and ${x_{\mathrm{dw}}}^A = \xi^A, {x_{\mathrm{dw}}}^3 = z_{\mathrm{dw}} = \epsilon(\xi^A)$, we may insert

$$\Delta^{\mu}{}_{A} = \begin{cases} \delta^{\mu}{}_{A}, & \mu \neq 3 \\ \partial\epsilon\big/\partial\xi^A, & \mu = 3 \end{cases} \tag{1.8}$$

into Eq. (1.4) to compute $g^{(3)}$ and $\delta g^{(3)}/\delta g_{\rho\sigma}$. The result of the latter is a symmetric tensor of type (2,0);

$$\left[\frac{\delta g^{(3)}}{\delta g_{\rho\sigma}}\right] = \\ \begin{pmatrix} \left(\iota_1{}^2 + \iota_2{}^2 + 1\right)a^4 & -\iota_0\iota_1 a^4 & -\iota_0\iota_2 a^4 & \iota_0 a^4 \\ -\iota_0\iota_1 a^4 & \iota_0{}^2 a^4 - \left(\iota_2{}^2 + 1\right)a^2 & \iota_1\iota_2 a^2 & -\iota_1 a^2 \\ -\iota_0\iota_2 a^4 & \iota_1\iota_2 a^2 & \iota_0{}^2 a^4 - \left(\iota_1{}^2 + 1\right)a^2 & -\iota_2 a^2 \\ \iota_0 a^4 & -\iota_1 a^2 & -\iota_2 a^2 & \iota_0{}^2 a^4 - \left(\iota_1{}^2 + \iota_2{}^2\right)a^2 \end{pmatrix}, \tag{1.9}$$

where we defined $\iota_A \equiv \partial\epsilon\big/\partial\xi^A$.

Symmetron potential. We let $V_{\mathrm{eff}}(\phi) = V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2$. As such, $\phi_\pm = \pm\phi_0$ and $V(\phi_\pm) = 0$. Now,

$$\begin{aligned} \sigma\delta(z - z_{\mathrm{dw}}) &= \int_{-\infty}^{\infty}\mathrm{d}z\,\frac{\mathrm{d}\phi}{\mathrm{d}z}\sqrt{2V_{\mathrm{eff}}(\phi(z)) - 2V_{\mathrm{eff}}(\phi_\pm)}\,\delta(z - z_{\mathrm{dw}}) \\ &= \sqrt{2}\int_{-\infty}^{\infty}\mathrm{d}z\,\frac{\mathrm{d}\phi}{\mathrm{d}z}\sqrt{V_{\mathrm{eff}}(\phi(z))}\,\delta(z - z_{\mathrm{dw}}) \\ &= \sqrt{2}\sqrt{V_{\mathrm{eff}}(\phi(z_{\mathrm{dw}}))}\left.\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|_{z=z_{\mathrm{dw}}} \\ &= \sqrt{\frac{\lambda}{2}}\left(\phi(z_{\mathrm{dw}})^2 - \phi_0^2\right)\left.\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|_{z=z_{\mathrm{dw}}}. \end{aligned} \tag{1.10}$$

Have I completely misunderstood something here?

5[th] December 2023

1.2 Dynamics of Domain Walls in The Thin Wall Approximation, cont.

It is safe to assume that the wall thickness is much smaller than the horizon. The adiabatically static solution to the e.o.m. for ϕ is $\phi(t,z) = \phi_0 \tanh\left\{\frac{a(t)}{w_0}z\right\}$, where $w_0 = \phi_0^{-1}\sqrt{2/\lambda}$ is the wall thickness ((Press, Ryden, and Spergel, 1989)). Dismissing Eq. (1.10), we find the following:

$$\begin{aligned}
\sigma\delta(z-z_{\mathrm{dw}}) &= \int_{-\infty}^{\infty}\mathrm{d}z'\,\frac{\mathrm{d}\phi}{\mathrm{d}z'}\sqrt{2V_{\mathrm{eff}}(\phi)-2V_{\mathrm{eff}}(\phi_\pm)}\times\delta(z-z_{\mathrm{dw}})\\
&= \delta(z-z_{\mathrm{dw}})\sqrt{\frac{\lambda}{2}}\int_{\mathbb{R}}\mathrm{d}z'\,\frac{\mathrm{d}\phi}{\mathrm{d}z'}\left(\phi^2-\phi_0^2\right)\\
&= \delta(z-z_{\mathrm{dw}})\sqrt{\frac{\lambda}{2}}\frac{a\phi_0^3}{w_0}\int_{\mathbb{R}}\mathrm{d}z'\,\cosh^{-2}\left\{\frac{az'}{w_0}\right\}\left(\tanh^2\left\{\frac{az'}{w_0}\right\}-1\right)\\
&= \delta(z-z_{\mathrm{dw}})\sqrt{\frac{\lambda}{2}}\frac{a\phi_0^3}{w_0}\int_{\mathbb{R}}\mathrm{d}z'\,\cosh^{-4}\left\{\frac{az'}{w_0}\right\}\cdot(-1)\\
&= \dots\\
&= (-1)\times\frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3\times\delta(z-z_{\mathrm{dw}})
\end{aligned}\tag{1.11}$$

⌜There might be a sign error somewhere as I believe the tension should be positive.⌟ Gathering it all, we have

$$T^{\mu\nu} = (\pm)\,\delta(z-\epsilon)\cdot\frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3\cdot a^{-6}\left[\left(\frac{\partial\epsilon}{\partial\xi^0}\right)^2 - a^{-2}\left(\left(\frac{\partial\epsilon}{\partial\xi^1}\right)^2+\left(\frac{\partial\epsilon}{\partial\xi^2}\right)^2+1\right)\right]^{-1/2}\cdot\frac{\delta g^{(3)}}{\delta g_{\mu\nu}},\tag{1.12}$$

where the last factor is found from Eq. (1.9).

Perturbation. The small perturbation ϵ obeys

$$\ddot{\epsilon}+4\frac{\dot a}{a}\dot\epsilon-\frac{1}{a^2}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\epsilon = 0;\tag{1.13}$$

a linear PDE with solutions of the form $\epsilon(\xi^A) = \tilde\epsilon(\xi^0)\mathrm{e}^{\mathrm{i}(u_1\xi^1+u_2\xi^2)}$, where $\tilde\epsilon$ satisfies

$$\ddot{\tilde\epsilon}+4\frac{\dot a}{a}\dot{\tilde\epsilon}+\frac{u^2}{a^2}\tilde\epsilon = 0;\; u^2 = u_1^2+u_2^2.\tag{1.14}$$

Do you have any suggestions as to how to solve this?

We can rewrite Eq. (1.12):

$$\begin{aligned}
T^{\mu\nu} &= (\pm)\,\delta(z-\epsilon)\cdot\frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3\cdot a^{-5}\left[a^2\dot\epsilon^2+u^2\epsilon-1\right]^{-1/2}\cdot\frac{\delta g^{(3)}}{\delta g_{\mu\nu}}\\
&= (\pm)\,\delta(z-\epsilon)\cdot\frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3\cdot a^{-5}\left[a^2\dot{\tilde\epsilon}^2\mathrm{e}^{2\mathrm{i}(u_1x+u_2y)}+u^2\tilde\epsilon\mathrm{e}^{\mathrm{i}(u_1x+u_2y)}-1\right]^{-1/2}\cdot\frac{\delta g^{(3)}}{\delta g_{\mu\nu}}
\end{aligned}\tag{1.15}$$

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1.3 Perturbation: Solving the e.o.m.

We have

$$T^{\mu\nu} = \frac{(\pm)2\,\sigma\delta(z-\epsilon)}{a^5\sqrt{a^2\dot{\epsilon}^2+u^2\epsilon^2-1}}\frac{\delta g^{(3)}}{\delta g_{\mu\nu}}. \tag{1.16}$$

Eq. (1.14) can be solved analytically for $a \propto t^\beta$. We let η denote conformal time s.t. $\mathrm{d}\eta = a^{-1}\mathrm{d}t$ and $a \propto \eta^\alpha$, where $\alpha = \beta/(1-\beta)$. We let $\epsilon(t,x,y) = \epsilon(t)f(x,y)$, and ignore the tilde from before. For a matter dominated universe, $\beta = 2/3$ and $\alpha = 2$.

We make use of the transformed Bessel's equation of the form

$$x^2y'' + (1-2\mathfrak{a})xy' + \left(\mathfrak{b}^2\mathfrak{c}^2x^{2\mathfrak{c}} + \mathfrak{a}^2 - \ell^2\mathfrak{c}^2\right)y = 0; \quad \mathfrak{a},\mathfrak{b},\mathfrak{c},\ell \in \mathbb{C}, \tag{1.17}$$

whose general solution is $y(x) = x^{\mathfrak{a}}\{c_1\mathcal{J}_\ell(\mathfrak{b}x^{\mathfrak{c}}) + c_2\mathcal{Y}_\ell(\mathfrak{b}x^{\mathfrak{c}})\}$ ((see Bowman, 1958, p.117–118)). The properties of the Bessel functions of the first and second kind gives

$$y(x) = \begin{cases} x^{\mathfrak{a}}\{c_1\mathcal{J}_\ell(\mathfrak{b}x^{\mathfrak{c}}) + c_2\mathcal{J}_{-\ell}(\mathfrak{b}x^{\mathfrak{c}})\}, & \ell \notin \mathbb{Z} \\ x^{\mathfrak{a}}\{c_1\mathcal{J}_\ell(\mathfrak{b}x^{\mathfrak{c}}) + c_2\mathcal{Y}_\ell(\mathfrak{b}x^{\mathfrak{c}})\}, & \ell \in \mathbb{Z} \end{cases}. \tag{1.18}$$

1.3.1 Conformal Time Frame

For $\epsilon(t) \to \epsilon(\eta)$, Eq. (1.14) reads

$$\eta^2\epsilon'' + 3\alpha\eta\epsilon' + u^2\eta^2\epsilon = 0, \tag{1.19}$$

where primed means conformal time derivative. The solution to this equation is

$$\epsilon(\eta) = \eta^\ell\{c_1\mathcal{J}_\ell(u\eta) + c_2\mathcal{Y}_\ell(u\eta)\}; \quad \ell = \frac{1-3\alpha}{2}. \tag{1.20}$$

Matter domination. With $\alpha = 2$, $\ell = -5/2$ and thus

$$\epsilon(\eta) = c\cdot\frac{\mathcal{J}_{5/2}(u\eta)}{\eta^{5/2}} \tag{1.21}$$

is a solution.

1.3.2 Cosmic Time Frame

When $a(t) = Kt^\beta$, Eq. (1.14) is simply

$$t^2\ddot{\epsilon} + 4\beta t\dot{\epsilon} + (u/K)^2t^{2(1-\beta)}\epsilon = 0, \tag{1.22}$$

with solution

$$\epsilon(t) = t^{\ell\gamma}\left\{c_1\mathcal{J}_\ell\left(\frac{u}{K\gamma}t^\gamma\right) + c_2\mathcal{Y}_\ell\left(\frac{u}{K\gamma}t^\gamma\right)\right\}; \quad \ell = \frac{1-4\beta}{2\gamma}, \gamma = 1-\beta. \tag{1.23}$$

Matter domination. With $\beta = 2/3$, $\ell = -5/2$ and a solution is

$$\epsilon(t) = c\cdot\frac{\mathcal{J}_{5/2}\left(\frac{3u}{K}t^{1/3}\right)}{t^{5/6}}. \tag{1.24}$$

1.3.3 Stress-Energy Tensor

We once again turn our attention to the stress-energy tensor, writing it out for $\epsilon(t, x, y) = \epsilon(t) f(x, y)$, where $f(x, y) = \mathrm{e}^{\mathrm{i}(u_1 x + u_2 y)}$:

$$\begin{aligned} T^{\mu\nu} &= (\pm) \frac{2\,\sigma\delta(z - f\epsilon)}{a^5 \sqrt{a^2 f^2 \dot{\epsilon}^2 + u^2 f^2 \epsilon^2 - 1}} \frac{\delta g^{(3)}}{\delta g_{\mu\nu}} \\ &= (\pm) \frac{2\sigma\delta(z - f\epsilon)}{a^5 \sqrt{f^2 \left[\epsilon'^2 + u^2 \epsilon^2\right] - 1}} \frac{\delta g^{(3)}}{\delta g_{\mu\nu}} \end{aligned} \tag{1.25}$$

The square root in the denominator is straight-forwardly computed when using e.g. Eq. (1.21).

1.4 Wall Profile

We still need to replace $\sigma\delta(z - \epsilon(t, x, y))$ in the expression for $T^{\mu\nu}$. **I have still not figured out how to work this out.**

2 Thin Wall Approximation; Vol. 2

16th January 2024

2.1 Gravitational Waves from Domain Walls in The Thin Wall Approximation

We have the spacetime metric $g_{\mu\nu}$ and the induced metric

$$\gamma_{ab} = g_{\mu\nu}\frac{\mathrm{d}x^\mu_{\mathrm{dw}}}{\mathrm{d}\xi^a}\frac{\mathrm{d}x^\nu_{\mathrm{dw}}}{\mathrm{d}\xi^b}; \quad \left[x^\mu_{\mathrm{dw}}\right] = (\xi^0, \xi^1, \xi^2, \epsilon(\xi^a)), \tag{2.1}$$

where we let $a, b = 0, 1, 2$. Consider $g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$. We define $\iota_a \equiv \partial\epsilon/\partial\xi^a$ and $\iota_3 \equiv -1$ for notational ease. The determinant of the induced metric is

$$\gamma = -a^4[-(a\iota_0)^2 + \underbrace{\iota_1{}^2 + \iota_2{}^2 + \iota_3{}^2}_{\equiv\iota^2}]. \tag{2.2}$$

In the thin wall approximation, the surface tension

$$\sigma = \int_{-\infty}^{+\infty}\mathrm{d}z\, T_{00} \simeq -\int_{\phi_-}^{\phi_+}\mathrm{d}\phi\, \sqrt{2V_{\mathrm{eff}}(\phi) - 2V_{\mathrm{eff}}(\phi_\pm)}. \tag{2.3}$$

The covariant action

$$S_{\mathrm{dw}} = \int d^4x\, \mathcal{L}_{\mathrm{dw}} = -\sigma\int d^3\xi\, \sqrt{-\gamma} = -\sigma\int d^4x\, \sqrt{-\gamma}\,\delta(z - z_{\mathrm{dw}}) \tag{2.4}$$

and thus, the stress–energy tensor

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta\mathcal{L}_{\mathrm{dw}}}{\delta g_{\mu\nu}} = \frac{2\sigma\delta(z-\epsilon)}{\sqrt{-g}\,\sqrt{-\gamma}}\frac{\delta\gamma}{\delta g_{\mu\nu}} = \frac{2\sigma\delta(z-\epsilon)}{a^5\,\sqrt{\iota^2 - (a\iota_0)^2}}\frac{\delta\gamma}{\delta g_{\mu\nu}}. \tag{2.5}$$

We have calculated the functional derivative $\delta\gamma\big/\delta g_{\mu\nu}$ before:

$$\frac{\delta\gamma}{\delta g_{00}} = a^4\iota^2 \qquad \frac{\delta\gamma}{\delta g_{0i}} = -a^4\iota_0\iota_i \qquad \frac{\delta\gamma}{\delta g_{ij}} = a^2\left[\iota_i\iota_j + \delta_{ij}\left((a\iota_0)^2 - \iota^2\right)\right] \tag{2.6}$$

With the ansatz $\epsilon(\xi^a) = \epsilon_t(\xi^0)\mathrm{e}^{-\mathrm{i}u_1\xi^1}\mathrm{e}^{-\mathrm{i}u_2\xi^2}$, solutions for the equations of motion for ϵ_t are known for $a \propto t^\beta$. In that case, $\iota_0 = \dot\epsilon$, $\iota_1 = -\mathrm{i}u_1\epsilon$, $\iota_2 = -\mathrm{i}u_2\epsilon$ and, of course, $\iota_3 = -1$.

2.1.1 The Symmetron Potential

It is easily shown that for $V_{\text{eff}}(\phi) = V_{\text{Sym}}(\phi) \equiv \frac{\lambda}{4}\left(\phi^2 - {\phi_0}^2\right)^2$, the surface tension reduces to $\sigma = \sigma_0 \equiv \frac{4}{3}{\phi_0}^3\sqrt{\lambda/2}$. We consider matter domination ($a \propto t^{2/3} \propto \eta^2$) and assume a solution $\epsilon_\eta = \epsilon_0\eta^{-5/2}\mathcal{J}_{5/2}(u\eta)$, where η is conformal time. Note that $\epsilon' = a\dot{\epsilon}$. We have

$$T^{\mu\nu}(\eta, \boldsymbol{x}) = \frac{2\sigma_0\delta(z-\epsilon)}{a^5\sqrt{I-(a\dot{\epsilon})^2}}\frac{\delta\gamma}{\delta g_{\mu\nu}} = \frac{2\sigma_0\delta(z-\epsilon)}{a^5\sqrt{1-(u\epsilon)^2-\epsilon'^2}}\frac{\delta\gamma}{\delta g_{\mu\nu}}. \tag{2.7}$$

Neglecting all $\mathcal{O}(\epsilon^2)$-terms, we get that the only non-vanishing contributions are:

$$\delta\gamma\big/\delta g_{00} = a^4 \qquad \delta\gamma\big/\delta g_{11} = -a^2 \qquad \delta\gamma\big/\delta g_{22} = -a^2 \tag{2.8}$$

$$\delta\gamma\big/\delta g_{03} = a^3\epsilon' \qquad \delta\gamma\big/\delta g_{13} = \mathrm{i}u_1a^2\epsilon \qquad \delta\gamma\big/\delta g_{23} = \mathrm{i}u_2a^2\epsilon \tag{2.9}$$

We have $T^{\mu\nu} = T^{(\mu\nu)}$. Let indices $A, B, C = 1, 2$ and $\kappa = 8\pi^2\sigma_0a^{-3}$. In Fourier space, we have the following:

$$\begin{aligned}
T^{00}(\eta, \boldsymbol{k}) &= \kappa a^2\left\{\delta^{(2)}\left(k_A\right) + \mathrm{i}k_3\epsilon_\eta\delta^{(2)}\left(k_A - u_A\right)\right\}\\
T^{0i}(\eta, \boldsymbol{k}) &= \delta^{i3}\cdot\kappa a\epsilon'_\eta\delta^{(2)}\left(k_A - u_A\right)\\
T^{AB}(\eta, \boldsymbol{k}) &= \delta^{AB}\cdot(-\kappa)\left\{\delta^{(2)}\left(k_C\right) + \mathrm{i}k_3\epsilon_\eta\delta^{(2)}\left(k_C - u_C\right)\right\}\\
T^{i3}(\eta, \boldsymbol{k}) &= \delta^{iA}\cdot\kappa\mathrm{i}u_A\epsilon_\eta\delta^{(2)}\left(k_B - u_B\right)
\end{aligned} \tag{2.10}$$

Gravitational Waves. The transverse, traceless tensor perturbation h_{ij}, showing up in the perturbed line element $ds^2 = a^2\left\{-\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right\}$, has the e.o.m.

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + 2\frac{a'}{a}\frac{\mathrm{d}}{\mathrm{d}\eta} + k^2\right]h_{ij}(\eta, \boldsymbol{k}) = 16\pi G_{\mathrm{N}}\Lambda_{ij.kl}(\boldsymbol{n})T_{kl}(\eta, \boldsymbol{k}); \quad \boldsymbol{k} = k\boldsymbol{n}, |\boldsymbol{n}| = 1. \tag{2.11}$$

We extracted the transverse, traceless (TT) part of the symmetric stress–energy tensor by use of the "Lambda tensor" $\Lambda_{ij.kl}$. We find that

$$\Lambda_{ij.kl}(\boldsymbol{n})T_{kl}(\eta, \boldsymbol{k}) = 4\pi^2\sigma_0a\cdot\mathrm{i}\epsilon_\eta k^{-4}\delta^{(2)}\left(k_C - u_C\right)\cdot k_3t_{ij} \tag{2.12a}$$

where

$$t_{AB} = \delta_{AB}k^2u^2 + \left(u^2 - 2k^2\right)u_Au_B \tag{2.12b}$$

$$t_{A3} = u^2u_Ak_3 \tag{2.12c}$$

$$t_{33} = -u^4 \tag{2.12d}$$

and $k_3 = \sqrt{k^2 - u^2}$ necessarily.

Solving the e.o.m. using Green's functions ((cf. Kawasaki and Saikawa, 2011)). Now, the tensor field $\mathsf{h}_{ij} \equiv ah_{ij}$ is given by

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{16\pi G_{\mathrm{N}}}{k^2}\int_{\tau_{\text{ini}}}^{\tau}\mathrm{d}\tau'\underbrace{\frac{\pi}{2}\sqrt{\tau\tau'}\{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\}}_{\times\Theta(\tau-\tau')=\mathcal{G}_\nu(\tau,\tau')}a(\tau')\Lambda_{ij.kl}(\boldsymbol{n})T_{kl}(\tau', \boldsymbol{k}) \tag{2.13}$$

with $\tau = k\eta$, where $\nu = \alpha - 1/2 = 3/2$ for matter domination. That is,

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{64\pi^3 G_{\mathrm{N}}\sigma_0}{k^6}\delta^{(2)}\big(k_A - u_A\big)k_3 t_{ij} \times \int_{\tau_{\mathrm{ini}}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_{3/2}(\tau, \tau')a^2(\tau')\mathrm{i}\epsilon_\eta(\tau');$$
$$\mathcal{G}_{3/2}(\tau, \tau') = \Theta(\tau - \tau')\frac{1}{\tau\tau'}[(\tau\tau' + 1)\sin(\tau - \tau') - (\tau - \tau')\cos(\tau - \tau')].^1 \quad (2.14)$$

We let $a(\eta) = a_{\mathrm{ini}}(\eta/\eta_{\mathrm{ini}})^\alpha$. Thus, if one solves the integral

$$I \equiv \int_{\tau_{\mathrm{ini}}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_{3/2}(\tau, \tau')\tau'^{3/2}\mathcal{J}_{5/2}(u\tau'), \quad (2.15)$$

one has an explicit expression for the tensor perturbation in Fourier space:

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{64\pi^3 G_{\mathrm{N}}\sigma_0\epsilon_0}{k^6}\frac{a_{\mathrm{ini}}{}^2}{\eta_{\mathrm{ini}}{}^4}\delta^{(2)}\big(k_A - u_A\big)k_3 t_{ij}(k)\cdot \mathrm{i}I(k\eta) \quad (2.16)$$

A closer look at I. Explicitly—still for a matter dominated universe—we have

$$I = \int_{\tau_{\mathrm{ini}}}^{\tau} \mathrm{d}\tau'\, \frac{\sqrt{\tau'}}{\tau}[(\tau\tau' + 1)\sin(\tau - \tau') - (\tau - \tau')\cos(\tau - \tau')]$$
$$\times \sqrt{\frac{2}{\pi}}\left[\left(-1 + \frac{3}{u^2\tau'^2}\right)\frac{\cos(u\tau')}{\sqrt{u\tau'}} + \frac{3\sin(u\tau')}{u\tau'\sqrt{u\tau'}}\right], \quad (2.17)$$

which we rewrite;

$$I = \sqrt{\frac{2}{u\pi}}\int_{\tau_{\mathrm{ini}}}^{\tau} \mathrm{d}\tau'\, \frac{1}{\tau}[(\tau\tau' + 1)\sin(\tau - \tau') - (\tau - \tau')\cos(\tau - \tau')]$$
$$\times \left[\left(-1 + \frac{3}{u^2\tau'^2}\right)\cos(u\tau') + \frac{3}{u\tau'}\sin(u\tau')\right]. \quad (2.18)$$

In the special case where $\tau \gg 1$ (wavelength of GWs well inside the Hubble horizon), we can simplify to be left with

$$I \overset{\tau,\tau'\gg 1}{\simeq} \sqrt{\frac{2}{u\pi}}\int_{\tau_{\mathrm{ini}}}^{\tau} \mathrm{d}\tau'\, \Big\{-\tau'\cos(u\tau')\sin(\tau - \tau') - (\tau'/\tau)\cos(u\tau')\cos(\tau - \tau')$$
$$+ (3/u)\sin(u\tau')\sin(\tau - \tau') + \cos(u\tau')\cos(\tau - \tau')\Big\}, \quad (2.19)$$

an integral with a well-defined algebraic solution, though rather ugly and long.

Comment. First of all, there is a propagating sign error somewhere, stemming from the surface tension/Lagrangian. I am somewhat confused about the imaginary factor apparently surviving all steps. Especially worrisome in the final expression for the tensor field in Fourier space, Eq. (2.16). If these results are error-free, it should not be a very difficult task generalising to a framework with arbitrary (likely power-law) scale factor and perturbation.

I have not written the inverse F.T. of Eq. (2.16). It seems possible to do by hand, but rather complicated as I and t_{ij} depend on k (or k_3).

[1] Note that if $\tau = k\eta \gg 1$, $\mathcal{G}_{3/2}(\tau - \tau') \simeq \Theta(\tau - \tau')\sin(\tau - \tau')$.

17[th] January 2024

2.2 Gravitational Waves from Thin Domain Walls: Symmetron Model

We let $\epsilon(\eta, x, y) \rightarrow \epsilon(\eta, x)$ represent a plane wave perturbation to the thin, infinite wall in the xy–plane. Impose $\epsilon(\eta, x) = \epsilon_\eta(\eta)\epsilon_x(x)$. After correcting some mistakes, we have the spatial part of the stress–energy tensor in *Fourier* space:

$$T_{11}(\eta, \boldsymbol{k}) = T_{22}(\eta, \boldsymbol{k}) = 4\pi\sigma_0 a \delta\left(k_2\right) \cdot (-1)\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.20}$$

$$T_{13}(\eta, \boldsymbol{k}) = T_{31}(\eta, \boldsymbol{k}) = 4\pi\sigma_0 a \delta\left(k_2\right) \cdot \mathrm{i}k_1\epsilon_\eta\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.21}$$

Let $\Lambda_{ij.kl}T_{kl} \equiv 2\pi\sigma_0 a\delta\left(k_2\right)k^{-4}t_{ij}$. Then:

$$t_{11} = -{k_1}^2{k_3}^2 \cdot (-1)\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] - k_1{k_3}^3 \cdot 2\mathrm{i}k_1\epsilon_\eta\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.22a}$$

$$t_{22} = k^2{k_1}^2 \cdot (-1)\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] + k^2k_1k_3 \cdot 2\mathrm{i}k_1\epsilon_\eta\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.22b}$$

$$t_{33} = -{k_1}^4 \cdot (-1)\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] - {k_1}^3k_3 \cdot 2\mathrm{i}k_1\epsilon_\eta\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.22c}$$

$$t_{13} = {k_1}^3k_3 \cdot (-1)\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] + {k_1}^2{k_3}^2 \cdot 2\mathrm{i}k_1\epsilon_\eta\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.22d}$$

The GWs generated from this system are given by

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{16\pi G_\mathrm{N}}{k^2}\int_{\tau_\mathrm{ini}}^{\tau}\mathrm{d}\tau'\,\mathcal{G}_\nu(\tau,\tau')a(\tau')\Lambda_{ij.kl}(\boldsymbol{n})T_{kl}(\tau',\boldsymbol{k});$$
$$\mathcal{G}_\nu(\tau,\tau') = \Theta(\tau-\tau')\frac{\pi}{2}\sqrt{\tau\tau'}\{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\}, \tag{2.23}$$

where $\mathsf{h}_{ij} = ah_{ij}$ and $\nu = \alpha - 1/2$ for a universe with $a \propto \eta^\alpha$. Assuming $\alpha = 2$, we have from before that $\epsilon_\eta(\eta) = \epsilon_0\eta^{-5/2}\mathcal{J}_{5/2}(u\eta)$, where u is the wavenumber associated with $\epsilon_x(x)$ (e.g. $\epsilon_x(x) = \sin(ux)$). Thus,

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{32\pi^2 G_\mathrm{N}\sigma_0}{k^6}\delta\left(k_2\right)\int_{\tau_\mathrm{ini}}^{\tau}\mathrm{d}\tau'\,\mathcal{G}_{3/2}(\tau,\tau')a^2(\tau')t_{ij}(\tau',k_1,k_3); \quad \tau = k\eta. \tag{2.24}$$

2.2.1 Two Fourier transforms

To get explicit expressions in t_{ij}, we need to calculate **(a)** $\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right]$ and **(b)** $\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right]$. We impose $\epsilon_x = \sin(ux)$. Then, we effectively have to solve **(a)** $\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}C\sin(ux)}\right]$ and **(b)** $\mathcal{F}_{k_1}\left[\sin(ux)\mathrm{e}^{\mathrm{i}C\sin(ux)}\right]$.

(a)

$$\begin{aligned}\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}C\sin(ux)}\right] &= \int\mathrm{d}x\,\mathrm{e}^{\mathrm{i}C\sin(ux)}\mathrm{e}^{\mathrm{i}k_1x}\\ &= 2\pi\delta\left(k_1\right)\mathcal{J}_0(C) - 2\pi\sum_{n=1}^{\infty}\mathcal{J}_n(C)\left[\delta\left(k_1-nu\right) + (-1)^n\delta\left(k_1+nu\right)\right]\\ &= 2\pi\sum_{n=-\infty}^{+\infty}\mathcal{J}_n(C)\delta\left(k_1+nu\right)\end{aligned}$$

(b)

$$\begin{aligned}\mathcal{F}_{k_1}\left[\sin(ux)\mathrm{e}^{\mathrm{i}C\sin(ux)}\right] &= -\int \mathrm{d}x\ \sin(ux)\mathrm{e}^{\mathrm{i}C\sin(ux)}\mathrm{e}^{\mathrm{i}k_1 x} \\ &= \mathrm{i}\pi \sum_{n=-\infty}^{+\infty} [\mathcal{J}_{n-1}(C) - \mathcal{J}_{n+1}(C)]\delta\left(k_1 - nu\right)\end{aligned}$$

19th January 2024

2.3 Stress–Energy Tensor: A Revised Calculation

We focus of the spatial components of the stress–energy tensor, which to first order in ϵ is

$$T_{ij}(\eta, \boldsymbol{x}) = a^4 T^{ij}(\eta, \boldsymbol{x}) = 2\sigma_0 a^{-1}\delta(z-\epsilon)\delta\gamma\big/\delta g_{ij}, \tag{2.25}$$

where for $\epsilon = \epsilon(\eta, x)$ we have the only non-vanishing contributions $\delta\gamma\big/\delta g_{11} = \delta\gamma\big/\delta g_{22} = -a^2$ and $\delta\gamma\big/\delta g_{13} = \delta\gamma\big/\delta g_{31} = -a^2\partial_1\epsilon$.

2.3.1 Fourier Space

Now, we find $T_{ij}(\eta, \boldsymbol{k})$:

$$\begin{aligned}
\mathcal{F}_{\boldsymbol{k}}\Big[T_{ij}(\eta,\boldsymbol{x})\Big] &= \int d^3x\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} T_{ij}(\eta,\boldsymbol{x}) \\
&= 2\sigma_0 a^{-1}\cdot 2\pi\delta\big(k_2\big)\cdot \int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}k_1 x}\delta\gamma\big/\delta g_{ij}\cdot \int \mathrm{d}z\, \mathrm{e}^{\mathrm{i}k_3 z}\delta(z-\epsilon) \\
&= 2\sigma_0 a^{-1}\cdot 2\pi\delta\big(k_2\big)\cdot \int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}k_1 x}\delta\gamma\big/\delta g_{ij}\cdot \mathrm{e}^{\mathrm{i}k_3\epsilon} \\
&= 2\sigma_0 a^{-1}\cdot 2\pi\delta\big(k_2\big)\cdot \mathcal{F}_{k_1}\Big[\delta\gamma\big/\delta g_{ij}\cdot \mathrm{e}^{\mathrm{i}k_3\epsilon}\Big]
\end{aligned} \tag{2.26}$$

Thus,

$$T_{11}(\eta,\boldsymbol{k}) = T_{22}(\eta,\boldsymbol{k}) = -4\pi\sigma_0 a\delta\big(k_2\big)\mathcal{F}_{k_1}\Big[\mathrm{e}^{\mathrm{i}k_3\epsilon}\Big] \tag{2.27a}$$

$$T_{13}(\eta,\boldsymbol{k}) = T_{31}(\eta,\boldsymbol{k}) = -4\pi\sigma_0 a\delta\big(k_2\big)\mathcal{F}_{k_1}\Big[(\partial_1\epsilon)\mathrm{e}^{\mathrm{i}k_3\epsilon}\Big] \tag{2.27b}$$

are the non-vanishing components.

We impose $\epsilon = \bar{\epsilon}(\eta)\sin(ux)$. The Jacobi-Anger expansion,

$$\mathrm{e}^{\mathrm{i}\xi\sin\theta} = \mathrm{e}^{\mathrm{i}\xi\cos\theta'} = \sum_{n=-\infty}^{\infty} \mathrm{i}^n \mathcal{J}_n(\xi)\mathrm{e}^{\mathrm{i}n\theta'} = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(\xi)\mathrm{e}^{\mathrm{i}n\theta}; \quad \theta' = \theta - \frac{\pi}{2}, \tag{2.28}$$

is essential to the coming calculations. In addition, we make use of $\mathcal{J}_{-n}(\xi) = (-1)^n\mathcal{J}_n(\xi)$ and $\mathcal{J}_{n-1}(\xi) + \mathcal{J}_{n+1}(\xi) = (2n/\xi)\mathcal{J}_n(\xi)$.

(a) $\mathcal{F}_{k_1}\Big[\mathrm{e}^{\mathrm{i}k_3\epsilon}\Big] = \mathcal{F}_{k_1}\Big[\mathrm{e}^{\mathrm{i}k_3\bar{\epsilon}\sin(ux)}\Big]$**.**

$$\begin{aligned}
\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}c\sin(ux)}\mathrm{e}^{\mathrm{i}\omega x} &= \sum_{n\in\mathbb{Z}} \mathcal{J}_n(c)\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}nux}\mathrm{e}^{\mathrm{i}\omega x} \\
&= 2\pi\sum_{n\in\mathbb{Z}} \mathcal{J}_n(c)\delta(\omega + nu)
\end{aligned}$$

$$\Longrightarrow \mathcal{F}_{k_1}\Big[\mathrm{e}^{\mathrm{i}k_3\bar{\epsilon}\sin(ux)}\Big] = \begin{cases} 2\pi\mathcal{J}_\ell(k_3\bar{\epsilon}) & \text{if } \ell = -k_1/u \in \mathbb{Z}, \\ 0 & \text{else.}\end{cases} \tag{2.29}$$

(b) $\mathcal{F}_{k_1}\left[(\partial_1\epsilon)\mathrm{e}^{ik_3\epsilon}\right] = u\bar{\epsilon}\mathcal{F}_{k_1}\left[\cos(ux)\mathrm{e}^{ik_3\bar{\epsilon}\sin(ux)}\right]$**.**

$$
\begin{aligned}
\int \mathrm{d}x\, \cos(ux)\mathrm{e}^{\mathrm{i}c\sin(ux)}\mathrm{e}^{\mathrm{i}\omega x} &= \frac{1}{2}\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}c\sin(ux)}\left[\mathrm{e}^{\mathrm{i}ux} + \mathrm{e}^{-\mathrm{i}ux}\right]\mathrm{e}^{\mathrm{i}\omega x} \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathcal{J}_n(c)\int \mathrm{d}x \left[\mathrm{e}^{\mathrm{i}(n+1)ux} + \mathrm{e}^{\mathrm{i}(n-1)ux}\right]\mathrm{e}^{\mathrm{i}\omega x} \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} [\mathcal{J}_{n+1}(c) + \mathcal{J}_{n-1}(c)]\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}nux}\mathrm{e}^{\mathrm{i}\omega x} \\
&= \pi\sum_{n\in\mathbb{Z}} \underbrace{[\mathcal{J}_{n+1}(c) + \mathcal{J}_{n-1}(c)]}_{2nc^{-1}\mathcal{J}_n(c)}\,\delta(\omega + nu) \\
&= \frac{2\pi}{c}\sum_{n\in\mathbb{Z}} n\mathcal{J}_n(c)\delta(\omega + nu)
\end{aligned}
$$

$$
\Longrightarrow u\bar{\epsilon}\mathcal{F}_{k_1}\left[\cos(ux)\mathrm{e}^{ik_3\bar{\epsilon}\sin(ux)}\right] = \begin{cases} -2\pi(k_1/k_3)\mathcal{J}_\ell(k_3\bar{\epsilon}) & \text{if } \ell = -k_1/u \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases} \tag{2.30}
$$

2.3.2 Traceless–Transverse Gauge

We extract the transverse, traceless (TT) part of the stress–energy tensor by use of the "Lambda tensor", i.e. $T^{\mathrm{TT}}_{ij}(\eta, \boldsymbol{k}) = \Lambda_{ij.kl}(\boldsymbol{k}/k)T_{kl}(\eta, \boldsymbol{k})$. The non-vanishing contributions are given by:

$$
\begin{aligned}
2k^4T^{\mathrm{TT}}_{11} &= -{k_1}^2{k_3}^2T_{11} - 2k_1{k_3}^3T_{13} && (2.31\text{a}) \\
2k^4T^{\mathrm{TT}}_{22} &= k^2{k_1}^2T_{11} + 2k^2k_1k_3T_{13} && (2.31\text{b}) \\
2k^4T^{\mathrm{TT}}_{33} &= -{k_1}^4T_{11} - 2{k_1}^3k_3T_{13} && (2.31\text{c}) \\
2k^4T^{\mathrm{TT}}_{13} &= {k_1}^3k_3T_{11} + 2{k_1}^2{k_3}^2T_{13} && (2.31\text{d})
\end{aligned}
$$

More compactly, we can use Eq. (2.29) and Eq. (2.30) and write

$$
\begin{aligned}
T^{\mathrm{TT}}_{ij} &= \frac{1}{2k^4}\left({k_1}^2T_{11} + 2k_1k_3T_{13}\right)\cdot\left[\delta_{ij}\left(k^2 - 2k_ik_j\right) + k_ik_j\right] \qquad \Bigg|\, \ell = -\frac{k_1}{u} \\
&= \frac{-4\pi\sigma_0 a\delta\left(k_2\right)}{2k^4}\left({k_1}^2 - 2k_1k_3\cdot(k_1/k_3)\right)2\pi\mathcal{J}_\ell(k_3\bar{\epsilon})\cdot\left[\delta_{ij}\left(k^2 - 2k_ik_j\right) + k_ik_j\right] \\
&= \frac{4\pi^2\sigma_0}{k^4}\delta\left(k_2\right){k_1}^2\cdot\left[\delta_{ij}\left(k^2 - 2k_ik_j\right) + k_ik_j\right]\cdot a\mathcal{J}_\ell(k_3\bar{\epsilon}),
\end{aligned} \tag{2.32}
$$

$\forall\, \ell \in \mathbb{Z}$, otherwise the solution is trivial.

2.3.3 Gravitational Waves

We define $\mathsf{h}_{ij} \equiv ah_{ij}$. We consider a universe where $a \propto \eta^\alpha$. Now

$$
\begin{aligned}
\mathsf{h}_{ij}(\eta, \boldsymbol{k}) &= \frac{16\pi G_{\mathrm{N}}}{k^2}\int_{\tau_{\mathrm{ini}}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_\nu(\tau, \tau')a(\tau')T^{\mathrm{TT}}_{ij}(\tau', \boldsymbol{k}); \qquad \nu = \alpha - \frac{1}{2}; \\
\mathcal{G}_\nu(\tau, \tau') &= \Theta(\tau - \tau')\frac{\pi}{2}\sqrt{\tau\tau'}\{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\}
\end{aligned} \tag{2.33}
$$

is the expression for the tensor perturbations. For our specific setup, this can be rewritten;

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{64\pi^3\sigma_0 G_\mathrm{N}}{k^6}\delta\big(k_2\big)k_1{}^2\Big[\delta_{ij}\big(k^2 - 2k_i k_j\big) + k_i k_j\Big] \times \sum_{n\in\mathbb{Z}} \delta(\ell - n)$$
$$\times \int_{\tau_\mathrm{ini}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_\nu(\tau, \tau') a^2(\tau') \mathcal{J}_\ell\big(k_3 \bar{\epsilon}(\tau')\big); \quad \tau = k\eta,\ \ell = -k_1/u \quad (2.34)$$

Matter Domination. We let $a(\eta) = a_\mathrm{ini}(\eta/\eta_\mathrm{ini})^\alpha$ and consider $\alpha = 2$. Now $\bar{\epsilon}(\eta) = \epsilon_0 \eta^{-5/2} \mathcal{J}_{5/2}(u\eta)$ satisfies the e.o.m. for the time-dependence of ϵ. Furthermore,

$$\mathcal{G}_{3/2}(\tau, \tau') = \Theta(\tau - \tau') \frac{(\tau\tau' + 1)\sin(\tau - \tau') - (\tau - \tau')\cos(\tau - \tau')}{\tau\tau'}. \quad (2.35)$$

We essentially have to solve

$$\int_{\tau_\mathrm{ini}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_{3/2}(\tau, \tau') \tau'^4 \mathcal{J}_\ell\big(k_3 \bar{\epsilon}(\tau')\big), \quad \ell \in \mathbb{Z}, \quad (2.36)$$

and this is where I am stuck.

29th January 2024

2.4 The Complete Model

We consider a universe for which $a(\eta) \propto \eta^{\alpha}$ for $\eta \geq \eta_{\rm init}$. The line element is $ds^2 = a^2(\eta)\left\{-\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right\}$. The wall is represented by the Symmetron potential $V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2$ and "width" w_0. The location of the wall is $[X^{\mu}] = (\eta, x, y, \epsilon(\eta, x, y))$ where ϵ is a linear perturbation with e.o.m.

$$\ddot{\epsilon} + 3\mathcal{H}\dot{\epsilon} - \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\epsilon = 0; \quad \mathcal{H} = \frac{\dot{a}}{a}, \quad \dot{} \equiv \frac{\mathrm{d}}{\mathrm{d}\eta}. \tag{2.37}$$

We consider plane waves of eigenvalues $-u^2$. In particular, we impose $\epsilon(\eta, x, y) = \epsilon(\eta, x) = \bar{\epsilon}(\eta)\sin(u_1 x)$, such that $u = \left|u_1\right|$ and $\bar{\epsilon}(\eta) = \epsilon_0\eta^{\gamma}\left\{c_1\mathcal{J}_{\gamma}(u\eta) + c_2\mathcal{Y}_{\gamma}(u\eta)\right\}$, $\gamma = \frac{1}{2}(1 - 3\alpha)$. We may restrict $c_1^2 + c_2^2 = 1$ so that ϵ_0 properly reflects the order of the perturbation, $\mathscr{O}(\epsilon_0) \equiv \mathscr{O}(\epsilon)$.

For computational ease, we divide our final expression into parts, effectively parametrising $k_i \mapsto (k_3, \ell)$:

$$\begin{aligned} a(\eta)h_{ij}(\eta, \boldsymbol{k}) &= (\text{const.}) \cdot \delta\left(k_2\right)[\![\ell \in \mathbb{Z}]\!] \cdot K_{ij} \cdot \mathrm{e}^{-\frac{1}{2}(w_0 k_3)^2} \cdot I; \quad \ell = -k_1/u_1; \\ (\text{const.}) &= 32\pi^3 G_{\rm N}\sigma_0\left(a_{\rm init}/\eta_{\rm init}^{\alpha}\right)^2, \quad K_{ij} = k^{-6}{k_1}^2\left[\delta_{ij}\left(k^2 - 2k_i k_j\right) + k_i k_j\right], \\ I &= \int_{\tau_{\rm init}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_{\nu}(\tau, \tau'){\tau'}^{2\nu+1}\mathcal{J}_{\ell}\left(k_3\bar{\epsilon}(\tau'; u)\right); \quad \tau = k\eta,\ \nu = \alpha + 1/2 \end{aligned} \tag{2.38}$$

Furthermore, we have:

$$\sigma_0 = \frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3 \tag{2.39}$$

$$a(\eta) = a_{\rm init}\left(\frac{\eta}{\eta_{\rm init}}\right)^{\alpha} \tag{2.40}$$

$$a_{\rm init} = a(\eta_{\rm init}) \tag{2.41}$$

$$\mathcal{G}_{\nu}(\tau, \tau') = \frac{\pi}{2}\{\mathcal{Y}_{\nu}(\tau)\mathcal{J}_{\nu}(\tau') - \mathcal{J}_{\nu}(\tau)\mathcal{Y}_{\nu}(\tau')\} \tag{2.42}$$

Assuming a wall profile $\phi(\eta, z) = \phi_0\tanh\left(\sqrt{\lambda/2}\,a(\eta)z\right)$, we can restrain the width $w_0 \simeq \sqrt{2/\lambda}\,\phi_0^{-1}$, i.e. the (comoving) spatial variation of ϕ. Note that the wall should still be *thin*; $w_0 \ll \mathcal{H}$. Thus, given α and a specified perturbation—$\{u_1, \epsilon_0, c_1, c_2\}$—the model depends on parameters $\{\phi_0, \lambda, \eta_{\rm init}, a_{\rm init}\}$. In $\boldsymbol{k}$-space, non-vanishing h_{ij}'s arise for $i = j$ and $(ij) = (13)$ when $k_2 = 0$ and k_1 takes the values that are multiples of u_1, for any $k_3 \in \mathbb{R}$.

6th February 2024

2.5 The Complete Model

We consider a universe for which $a(\eta) \propto \eta^\alpha$ for $\eta \geq \eta_{\text{init}}$. The line element is $ds^2 = a^2(\eta)\left\{-\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right\}$. The wall is represented by the Symmetron potential $V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2$ and "width" w_0. The location of the wall is $[X^\mu] = (\eta, x, y, \epsilon(\eta, x, y))$ where ϵ is a linear perturbation with e.o.m.

$$\ddot{\epsilon} + 3\mathcal{H}\dot{\epsilon} - \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\epsilon = 0; \quad \mathcal{H} = \frac{\dot{a}}{a}, \quad \dot{} \equiv \frac{\mathrm{d}}{\mathrm{d}\eta}. \tag{2.43}$$

We consider plane waves of eigenvalues $-u^2$. In particular, we impose $\epsilon(\eta, x, y) = \epsilon(\eta, x) = \bar{\epsilon}(\eta)\sin(u_x x)$, such that $u = |u_x|$ and $\bar{\epsilon}(\eta) = \epsilon_0\eta^\gamma\left\{c_1\mathcal{J}_\gamma(u\eta) + c_2\mathcal{Y}_\gamma(u\eta)\right\}$, $\gamma = \frac{1}{2}(1 - 3\alpha)$. We may restrict $c_1^2 + c_2^2 = 1$ so that ϵ_0 properly reflects the order of the perturbation, $\mathcal{O}(\epsilon_0) \equiv \mathcal{O}(\epsilon)$.

In the limit where $w_0 \to 0$, we have

$$T_{ij} = g_{i\mu}g_{j\nu}T^{\mu\nu} = a^4(\eta)\delta_{ik}\delta_{jl}T^{kl} = a^4(\eta)T^{ij} \tag{2.44}$$

and

$$T^{\mu\nu}(\eta, \boldsymbol{x}) = \frac{\sigma\delta(z - \epsilon(\eta, x))}{a^7(\eta)}\frac{\delta\gamma}{\delta g_{\mu\nu}} + \mathcal{O}(\epsilon^2), \tag{2.45}$$

where $\delta\gamma/\delta g_{11} = \delta\gamma/\delta g_{22} = -a^4$ and $\delta\gamma/\delta g_{(13)} = -a^4\epsilon_{,1} = -a^4 u_x\cos(u_x x)\bar{\epsilon}$ are the only non-vanishing spatial components. In Fourier space this gives

$$T_{11}(\eta, \boldsymbol{k}) = T_{22}(\eta, \boldsymbol{k}) = -(2\pi)^2\sigma\delta\left(k_y\right)a(\eta)\mathcal{J}_\ell(k_z\bar{\epsilon}) \tag{2.46a}$$

$$T_{13}(\eta, \boldsymbol{k}) = T_{31}(\eta, \boldsymbol{k}) = -(2\pi)^2\sigma\delta\left(k_y\right)(\ell u_x/k_z)a(\eta)\mathcal{J}_\ell(k_z\bar{\epsilon}), \tag{2.46b}$$

for $\ell = -k_x/u_x \in \mathbb{Z}$. We consider the surface tension σ^2 to be constant in the thin wall limit; $\sigma = \sigma_0 \equiv 4/3\sqrt{\lambda/2}\phi_0^3$.

We can define a polarisation basis for a wave propagating along $\boldsymbol{k} = k\hat{\boldsymbol{\Omega}}$:

$$e_{ij}^+ = [\hat{\boldsymbol{m}} \otimes \hat{\boldsymbol{m}} - \hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}}]_{ij} \tag{2.47a}$$

$$e_{ij}^\times = [\hat{\boldsymbol{m}} \otimes \hat{\boldsymbol{n}} - \hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{m}}]_{ij} \tag{2.47b}$$

$\{\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{\Omega}}\}$ is an orthonormal basis, right-handed. We consider $\boldsymbol{k} = (-\ell u_x, 0, k_z)$ such that $k^2 = (\ell u)^2 + k_z^2$. In choosing $\hat{\boldsymbol{m}} = (0, 1, 0)$, we get $\hat{\boldsymbol{n}} = (-k_z, 0, -\ell u_x)/k$. Now,

$$\left[e_{ij}^+\right] = -\frac{1}{k^2}\begin{pmatrix} k_z^2 & 0 & \ell u_x k_z \\ 0 & -k^2 & 0 \\ \ell u_x k_z & 0 & (\ell u)^2 \end{pmatrix}, \quad \left[e_{ij}^\times\right] = \frac{1}{k}\begin{pmatrix} 0 & k_z & 0 \\ -k_z & 0 & -\ell u_x \\ 0 & \ell u_x & 0 \end{pmatrix}, \tag{2.48}$$

such that $T_{ij}^{\mathrm{TT}}(\eta, \boldsymbol{k}) = T_+^{\mathrm{TT}}(\eta, \boldsymbol{k})e_{ij}^+(\hat{\boldsymbol{\Omega}}) + T_\times^{\mathrm{TT}}(\eta, \boldsymbol{k})e_{ij}^\times(\hat{\boldsymbol{\Omega}})$.

In the TT frame, the non-zero components of T_{ij}^{TT} will be for $i = j$ and $(ij) = (13)$. We immediately see that $T_\times^{\mathrm{TT}} = 0$. Using the TT properties, we find that there is only one degree

[2]Using definition $\sigma \equiv \int_{-\infty}^{\infty}\mathrm{d}(az)\rho(z) = -a\int_{-\infty}^{\infty}\mathrm{d}z\, T^0_{\ 0}$.

of freedom here, and we can express all components as functions of $T_{33}^{\mathrm{TT}}(\eta, \boldsymbol{k})$. Furthermore, we find

$$\begin{aligned} T_{+}^{\mathrm{TT}}(\eta, \boldsymbol{k}) &= -(k/\ell u)^2 T_{33}^{\mathrm{TT}}(\eta, \boldsymbol{k}) \\ &= -(k/\ell u)^2 \cdot \frac{1}{2k^4}\Big(-k_x{}^4 T_{11}(\eta, \boldsymbol{k}) - 2k_x{}^3 k_z T_{13}(\eta, \boldsymbol{k})\Big) \\ &= 2\pi^2 \sigma_0 \delta\big(k_y\big)(\ell u/k)^2 a(\eta) \mathcal{J}_\ell(k_z \bar{\epsilon}). \end{aligned} \tag{2.49}$$

The comoving GWs, decomposed as $\mathsf{h}_{ij} \equiv a h_{ij} = \mathsf{h}_+ e_{ij}^+ + \mathsf{h}_\times e_{ij}^\times = \mathsf{h}_+ e_{ij}^+$, are obtained by

$$\mathsf{h}_+(\eta, \boldsymbol{k}) = \frac{16\pi G_\mathrm{N}}{k^2} \int_{\tau_\mathrm{ini}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_\nu(\tau, \tau') a(\tau'/k) T_+^{\mathrm{TT}}(\tau'/k, \boldsymbol{k}); \quad \nu = \alpha - \frac{1}{2};$$
$$\mathcal{G}_\nu(\tau, \tau') = \Theta(\tau - \tau')\frac{\pi}{2}\sqrt{\tau\tau'}\{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\}. \tag{2.50}$$

Explicitly,

$$\mathsf{h}_+(\eta, \boldsymbol{k}) = \frac{32\pi^3 G_\mathrm{N}\sigma_0}{k^2}\delta\big(k_y\big)\left(\frac{\ell u}{k}\right)^2 \int_{\tau_\mathrm{ini}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_\nu(\tau, \tau') a^2(\tau'/k) \mathcal{J}_\ell(k_z \bar{\epsilon}(\tau'/k)). \tag{2.51}$$

For computational ease, we divide the expression into parts:

$$\mathsf{h}_+(\eta, \boldsymbol{k}) = \delta\big(k_y\big)\delta(k_x + \ell u_x)[\![\ell \in \mathbb{Z}]\!] \cdot \mathsf{h}_+(\eta, \ell, k_z) \tag{2.52a}$$

$$\mathsf{h}_+(\eta, \ell, k_z) = (\text{const.}) \cdot k^{-2(\alpha+2)}\ell^2 \cdot I(\eta, \ell, k_z);$$
$$(\text{const.}) = 32\pi^3 G_\mathrm{N}\sigma_0 u^2\big(a_\mathrm{init}/\eta_\mathrm{init}^\alpha\big)^2; \quad \sigma_0 = \frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3,$$
$$I = \int_{\tau_\mathrm{init}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_\nu(\tau, \tau')\tau'^{2\nu+1}\mathcal{J}_\ell(k_z\bar{\epsilon}(\tau'/k)); \quad \mathcal{G}_\nu(\tau, \tau') = \frac{\pi}{2}\{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\} \tag{2.52b}$$

Beyond the thin-wall limit. We swap out the Dirac-delta distribution in the wall profile (call it $\Phi(z - \epsilon)$) with a Gaussian function of mean ϵ and standard deviation w_0, taken as the "width" of the wall, i.e.:

$$\delta(z - \epsilon) \to \Phi(z - \epsilon) = \frac{1}{\sqrt{2\pi}w_0}\exp\left\{-\frac{(z-\epsilon)^2}{2w_0^2}\right\} \rightsquigarrow \lim_{w_0 \to 0}\Phi(z - \epsilon) = \delta(z - \epsilon) \tag{2.53}$$

The ultimate effect of this change is simply an extra factor $\mathrm{e}^{-\frac{1}{2}(w_0 k_z)^2}$ in the expression for h_+, which naturally is unity when $w_0 \to 0$.

2.5.1 Analysis

Assuming a wall profile $\phi(\eta, z) = \phi_0 \tanh\big(\sqrt{\lambda/2}a(\eta)z\big)$, we can restrain the width $w_0 \simeq \sqrt{2/\lambda}\phi_0^{-1}$, i.e. the (comoving) spatial variation of ϕ. Note that the wall should still be *thin*; $w_0 \ll \mathcal{H}$. Thus, given α and a specified perturbation—$\{u_x, \epsilon_0, c_1, c_2\}$—the model depends on parameters $\{\phi_0, \lambda, \eta_\mathrm{init}\}$. In $\boldsymbol{k}$-space, non-vanishing h_{ij}'s arise for $i = j$ and $(ij) = (13)$ when $k_y = 0$ and k_x takes the values that are multiples of u_x, for any $k_z \in \mathbb{R}$.

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Consider expanding universe with domain wall in the xy-plane in the thin wall limit. Add perturbation $\epsilon(\eta, x)$. What do the GWs look like?

We let $\epsilon(\eta, x) = \bar{\epsilon}(\eta)\sin(ux)$ and

$$\mathcal{G}_\nu(\tau,\tau') = \Theta(\tau-\tau')\frac{\pi}{2}\sqrt{\tau\tau'}\{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\} \tag{1}$$

for $\nu = \alpha - 1/2$, where $a \propto \eta^\alpha$. Let $ah_{ij} = \mathsf{h}_{ij}$. After many steps, we get

$$\begin{aligned}\mathsf{h}_{ij}(\eta,\boldsymbol{k}) = (\pm)\frac{32\pi^3\sigma_0 G_\mathrm{N}}{k^6}\delta\big(k_2\big){k_1}^2\Big[\delta_{ij}\big(k^2 - 2k_ik_j\big) + k_ik_j\Big] \times \sum_{n\in\mathbb{Z}}\delta(\ell - n)\\ \times \int_{\tau_\mathrm{ini}}^{\tau}\mathrm{d}\tau'\,\mathcal{G}_\nu(\tau,\tau')a^2(\tau')\mathcal{J}_\ell\big(k_3\bar{\epsilon}(\tau')\big); \quad \tau = k\eta,\ \ell = -k_1/u.\end{aligned} \tag{2}$$

For $\alpha = 2$ we have $\nu = 3/2$ and $\bar{\epsilon}(\eta) \sim \eta^{-5/2}\mathcal{J}_{\pm 5/2}(u\eta)$, so this integral should probably be solved numerically. (The sign confusion stems from the variation of the domain wall action.)

Next step is to modify this to work beyond the thin wall limit.

A Old Texts

DRAFT

⌜

Consider a planar domain wall in the xy-plane in a flat FRW universe, represented by a scalar field $\phi(\eta, \boldsymbol{x})$ and a potential $V(\phi)$. The action of this theory is

$$S = \int d^4x \sqrt{-g} \left\{ 16\pi G_{\mathrm{N}} \mathcal{R} - \frac{1}{2}\phi^{;\mu}\phi_{;\mu} + V(\phi) \right\}. \tag{A.1}$$

The background metric is

$$d\overline{s}^2 = \overline{g}_{\mu\nu} \mathrm{d}\overline{x}^\mu \mathrm{d}\overline{x}^\nu = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j = a^2(\eta)\left\{-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right\}. \tag{A.2}$$

The solution to $\overline{\Box}\phi = \mathrm{d}V/\mathrm{d}\phi$ is denoted $\overline{\phi}(\eta, z)$. We let indices $a, b, c = 1, 2$ and $i, j, k, l, \ldots = 1, 2, 3$. Now we add a linear perturbation $\zeta(\eta, x^a)$ to the wall such that

$$\phi(\eta, \boldsymbol{x}) = \overline{\phi}(\eta, z; \zeta(\eta, x^a)) = \overline{\phi}(\eta, z; 0) + \zeta(\eta, x^a) \left.\frac{\partial\overline{\phi}}{\partial z}\right|_{\zeta=0} + \mathscr{O}(\zeta^2). \tag{A.3}$$

⌜Remember eqs for ζ!⌟ Furthermore, Fourier transforming [←show this!]▪ the spatial components gives

$$\phi(\eta, \boldsymbol{k}) = \int d^3x\, \mathrm{e}^{\mathrm{i}k_i x^i} \phi(\eta, \boldsymbol{x}) = \left[(2\pi)^2 \delta^{(2)}(k_a) - \mathrm{i}k_3\zeta(\eta, k_a)\right]\overline{\phi}(\eta, k_3; 0) + O\left(\zeta^2\right). \tag{A.4}$$

The TT-part of the energy-momentum tensor is [←refer to some section]▪ ⌜NB: g cannot have cross terms!!⌟

$$T_{ij}^{\mathrm{TT}}(\eta, \boldsymbol{k}) = \Lambda_{ij.kl}(\hat{\boldsymbol{k}}) \int \frac{d^3p}{(2\pi)^3}\, p_k p_l \phi(\eta, \boldsymbol{p})\phi(\eta, \boldsymbol{k} - \boldsymbol{p}). \tag{A.5}$$

We define a quantity t_{kl} by

$$T_{ij}^{\mathrm{TT}}(\eta, \boldsymbol{k}) = \Lambda_{ij.kl}(\hat{\boldsymbol{k}}) \left(\frac{1}{2\pi} \cdot t_{kl}(\eta, \boldsymbol{k}) + \mathscr{O}(\zeta^2)\right), \tag{A.6}$$

and the additional function

$$\mathfrak{I}_n(\eta, q_0) = \int_{\mathbb{R}} \mathrm{d}q\, q^n \overline{\phi}(\eta, q; 0)\overline{\phi}(\eta, q_0 - q; 0). \tag{A.7}$$

After some manipulation [←show this!]▪, we get the following:

$$t_{ab}(\eta, \boldsymbol{k}) = k_a k_b[-\mathrm{i}\zeta(\eta, k_c)]\mathfrak{I}_1(\eta, k_3) \tag{A.8a}$$
$$t_{a3}(\eta, \boldsymbol{k}) = k_a[-\mathrm{i}\zeta(\eta, k_c)]\mathfrak{I}_2(\eta, k_3) \tag{A.8b}$$
$$t_{33}(\eta, \boldsymbol{k}) = k_3[-\mathrm{i}\zeta(\eta, k_c)]\mathfrak{I}_2(\eta, k_3) + (2\pi)^2\delta^{(2)}(k_a)\mathfrak{I}_2(\eta, k_3) \tag{A.8c}$$

⌜There are some *small* constraint on the perturbation from this. Need to be commented!⌟

Gravitational waves sourced by this field is – to first order in ζ – given by

$$\begin{aligned} ah_{ij}(\eta,\boldsymbol{k}) &= \frac{16\pi G_\mathrm{N}}{k}\int_{\eta_\mathrm{i}}^{\eta}\mathrm{d}\eta'\ \sin\left(k[\eta-\eta']\right)a(\eta')T^\mathrm{TT}_{ij}(\eta',\boldsymbol{k}) \\ &= \frac{8G_\mathrm{N}}{k}\Lambda_{ij.kl}(\hat{\boldsymbol{k}})\int_{\eta_\mathrm{i}}^{\eta}\mathrm{d}\eta'\ \sin\left(k[\eta-\eta']\right)a(\eta')t_{kl}(\eta',\boldsymbol{k}) + \mathcal{O}(\zeta^2). \end{aligned} \tag{A.9}$$

Remaining are the $\Lambda_{ij.kl}t_{kl}$-elements, which in total are ⌈6⌋? terms per ij, due to symmetry in t_{kl}:

$$\begin{aligned} \Lambda_{ij.kl}(\hat{\boldsymbol{k}})t_{kl}(\eta,\boldsymbol{k}) = \Big\{&\Big(\Lambda_{ij.12}+\Lambda_{ij.21}\Big)t_{12} + \Big(\Lambda_{ij.13}+\Lambda_{ij.31}\Big)t_{13} + \Big(\Lambda_{ij.23}+\Lambda_{ij.32}\Big)t_{23}\Big\}(\eta,k\hat{\boldsymbol{k}}) \\ &+\Big\{\Lambda_{ij.11}t_{11}+\Lambda_{ij.22}t_{22}+\Lambda_{ij.33}t_{33}\Big\}(\eta,k\hat{\boldsymbol{k}}) \end{aligned} \tag{A.10}$$

All of these are on the form

$$-\mathrm{i}\zeta(\eta,k_a)\times\Big\{k^2k^2k^2\Im_1(\eta,k_3)A_{ij}(\hat{\boldsymbol{k}}) + k\Im_2(\eta,k_3)B_{ij}(\hat{\boldsymbol{k}})\Big\}, \tag{A.11}$$

leaving

$$ah_{ij}(\eta,\boldsymbol{k}) = 8G_\mathrm{N}\Big[kA_{ij}(\hat{\boldsymbol{k}})\mathcal{I}_1(\eta,\boldsymbol{k};\eta_\mathrm{i}) + B_{ij}(\hat{\boldsymbol{k}})\mathcal{I}_2(\eta,\boldsymbol{k};\eta_\mathrm{i})\Big] \tag{A.12}$$

where

$$\mathcal{I}_n(\eta,\boldsymbol{k};\eta_\mathrm{i}) = -\mathrm{i}\int_{\eta_\mathrm{i}}^{\eta}\mathrm{d}\eta'\,a(\eta')\sin\left(k(\eta-\eta')\right)\times\zeta(\eta',k_a)\Im_n(\eta',k_3). \tag{A.13}$$

Furthermore, we can show [←proof!] that $A_{ij}(\boldsymbol{n}) = -n_3B_{ij}(\boldsymbol{n}) \equiv +2n_3C_{ij}(\boldsymbol{n})$ for $|\boldsymbol{n}|^2 = n_1^2+n_2^2+n_3^2 = 1$, allowing for the slightly simpler expression

$$ah_{ij}(\eta,\boldsymbol{k}) = 4G_\mathrm{N}C_{ij}(\hat{\boldsymbol{k}})\Big[k_3\mathcal{I}_1(\eta,\boldsymbol{k};\eta_\mathrm{i}) - \mathcal{I}_2(\eta,\boldsymbol{k};\eta_\mathrm{i})\Big], \tag{A.14}$$

where⌈:

$$\begin{aligned} C_{ab}(\boldsymbol{n}) &= n_3\Big[n_an_b\Big(n_3^2+1\Big) - \delta_{ab}\Big(1-n_3^2\Big)\Big] \\ C_{a3}(\boldsymbol{n}) &= -n_an_3^2\Big(1-n_3^2\Big) \\ C_{33}(\boldsymbol{n}) &= n_3^2\Big(1-n_3^2\Big)^2 \end{aligned} \tag{A.15}$$

⌋?

Redshift $\mathfrak{z}_* = 2 \therefore a(\eta_\mathrm{i}) = (1+\mathfrak{z}_*)^{-1} = 1/3$
$ds^2 = a^2(\eta)\Big(\delta_{\mu\nu}+h_{\mu\nu}\Big)\mathrm{d}x^\mu\mathrm{d}x^\nu, x^0 = \eta$
$u_\mathfrak{a}x^\mathfrak{a}, \mathfrak{a} = 0,1,2$
$u_\iota x^\iota, \iota = 0,1,2\mathfrak{b}\mathfrak{b}$

Important references: ((Vachaspati, 2006, p. 145)), ((Vilenkin, 1985, p. 291)), ((Vilenkin and Shellard, 1994, p. 375))