# 1 Thin Wall Approximation

$24^{\text{th}}$ November 2023

## 1.1 Dynamics of Domain Walls in The Thin Wall Approximation

Preceding Julian's notes ('Dynamics of Domain Walls in the Thin Wall approximation'). I could not get a symmetric stress-energy tensor from equations (18) and (4) in said notes. I then calculated the determinant ($g^{(3)}$) for myself, and by using that expression (Eq. (1.3) and Eq. (1.4) below) the functional derivative $\delta g^{(3)}/\delta g_{\rho\sigma}$ becomes symmetric, hence $T^{\rho\sigma}$ is symmetric.

**Covariant action.** Consider symmetron potential, thin wall limit. Surface tension is

$$\sigma \simeq \int_{\phi_-}^{\phi_+} \mathrm{d}\phi \, \sqrt{2V_{\mathrm{eff}}(\phi) - 2V_{\mathrm{eff}}(\phi_\pm)}, \tag{1.1}$$

where $\phi_\pm = \phi(z \to \pm\infty)$. We write the covariant action as $S_{\mathrm{dw}} = -\sigma \int d^3\xi \, \sqrt{-g^{(3)}}$. The induced metric is

$$g^{(3)}_{AB} = g_{\mu\nu} \frac{\mathrm{d}x_{\mathrm{dw}}{}^\mu}{\mathrm{d}\xi^A} \frac{\mathrm{d}x_{\mathrm{dw}}{}^\nu}{\mathrm{d}\xi^B}; \quad A, B = 0, 1, 2, \tag{1.2}$$

where $x_{\mathrm{dw}}{}^\mu(\xi^A)$ is the embedding function. The determinant of the world volume metric is

$$g^{(3)} = \tilde{\varepsilon}_{ABC} g^{(3)}_{0A} g^{(3)}_{1B} g^{(3)}_{2C} = g_{\mu\nu} g_{\kappa\lambda} g_{\alpha\beta} \mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta}, \tag{1.3}$$

where $\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} = \tilde{\varepsilon}_{ABC} \Delta^\mu{}_0 \Delta^\nu{}_A \Delta^\kappa{}_1 \Delta^\lambda{}_B \Delta^\alpha{}_2 \Delta^\beta{}_C$; $\Delta^\mu{}_A \equiv \mathrm{d}x_{\mathrm{dw}}{}^\mu \big/ \mathrm{d}\xi^A$. In particular,

$$\begin{aligned} \mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} = {} & \Delta^\mu{}_0 \Delta^\nu{}_0 \Delta^\kappa{}_1 \Delta^\lambda{}_1 \Delta^\alpha{}_2 \Delta^\beta{}_2 + \Delta^\mu{}_0 \Delta^\nu{}_1 \Delta^\kappa{}_1 \Delta^\lambda{}_2 \Delta^\alpha{}_2 \Delta^\beta{}_0 + \Delta^\mu{}_0 \Delta^\nu{}_2 \Delta^\kappa{}_1 \Delta^\lambda{}_0 \Delta^\alpha{}_2 \Delta^\beta{}_1 \\ & - \Delta^\mu{}_0 \Delta^\nu{}_0 \Delta^\kappa{}_1 \Delta^\lambda{}_2 \Delta^\alpha{}_2 \Delta^\beta{}_1 - \Delta^\mu{}_0 \Delta^\nu{}_1 \Delta^\kappa{}_1 \Delta^\lambda{}_0 \Delta^\alpha{}_2 \Delta^\beta{}_2 - \Delta^\mu{}_0 \Delta^\nu{}_2 \Delta^\kappa{}_1 \Delta^\lambda{}_1 \Delta^\alpha{}_2 \Delta^\beta{}_0. \end{aligned} \tag{1.4}$$

**Stress-energy tensor.** We consider a planar wall lying in the $xy$-plane with a small perturbation in the $z$-direction. The stress-energy tensor is given by

$$T^{\rho\sigma} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g_{\rho\sigma}} = \frac{\sigma \delta z - z_{\mathrm{dw}}}{\sqrt{-g}\sqrt{-g^{(3)}}} \frac{\delta g^{(3)}}{\delta g_{\rho\sigma}}. \tag{1.5}$$

We need the functional derivative of $g^{(3)}$ and the quantity $\sigma\delta z - z_{\mathrm{dw}}$.

### 1.1.1 My Calculation

We vary $g^{(3)}$ w.r.t. $g_{\rho\sigma}$, ignoring $\mathscr{O}((\delta g_{\rho\sigma})^2)$-terms:

$$\begin{aligned}
g^{(3)} + \delta g^{(3)} &= \left(g_{\mu\nu} + \delta g_{\mu\nu}\right)\left(g_{\kappa\lambda} + \delta g_{\kappa\lambda}\right)\left(g_{\alpha\beta} + \delta g_{\alpha\beta}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} \\
&= g^{(3)} + \left(\delta g_{\mu\nu} g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu}\delta g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu} g_{\kappa\lambda}\delta g_{\alpha\beta}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} \\
&= g^{(3)} + \left(\frac{\partial g_{\mu\nu}}{\partial g_{\rho\sigma}}\delta g_{\rho\sigma} g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu}\frac{\partial g_{\kappa\lambda}}{\partial g_{\rho\sigma}}\delta g_{\rho\sigma} g_{\alpha\beta} + g_{\mu\nu} g_{\kappa\lambda}\frac{\partial g_{\alpha\beta}}{\partial g_{\rho\sigma}}\delta g_{\rho\sigma}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} \\
&= g^{(3)} + \left(\delta^{\rho}{}_{\mu}\delta^{\sigma}{}_{\nu} g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu}\delta^{\rho}{}_{\kappa}\delta^{\sigma}{}_{\lambda} g_{\alpha\beta} + g_{\mu\nu} g_{\kappa\lambda}\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta}\cdot\delta g_{\rho\sigma} \\
&= g^{(3)} + \left(g_{\kappa\lambda} g_{\alpha\beta}\mathbf{Q}^{\rho\sigma\kappa\lambda\alpha\beta} + g_{\mu\nu} g_{\alpha\beta}\mathbf{Q}^{\mu\nu\rho\sigma\alpha\beta} + + g_{\mu\nu} g_{\kappa\lambda}\mathbf{Q}^{\mu\nu\kappa\lambda\rho\sigma}\right)\cdot\delta g_{\rho\sigma}
\end{aligned} \tag{1.6}$$

Thus,

$$\frac{\delta g^{(3)}}{\delta g_{\rho\sigma}} = g_{\kappa\lambda} g_{\alpha\beta}\left\{\mathbf{Q}^{\rho\sigma\kappa\lambda\alpha\beta} + \mathbf{Q}^{\alpha\beta\rho\sigma\kappa\lambda} + \mathbf{Q}^{\kappa\lambda\alpha\beta\rho\sigma}\right\}. \tag{1.7}$$

**Flat FRW universe.** With $g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ and ${x_{\mathrm{dw}}}^A = \xi^A, {x_{\mathrm{dw}}}^3 = z_{\mathrm{dw}} = \epsilon(\xi^A)$, we may insert

$$\Delta^{\mu}{}_{A} = \begin{cases} \delta^{\mu}{}_{A}, & \mu \neq 3 \\ \partial\epsilon\big/\partial\xi^A\,, & \mu = 3 \end{cases} \tag{1.8}$$

into Eq. (1.4) to compute $g^{(3)}$ and $\delta g^{(3)}/\delta g_{\rho\sigma}$. The result of the latter is a symmetric tensor of type (2,0);

$$\left[\frac{\delta g^{(3)}}{\delta g_{\rho\sigma}}\right] = \\
\begin{pmatrix}
\left(\iota_1{}^2 + \iota_2{}^2 + 1\right)a^4 & -\iota_0\iota_1 a^4 & -\iota_0\iota_2 a^4 & \iota_0 a^4 \\
-\iota_0\iota_1 a^4 & \iota_0{}^2 a^4 - \left(\iota_2{}^2 + 1\right)a^2 & \iota_1\iota_2 a^2 & -\iota_1 a^2 \\
-\iota_0\iota_2 a^4 & \iota_1\iota_2 a^2 & \iota_0{}^2 a^4 - \left(\iota_1{}^2 + 1\right)a^2 & -\iota_2 a^2 \\
\iota_0 a^4 & -\iota_1 a^2 & -\iota_2 a^2 & \iota_0{}^2 a^4 - \left(\iota_1{}^2 + \iota_2{}^2\right)a^2
\end{pmatrix}, \tag{1.9}$$

where we defined $\iota_A \equiv \partial\epsilon\big/\partial\xi^A$.

**Symmetron potential.** We let $V_{\mathrm{eff}}(\phi) = V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2$. As such, $\phi_\pm = \pm\phi_0$ and $V(\phi_\pm) = 0$. Now,

$$\begin{aligned}
\sigma\delta z - z_{\mathrm{dw}} &= \int_{-\infty}^{\infty}\mathrm{d}z\,\frac{\mathrm{d}\phi}{\mathrm{d}z}\sqrt{2V_{\mathrm{eff}}(\phi(z)) - 2V_{\mathrm{eff}}(\phi_\pm)}\delta z - z_{\mathrm{dw}} \\
&= \sqrt{2}\int_{-\infty}^{\infty}\mathrm{d}z\,\frac{\mathrm{d}\phi}{\mathrm{d}z}\sqrt{V_{\mathrm{eff}}(\phi(z))}\delta z - z_{\mathrm{dw}} \\
&= \sqrt{2}\sqrt{V_{\mathrm{eff}}(\phi(z_{\mathrm{dw}}))}\frac{\mathrm{d}\phi}{\mathrm{d}z}\bigg|_{z=z_{\mathrm{dw}}} \\
&= \sqrt{\frac{\lambda}{2}}\left(\phi(z_{\mathrm{dw}})^2 - \phi_0^2\right)\frac{\mathrm{d}\phi}{\mathrm{d}z}\bigg|_{z=z_{\mathrm{dw}}}.
\end{aligned} \tag{1.10}$$

**Have I completely misunderstood something here?**

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## 1.2 Dynamics of Domain Walls in The Thin Wall Approximation, cont.

It is safe to assume that the wall thickness is much smaller than the horizon. The adiabatically static solution to the e.o.m. for $\phi$ is $\phi(t,z) = \phi_0 \tanh\left\{\frac{a(t)}{w_0}z\right\}$, where $w_0 = \phi_0^{-1}\sqrt{2/\lambda}$ is the wall thickness ((Press, Ryden, and Spergel, 1989)). Dismissing Eq. (1.10), we find the following:

$$\begin{aligned}
\sigma\delta z - z_{\mathrm{dw}} &= \int_{-\infty}^{\infty}\mathrm{d}z'\,\frac{\mathrm{d}\phi}{\mathrm{d}z'}\sqrt{2V_{\mathrm{eff}}(\phi) - 2V_{\mathrm{eff}}(\phi_\pm)} \times \delta z - z_{\mathrm{dw}} \\
&= \delta z - z_{\mathrm{dw}}\sqrt{\frac{\lambda}{2}}\int_{\mathbb{R}}\mathrm{d}z'\,\frac{\mathrm{d}\phi}{\mathrm{d}z'}\left(\phi^2 - \phi_0^2\right) \\
&= \delta z - z_{\mathrm{dw}}\sqrt{\frac{\lambda}{2}}\frac{a\phi_0^3}{w_0}\int_{\mathbb{R}}\mathrm{d}z'\,\cosh^{-2}\left\{\frac{az'}{w_0}\right\}\left(\tanh^2\left\{\frac{az'}{w_0}\right\} - 1\right) \\
&= \delta z - z_{\mathrm{dw}}\sqrt{\frac{\lambda}{2}}\frac{a\phi_0^3}{w_0}\int_{\mathbb{R}}\mathrm{d}z'\,\cosh^{-4}\left\{\frac{az'}{w_0}\right\}\cdot(-1) \\
&= \dots \\
&= (-1) \times \frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3 \times \delta z - z_{\mathrm{dw}}
\end{aligned} \tag{1.11}$$

⌜There might be a sign error somewhere as I believe the tension should be positive.⌟ Gathering it all, we have

$$T^{\mu\nu} = (\pm)\,\delta z - \epsilon \cdot \frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3 \cdot a^{-6}\left[\left(\frac{\partial\epsilon}{\partial\xi^0}\right)^2 - a^{-2}\left(\left(\frac{\partial\epsilon}{\partial\xi^1}\right)^2 + \left(\frac{\partial\epsilon}{\partial\xi^2}\right)^2 + 1\right)\right]^{-1/2} \cdot \frac{\delta g^{(3)}}{\delta g_{\mu\nu}}, \tag{1.12}$$

where the last factor is found from Eq. (1.9).

**Perturbation.** The small perturbation $\epsilon$ obeys

$$\ddot{\epsilon} + 4\frac{\dot{a}}{a}\dot{\epsilon} - \frac{1}{a^2}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\epsilon = 0; \tag{1.13}$$

a linear PDE with solutions of the form $\epsilon(\xi^A) = \tilde{\epsilon}(\xi^0)\mathrm{e}^{\mathrm{i}(u_1\xi^1 + u_2\xi^2)}$, where $\tilde{\epsilon}$ satisfies

$$\ddot{\tilde{\epsilon}} + 4\frac{\dot{a}}{a}\dot{\tilde{\epsilon}} + \frac{u^2}{a^2}\tilde{\epsilon} = 0;\ u^2 = u_1^2 + u_2^2. \tag{1.14}$$

**Do you have any suggestions as to how to solve this?**

We can rewrite Eq. (1.12):

$$\begin{aligned}
T^{\mu\nu} &= (\pm)\,\delta z - \epsilon \cdot \frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3 \cdot a^{-5}\left[a^2\dot{\epsilon}^2 + u^2\epsilon - 1\right]^{-1/2} \cdot \frac{\delta g^{(3)}}{\delta g_{\mu\nu}} \\
&= (\pm)\,\delta z - \epsilon \cdot \frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3 \cdot a^{-5}\left[a^2\dot{\tilde{\epsilon}}^2\mathrm{e}^{2\mathrm{i}(u_1x + u_2y)} + u^2\tilde{\epsilon}\mathrm{e}^{\mathrm{i}(u_1x+u_2y)} - 1\right]^{-1/2} \cdot \frac{\delta g^{(3)}}{\delta g_{\mu\nu}}
\end{aligned} \tag{1.15}$$

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## 1.3 Perturbation: Solving the e.o.m.

We have

$$T^{\mu\nu} = \frac{(\pm)2\,\sigma\delta z - \epsilon}{a^5\sqrt{a^2\dot{\epsilon}^2 + u^2\epsilon^2 - 1}}\frac{\delta g^{(3)}}{\delta g_{\mu\nu}}. \tag{1.16}$$

Eq. (1.14) can be solved analytically for $a \propto t^\beta$. We let $\eta$ denote conformal time s.t. $\mathrm{d}\eta = a^{-1}\mathrm{d}t$ and $a \propto \eta^\alpha$, where $\alpha = \beta/(1-\beta)$. We let $\epsilon(t,x,y) = \epsilon(t)f(x,y)$, and ignore the tilde from before. For a matter dominated universe, $\beta = 2/3$ and $\alpha = 2$.

We make use of the transformed Bessel's equation of the form

$$x^2y'' + (1-2\mathfrak{a})xy' + \left(\mathfrak{b}^2\mathfrak{c}^2x^{2\mathfrak{c}} + \mathfrak{a}^2 - \ell^2\mathfrak{c}^2\right)y = 0; \quad \mathfrak{a},\mathfrak{b},\mathfrak{c},\ell \in \mathbb{C}, \tag{1.17}$$

whose general solution is $y(x) = x^{\mathfrak{a}}\{c_1\mathcal{J}_\ell(\mathfrak{b}x^{\mathfrak{c}}) + c_2\mathcal{Y}_\ell(\mathfrak{b}x^{\mathfrak{c}})\}$ ((see Bowman, 1958, p.117–118)). The properties of the Bessel functions of the first and second kind gives

$$y(x) = \begin{cases} x^{\mathfrak{a}}\{c_1\mathcal{J}_\ell(\mathfrak{b}x^{\mathfrak{c}}) + c_2\mathcal{J}_{-\ell}(\mathfrak{b}x^{\mathfrak{c}})\}, & \ell \notin \mathbb{Z} \\ x^{\mathfrak{a}}\{c_1\mathcal{J}_\ell(\mathfrak{b}x^{\mathfrak{c}}) + c_2\mathcal{Y}_\ell(\mathfrak{b}x^{\mathfrak{c}})\}, & \ell \in \mathbb{Z} \end{cases}. \tag{1.18}$$

### 1.3.1 Conformal Time Frame

For $\epsilon(t) \to \epsilon(\eta)$, Eq. (1.14) reads

$$\eta^2\epsilon'' + 3\alpha\eta\epsilon' + u^2\eta^2\epsilon = 0, \tag{1.19}$$

where primed means conformal time derivative. The solution to this equation is

$$\epsilon(\eta) = \eta^\ell\{c_1\mathcal{J}_\ell(u\eta) + c_2\mathcal{Y}_\ell(u\eta)\}; \quad \ell = \frac{1-3\alpha}{2}. \tag{1.20}$$

**Matter domination.** With $\alpha = 2$, $\ell = -5/2$ and thus

$$\epsilon(\eta) = c \cdot \frac{\mathcal{J}_{5/2}(u\eta)}{\eta^{5/2}} \tag{1.21}$$

is a solution.

### 1.3.2 Cosmic Time Frame

When $a(t) = Kt^\beta$, Eq. (1.14) is simply

$$t^2\ddot{\epsilon} + 4\beta t\dot{\epsilon} + (u/K)^2t^{2(1-\beta)}\epsilon = 0, \tag{1.22}$$

with solution

$$\epsilon(t) = t^{\ell\gamma}\left\{c_1\mathcal{J}_\ell\left(\frac{u}{K\gamma}t^\gamma\right) + c_2\mathcal{Y}_\ell\left(\frac{u}{K\gamma}t^\gamma\right)\right\}; \quad \ell = \frac{1-4\beta}{2\gamma}, \gamma = 1-\beta. \tag{1.23}$$

**Matter domination.** With $\beta = 2/3$, $\ell = -5/2$ and a solution is

$$\epsilon(t) = c \cdot \frac{\mathcal{J}_{5/2}\left(\frac{3u}{K}t^{1/3}\right)}{t^{5/6}}. \tag{1.24}$$

### 1.3.3 Stress-Energy Tensor

We once again turn our attention to the stress-energy tensor, writing it out for $\epsilon(t,x,y) = \epsilon(t)f(x,y)$, where $f(x,y) = \mathrm{e}^{\mathrm{i}(u_1 x + u_2 y)}$:

$$\begin{aligned} T^{\mu\nu} &= (\pm)\frac{2\,\sigma\delta z - f\epsilon}{a^5\sqrt{a^2 f^2 \dot{\epsilon}^2 + u^2 f^2 \epsilon^2 - 1}}\frac{\delta g^{(3)}}{\delta g_{\mu\nu}} \\ &= (\pm)\frac{2\sigma\delta z - f\epsilon}{a^5\sqrt{f^2\left[\epsilon'^2 + u^2\epsilon^2\right] - 1}}\frac{\delta g^{(3)}}{\delta g_{\mu\nu}} \end{aligned} \tag{1.25}$$

The square root in the denominator is straight-forwardly computed when using e.g. Eq. (1.21).

## 1.4 Wall Profile

We still need to replace $\sigma\delta z - \epsilon(t,x,y)$ in the expression for $T^{\mu\nu}$. **I have still not figured out how to work this out.**

# 2 Thin Wall Approximation; Vol. 2

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## 2.1 Gravitational Waves from Domain Walls in The Thin Wall Approximation

We have the spacetime metric $g_{\mu\nu}$ and the induced metric

$$\gamma_{ab} = g_{\mu\nu}\frac{\mathrm{d}x^{\mu}_{\mathrm{dw}}}{\mathrm{d}\xi^{a}}\frac{\mathrm{d}x^{\nu}_{\mathrm{dw}}}{\mathrm{d}\xi^{b}}; \quad \left[x^{\mu}_{\mathrm{dw}}\right] = (\xi^0, \xi^1, \xi^2, \epsilon(\xi^a)), \tag{2.1}$$

where we let $a, b = 0, 1, 2$. Consider $g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$. We define $\iota_a \equiv \partial\epsilon/\partial\xi^a$ and $\iota_3 \equiv -1$ for notational ease. The determinant of the induced metric is

$$\gamma = -a^4[-(a\iota_0)^2 + \underbrace{\iota_1{}^2 + \iota_2{}^2 + \iota_3{}^2}_{\equiv\iota^2}]. \tag{2.2}$$

In the thin wall approximation, the surface tension

$$\sigma = \int_{-\infty}^{+\infty} \mathrm{d}z\, T_{00} \simeq - \int_{\phi_-}^{\phi_+} \mathrm{d}\phi\, \sqrt{2V_{\mathrm{eff}}(\phi) - 2V_{\mathrm{eff}}(\phi_\pm)}. \tag{2.3}$$

The covariant action

$$S_{\mathrm{dw}} = \int d^4x\, \mathcal{L}_{\mathrm{dw}} = -\sigma \int d^3\xi\, \sqrt{-\gamma} = -\sigma \int d^4x\, \sqrt{-\gamma}\, \delta z - z_{\mathrm{dw}} \tag{2.4}$$

and thus, the stress–energy tensor

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta\mathcal{L}_{\mathrm{dw}}}{\delta g_{\mu\nu}} = \frac{2\sigma\delta z - \epsilon}{\sqrt{-g}\,\sqrt{-\gamma}}\frac{\delta\gamma}{\delta g_{\mu\nu}} = \frac{2\sigma\delta z - \epsilon}{a^5\,\sqrt{\iota^2 - (a\iota_0)^2}}\frac{\delta\gamma}{\delta g_{\mu\nu}}. \tag{2.5}$$

We have calculated the functional derivative $\delta\gamma\big/\delta g_{\mu\nu}$ before:

$$\frac{\delta\gamma}{\delta g_{00}} = a^4\iota^2 \qquad \frac{\delta\gamma}{\delta g_{0i}} = -a^4\iota_0\iota_i \qquad \frac{\delta\gamma}{\delta g_{ij}} = a^2\Big[\iota_i\iota_j + \delta_{ij}\Big((a\iota_0)^2 - \iota^2\Big)\Big] \tag{2.6}$$

With the ansatz $\epsilon(\xi^a) = \epsilon_t(\xi^0)\mathrm{e}^{-\mathrm{i}u_1\xi^1}\mathrm{e}^{-\mathrm{i}u_2\xi^2}$, solutions for the equations of motion for $\epsilon_t$ are known for $a \propto t^\beta$. In that case, $\iota_0 = \dot{\epsilon}$, $\iota_1 = -\mathrm{i}u_1\epsilon$, $\iota_2 = -\mathrm{i}u_2\epsilon$ and, of course, $\iota_3 = -1$.

### 2.1.1 The Symmetron Potential

It is easily shown that for $V_{\text{eff}}(\phi) = V_{\text{Sym}}(\phi) \equiv \frac{\lambda}{4}\left(\phi^2 - \phi_0{}^2\right)^2$, the surface tension reduces to $\sigma = \sigma_0 \equiv \frac{4}{3}\phi_0{}^3\sqrt{\lambda/2}$. We consider matter domination ($a \propto t^{2/3} \propto \eta^2$) and assume a solution $\epsilon_\eta = \epsilon_0\eta^{-5/2}\mathcal{J}_{5/2}(u\eta)$, where $\eta$ is conformal time. Note that $\epsilon' = a\dot{\epsilon}$. We have

$$T^{\mu\nu}(\eta, \boldsymbol{x}) = \frac{2\sigma_0\delta z - \epsilon}{a^5\sqrt{I - (a\dot{\epsilon})^2}}\frac{\delta\gamma}{\delta g_{\mu\nu}} = \frac{2\sigma_0\delta z - \epsilon}{a^5\sqrt{1 - (u\epsilon)^2 - \epsilon'^2}}\frac{\delta\gamma}{\delta g_{\mu\nu}}. \tag{2.7}$$

Neglecting all $\mathscr{O}(\epsilon^2)$-terms, we get that the only non-vanishing contributions are:

$$\delta\gamma\big/\delta g_{00} = a^4 \qquad \delta\gamma\big/\delta g_{11} = -a^2 \qquad \delta\gamma\big/\delta g_{22} = -a^2 \tag{2.8}$$

$$\delta\gamma\big/\delta g_{03} = a^3\epsilon' \qquad \delta\gamma\big/\delta g_{13} = \mathrm{i}u_1 a^2\epsilon \qquad \delta\gamma\big/\delta g_{23} = \mathrm{i}u_2 a^2\epsilon \tag{2.9}$$

We have $T^{\mu\nu} = T^{(\mu\nu)}$. Let indices $A, B, C = 1, 2$ and $\kappa = 8\pi^2\sigma_0 a^{-3}$. In Fourier space, we have the following:

$$\begin{aligned}
T^{00}(\eta, \boldsymbol{k}) &= \kappa a^2\left\{\delta^{(2)}k_A + \mathrm{i}k_3\epsilon_\eta\delta^{(2)}k_A - u_A\right\} \\
T^{0i}(\eta, \boldsymbol{k}) &= \delta^{i3}\cdot\kappa a\epsilon'_\eta\delta^{(2)}k_A - u_A \\
T^{AB}(\eta, \boldsymbol{k}) &= \delta^{AB}\cdot(-\kappa)\left\{\delta^{(2)}k_C + \mathrm{i}k_3\epsilon_\eta\delta^{(2)}k_C - u_C\right\} \\
T^{i3}(\eta, \boldsymbol{k}) &= \delta^{iA}\cdot\kappa\mathrm{i}u_A\epsilon_\eta\delta^{(2)}k_B - u_B
\end{aligned} \tag{2.10}$$

**Gravitational Waves.** The transverse, traceless tensor perturbation $h_{ij}$, showing up in the perturbed line element $ds^2 = a^2\left\{-\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right\}$, has the e.o.m.

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + 2\frac{a'}{a}\frac{\mathrm{d}}{\mathrm{d}\eta} + k^2\right]h_{ij}(\eta, \boldsymbol{k}) = 16\pi G_{\mathrm{N}}\Lambda_{ij.kl}(\boldsymbol{n})T_{kl}(\eta, \boldsymbol{k}); \quad \boldsymbol{k} = k\boldsymbol{n}, |\boldsymbol{n}| = 1. \tag{2.11}$$

We extracted the transverse, traceless (TT) part of the symmetric stress–energy tensor by use of the "Lambda tensor" $\Lambda_{ij.kl}$. We find that

$$\Lambda_{ij.kl}(\boldsymbol{n})T_{kl}(\eta, \boldsymbol{k}) = 4\pi^2\sigma_0 a\cdot\mathrm{i}\epsilon_\eta k^{-4}\delta^{(2)}k_C - u_C\cdot k_3 t_{ij} \tag{2.12a}$$

where

$$t_{AB} = \delta_{AB}k^2u^2 + \left(u^2 - 2k^2\right)u_A u_B \tag{2.12b}$$

$$t_{A3} = u^2 u_A k_3 \tag{2.12c}$$

$$t_{33} = -u^4 \tag{2.12d}$$

and $k_3 = \sqrt{k^2 - u^2}$ necessarily.

Solving the e.o.m. using Green's functions ((cf. Kawasaki and Saikawa, 2011)). Now, the tensor field $\mathsf{h}_{ij} \equiv ah_{ij}$ is given by

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{16\pi G_{\mathrm{N}}}{k^2}\int_{\tau_{\text{ini}}}^{\tau}\mathrm{d}\tau'\,\underbrace{\frac{\pi}{2}\sqrt{\tau\tau'}\{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\}}_{\times\Theta\tau-\tau'=\mathcal{G}_\nu(\tau,\tau')}a(\tau')\Lambda_{ij.kl}(\boldsymbol{n})T_{kl}(\tau', \boldsymbol{k}) \tag{2.13}$$

with $\tau = k\eta$, where $\nu = \alpha - 1/2 = 3/2$ for matter domination. That is,

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{64\pi^3 G_\mathrm{N}\sigma_0}{k^6}\delta^{(2)}k_A - u_A k_3 t_{ij} \times \int_{\tau_\mathrm{ini}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_{3/2}(\tau, \tau') a^2(\tau')\mathrm{i}\epsilon_\eta(\tau');$$
$$\mathcal{G}_{3/2}(\tau, \tau') = \Theta\tau - \tau' \frac{1}{\tau\tau'}[(\tau\tau' + 1)\sin(\tau - \tau') - (\tau - \tau')\cos(\tau - \tau')].^1 \quad (2.14)$$

We let $a(\eta) = a_\mathrm{ini}(\eta/\eta_\mathrm{ini})^\alpha$. Thus, if one solves the integral

$$I \equiv \int_{\tau_\mathrm{ini}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_{3/2}(\tau, \tau')\tau'^{3/2}\mathcal{J}_{5/2}(u\tau'), \quad (2.15)$$

one has an explicit expression for the tensor perturbation in Fourier space:

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{64\pi^3 G_\mathrm{N}\sigma_0\epsilon_0}{k^6}\frac{a_\mathrm{ini}{}^2}{\eta_\mathrm{ini}{}^4}\delta^{(2)}k_A - u_A k_3 t_{ij}(k)\cdot \mathrm{i}I(k\eta) \quad (2.16)$$

**A closer look at $I$.** Explicitly—still for a matter dominated universe—we have

$$I = \int_{\tau_\mathrm{ini}}^{\tau} \mathrm{d}\tau'\, \frac{\sqrt{\tau'}}{\tau}[(\tau\tau' + 1)\sin(\tau - \tau') - (\tau - \tau')\cos(\tau - \tau')]$$
$$\times \sqrt{\frac{2}{\pi}}\left[\left(-1 + \frac{3}{u^2\tau'^2}\right)\frac{\cos(u\tau')}{\sqrt{u\tau'}} + \frac{3\sin(u\tau')}{u\tau'\sqrt{u\tau'}}\right], \quad (2.17)$$

which we rewrite;

$$I = \sqrt{\frac{2}{u\pi}}\int_{\tau_\mathrm{ini}}^{\tau} \mathrm{d}\tau'\, \frac{1}{\tau}[(\tau\tau' + 1)\sin(\tau - \tau') - (\tau - \tau')\cos(\tau - \tau')]$$
$$\times\left[\left(-1 + \frac{3}{u^2\tau'^2}\right)\cos(u\tau') + \frac{3}{u\tau'}\sin(u\tau')\right]. \quad (2.18)$$

In the special case where $\tau \gg 1$ (wavelength of GWs well inside the Hubble horizon), we can simplify to be left with

$$I \overset{\tau,\tau'\gg 1}{\simeq} \sqrt{\frac{2}{u\pi}}\int_{\tau_\mathrm{ini}}^{\tau} \mathrm{d}\tau'\, \Big\{ -\tau'\cos(u\tau')\sin(\tau - \tau') - (\tau'/\tau)\cos(u\tau')\cos(\tau - \tau')$$
$$+ (3/u)\sin(u\tau')\sin(\tau - \tau') + \cos(u\tau')\cos(\tau - \tau')\Big\}, \quad (2.19)$$

an integral with a well-defined algebraic solution, though rather ugly and long.

**Comment.** **First of all, there is a propagating sign error somewhere, stemming from the surface tension/Lagrangian. I am somewhat confused about the imaginary factor apparently surviving all steps. Especially worrisome in the final expression for the tensor field in Fourier space, Eq. (2.16). If these results are error-free, it should not be a very difficult task generalising to a framework with arbitrary (likely power-law) scale factor and perturbation.**

**I have not written the inverse F.T. of Eq. (2.16). It seems possible to do by hand, but rather complicated as $I$ and $t_{ij}$ depend on $k$ (or $k_3$).**

[1] Note that if $\tau = k\eta \gg 1$, $\mathcal{G}_{3/2}(\tau - \tau') \simeq \Theta\tau - \tau' \sin(\tau - \tau')$.

17th January 2024

## 2.2 Gravitational Waves from Thin Domain Walls: Symmetron Model

We let $\epsilon(\eta, x, y) \to \epsilon(\eta, x)$ represent a plane wave perturbation to the thin, infinite wall in the $xy$–plane. Impose $\epsilon(\eta, x) = \epsilon_\eta(\eta)\epsilon_x(x)$. After correcting some mistakes, we have the spatial part of the stress–energy tensor in *Fourier* space:

$$T_{11}(\eta, \boldsymbol{k}) = T_{22}(\eta, \boldsymbol{k}) = 4\pi\sigma_0 a\delta k_2 \cdot (-1)\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.20}$$

$$T_{13}(\eta, \boldsymbol{k}) = T_{31}(\eta, \boldsymbol{k}) = 4\pi\sigma_0 a\delta k_2 \cdot \mathrm{i}k_1\epsilon_\eta\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.21}$$

Let $\Lambda_{ij.kl}T_{kl} \equiv 2\pi\sigma_0 a\delta k_2 k^{-4} t_{ij}$. Then:

$$t_{11} = -{k_1}^2{k_3}^2 \cdot (-1)\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] - k_1{k_3}^3 \cdot 2\mathrm{i}k_1\epsilon_\eta\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.22a}$$

$$t_{22} = k^2{k_1}^2 \cdot (-1)\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] + k^2k_1k_3 \cdot 2\mathrm{i}k_1\epsilon_\eta\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.22b}$$

$$t_{33} = -{k_1}^4 \cdot (-1)\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] - {k_1}^3k_3 \cdot 2\mathrm{i}k_1\epsilon_\eta\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.22c}$$

$$t_{13} = {k_1}^3k_3 \cdot (-1)\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] + {k_1}^2{k_3}^2 \cdot 2\mathrm{i}k_1\epsilon_\eta\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right] \tag{2.22d}$$

The GWs generated from this system are given by

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{16\pi G_\mathrm{N}}{k^2}\int_{\tau_\mathrm{ini}}^{\tau}\mathrm{d}\tau'\,\mathcal{G}_\nu(\tau, \tau')a(\tau')\Lambda_{ij.kl}(\boldsymbol{n})T_{kl}(\tau', \boldsymbol{k});$$
$$\mathcal{G}_\nu(\tau, \tau') = \Theta\tau - \tau'\frac{\pi}{2}\sqrt{\tau\tau'}\{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\}, \tag{2.23}$$

where $\mathsf{h}_{ij} = ah_{ij}$ and $\nu = \alpha - 1/2$ for a universe with $a \propto \eta^\alpha$. Assuming $\alpha = 2$, we have from before that $\epsilon_\eta(\eta) = \epsilon_0\eta^{-5/2}\mathcal{J}_{5/2}(u\eta)$, where $u$ is the wavenumber associated with $\epsilon_x(x)$ (e.g. $\epsilon_x(x) = \sin(ux)$). Thus,

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{32\pi^2 G_\mathrm{N}\sigma_0}{k^6}\delta k_2\int_{\tau_\mathrm{ini}}^{\tau}\mathrm{d}\tau'\,\mathcal{G}_{3/2}(\tau, \tau')a^2(\tau')t_{ij}(\tau', k_1, k_3); \quad \tau = k\eta. \tag{2.24}$$

### 2.2.1 Two Fourier transforms

To get explicit expressions in $t_{ij}$, we need to calculate **(a)** $\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right]$ and **(b)** $\mathcal{F}_{k_1}\left[\epsilon_x\mathrm{e}^{\mathrm{i}k_3\epsilon_\eta\epsilon_x}\right]$. We impose $\epsilon_x = \sin(ux)$. Then, we effectively have to solve **(a)** $\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}C\sin(ux)}\right]$ and **(b)** $\mathcal{F}_{k_1}\left[\sin(ux)\mathrm{e}^{\mathrm{i}C\sin(ux)}\right]$.

**(a)**

$$\begin{aligned}\mathcal{F}_{k_1}\left[\mathrm{e}^{\mathrm{i}C\sin(ux)}\right] &= \int\mathrm{d}x\,\mathrm{e}^{\mathrm{i}C\sin(ux)}\mathrm{e}^{\mathrm{i}k_1x} \\ &= 2\pi\delta k_1\mathcal{J}_0(C) - 2\pi\sum_{n=1}^{\infty}\mathcal{J}_n(C)\left[\delta k_1 - nu + (-1)^n\delta k_1 + nu\right] \\ &= 2\pi\sum_{n=-\infty}^{+\infty}\mathcal{J}_n(C)\delta k_1 + nu\end{aligned}$$

**(b)**

$$
\begin{aligned}
\mathcal{F}_{k_1}\left[\sin(ux)\mathrm{e}^{\mathrm{i}C\sin(ux)}\right] &= -\int \mathrm{d}x \ \sin(ux)\mathrm{e}^{\mathrm{i}C\sin(ux)}\mathrm{e}^{\mathrm{i}k_1 x} \\
&= \mathrm{i}\pi \sum_{n=-\infty}^{+\infty} [\mathcal{J}_{n-1}(C) - \mathcal{J}_{n+1}(C)]\delta k_1 - nu
\end{aligned}
$$

19th January 2024

## 2.3 Stress–Energy Tensor: A Revised Calculation

We focus of the spatial components of the stress–energy tensor, which to first order in $\epsilon$ is

$$T_{ij}(\eta, \boldsymbol{x}) = a^4 T^{ij}(\eta, \boldsymbol{x}) = 2\sigma_0 a^{-1}\delta z - \epsilon\delta\gamma\Big/\delta g_{ij}, \tag{2.25}$$

where for $\epsilon = \epsilon(\eta, x)$ we have the only non-vanishing contributions $\delta\gamma\Big/\delta g_{11} = \delta\gamma\Big/\delta g_{22} = -a^2$ and $\delta\gamma\Big/\delta g_{13} = \delta\gamma\Big/\delta g_{31} = -a^2\partial_1\epsilon$.

### 2.3.1 Fourier Space

Now, we find $T_{ij}(\eta, \boldsymbol{k})$:

$$\begin{aligned}\mathcal{F}_{\boldsymbol{k}}\Big[T_{ij}(\eta, \boldsymbol{x})\Big] &= \int d^3x\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} T_{ij}(\eta, \boldsymbol{x}) \\ &= 2\sigma_0 a^{-1} \cdot 2\pi\delta k_2 \cdot \int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}k_1 x}\delta\gamma\Big/\delta g_{ij} \cdot \int \mathrm{d}z\, \mathrm{e}^{\mathrm{i}k_3 z}\delta z - \epsilon \\ &= 2\sigma_0 a^{-1} \cdot 2\pi\delta k_2 \cdot \int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}k_1 x}\delta\gamma\Big/\delta g_{ij} \cdot \mathrm{e}^{\mathrm{i}k_3\epsilon} \\ &= 2\sigma_0 a^{-1} \cdot 2\pi\delta k_2 \cdot \mathcal{F}_{k_1}\Big[\delta\gamma\Big/\delta g_{ij} \cdot \mathrm{e}^{\mathrm{i}k_3\epsilon}\Big]\end{aligned} \tag{2.26}$$

Thus,

$$T_{11}(\eta, \boldsymbol{k}) = T_{22}(\eta, \boldsymbol{k}) = -4\pi\sigma_0 a\delta k_2 \mathcal{F}_{k_1}\Big[\mathrm{e}^{\mathrm{i}k_3\epsilon}\Big] \tag{2.27a}$$

$$T_{13}(\eta, \boldsymbol{k}) = T_{31}(\eta, \boldsymbol{k}) = -4\pi\sigma_0 a\delta k_2 \mathcal{F}_{k_1}\Big[(\partial_1\epsilon)\mathrm{e}^{\mathrm{i}k_3\epsilon}\Big] \tag{2.27b}$$

are the non-vanishing components.

We impose $\epsilon = \bar{\epsilon}(\eta)\sin(ux)$. The Jacobi-Anger expansion,

$$\mathrm{e}^{\mathrm{i}\xi\sin\theta} = \mathrm{e}^{\mathrm{i}\xi\cos\theta'} = \sum_{n=-\infty}^{\infty} \mathrm{i}^n \mathcal{J}_n(\xi)\mathrm{e}^{\mathrm{i}n\theta'} = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(\xi)\mathrm{e}^{\mathrm{i}n\theta}; \quad \theta' = \theta - \frac{\pi}{2}, \tag{2.28}$$

is essential to the coming calculations. In addition, we make use of $\mathcal{J}_{-n}(\xi) = (-1)^n\mathcal{J}_n(\xi)$ and $\mathcal{J}_{n-1}(\xi) + \mathcal{J}_{n+1}(\xi) = (2n/\xi)\mathcal{J}_n(\xi)$.

**(a)** $\mathcal{F}_{k_1}\Big[\mathrm{e}^{\mathrm{i}k_3\epsilon}\Big] = \mathcal{F}_{k_1}\Big[\mathrm{e}^{\mathrm{i}k_3\bar{\epsilon}\sin(ux)}\Big]$**.**

$$\begin{aligned}\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}c\sin(ux)}\mathrm{e}^{\mathrm{i}\omega x} &= \sum_{n\in\mathbb{Z}} \mathcal{J}_n(c) \int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}nux}\mathrm{e}^{\mathrm{i}\omega x} \\ &= 2\pi \sum_{n\in\mathbb{Z}} \mathcal{J}_n(c)\delta\omega + nu\end{aligned}$$

$$\Longrightarrow \mathcal{F}_{k_1}\Big[\mathrm{e}^{\mathrm{i}k_3\bar{\epsilon}\sin(ux)}\Big] = \begin{cases} 2\pi\mathcal{J}_\ell(k_3\bar{\epsilon}) & \text{if } \ell = -k_1/u \in \mathbb{Z}, \\ 0 & \text{else.}\end{cases} \tag{2.29}$$

**(b)** $\mathcal{F}_{k_1}\left[(\partial_1\epsilon)\mathrm{e}^{\mathrm{i}k_3\epsilon}\right] = u\bar{\epsilon}\mathcal{F}_{k_1}\left[\cos(ux)\mathrm{e}^{\mathrm{i}k_3\bar{\epsilon}\sin(ux)}\right]$**.**

$$\begin{aligned}
\int \mathrm{d}x\, \cos(ux)\mathrm{e}^{\mathrm{i}c\sin(ux)}\mathrm{e}^{\mathrm{i}\omega x} &= \frac{1}{2}\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}c\sin(ux)}\left[\mathrm{e}^{\mathrm{i}ux} + \mathrm{e}^{-\mathrm{i}ux}\right]\mathrm{e}^{\mathrm{i}\omega x} \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathcal{J}_n(c)\int \mathrm{d}x\left[\mathrm{e}^{\mathrm{i}(n+1)ux} + \mathrm{e}^{\mathrm{i}(n-1)ux}\right]\mathrm{e}^{\mathrm{i}\omega x} \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} [\mathcal{J}_{n+1}(c) + \mathcal{J}_{n-1}(c)]\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}nux}\mathrm{e}^{\mathrm{i}\omega x} \\
&= \pi\sum_{n\in\mathbb{Z}} \underbrace{[\mathcal{J}_{n+1}(c) + \mathcal{J}_{n-1}(c)]}_{2nc^{-1}\mathcal{J}_n(c)}\delta\omega + nu \\
&= \frac{2\pi}{c}\sum_{n\in\mathbb{Z}} n\mathcal{J}_n(c)\delta\omega + nu
\end{aligned}$$

$$\Longrightarrow u\bar{\epsilon}\mathcal{F}_{k_1}\left[\cos(ux)\mathrm{e}^{\mathrm{i}k_3\bar{\epsilon}\sin(ux)}\right] = \begin{cases} -2\pi(k_1/k_3)\mathcal{J}_\ell(k_3\bar{\epsilon}) & \text{if } \ell = -k_1/u \in \mathbb{Z}, \\ 0 & \text{else.}\end{cases} \tag{2.30}$$

### 2.3.2 Traceless–Transverse Gauge

We extract the transverse, traceless (TT) part of the stress–energy tensor by use of the "Lambda tensor", i.e. $T_{ij}^{\mathrm{TT}}(\eta, \boldsymbol{k}) = \Lambda_{ij.kl}(\boldsymbol{k}/k)T_{kl}(\eta, \boldsymbol{k})$. The non-vanishing contributions are given by:

$$2k^4T_{11}^{\mathrm{TT}} = -k_1{}^2k_3{}^2T_{11} - 2k_1k_3{}^3T_{13} \tag{2.31a}$$
$$2k^4T_{22}^{\mathrm{TT}} = k^2k_1{}^2T_{11} + 2k^2k_1k_3T_{13} \tag{2.31b}$$
$$2k^4T_{33}^{\mathrm{TT}} = -k_1{}^4T_{11} - 2k_1{}^3k_3T_{13} \tag{2.31c}$$
$$2k^4T_{13}^{\mathrm{TT}} = k_1{}^3k_3T_{11} + 2k_1{}^2k_3{}^2T_{13} \tag{2.31d}$$

More compactly, we can use Eq. (2.29) and Eq. (2.30) and write

$$\begin{aligned}
T_{ij}^{\mathrm{TT}} &= \frac{1}{2k^4}\left(k_1{}^2T_{11} + 2k_1k_3T_{13}\right)\cdot\left[\delta_{ij}\left(k^2 - 2k_ik_j\right) + k_ik_j\right] \qquad \Bigg|\, \ell = -\frac{k_1}{u} \\
&= \frac{-4\pi\sigma_0 a\delta k_2}{2k^4}\left(k_1{}^2 - 2k_1k_3\cdot(k_1/k_3)\right)2\pi\mathcal{J}_\ell(k_3\bar{\epsilon})\cdot\left[\delta_{ij}\left(k^2 - 2k_ik_j\right) + k_ik_j\right] \\
&= \frac{4\pi^2\sigma_0}{k^4}\delta k_2k_1{}^2\cdot\left[\delta_{ij}\left(k^2 - 2k_ik_j\right) + k_ik_j\right]\cdot a\mathcal{J}_\ell(k_3\bar{\epsilon}),
\end{aligned} \tag{2.32}$$

$\forall\, \ell \in \mathbb{Z}$, otherwise the solution is trivial.

### 2.3.3 Gravitational Waves

We define $\mathsf{h}_{ij} \equiv ah_{ij}$. We consider a universe where $a \propto \eta^\alpha$. Now

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{16\pi G_{\mathrm{N}}}{k^2}\int_{\tau_{\mathrm{ini}}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_\nu(\tau, \tau')a(\tau')T_{ij}^{\mathrm{TT}}(\tau', \boldsymbol{k}); \quad \nu = \alpha - \frac{1}{2};$$
$$\mathcal{G}_\nu(\tau, \tau') = \Theta\tau - \tau'\frac{\pi}{2}\sqrt{\tau\tau'}\{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\} \tag{2.33}$$

is the expression for the tensor perturbations. For our specific setup, this can be rewritten;

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = \frac{64\pi^3\sigma_0 G_\mathrm{N}}{k^6}\delta k_2 {k_1}^2\Big[\delta_{ij}\Big(k^2 - 2k_ik_j\Big) + k_ik_j\Big] \times \sum_{n\in\mathbb{Z}} \delta\ell - n$$
$$\times \int_{\tau_\mathrm{ini}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_\nu(\tau,\tau') a^2(\tau')\mathcal{J}_\ell\Big(k_3\bar{\epsilon}(\tau')\Big); \quad \tau = k\eta,\ \ell = -k_1/u \tag{2.34}$$

**Matter Domination.** We let $a(\eta) = a_\mathrm{ini}(\eta/\eta_\mathrm{ini})^\alpha$ and consider $\alpha = 2$. Now $\bar{\epsilon}(\eta) = \epsilon_0\eta^{-5/2}\mathcal{J}_{5/2}(u\eta)$ satisfies the e.o.m. for the time-dependence of $\epsilon$. Furthermore,

$$\mathcal{G}_{3/2}(\tau,\tau') = \Theta\tau - \tau' \frac{(\tau\tau' + 1)\sin(\tau - \tau') - (\tau - \tau')\cos(\tau - \tau')}{\tau\tau'}. \tag{2.35}$$

We essentially have to solve

$$\int_{\tau_\mathrm{ini}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_{3/2}(\tau,\tau')\tau'^4\mathcal{J}_\ell\Big(k_3\bar{\epsilon}(\tau')\Big), \quad \ell \in \mathbb{Z}, \tag{2.36}$$

**and this is where I am stuck.**

29th January 2024

## 2.4 The Complete Model

We consider a universe for which $a(\eta) \propto \eta^{\alpha}$ for $\eta \geq \eta_{\text{init}}$. The line element is $ds^2 = a^2(\eta)\left\{-\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right\}$. The wall is represented by the Symmetron potential $V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2$ and "width" $w_0$. The location of the wall is $[X^\mu] = (\eta, x, y, \epsilon(\eta, x, y))$ where $\epsilon$ is a linear perturbation with e.o.m.

$$\ddot{\epsilon} + 3\mathcal{H}\dot{\epsilon} - \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\epsilon = 0; \quad \mathcal{H} = \frac{\dot{a}}{a}, \quad \dot{} \equiv \frac{\mathrm{d}}{\mathrm{d}\eta}. \tag{2.37}$$

We consider plane waves of eigenvalues $-u^2$. In particular, we impose $\epsilon(\eta, x, y) = \epsilon(\eta, x) = \bar{\epsilon}(\eta)\sin(u_1 x)$, such that $u = \left|u_1\right|$ and $\bar{\epsilon}(\eta) = \epsilon_0\eta^{\gamma}\left\{c_1\mathcal{J}_\gamma(u\eta) + c_2\mathcal{Y}_\gamma(u\eta)\right\}$, $\gamma = \frac{1}{2}(1 - 3\alpha)$. We may restrict $c_1^2 + c_2^2 = 1$ so that $\epsilon_0$ properly reflects the order of the perturbation, $\mathcal{O}(\epsilon_0) \equiv \mathcal{O}(\epsilon)$.

For computational ease, we divide our final expression into parts, effectively parametrising $k_i \mapsto (k_3, \ell)$:

$$\begin{aligned}
a(\eta)h_{ij}(\eta, \boldsymbol{k}) &= (\text{const.}) \cdot \delta k_2 [\![\ell \in \mathbb{Z}]\!] \cdot K_{ij} \cdot \mathrm{e}^{-\frac{1}{2}(w_0 k_3)^2} \cdot I; \quad \ell = -k_1/u_1; \\
(\text{const.}) &= 32\pi^3 G_{\mathrm{N}}\sigma_0\left(a_{\text{init}}/\eta_{\text{init}}^{\alpha}\right)^2, \quad K_{ij} = k^{-6}k_1{}^2\left[\delta_{ij}\left(k^2 - 2k_ik_j\right) + k_ik_j\right], \\
I &= \int_{\tau_{\text{init}}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_\nu(\tau, \tau')\tau'^{2\nu+1}\mathcal{J}_\ell\left(k_3\bar{\epsilon}(\tau'; u)\right); \quad \tau = k\eta,\ \nu = \alpha + 1/2
\end{aligned} \tag{2.38}$$

Furthermore, we have:

$$\sigma_0 = \frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3 \tag{2.39}$$

$$a(\eta) = a_{\text{init}}\left(\frac{\eta}{\eta_{\text{init}}}\right)^{\alpha} \tag{2.40}$$

$$a_{\text{init}} = a(\eta_{\text{init}}) \tag{2.41}$$

$$\mathcal{G}_\nu(\tau, \tau') = \frac{\pi}{2}\{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\} \tag{2.42}$$

Assuming a wall profile $\phi(\eta, z) = \phi_0 \tanh\left(\sqrt{\lambda/2}a(\eta)z\right)$, we can restrain the width $w_0 \simeq \sqrt{2/\lambda}\phi_0^{-1}$, i.e. the (comoving) spatial variation of $\phi$. Note that the wall should still be *thin*; $w_0 \ll \mathcal{H}$. Thus, given $\alpha$ and a specified perturbation—$\{u_1, \epsilon_0, c_1, c_2\}$—the model depends on parameters $\{\phi_0, \lambda, \eta_{\text{init}}, a_{\text{init}}\}$. In $\boldsymbol{k}$-space, non-vanishing $h_{ij}$'s arise for $i = j$ and $(ij) = (13)$ when $k_2 = 0$ and $k_1$ takes the values that are multiples of $u_1$, for any $k_3 \in \mathbb{R}$.

6th February 2024

## 2.5 The Complete Model

We consider a universe for which $a(\eta) \propto \eta^\alpha$ for $\eta \geq \eta_{\text{init}}$. The line element is $ds^2 = a^2(\eta)\left\{-\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right\}$. The wall is represented by the Symmetron potential $V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2$ and "width" $w_0$. The location of the wall is $[X^\mu] = (\eta, x, y, \epsilon(\eta, x, y))$ where $\epsilon$ is a linear perturbation with e.o.m.

$$\ddot{\epsilon} + 3\mathcal{H}\dot{\epsilon} - \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\epsilon = 0; \quad \mathcal{H} = \frac{\dot{a}}{a}, \quad \dot{} \equiv \frac{\mathrm{d}}{\mathrm{d}\eta}. \tag{2.43}$$

We consider plane waves of eigenvalues $-u^2$. In particular, we impose $\epsilon(\eta, x, y) = \epsilon(\eta, x) = \bar{\epsilon}(\eta)\sin(u_x x)$, such that $u = |u_x|$ and $\bar{\epsilon}(\eta) = \epsilon_0\eta^\gamma\left\{c_1\mathcal{J}_\gamma(u\eta) + c_2\mathcal{Y}_\gamma(u\eta)\right\}$, $\gamma = \frac{1}{2}(1 - 3\alpha)$. We may restrict $c_1^2 + c_2^2 = 1$ so that $\epsilon_0$ properly reflects the order of the perturbation, $\mathcal{O}(\epsilon_0) \equiv \mathcal{O}(\epsilon)$.

In the limit where $w_0 \to 0$, we have

$$T_{ij} = g_{i\mu}g_{j\nu}T^{\mu\nu} = a^4(\eta)\delta_{ik}\delta_{jl}T^{kl} = a^4(\eta)T^{ij} \tag{2.44}$$

and

$$T^{\mu\nu}(\eta, \boldsymbol{x}) = \frac{\sigma\delta z - \epsilon(\eta, x)}{a^7(\eta)}\frac{\delta\gamma}{\delta g_{\mu\nu}} + \mathcal{O}(\epsilon^2), \tag{2.45}$$

where $\delta\gamma/\delta g_{11} = \delta\gamma/\delta g_{22} = -a^4$ and $\delta\gamma/\delta g_{(13)} = -a^4\epsilon_{,1} = -a^4u_x\cos(u_x x)\bar{\epsilon}$ are the only non-vanishing spatial components. In Fourier space this gives

$$T_{11}(\eta, \boldsymbol{k}) = T_{22}(\eta, \boldsymbol{k}) = -(2\pi)^2\sigma\delta k_y a(\eta)\mathcal{J}_\ell(k_z\bar{\epsilon}) \tag{2.46a}$$

$$T_{13}(\eta, \boldsymbol{k}) = T_{31}(\eta, \boldsymbol{k}) = -(2\pi)^2\sigma\delta k_y(\ell u_x/k_z)a(\eta)\mathcal{J}_\ell(k_z\bar{\epsilon}), \tag{2.46b}$$

for $\ell = -k_x/u_x \in \mathbb{Z}$. We consider the surface tension $\sigma$[2] to be constant in the thin wall limit; $\sigma = \sigma_0 \equiv 4/3\sqrt{\lambda/2}\phi_0^3$.

We can define a polarisation basis for a wave propagating along $\boldsymbol{k} = k\hat{\boldsymbol{\Omega}}$:

$$e^+_{ij} = [\hat{\boldsymbol{m}} \otimes \hat{\boldsymbol{m}} - \hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}}]_{ij} \tag{2.47a}$$

$$e^\times_{ij} = [\hat{\boldsymbol{m}} \otimes \hat{\boldsymbol{n}} - \hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{m}}]_{ij} \tag{2.47b}$$

$\{\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{\Omega}}\}$ is an orthonormal basis, right-handed. We consider $\boldsymbol{k} = (-\ell u_x, 0, k_z)$ such that $k^2 = (\ell u)^2 + k_z^2$. In choosing $\hat{\boldsymbol{m}} = (0, 1, 0)$, we get $\hat{\boldsymbol{n}} = (-k_z, 0, -\ell u_x)/k$. Now,

$$\left[e^+_{ij}\right] = -\frac{1}{k^2}\begin{pmatrix} k_z^2 & 0 & \ell u_x k_z \\ 0 & -k^2 & 0 \\ \ell u_x k_z & 0 & (\ell u)^2 \end{pmatrix}, \quad \left[e^\times_{ij}\right] = \frac{1}{k}\begin{pmatrix} 0 & k_z & 0 \\ -k_z & 0 & -\ell u_x \\ 0 & \ell u_x & 0 \end{pmatrix}, \tag{2.48}$$

such that $T^{\mathrm{TT}}_{ij}(\eta, \boldsymbol{k}) = T^{\mathrm{TT}}_+(\eta, \boldsymbol{k})e^+_{ij}(\hat{\boldsymbol{\Omega}}) + T^{\mathrm{TT}}_\times(\eta, \boldsymbol{k})e^\times_{ij}(\hat{\boldsymbol{\Omega}})$.

In the TT frame, the non-zero components of $T^{\mathrm{TT}}_{ij}$ will be for $i = j$ and $(ij) = (13)$. We immediately see that $T^{\mathrm{TT}}_\times = 0$. Using the TT properties, we find that there is only one degree

[2] Using definition $\sigma \equiv \int_{-\infty}^{\infty}\mathrm{d}(az)\rho(z) = -a\int_{-\infty}^{\infty}\mathrm{d}z\, T^0{}_0$.

of freedom here, and we can express all components as functions of $T_{33}^{\mathrm{TT}}(\eta, \boldsymbol{k})$. Furthermore, we find

$$\begin{aligned} T_{+}^{\mathrm{TT}}(\eta, \boldsymbol{k}) &= -(k/\ell u)^2 T_{33}^{\mathrm{TT}}(\eta, \boldsymbol{k}) \\ &= -(k/\ell u)^2 \cdot \frac{1}{2k^4}\left(-{k_x}^4 T_{11}(\eta, \boldsymbol{k}) - 2{k_x}^3 k_z T_{13}(\eta, \boldsymbol{k})\right) \\ &= 2\pi^2 \sigma_0 \delta k_y (\ell u/k)^2 a(\eta) \mathcal{J}_\ell(k_z \bar{\epsilon}). \end{aligned} \tag{2.49}$$

The comoving GWs, decomposed as $\mathsf{h}_{ij} \equiv a h_{ij} = \mathsf{h}_+ e^+_{ij} + \mathsf{h}_\times e^\times_{ij} = \mathsf{h}_+ e^+_{ij}$, are obtained by

$$\begin{aligned} \mathsf{h}_+(\eta, \boldsymbol{k}) = \frac{16\pi G_{\mathrm{N}}}{k^2} \int_{\tau_{\mathrm{ini}}}^{\tau} \mathrm{d}\tau' \, \mathcal{G}_\nu(\tau, \tau') a(\tau'/k) T_{+}^{\mathrm{TT}}(\tau'/k, \boldsymbol{k}); \quad \nu = \alpha - \frac{1}{2}; \\ \mathcal{G}_\nu(\tau, \tau') = \Theta\tau - \tau' \frac{\pi}{2} \sqrt{\tau\tau'} \{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\}. \end{aligned} \tag{2.50}$$

Explicitly,

$$\mathsf{h}_+(\eta, \boldsymbol{k}) = \frac{32\pi^3 G_{\mathrm{N}} \sigma_0}{k^2} \delta k_y \left(\frac{\ell u}{k}\right)^2 \int_{\tau_{\mathrm{ini}}}^{\tau} \mathrm{d}\tau' \, \mathcal{G}_\nu(\tau, \tau') a^2(\tau'/k) \mathcal{J}_\ell(k_z \bar{\epsilon}(\tau'/k)). \tag{2.51}$$

For computational ease, we divide the expression into parts:

$$\mathsf{h}_+(\eta, \boldsymbol{k}) = \delta k_y \delta k_x + \ell u_x [\![\ell \in \mathbb{Z}]\!] \cdot \mathsf{h}_+(\eta, \ell, k_z) \tag{2.52a}$$

$$\begin{aligned} \mathsf{h}_+(\eta, \ell, k_z) = (\text{const.}) \cdot k^{-2(\alpha+2)} \ell^2 \cdot I(\eta, \ell, k_z); \\ (\text{const.}) = 32\pi^3 G_{\mathrm{N}} \sigma_0 u^2 \left(a_{\mathrm{init}}/\eta_{\mathrm{init}}^{\alpha}\right)^2; \quad \sigma_0 = \frac{4}{3}\sqrt{\frac{\lambda}{2}} \phi_0^3, \\ I = \int_{\tau_{\mathrm{init}}}^{\tau} \mathrm{d}\tau' \, \mathcal{G}_\nu(\tau, \tau') \tau'^{2\nu+1} \mathcal{J}_\ell(k_z \bar{\epsilon}(\tau'/k)); \quad \mathcal{G}_\nu(\tau, \tau') = \frac{\pi}{2} \{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\} \end{aligned} \tag{2.52b}$$

**Beyond the thin-wall limit.** We swap out the Dirac-delta distribution in the wall profile (call it $\Phi(z - \epsilon)$) with a Gaussian function of mean $\epsilon$ and standard deviation $w_0$, taken as the "width" of the wall, i.e.:

$$\delta z - \epsilon \to \Phi(z - \epsilon) = \frac{1}{\sqrt{2\pi} w_0} \exp\left\{-\frac{(z - \epsilon)^2}{2w_0^2}\right\} \rightsquigarrow \lim_{w_0 \to 0} \Phi(z - \epsilon) = \delta z - \epsilon \tag{2.53}$$

The ultimate effect of this change is simply an extra factor $\mathrm{e}^{-\frac{1}{2}(w_0 k_z)^2}$ in the expression for $\mathsf{h}_+$, which naturally is unity when $w_0 \to 0$.

### 2.5.1 Analysis

Assuming a wall profile $\phi(\eta, z) = \phi_0 \tanh\left(\sqrt{\lambda/2} a(\eta) z\right)$, we can restrain the width $w_0 \simeq \sqrt{2/\lambda}\phi_0^{-1}$, i.e. the (comoving) spatial variation of $\phi$. Note that the wall should still be *thin*; $w_0 \ll \mathcal{H}$. Thus, given $\alpha$ and a specified perturbation—$\{u_x, \epsilon_0, c_1, c_2\}$—the model depends on parameters $\{\phi_0, \lambda, \eta_{\mathrm{init}}\}$. In $\boldsymbol{k}$-space, non-vanishing $h_{ij}$'s arise for $i = j$ and $(ij) = (13)$ when $k_y = 0$ and $k_x$ takes the values that are multiples of $u_x$, for any $k_z \in \mathbb{R}$.

# 3 First Results

$11^{\text{th}}$ March 2024

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**units:** geometrised units where $c = G_{\mathrm{N}} = 1$; [length] $\equiv$ m = Mpc$/h_0$

**general variables:** $\eta$ = conformal time, $\boldsymbol{k} = (k_x, k_y, k_z) = k\boldsymbol{\Omega_k}$ = comoving wavevector, $a = a_{\text{init}}(\eta/\eta_{\text{init}})^{\alpha}$ = scale factor, $ds^2 = a^2(\eta)\eta_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}$ = unperturbed line element $\rightarrow$ $ds^2 = a^2(\eta)\Big(\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij}(\eta, \boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j)\Big)$ = perturbed line element

**additional symbols:** $\bar{h}_{\circ} = ah_{\circ}$ = scaled strain; $\circ = +, \times$ = GW polarisation

---

Domain wall in $xy$-plane, surface tension obtained from Symmetron parameters (potential in vacuum being $V(\phi) = \frac{\lambda}{4}(\phi - \phi_0)^2$),

$$\sigma = \sigma_0\sqrt{1 - \frac{\rho}{\rho_{\text{SSB}}}}\left[1 + \frac{\rho}{2\rho_{\text{SSB}}}\right] = \sigma_0\sqrt{1 - \left(\frac{a}{a_{\text{SSB}}}\right)^{3(1+w)}}\left[1 + \frac{1}{2}\left(\frac{a}{a_{\text{SSB}}}\right)^{3(1+w)}\right] \tag{3.1}$$

where $\sigma_0 = \frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3$ ($[\sigma_0] = \mathrm{m}^{-1}$) and $w$ is the eq. of state parameter for the perfect fluid component dominating the universe.

The wall is initially located at $X^{\mu} = (\eta, x, y, 0)$. At $\eta = \eta_{\text{init}}$, we add a perturbation s.t. $X^{\mu} = (\eta, x, y, \epsilon(\eta, x, y))$. We solve the eom by decomposing $\epsilon(\eta, x, y) = \epsilon_0 \cdot \varepsilon_u(\eta)\sin(u_x x)$ (letting $\epsilon_0$ carry the dimensionality), and get $\varepsilon_u(\eta) = (\eta/\eta_{\text{init}})^{\gamma}\Big\{c_1\mathcal{J}_{\gamma}(u\eta) + c_2\mathcal{Y}_{\gamma}(u\eta)\Big\}$ with $\gamma = \frac{1}{2}(1 - 3\alpha)$.

Let $w_0$ be a length scale describing the "width" of the wall and

$$\Phi(z - \epsilon) = \frac{1}{\sqrt{2\pi}w_0}\exp\left\{-\frac{(z - \epsilon)^2}{2w_0^2}\right\}. \tag{3.2}$$

Now $[\Phi] = \mathrm{m}^{-1}$, and $\lim_{w_0\to 0}\Phi(z - \epsilon) = \delta(z - \epsilon)$ restores the thin-wall setup. The non-zero spatial components of the (Hilbert) stress–energy tensor are

$$\begin{aligned} T_{xx}(\eta,\ \boldsymbol{x}) = T_{yy}(\eta,\ \boldsymbol{x}) &= \sigma\Phi(z - \epsilon) \quad \text{and} \\ T_{xz}(\eta,\ \boldsymbol{x}) = T_{zx}(\eta,\ \boldsymbol{x}) &= \sigma\Phi(z - \epsilon)\partial_x\epsilon. \end{aligned} \tag{3.3}$$

In Fourier space,

$$\begin{aligned} T_{xx}(\eta, \boldsymbol{k}) &= (2\pi)^2\sigma W(k_z^2)\delta(k_y)\sum_{n\in\mathbb{Z}}\delta(k_x + nu_x)\mathcal{J}_n[\epsilon_0 k_z \cdot \varepsilon_u(\eta)] \quad \text{and} \\ T_{xz}(\eta, \boldsymbol{k}) &= -\frac{k_x}{k_z}T_{xx}(\eta, \boldsymbol{k}), \end{aligned} \tag{3.4}$$

where $W(k_z^2) = \mathrm{e}^{-\frac{1}{2}(w_0 k_z)^2}$. We find the TT-part of $T_{ij}$, all components of which can be obtained from the TT-conditions and

$$T^{\mathrm{TT}}{}_{zz}(\eta, \boldsymbol{k}) = \frac{k_x^4}{2k^4} T_{xx}(\eta, \boldsymbol{k}). \tag{3.5}$$

**Polarisation basis.** From the right-handed orthonormal basis $\{\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{\Omega}}\}$—for which $\hat{\boldsymbol{\Omega}} \parallel \boldsymbol{k}$—we may construct a linear polarisation basis from the polarisation tensors

$$\begin{aligned} e^{+}(\hat{\boldsymbol{\Omega}}) &= \hat{\boldsymbol{m}} \otimes \hat{\boldsymbol{m}} - \hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}} \quad \text{and} \\ e^{\times}(\hat{\boldsymbol{\Omega}}) &= \hat{\boldsymbol{m}} \otimes \hat{\boldsymbol{n}} - \hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{m}}. \end{aligned} \tag{3.6}$$

We decompose $T^{\mathrm{TT}}{}_{ij}(\eta, \boldsymbol{k} = k\hat{\boldsymbol{\Omega}}) = T_{+}(\eta, \boldsymbol{k}) e^{+}(\hat{\boldsymbol{\Omega}}) + T_{\times}(\eta, \boldsymbol{k}) e^{\times}(\hat{\boldsymbol{\Omega}})$.

We can show that in our scenario, with $\hat{\boldsymbol{m}} = (0, 1, 0)$ and $\hat{\boldsymbol{n}} = (-k_z, 0, -k_x)/k$, we get $T_{+}(\eta, \boldsymbol{k}) = -\frac{k_x^2}{2k^2} T_{xx}(\eta, \boldsymbol{k})$ and $T_{\times}(\eta, \boldsymbol{k}) = 0$.

**Sourced gravitational waves.** The equation of motion for the tensor perturbation to the metric is $\Box h_\circ = 16\pi G_{\mathrm{N}} a^{-2} T_\circ$, where $\Box$ is the d'Alembertian in the flat FRW universe. The general solution for the sourced gravitational waves in Fourier space is

$$\begin{aligned} \bar{h}_\circ(\eta, \boldsymbol{k}) &= \frac{16\pi G_{\mathrm{N}}}{k^2} \int_{\tau_{\mathrm{init}}}^{\tau} \mathrm{d}\tau'\, G_{\mathrm{r}}(\tau, \tau') a(\tau'/k) T_\circ(\tau'/k, \boldsymbol{k}); \quad \tau = k\eta, \\ G_{\mathrm{r}}(\tau, \tau') &= \Theta\tau - \tau' \frac{\pi}{2} \sqrt{\tau\tau'} \{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\}; \qquad \nu = \alpha - \frac{1}{2}. \end{aligned} \tag{3.7}$$

We decompose $\bar{h}_\circ(\eta, \boldsymbol{k}) = \sqrt{k\eta}\mathcal{J}_\nu(k\eta) A_\circ(k\eta, \boldsymbol{k}) + \sqrt{k\eta}\mathcal{Y}_\nu(k\eta) B_\circ(k\eta, \boldsymbol{k})$:

$$\begin{aligned} A_\circ(\tau, \boldsymbol{k}) &= -\frac{8\pi G_{\mathrm{N}}}{k^2} \int_{\tau_{\mathrm{init}}}^{\tau} \mathrm{d}\tau'\ \sqrt{\tau'}\mathcal{Y}_\nu(\tau') a(\tau'/k) T_\circ(\tau'/k, \boldsymbol{k}) \\ B_\circ(\tau, \boldsymbol{k}) &= +\frac{8\pi G_{\mathrm{N}}}{k^2} \int_{\tau_{\mathrm{init}}}^{\tau} \mathrm{d}\tau'\ \sqrt{\tau'}\mathcal{J}_\nu(\tau') a(\tau'/k) T_\circ(\tau'/k, \boldsymbol{k}) \end{aligned} \tag{3.8}$$

Note that $k = \sqrt{\ell^2 u^2 + k_z^2}$ is implicit. We can show that the conformal time derivative becomes

$$\dot{\bar{h}}(\eta, \boldsymbol{k}) = \frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\sqrt{\tau}\mathcal{J}_\nu(\tau)\Big] k A_\circ(\tau, \boldsymbol{k}) + \frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\sqrt{\tau}\mathcal{Y}_\nu(\tau)\Big] k B_\circ(\tau, \boldsymbol{k}). \tag{3.9}$$

For our scenario, this means that

$$\begin{aligned} \bar{h}_{+}(\eta, \boldsymbol{k}) &= -\frac{k_x^2}{2k^2} \frac{8\pi^2 G_{\mathrm{N}}}{k^2} \int_{\tau_{\mathrm{init}}}^{\tau} \mathrm{d}\tau'\ \sqrt{\tau\tau'} \{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\} a(\tau'/k) T_{xx}(\tau'/k, \boldsymbol{k}) \\ &= -16\pi^4 G_{\mathrm{N}} \Big(k_x^2/k^4\Big) W(k_z^2) \delta(k_y) \sum_{n\in\mathbb{Z}} \delta(k_x + n u_x) \mathcal{I}(k\eta, n, k_z) \end{aligned} \tag{3.10a}$$

where $\mathcal{I}(\tau, n, k_z) = \sqrt{\tau}\mathcal{J}_\nu(\tau)\mathcal{I}_A(\tau, n, k_z) + \sqrt{\tau}\mathcal{Y}_\nu(\tau)\mathcal{I}_B(\tau, n, k_z)$;

$$\begin{aligned} \mathcal{I}_A(\tau, n, k_z) &= -\int_{\tau_{\mathrm{init}}}^{\tau} \mathrm{d}\tau'\ \sqrt{\tau'}\mathcal{Y}_\nu(\tau') a(\tau'/k) \sigma(a) \mathcal{J}_n[\epsilon_0 k_z \cdot \varepsilon_u(\tau'/k)] \\ \mathcal{I}_B(\tau, n, k_z) &= +\int_{\tau_{\mathrm{init}}}^{\tau} \mathrm{d}\tau'\ \sqrt{\tau'}\mathcal{J}_\nu(\tau') a(\tau'/k) \sigma(a) \mathcal{J}_n[\epsilon_0 k_z \cdot \varepsilon_u(\tau'/k)] \end{aligned} \tag{3.10b}$$

We let $\bar{h}_+(\eta, \boldsymbol{k}) = \delta(k_y)\delta(k_x + \ell u_x)[\![\ell \in \mathbb{Z}]\!] \times \bar{h}_+(\eta, \ell, k_z)$, s.t. $[\bar{h}_+(\eta, \ell, k_z)] = \mathrm{m}$. Note that $\bar{h}_+(\eta, -\boldsymbol{k}) = \bar{h}_+(\eta, \boldsymbol{k})$. We may then write the inverse Fourier transform:

$$\begin{aligned}
\bar{h}_+(\eta, \boldsymbol{x}) &= \int \frac{d^3k}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \bar{h}_+(\eta, \boldsymbol{k}) \\
&= \sum_{\ell\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\ell u_x x} \int_{\mathbb{R}} \frac{\mathrm{d}k_z}{2\pi}\, \mathrm{e}^{-\mathrm{i}k_z z} \bar{h}_+(\eta, \ell, k_z) \\
&= -8\pi^3 G_\mathrm{N} u^2 \sum_{\ell\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\ell u_x x} \ell^2 \int_{\mathbb{R}} \mathrm{d}k_z\, \mathrm{e}^{-\mathrm{i}k_z z} \frac{W(k_z^2)}{k^4} \mathcal{I}(k\eta, \ell, k_z)
\end{aligned} \tag{3.11}$$

**Free gravitational waves.** In the event that $T_\circ(\eta, \boldsymbol{k})$ becomes negligible at some conformal time $\eta_\text{final}$, the tensor perturbations will propagate freely, obeying

$$\bar{h}_\circ(\eta \geq \eta_\text{final}, \boldsymbol{k}) = \sqrt{\tau}\mathcal{J}_\nu(\tau) A_\circ(\boldsymbol{k}) + \sqrt{\tau}\mathcal{Y}_\nu(\tau) B_\circ(\boldsymbol{k}); \quad \tau = k\eta \tag{3.12}$$

where $A_\circ(\boldsymbol{k}) \equiv A_\circ(\tau_\text{final}, \boldsymbol{k})$ and $B_\circ(\boldsymbol{k}) \equiv B_\circ(\tau_\text{final}, \boldsymbol{k})$ from Eq. (3.8). The time derivative is then easily computed:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\eta}[a(\eta) h_\circ(\eta, \boldsymbol{k})] = \dot{\bar{h}}_\circ(\eta, \boldsymbol{k}) = &\frac{\sqrt{\tau}}{4\nu}[(1 + 2\nu)\mathcal{J}_{\nu-1}(\tau) + (1 - 2\nu)\mathcal{J}_{\nu+1}(\tau)] k A_\circ(\boldsymbol{k}) \\
&+ \frac{\sqrt{\tau}}{4\nu}[(1 + 2\nu)\mathcal{Y}_{\nu-1}(\tau) + (1 - 2\nu)\mathcal{Y}_{\nu+1}(\tau)] k B_\circ(\boldsymbol{k}); \quad \eta \geq \eta_\text{final}
\end{aligned} \tag{3.13}$$

## 3.1 Comparison with other results

We define the dimensionless energy density per logarithmic frequency ((Christiansen et al., 2024))

$$\Omega_\text{gw}(f) = \frac{1}{\rho_\text{cr0}} \frac{\mathrm{d}\rho_\text{gw}}{\mathrm{d}\ln f} = \frac{4\pi^2}{3H_0^2} f^3 S_h(f), \tag{3.14}$$

where $S_h(f) = P_{\dot{h}}(k = 2\pi f)/2\pi$ is the spectral density and $P_{\dot{h}}(k)$ the power spectrum of $\dot{h}_{ij}$,

$$\sum_{ij} \left\langle \dot{h}_{ij}(\eta, \boldsymbol{k}) \dot{h}_{ij}(\eta, \boldsymbol{k}') \right\rangle = 2 \sum_{\circ=+,\times} \left\langle \dot{h}_\circ(\eta, \boldsymbol{k}) \dot{h}_\circ(\eta, \boldsymbol{k}') \right\rangle = \delta^{(3)}(\boldsymbol{k} + \boldsymbol{k}') P_{\dot{h}}(k) \tag{3.15}$$

Set $G_\mathrm{N} = 1$, $H_0 = \frac{1}{2998} h_0/\text{Mpc}$, $\Omega_\text{m0} = 0.279$. Also set $\alpha = 2$ (matter domination). Starting from $(\xi_*, a_*, \beta_*) = (3.3 \cdot 10^4, 0.33, 1)$ in Christiansen et al. ((2023, Fig. 10)). See attachment `someresults.tgz`.

# 4 Stable Field Configuration

21st May 2024

## 4.1 Symmetron field

We have the eom $\Box\phi = V_{\text{eff},\phi}$ where the effective Symmetron potential is

$$V_{\text{eff}}(\phi) = \frac{\lambda}{4}\phi^4 - \frac{\mu^2}{2}\phi^2\left(1 - \frac{\rho_{\text{m}}}{\mu^2 M^2}\right). \tag{4.1}$$

The expanding background is such that $a \propto \eta^\alpha$, where $\eta$ is conformal time. The matter density at symmetry breaking (SB) is $\rho_{\text{m},*} = \mu^2 M^2$ and thus

$$\phi_\pm = \pm\sqrt{\frac{\mu^2 M^2 - \rho_{\text{m}}}{\lambda M^2}} = \pm\phi_\infty\sqrt{1 - \frac{\rho_{\text{m}}}{\rho_{\text{m},*}}} \tag{4.2}$$

We define $\chi \equiv \phi/\phi_\infty$ where $\phi_\infty = \mu/\sqrt{\lambda}$ corresponds to the asymptotic minima after SB. Since $\rho_{\text{m}} = \rho_{\text{m},*}(a/a_*)^{-3}$, we get

$$\chi_\pm = \phi_\pm/\phi_\infty = \pm\sqrt{1 - (a_*/a)^3} = \pm\sqrt{1 - (\eta_*/\eta)^{3\alpha}} = \pm\sqrt{1 - s^{-3\alpha}} \tag{4.3}$$

where we defined the dimensionless time variable $s \equiv \eta/\eta_*$. The eom for $\phi$,

$$-a^{-2}\left[\ddot{\phi} + \frac{2\alpha}{\eta}\dot{\phi} - \boldsymbol{\nabla}^2\phi\right] = \lambda\phi^3 - \mu^2\left(1 - \frac{\rho_{\text{m}}}{\rho_{\text{m},*}}\right), \tag{4.4}$$

is rewritten

$$\chi'' + \frac{2\alpha}{s}\chi' - \eta_*^2\boldsymbol{\nabla}^2\chi = -\frac{a_*^2}{2\xi_*^2}H_0^2\eta_*^2 \cdot s^{2\alpha}\left\{\chi^2 - \chi_+^2\right\}\chi, \tag{4.5}$$

where dot and prime means derivative w.r.t. conformal and dimensionless time, respectively. We used $\xi_* = H_0/(\sqrt{2}\mu)$.

**Inside the domain.** Well inside the domain, spatial gradients are negligible. We let $\check{\chi}$ be the positive solution to Eq. (4.5) when $\boldsymbol{\nabla}^2\chi \simeq 0$, and use $\chi_+$ as time coordinate. From here, we consider **matter domination** ($\alpha = 2$). Now, the eom for $\check{\chi}$ is

$$\frac{\mathrm{d}^2\check{\chi}}{\mathrm{d}\chi_+^2} - \frac{1}{\chi_+\left(1 - \chi_+^2\right)}\frac{\mathrm{d}\check{\chi}}{\mathrm{d}\chi_+} + m^2\frac{\chi_+^2\left(\check{\chi}^2 - \chi_+^2\right)}{\left(1 - \chi_+^2\right)^3}\check{\chi} = 0 \tag{4.6}$$

where

$$m = \frac{2\mu}{3\mathcal{H}_*(1+z_*)} = \frac{\sqrt{2}}{3}\frac{a_*^{3/2}}{\xi_*}. \tag{4.7}$$

This solution is to be used as boundary conditions for $\chi$:

$$\chi(s, z \to \pm\infty) = \pm\breve{\chi}(s) \tag{4.8a}$$
$$\chi'(s, z \to \pm\infty) = \pm\breve{\chi}'(s) \tag{4.8b}$$

We solve Eq. (4.6) in two regimes, each solution expanded around the extremal values of $\chi_+ \in [0, 1]$:

$$\breve{\chi} \overset{\chi_+\sim 1}{\simeq} \chi_+ + \frac{8(3-m^2)}{m^4}(\chi_+ - 1)^3 + \frac{1440 - 636m^2 + 41m^4}{2m^6}(\chi_+ - 1)^4 \tag{4.9a}$$
$$\breve{\chi} \overset{\chi_+\sim 0}{\simeq} \chi_* + \frac{C}{2}\chi_+^2 + \frac{C - \chi_*^3 m^2}{8}\chi_+^4 \tag{4.9b}$$

By matching these solutions at some point in between where they overlap, we can find $\chi_*$ and $C$, at least approximately. I have written a `Python` code that finds $\breve{\chi}$ for any given $m \in [10, \sim 4000]$.

We also want to study the field $q = a^2\dot{\chi} = a^2/\eta_* \chi' = a^2/\eta_* \chi_+' \, \mathrm{d}\chi/\mathrm{d}\chi_+ \equiv q_+ \, \mathrm{d}\chi/\mathrm{d}\chi_+$. We let $\breve{q} = q_+ \, \mathrm{d}\breve{\chi}/\mathrm{d}\chi_+$.

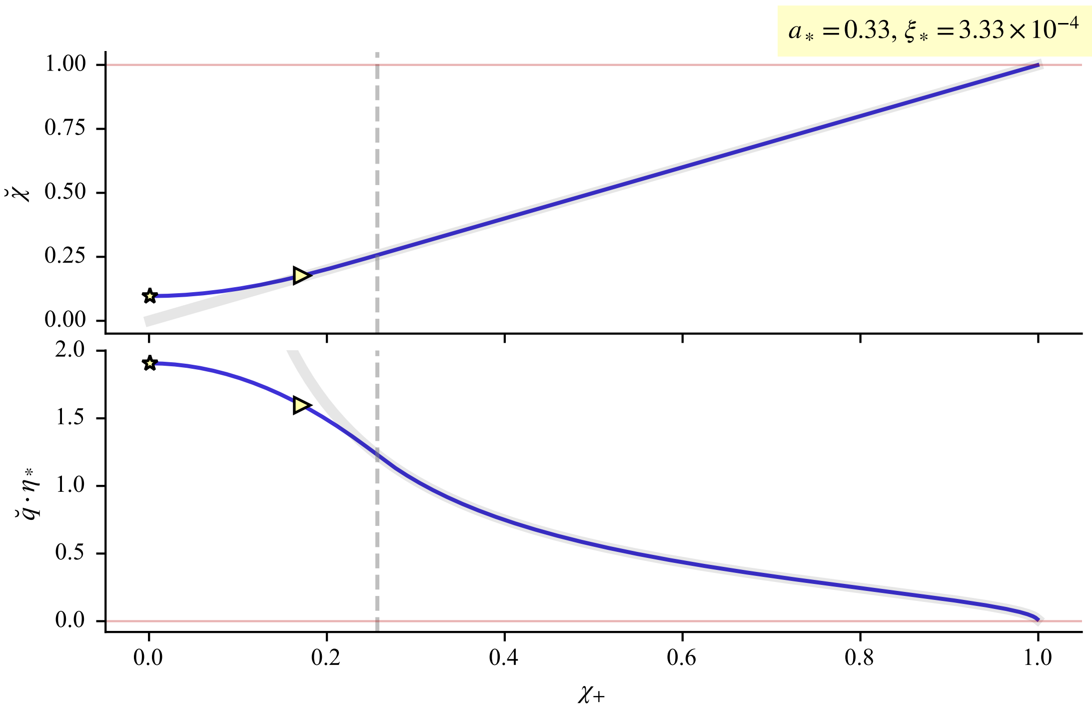



**Figure 4.1:** The asymptotic scalar fields $\breve{\chi}$ and $\breve{q}$ (blue solid) as functions of $\chi_+$, with $\chi_+$ and $q_+$ (grey solid) for comparison. The dashed grey vertical line marks the value of $\chi_+$ for which the solutions in Eq. (4.9) are matched. The markers ⋆ and ▸ indicate the field values at SB and simulation start (here: redshift $\mathfrak{z} = 2$), respectively.

### 4.1.1 Full field

We have previously used $\chi_\pm$ as boundary conditions for the Symmetron field, which gave the quasi-static solution to Eq. (4.5)

$$\chi(s,z) = \chi_+(s)\tanh\left\{\frac{a_* H_0}{2\xi_*} s^2 \chi_+(s)(z - z_{\mathrm{dw}})\right\}.^{1} \tag{4.10}$$

If we instead naively guess that

$$\chi(s,z) = \check{\chi}(s)\tanh\left\{\frac{a_* H_0}{2\xi_*} s^2 \check{\chi}(s)(z - z_{\mathrm{dw}})\right\} \tag{4.11}$$

solves the eom with new boundary conditions (Eq. (4.8)), we see that the field moves "slower":
**But does it still solve the eom?**

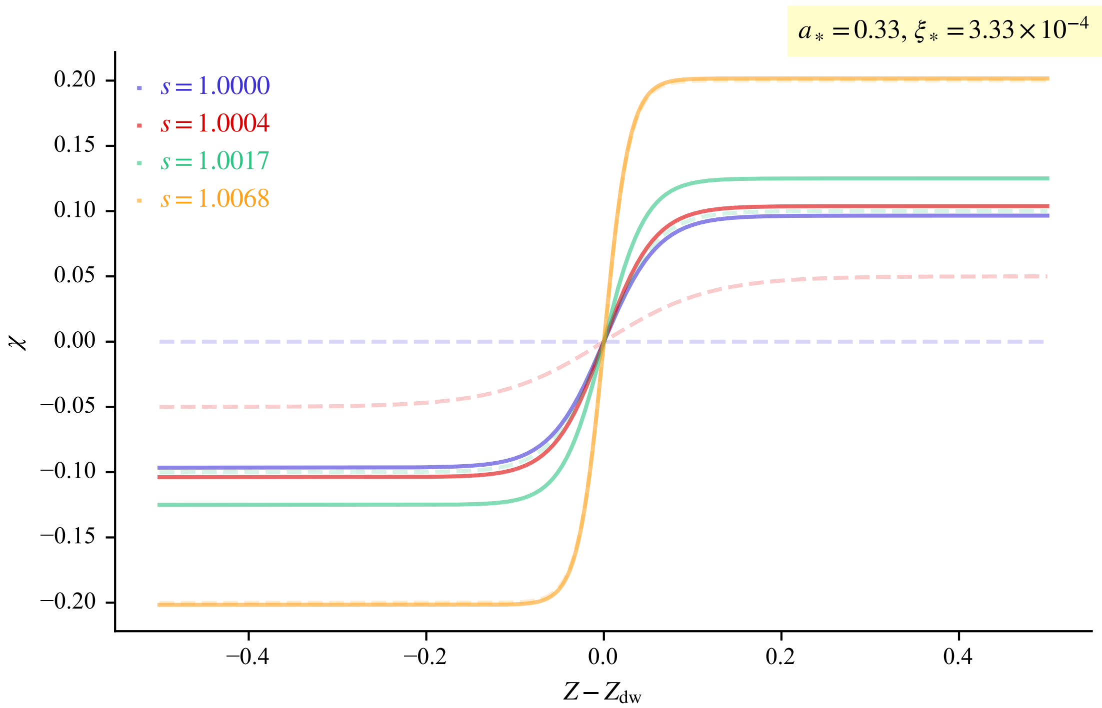



**Figure 4.2:** The field $\chi$ as function of spatial coordinate $Z \propto z$ for time points close to SB ($s = 1$). Both solutions Eq. (4.10) (dashed) and Eq. (4.11) (solid) are represented.

[1] Setting time derivatives to zero.

4th June 2024

## 4.2 Analytic work vs. gwasevolution

In the calculations, I named the dimensionless Symmetron field $\chi$ and its corresponding asymptotic field (where spatial gradients are neglected) $\breve{\chi}$. I use $\mathfrak{z} = 1/a - 1$ to denote cosmic redshift, to avoid confusion with the much-used spatial coordinate $z$.

### 4.2.1 Simulation setup

I have studied simulations with $\alpha = 2$ (matter dominated universe), keeping the following parameters (more or less) constant:

| | | |
|---|---|---|
| $(a_*, \xi_*, \beta_*)$ | $= (\texttt{astar}, \texttt{xistar}, \texttt{betastar})$ | $= (0.33, 3.33 \times 10^{-4}, 1)$ |
| $(L\,[\mathrm{Mpc}/h_0], N)$ | $= (\texttt{boxsize}, \texttt{Ngrid})$ | $= (1024, 512)$ |

I have been varying initial time $\mathfrak{z}_{\text{init}}$ = `initial redshift` and perturbation parameters $(\epsilon_*, u) \sim (L/4, 4 \cdot 2\pi\eta_*/L)$, giving initial conditions:

| | | |
|---|---|---|
| $\mathfrak{z}_{\text{init}}$ | = `initial redshift` | $< \mathfrak{z}_*$ |
| $(\epsilon_{\text{init}}, \dot{\epsilon}_{\text{init}})$ | = (`pert epsilon IC`, `pert epsilonprime IC`) | [dep. on $\mathfrak{z}_{\text{init}}$] |
| $(m_x, m_y)$ | = (`pert num osc x`, `pert num osc y`) | $(= 0, \in [2, 8])$ |

**The problem:** For the simulated perturbed domain wall position as function of time, whilst resembling the analytical prediction, the results *in general* show a significant difference. The amplitude the perturbation as seen for each individual "period" is very close to its prediction, and after some time, the frequency seem to coincide with the perturbation wavenumber. In the beginning of each simulation, this position graph is "slowed down" with respect to the analytical prediction, and this pause seems to occur at a critical point in each simulation; the point where the scalar field `achi`, as seen at along a slit in the $z$-direction, gets a "bump" around where the field tends to its minima (wall edges). This bump occurs in any simulation a short time after simulation start, regardless of perturbation or not.[2] A consequence of this is that `achimax`(`achimin`) is a bad estimate of $(-)\breve{\chi}$.

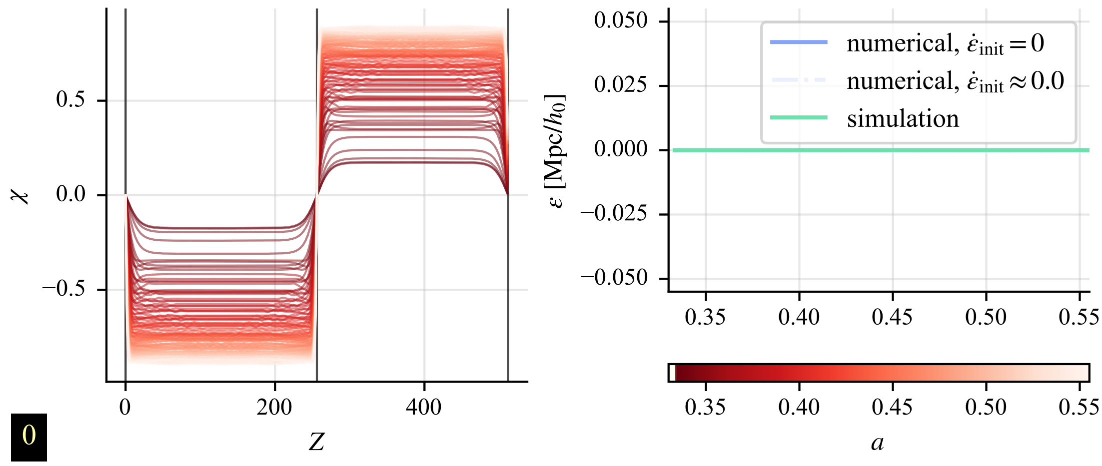
**Possible solutions:**

(i) Reduce oscillations around minima for `achi`. With correct initial conditions, we can make $\breve{\chi}/\chi_+ \to 1$. How?

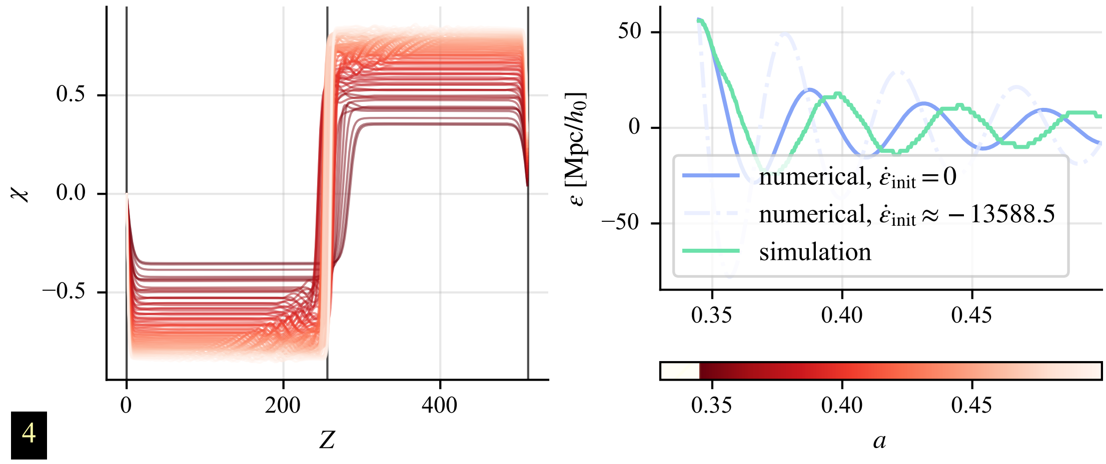
- Initialising the simulation well after $a_*$ (i.e. cheating) should produce smaller oscillations as one avoids the initially large $\breve{\chi}'$.
- Otherwise, one should be able to "tweak" the initial conditions at any point in time *after* PT to induce the smallest oscillation amplitude possible. See Sect. 4.1.

(ii) Consider non-trivial evolution of surface tension $\sigma$ in eom for $\epsilon$. Would this give a behaviour more like the simulation?

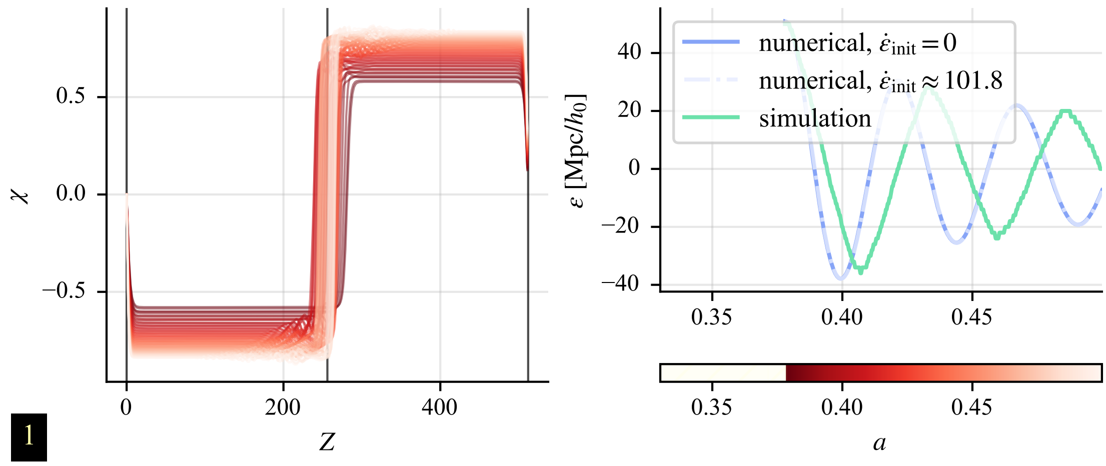
- Analytical approach: Use far-away approximation (Sect. 4.1).
- Brute force: Extract asymptotic field value from simulation and use this to solve eom for $\epsilon$ numerically.




**(a)** Simulation 0: $(\epsilon_*, m_y) = (0, 0)$, $(\mathfrak{z}_{\mathrm{init}}, \mathfrak{z}_{\mathrm{break}}) = (2.0, 0.8)$.




**(b)** Simulation 4: $(\epsilon_*, m_y) = (0.28L, 5)$, $(\mathfrak{z}_{\mathrm{init}}, \mathfrak{z}_{\mathrm{break}}) = (1.9, 1.0)$.




**(c)** Simulation 1: $(\epsilon_*, m_y) = (0.28L, 5)$, $(\mathfrak{z}_{\mathrm{init}}, \mathfrak{z}_{\mathrm{break}}) = (1.646, 1.0)$.

**Figure 4.3:** Fields at along a slit in the $z$-direction from three simulations. *Left & lower right panels:* The field value `achi` along a slit in the $z$ direction, at lattice coordinate $(X_0, Y_0) = (4, 128)$. Higher redshift is darker, see colour bars. *Upper right panels:* The displacement from original wall position as function of time. For the simulation, this is considered as the $Z$ coordinate where `abs(achi)` has its minimum value, minus the original position at $Z_0 = 256$, excluding the secondary wall.

We see from Fig. 4.3 that whilst delaying the simulation start (here: from $\mathfrak{z}_{\text{init}} = 2.0$ to $\mathfrak{z}_{\text{init}} = 1.9$) indeed minimises the asymptotic field's oscillations around the minima, introducing a perturbation to the middle wall gives a similar effect: there comes a "bump" that in turn gives wiggles on the field.

### 4.2.2 Equation of motion for $\epsilon$

We estimate the surface tension

$$\sigma = \int_{\phi(z\to-\infty)}^{\phi(z\to\infty)} \mathrm{d}\phi\, \sqrt{2V_{\text{eff}}(\phi) - 2V_{\text{eff}}(\phi_\pm)}, \tag{4.12}$$

now using $\phi(z \to \pm\infty) = \pm\phi_\infty \breve{\chi}$ instead of $\phi_\pm$. We get

$$\sigma = \sigma_0 \cdot \frac{1}{2}\Big( 3\chi_+^2 - \breve{\chi}^2\Big)\breve{\chi}. \tag{4.13}$$

The eom for $\epsilon$ when decomposed in terms of eigenfunctions with eigenvalues $-(u/\eta_*)^2$

$$\epsilon'' + \left(3\frac{a'}{a} + \frac{\sigma'}{\sigma}\right)\epsilon' + u^2\epsilon = 0; \quad ' \equiv \frac{\mathrm{d}}{\mathrm{d}s} \tag{4.14}$$

where we use the dimensionless (conformal) time variable $s = \eta/\eta_*$. The perturbation becomes the solution to this equation, $\epsilon(s)$, times a spatial part:

$$\epsilon(s, x, y) = \epsilon(s) \sin\left\{\frac{u_x x + u_y y}{\eta_*}\right\} \tag{4.15}$$

Going back to Aim (ii): In using Eq. (4.13) in Eq. (4.14), one may play around with various versions of $\breve{\chi}$ to solve it numerically. I have attempted using $\breve{\chi}$ = `achimax` from simulation, but also $\sqrt{\texttt{achi2av}}$ and by brute force; retrieving the field at some $z$-coordinate far enough from the walls. $\sigma'/\sigma$ involves $\breve{\chi}'$, which is a bit tricky as `aq` turns out to be extremely “noisy”.

**Result:** I see no significant change in the evolution of $\epsilon$ when adjusting the asymptotic scalar field, and nothing that seems to make the solution more similar to that of the simulated result.

**Possible explaination:** There are many possible explanations to this. Perhaps the field description and thin-wall approximation differ in this sense, and both of them are correct in their own applications. Eq. (4.12) might be just a bad estimate for this setup, or the naive limit substitution is the problem.

### 4.2.3 Initialising fields in gwasevolution

With Symmetron parameters $(a_*, \xi_*, \beta_*)$, we choose a value of $\epsilon_*$ and a scale $u$ for the perturbation. For simplicity, we let $\boldsymbol{u} = (0, u)$. To preserve BCs, we demand that $u/\eta_* = 2\pi/L{\cdot}m$ holds for integer $m$ (so that the corresponding wavelength is $L/m$), where $L$ is the simulation box size. First, we find the analytical prediction of $\epsilon(s)$. The fields $\chi$ (`achi`) and $q = a^2/\eta_*\chi'$ (`aq`) are initialised in accordance with the analytical profile:

$$\chi = \pm\chi_+ \cdot \mathsf{T}; \quad \mathsf{T} = \tanh\left\{\chi_+ \frac{a(z - z_{\text{dw}})}{2}\right\} \tag{4.16a}$$

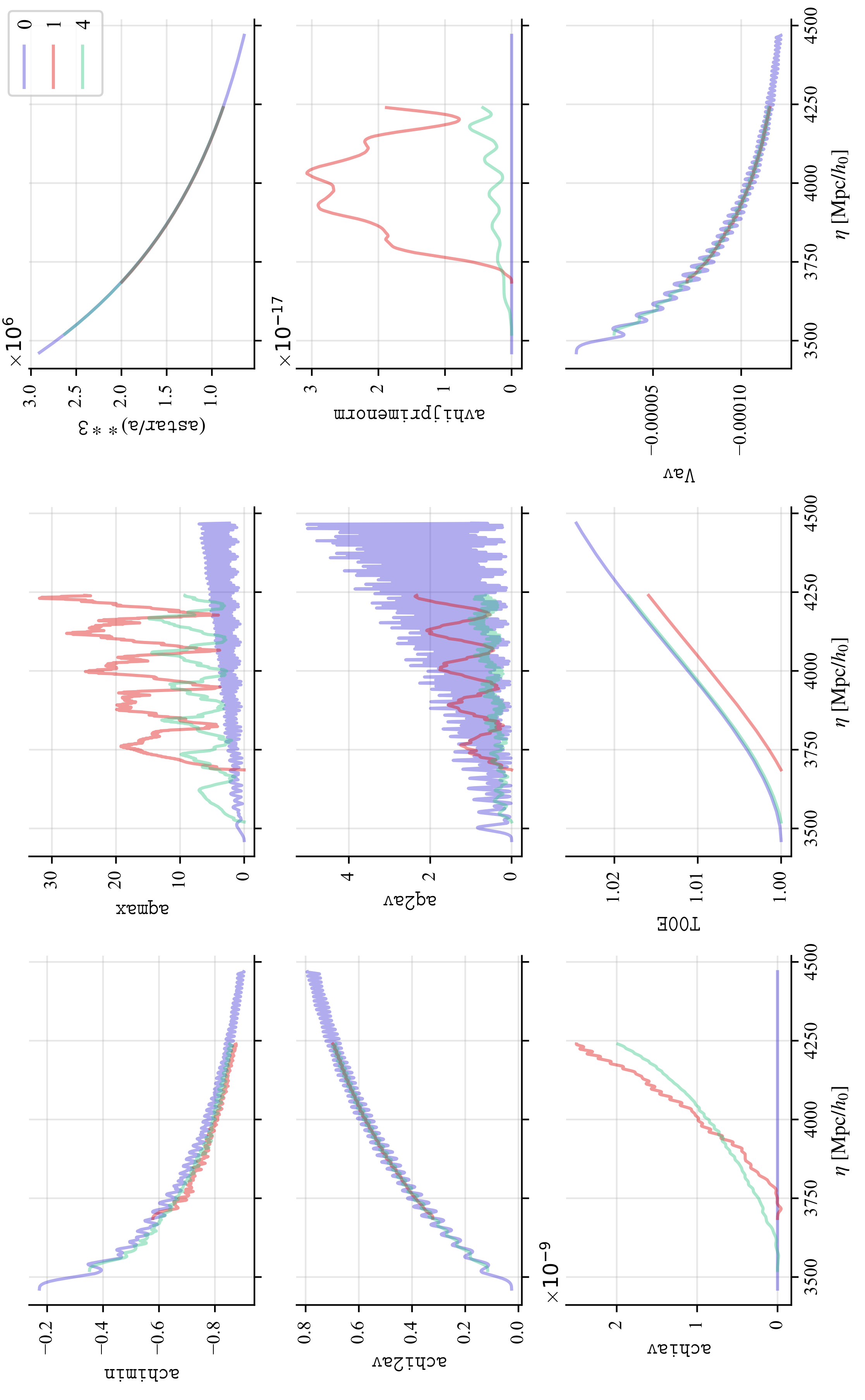
$$q = \pm a^2\left[\mathcal{H}\left\{\frac{3}{2}\frac{1-\chi_+^2}{\chi_+} \cdot \mathsf{T} + \frac{a}{4}(3-\chi_+^2)(z - z_{\text{dw}}) \cdot \mathsf{S}\right\} - \frac{a}{2}\chi_+^2 z_{\text{dw}} \cdot \mathsf{S}\right]; \quad \mathsf{S} = 1 - \mathsf{T}^2 \tag{4.16b}$$

[2] Unless—it seems—one begins the simulation at a later time, where $\breve{\chi}, \breve{q} \to \chi_+, q_+$ from Sect. 4.1.

There seems to be little impact on the result by initialising aq with Eq. (4.16b) instead of 0. The wall position for the secondary wall (located at the boundary) is trivial, but the middle wall has position $z_{\mathrm{dw}} = z_{\mathrm{dw}}(s,x,y) = z_0 + \epsilon(s,x,y)$, where $z_0 = L/2$. Then, $\dot{z_{\mathrm{dw}}} = \dot{\epsilon} = \epsilon'/\eta_*$.

**Observation:** It does not seem that the steepness of $\epsilon(s)$ is captured by this. Plots of Z[np.argmin(np.abs(achi))] indicate that the initial speed at which the amplitude is moving is zero, or very close to zero, no matter what initial time is chosen. A solution that solves both this issue and minimises oscillations around minima would be to set the initial time to a time point where $\dot{\epsilon} = 0$ and $a \gg a_*$. Such a simulation is labelled 1 here (see Fig. 4.3c), and shows more or less the same delation wrt. the thin-wall approximation in the wall position.

I added a figure of a sample of background quantities from the three simulations. The units are not included, but they are the same as in the code.


achimin
aqmax
(astar/a) * * 3
achi2av
aq2av
avhijprimenorm
achiav
TOOE
Vav
η [Mpc/h0]
0
1
4

# 5 Analysis and Debugging

15th July 2024

## 5.1 Calculations vs. simulations

Using wall position as input to the SE-tensor that sources the GWs, we can predict by calculations that the FT image of the GWs is purely real. However, simulations predict both real and imaginary parts. It turns out that the predicted real components matches very good with the simulated imaginary component.

### 5.1.1 Stress–Energy tensor

We consider the perturbation $\epsilon = \epsilon(\eta, y) = \epsilon(\eta)\sin uy$. The real-space SE-tensor that we get from varying the Nambu–Goto action is

$$\left[T_{\mu\nu}(\eta, \boldsymbol{x})\right] = -a(\eta)\sigma(\eta)\Phi(z-\epsilon)\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \partial_y\epsilon \\ 0 & 0 & \partial_y\epsilon & 0 \end{bmatrix}, \tag{5.1}$$

where $\partial_y\ \epsilon(\eta, y) = u\epsilon(\eta)\cos uy$. We use $\Phi(z-\epsilon) = \delta z - \epsilon(x, y)$. In Fourier space, we get the following for the non-zero components:

$$T_{xx}(\eta, \boldsymbol{k}) = -a(\eta)\sigma(\eta)\cdot 2\pi\delta k_x \int \mathrm{d}y\, \mathrm{e}^{\mathrm{i}k_z\epsilon}\mathrm{e}^{\mathrm{i}k_y y} \tag{5.2a}$$

$$T_{yz}(\eta, \boldsymbol{k}) = -a(\eta)\sigma(\eta)\cdot 2\pi\delta k_x\ u\epsilon(s) \int \mathrm{d}y\ \cos uy\ \mathrm{e}^{\mathrm{i}k_z\epsilon}\mathrm{e}^{\mathrm{i}k_y y} \tag{5.2b}$$

We immediately see that the imaginary parts vanish.

**TT gauge.** Writing out the linear polarisation basis for when $\boldsymbol{k} = (0, k_y, k_z)$, one sees that we only need e.g. $T^{\mathrm{TT}}_{xx}(\eta, \boldsymbol{k})$ to get $T^{\mathrm{TT}}_{+}$ and that $T^{\mathrm{TT}}_{\times} = 0$. We find that $T^{\mathrm{TT}}_{+} = T^{\mathrm{TT}}_{xx}(\eta, \boldsymbol{k})$ and

$$T^{\mathrm{TT}}_{+} = \frac{1}{2k^2}\left\{{k_y}^2 T_{xx}(\eta, \boldsymbol{k}) + 2k_y k_z T_{yz}(\eta, \boldsymbol{k})\right\} \tag{5.3}$$

**Explicit calculation.** We can use the Jacobi–Anger expansion to show the following relations:

$$\int \mathrm{d}y\, \mathrm{e}^{\mathrm{i}k_z\epsilon(\eta)\sin uy}\mathrm{e}^{\mathrm{i}k_y y} = \int \mathrm{d}x \sum_{n=-\infty}^{\infty} \mathcal{J}_n(k_z\epsilon(\eta))\, \mathrm{e}^{\mathrm{i}(k_y+nu)y} \tag{5.4a}$$

$$\begin{aligned}\int \mathrm{d}y\, \cos uy\, \mathrm{e}^{\mathrm{i}k_z\epsilon(\eta)\sin uy}\mathrm{e}^{\mathrm{i}k_y y} &= \frac{1}{2}\int \mathrm{d}x \sum_{n=-\infty}^{\infty} \left\{\mathcal{J}_{n+1}(k_z\epsilon(\eta)) + \mathcal{J}_{n-1}(k_z\epsilon(\eta))\right\} \mathrm{e}^{\mathrm{i}(k_y+nu)y} \\ &= \left(k_z\epsilon(\eta)\right)^{-1} \int \mathrm{d}x \sum_{n=-\infty}^{\infty} n\mathcal{J}_n(k_z\epsilon(\eta))\, \mathrm{e}^{\mathrm{i}(k_y+nu)y} \end{aligned} \tag{5.4b}$$

Assuming we can extract the summation out of the integration, we get

$$T_+^{\mathrm{TT}} = a(\eta)\sigma(\eta)\cdot(2\pi)^2\delta k_x \frac{{k_y}^2}{2k^2}\sum_{n\in\mathbb{Z}}\mathcal{J}_n(k_z\epsilon(\eta))\delta(k_y+nu). \tag{5.5}$$

This assumption may not hold as the condition for switching the order is that both sums/integrals converge.

**Parametrisation.** We choose to parametrise the wavevector $(0, k_y, k_z) \to (\ell, \zeta)$ with $\ell = k_y/u$ and $\zeta = k_z/k_y$, where only $\ell \in \mathbb{Z}$ gives non-zero $T_+^{\mathrm{TT}}$.

$$T_+^{\mathrm{TT}}(\eta, \ell, \zeta) = 4\pi^2 a(\eta)\sigma(\eta)\frac{(-1)^\ell \mathcal{J}_\ell(u\epsilon(\eta)\ell\zeta)}{1+\zeta^2} \tag{5.6}$$

### 5.1.2 Gravitational waves

The equation $\Box h_+ = S_+$, where $S_+ = 16\pi G_{\mathrm{N}} T_+^{\mathrm{TT}}$, can be solved through Green's functions to obtain the expression

$$a(\eta)h_+(\eta, \boldsymbol{k}) = \frac{1}{k}\int_{\eta_*}^{\eta}\mathrm{d}\eta'\, G_{\mathrm{r}}(k\eta, k\eta')a^3(\eta')S_+(\eta', \boldsymbol{k}) \tag{5.7}$$

where

$$G_{\mathrm{r}}(p,q) = \frac{\pi}{2}\Theta p - q\,\sqrt{pq}\{\mathcal{Y}_\nu(p)\mathcal{J}_\nu(q) - \mathcal{J}_\nu(p)\mathcal{Y}_\nu(q)\}; \quad \nu = \alpha - \frac{1}{2}. \tag{5.8}$$

20th August 2024

## 5.2 Dilemmas and paradoxes

Using *gwasevolution* to simulate the perturbed domain wall has proven to be a challenge. The general setup consists of two walls (necessary to preserve boundary conditions) parallel to the $xy$-plane, one of which is to be perturbed. Instead of being flat, the middle wall is given a sinus-like profile.

We consider matter domination and the Symmetron effective potential

$$V_{\text{eff}} = \frac{\lambda}{4}\phi^4 - \frac{\mu^2}{2}(1-\upsilon)\phi^2; \quad \upsilon = -T_{\text{m}}/\rho_* = \rho_{\text{m}}/\rho_* = (a_*/a)^3. \tag{5.9}$$

**Wall position.** By thin-wall approximation, the wall position—independent of spatial part—obeys

$$\ddot{\epsilon} + \left(3\frac{\dot{a}}{a} + \frac{\dot{\sigma}}{\sigma}\right)\dot{\epsilon} - a^{-2}\boldsymbol{\nabla}^2\epsilon = 0, \tag{5.10}$$

where dot means conformal time derivative. The surface tension was initially given as

$$\sigma = \int_{-f}^{+f} \mathrm{d}\phi\, \sqrt{2V_{\text{eff}}(\phi) - 2V_{\text{eff}}(\phi_\pm)} = \sqrt{\frac{\lambda}{2}} \cdot \frac{2}{3}\left[3\phi_+^2 - f^2\right]f \tag{5.11}$$

where $f = \phi_\pm = \pm\phi_\infty\sqrt{1-\upsilon}$ was considered to be the minima. In truth, however, the field will oscillate around each minimum, and so we should change the limits from $\pm\phi_+$ to the "true" minima $\pm\breve{\phi}$. This is a simple matter of letting $f = \breve{\phi}$. **Are we sure this expression makes sense when the field profile is tweaked?**

| **option** | **pro** | **con** |
|---|---|---|
| initial redshift $\mathfrak{z}_{\mathrm{init}}$ (`initial redshift`) | | |
| $\mathfrak{z}_{\mathrm{init}} = \mathfrak{z}_*$ | + capture entire evolution<br>+ wall coordinate zero velocity<br>+ wall evolution might be foreseeable | – difficult to recreate analytical setup (walls "crash" and initialisation is generally complicated) |
| $\mathfrak{z}_{\mathrm{init}} \gtrsim \mathfrak{z}_*$ | + more stable asymptotic evolution of scalar field (the further from PT, the better)<br>+ easier to tailor the system to our desired design | – loss of possibly important information that in turn affects tensor perturbations |
| number of grid points $N$ (`Ngrid`) | | |
| $N > N_0$ | + capture movement more precisely<br>+ allows smaller perturbation | – computational expense increased by $(N - N_0)^3$ |
| box size $L$ (`boxsize`) | | |
| $L > L_0$ | + propagation "information" collides later<br>+ wall's self-interaction, if relevant, happens later | – need for longer simulation time<br>– allows larger $\epsilon_*$, but smaller $u_y = 2\pi m_y/L$ |
| perturbation amplitude $\epsilon_*$ (`pert epsilonstar`) | | |
| $\epsilon_* \ll L/4$ | + more realistic<br>+ wall less likely to self-interact in $y$-direction, which would cause higher-order effects | – high (!) spatial resolution needed<br>– less intense GWs produces |
| perturbation scale $m_Y$ (`pert wavenumber_y`) | | |
| $m_Y = 2$ | + limit case with need for lowest possible spatial resolution | – long time passes before interesting behaviour occurs |
| $m_Y \geq 3$ | + allows shorter simulation time<br>+ introduces fundamentally different GW signature | – need for higher spatial resolution<br>– scalar field very sensitive to self-interaction in $y$-direction |

| **option** | **pro** | **con** |
|---|---|---|
| ICs (`achi`, `aq`) | | |
| $\chi_+, \chi'_+ \to \breve{\chi}, \breve{\chi}'$ | + oscillations around minima are reduced<br>+ may initialise fields as close to SSB as we would like | – harder to argue for the new solution in the non-asymptotic limit<br>– presence of wall seems to enhance oscillations either way<br>– challenging to find suitable expression for $\sigma'/\sigma$ to use in eom for $\epsilon$ |
| $\chi_+, \chi'_+ \to \chi_+, 0$ | + still (possibly) able to solve eom for $\epsilon$ | – max. amplitude of oscillations |
| Nature of perturbation | | |
| $\sin(2\pi m_y/L + \theta)$ | + simple, few parameters<br>+ purely analytical expression for SE-tensor<br>+ should be able to ignore all modes for which $K_x > 0$ or $K_y \neq nm_y$ | – gives rise to Dirac–Delta distr., may cause computational disadvantage |
| Gaussian | + avoid Dirac–Delta distributions (?) | – lack of analytical $T_{ij}(\eta, \boldsymbol{k})$ |

# Bibliography


F. Bowman. *Introduction to Bessel Functions*. January 1958. URL https://ui.adsabs.harvard.edu/abs/1958ibf..book.....B.

Øyvind Christiansen, Farbod Hassani, Mona Jalilvand, and David F. Mota. Asevolution: A relativistic N-body implementation of the (a)symmetron. *Journal of Cosmology and Astroparticle Physics*, 2023(05):009, May 2023. ISSN 1475-7516. doi: 10.1088/1475-7516/2023/05/009. URL https://iopscience.iop.org/article/10.1088/1475-7516/2023/05/009.

Øyvind Christiansen, Julian Adamek, Farbod Hassani, and David F. Mota. Gravitational waves from dark domain walls, January 2024. URL https://ui.adsabs.harvard.edu/abs/2024arXiv240102409C.

Masahiro Kawasaki and Ken'ichi Saikawa. Study of gravitational radiation from cosmic domain walls. *Journal of Cosmology and Astroparticle Physics*, 2011(09):008–008, September 2011. ISSN 1475-7516. doi: 10.1088/1475-7516/2011/09/008. URL http://arxiv.org/abs/1102.5628.

William H. Press, Barbara S. Ryden, and David N. Spergel. Dynamical Evolution of Domain Walls in an Expanding Universe. *The Astrophysical Journal*, 347:590, December 1989. ISSN 0004-637X. doi: 10.1086/168151. URL https://ui.adsabs.harvard.edu/abs/1989ApJ...347..590P.

Tanmay Vachaspati. *Kinks and Domain Walls: An Introduction to Classical and Quantum Solitons*. Cambridge University Press, Cambridge, 2006. ISBN 978-0-521-83605-0. doi: 10.1017/CBO9780511535192. URL https://www.cambridge.org/core/books/kinks-and-domain-walls/98D525CCD885D53F51BDFC3B08A711A6.

A. Vilenkin. Cosmic strings and domain walls. *Physics Reports*, 121:263–315, January 1985. ISSN 0370-1573. doi: 10.1016/0370-1573(85)90033-X. URL https://ui.adsabs.harvard.edu/abs/1985PhR...121..263V.

Alexander Vilenkin and E. Paul S. Shellard. *Cosmic Strings and Other Topological Defects*. January 1994. URL https://ui.adsabs.harvard.edu/abs/1994csot.book.....V.

Consider expanding universe with domain wall in the $xy$-plane in the thin wall limit. Add perturbation $\epsilon(\eta, x)$. What do the GWs look like?

We let $\epsilon(\eta, x) = \bar{\epsilon}(\eta)\sin(ux)$ and

$$\mathcal{G}_\nu(\tau, \tau') = \Theta\tau - \tau' \frac{\pi}{2}\sqrt{\tau\tau'}\{\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')\} \tag{1}$$

for $\nu = \alpha - 1/2$, where $a \propto \eta^\alpha$. Let $ah_{ij} = \mathsf{h}_{ij}$. After many steps, we get

$$\mathsf{h}_{ij}(\eta, \boldsymbol{k}) = (\pm)\frac{32\pi^3\sigma_0 G_{\mathrm{N}}}{k^6}\delta k_2 {k_1}^2\Big[\delta_{ij}\Big(k^2 - 2k_ik_j\Big) + k_ik_j\Big] \times \sum_{n\in\mathbb{Z}} \delta\ell - n$$
$$\times \int_{\tau_{\mathrm{ini}}}^{\tau} \mathrm{d}\tau'\, \mathcal{G}_\nu(\tau, \tau')a^2(\tau')\mathcal{J}_\ell\Big(k_3\bar{\epsilon}(\tau')\Big); \quad \tau = k\eta,\ \ell = -k_1/u. \tag{2}$$

For $\alpha = 2$ we have $\nu = 3/2$ and $\bar{\epsilon}(\eta) \sim \eta^{-5/2}\mathcal{J}_{\pm 5/2}(u\eta)$, so this integral should probably be solved numerically. (The sign confusion stems from the variation of the domain wall action.)

Next step is to modify this to work beyond the thin wall limit.

# A Old Texts

**DRAFT**

⌜

Consider a planar domain wall in the $xy$-plane in a flat FRW universe, represented by a scalar field $\phi(\eta, \boldsymbol{x})$ and a potential $V(\phi)$. The action of this theory is

$$S = \int d^4x \sqrt{-g} \left\{ 16\pi G_{\mathrm{N}} \mathcal{R} - \frac{1}{2} \phi^{;\mu} \phi_{;\mu} + V(\phi) \right\}. \tag{A.1}$$

The background metric is

$$d\overline{s}^2 = \overline{g}_{\mu\nu} \mathrm{d}\overline{x}^\mu \mathrm{d}\overline{x}^\nu = -\mathrm{d}t^2 + a^2(t)\delta_{ij} \mathrm{d}x^i \mathrm{d}x^j = a^2(\eta)\left\{-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right\}. \tag{A.2}$$

The solution to $\Box\phi = \mathrm{d}V/\mathrm{d}\phi$ is denoted $\overline{\phi}(\eta, z)$. We let indices $a, b, c = 1, 2$ and $i, j, k, l, \ldots = 1, 2, 3$. Now we add a linear perturbation $\zeta(\eta, x^a)$ to the wall such that

$$\phi(\eta, \boldsymbol{x}) = \overline{\phi}(\eta, z; \zeta(\eta, x^a)) = \overline{\phi}(\eta, z; 0) + \zeta(\eta, x^a) \frac{\partial \overline{\phi}}{\partial z}\bigg|_{\zeta=0} + \mathcal{O}(\zeta^2). \tag{A.3}$$

⌜Remember eqs for $\zeta$!⌟ Furthermore, Fourier transforming [←show this!]■ the spatial components gives

$$\phi(\eta, \boldsymbol{k}) = \int d^3x \, \mathrm{e}^{\mathrm{i}k_i x^i} \phi(\eta, \boldsymbol{x}) = \left[(2\pi)^2 \delta^{(2)} k_a - \mathrm{i}k_3 \zeta(\eta, k_a)\right] \overline{\phi}(\eta, k_3; 0) + O\left(\zeta^2\right). \tag{A.4}$$

The TT-part of the energy-momentum tensor is [←refer to some section]■ ⌜NB: $g$ cannot have cross terms!!⌟

$$T_{ij}^{\mathrm{TT}}(\eta, \boldsymbol{k}) = \Lambda_{ij.kl}(\hat{\boldsymbol{k}}) \int \frac{d^3p}{(2\pi)^3} \, p_k p_l \phi(\eta, \boldsymbol{p}) \phi(\eta, \boldsymbol{k} - \boldsymbol{p}). \tag{A.5}$$

We define a quantity $t_{kl}$ by

$$T_{ij}^{\mathrm{TT}}(\eta, \boldsymbol{k}) = \Lambda_{ij.kl}(\hat{\boldsymbol{k}}) \left( \frac{1}{2\pi} \cdot t_{kl}(\eta, \boldsymbol{k}) + \mathcal{O}(\zeta^2) \right), \tag{A.6}$$

and the additional function

$$\mathfrak{I}_n(\eta, q_0) = \int_{\mathbb{R}} \mathrm{d}q \, q^n \overline{\phi}(\eta, q; 0) \overline{\phi}(\eta, q_0 - q; 0). \tag{A.7}$$

After some manipulation [←show this!]■, we get the following:

$$t_{ab}(\eta, \boldsymbol{k}) = k_a k_b [-\mathrm{i}\zeta(\eta, k_c)] \mathfrak{I}_1(\eta, k_3) \tag{A.8a}$$
$$t_{a3}(\eta, \boldsymbol{k}) = k_a [-\mathrm{i}\zeta(\eta, k_c)] \mathfrak{I}_2(\eta, k_3) \tag{A.8b}$$
$$t_{33}(\eta, \boldsymbol{k}) = k_3 [-\mathrm{i}\zeta(\eta, k_c)] \mathfrak{I}_2(\eta, k_3) + (2\pi)^2 \delta^{(2)} k_a \mathfrak{I}_2(\eta, k_3) \tag{A.8c}$$

⌜There are some *small* constraint on the perturbation from this. Need to be commented!⌟

Gravitational waves sourced by this field is – to first order in $\zeta$ – given by

$$\begin{aligned} ah_{ij}(\eta,\boldsymbol{k}) &= \frac{16\pi G_\mathrm{N}}{k}\int_{\eta_\mathrm{i}}^{\eta}\mathrm{d}\eta'\;\sin\left(k[\eta-\eta']\right)a(\eta')T^{\mathrm{TT}}_{ij}(\eta',\boldsymbol{k}) \\ &= \frac{8G_\mathrm{N}}{k}\Lambda_{ij.kl}(\hat{\boldsymbol{k}})\int_{\eta_\mathrm{i}}^{\eta}\mathrm{d}\eta'\;\sin\left(k[\eta-\eta']\right)a(\eta')t_{kl}(\eta',\boldsymbol{k}) + \mathscr{O}(\zeta^2). \end{aligned} \tag{A.9}$$

Remaining are the $\Lambda_{ij.kl}t_{kl}$-elements, which in total are ⌈6⌋? terms per $ij$, due to symmetry in $t_{kl}$:

$$\begin{aligned} \Lambda_{ij.kl}(\hat{\boldsymbol{k}})t_{kl}(\eta,\boldsymbol{k}) = \Big\{&\left(\Lambda_{ij.12}+\Lambda_{ij.21}\right)t_{12} + \left(\Lambda_{ij.13}+\Lambda_{ij.31}\right)t_{13} + \left(\Lambda_{ij.23}+\Lambda_{ij.32}\right)t_{23}\Big\}(\eta,k\hat{\boldsymbol{k}}) \\ &+ \Big\{\Lambda_{ij.11}t_{11}+\Lambda_{ij.22}t_{22}+\Lambda_{ij.33}t_{33}\Big\}(\eta,k\hat{\boldsymbol{k}}) \end{aligned} \tag{A.10}$$

All of these are on the form

$$-\mathrm{i}\zeta(\eta,k_a)\times\left\{k^2k^2k^2\Im_1(\eta,k_3)A_{ij}(\hat{\boldsymbol{k}}) + k\Im_2(\eta,k_3)B_{ij}(\hat{\boldsymbol{k}})\right\}, \tag{A.11}$$

leaving

$$ah_{ij}(\eta,\boldsymbol{k}) = 8G_\mathrm{N}\Big[kA_{ij}(\hat{\boldsymbol{k}})\mathcal{I}_1(\eta,\boldsymbol{k};\eta_\mathrm{i}) + B_{ij}(\hat{\boldsymbol{k}})\mathcal{I}_2(\eta,\boldsymbol{k};\eta_\mathrm{i})\Big] \tag{A.12}$$

where

$$\mathcal{I}_n(\eta,\boldsymbol{k};\eta_\mathrm{i}) = -\mathrm{i}\int_{\eta_\mathrm{i}}^{\eta}\mathrm{d}\eta'\,a(\eta')\sin\left(k(\eta-\eta')\right)\times\zeta(\eta',k_a)\Im_n(\eta',k_3). \tag{A.13}$$

Furthermore, we can show [←proof!]■ that $A_{ij}(\boldsymbol{n}) = -n_3B_{ij}(\boldsymbol{n}) \equiv +2n_3C_{ij}(\boldsymbol{n})$ for $|\boldsymbol{n}|^2 = n_1^2+n_2^2+n_3^2 = 1$, allowing for the slightly simpler expression

$$ah_{ij}(\eta,\boldsymbol{k}) = 4G_\mathrm{N}C_{ij}(\hat{\boldsymbol{k}})\Big[k_3\mathcal{I}_1(\eta,\boldsymbol{k};\eta_\mathrm{i}) - \mathcal{I}_2(\eta,\boldsymbol{k};\eta_\mathrm{i})\Big], \tag{A.14}$$

where⌈:

$$\begin{aligned} C_{ab}(\boldsymbol{n}) &= n_3\Big[n_an_b\left(n_3^2+1\right) - \delta_{ab}\left(1-n_3^2\right)\Big] \\ C_{a3}(\boldsymbol{n}) &= -n_an_3^2\left(1-n_3^2\right) \\ C_{33}(\boldsymbol{n}) &= n_3^2\left(1-n_3^2\right)^2 \end{aligned} \tag{A.15}$$

⌋?

Redshift $\mathfrak{z}_* = 2 \therefore a(\eta_\mathrm{i}) = (1+\mathfrak{z}_*)^{-1} = 1/3$
$ds^2 = a^2(\eta)\left(\delta_{\mu\nu}+h_{\mu\nu}\right)\mathrm{d}x^\mu\mathrm{d}x^\nu,\, x^0 = \eta$
$u_\mathfrak{a}x^\mathfrak{a},\, \mathfrak{a} = 0,1,2$
$u_\iota x^\iota,\, \iota = 0,1,2\flat\flat$

**Important references:** ((Vachaspati, 2006, p. 145)), ((Vilenkin, 1985, p. 291)), ((Vilenkin and Shellard, 1994, p. 375))