UNIVERSITY
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Master's thesis

Gravitational waves from topological defects

Any short subtitle

Nanna Bryne

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Institute of Theoretical Astrophysics, Department of Physics
Faculty of Mathematics and Natural Sciences

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Supervisor:
David Fonseca Mota

Contents

Notation

Constants and units. We use ⌈'natural units'⌋? where $\hbar = c = 1$, where $\hbar$ is the reduced Planck constant and c is the speed of light in vacuum.⌜Planck units? Set $k_B = G_N = 1$?⌟ The Newtonian constant of gravitation G_N is referenced explicitly, and we use Planck units such as the Planck mass $M_{Pl} = (\hbar c/G_N)^{1/2} = G_N^{-1/2} \sim 10^{-8}$ kg.

Tensors. The metric signature $(-,+,+,+)$ is considered, i.e. $\det[g_{\mu\nu}] \equiv |g| < 0$. The Minkowski metric is denoted $\eta_{\mu\nu}$, whereas a general metric is denoted $g_{\mu\nu}$. A four-vector $p^\mu =$

$[\eta_{\mu\nu}] = \mathrm{diag}(-1, 1, 1, 1)$

Christophel symbols:

$$\Gamma^\rho_{\mu\nu} = \ldots \tag{1}$$

"Lambda tensor":

$$\Lambda_{ij,kl} = \ldots \tag{2}$$

Fourier transforms. We use the following convention for the Fourier transform of $f(x)$, $\tilde{f}(k)$, and its inverse, where x and k are Lorentz four-vectors:

$$\begin{aligned} f(x) &= \int \frac{d^4k}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}k\cdot x} \tilde{f}(k) \\ \tilde{f}(k) &= \int d^4x\, \mathrm{e}^{\mathrm{i}k\cdot x} f(x) \end{aligned} \tag{3}$$

Here, $k \cdot x = k_\sigma x^\sigma = g_{\rho\sigma} k^\rho x^\sigma$.

Acronyms

CDM	cold dark matter
CMB	cosmic microwave background (radiation)
DW	domain wall
GR	general relativity
GW	gravitational wave
ΛCDM	Lambda (Greek Λ) cold dark matter model; Standard cosmological model

Nomenclature

In the table below is listed the most frequently used symbols in this paper, for reference.

Table 1: helo

Symbol	Referent	SI-value or definition
Natural constants		
G_{N}	Newtonian constant of gravitation	1.2 kg
k_{B}	Boltzmann's constant	1.2 K
Fiducial quantities		
h_0	Reduced Hubble constant	0.67
Subscripts		
Q_{gw}	Quantity Q related to gravitational wave	
Functions and operators		
$\Theta(\xi)$	Heaviside step function	$\begin{cases} 1 & \xi > 0 \\ 0 & \xi < 0 \end{cases}$
$\mathrm{sgn}(\xi)$	Signum function	$2\Theta(\xi) - 1$
$\delta^{(n)}(\xi)$	Dirac-Delta function of $\xi \in \mathbb{R}^n$, $n \in \mathbb{N}$.	
$\delta^{\mu\nu}$	Kronecker delta.	

Chapter 1

Introduction

- GOALS:
  - Gather framework about GWs from DWs

$$\tilde{h}''_{\circledast} + 2\mathcal{H}\tilde{h}'_{\circledast} + k^2\tilde{h}_{\circledast} = 16\pi G_{\mathrm{N}} a^2 \tilde{\sigma}_{\circledast}; \quad \circledast = +, \times \tag{1.1}$$

$$\left(\tilde{h}^{\mathrm{TT}}\right)_{ij}(\eta, \boldsymbol{k}) = \sum_{\circledast=+,\times} e^{\circledast}_{ij}(\hat{\boldsymbol{k}})\tilde{h}_{\circledast}(\eta, \boldsymbol{k}) \tag{1.2}$$

$$\tilde{h}^{\mathrm{TT}}{}_{ij}(\eta, \boldsymbol{k}) = \sum_{\circledast=+,\times} e^{\circledast}_{ij}(\hat{\boldsymbol{k}})\tilde{h}_{\circledast}(\eta, \boldsymbol{k}) \tag{1.3}$$

1.1 Preliminaries

- variational calculus/ varying action
- action
- pert. theory?
- line element
- gauge invariance
- FRW cosmology
- classical field theory

$$G_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu} \tag{1.4}$$

The action

$$S_{\mathrm{ST}} = S_{\mathrm{EH}} + S_{\phi} + S_{\mathrm{m}} = \int d^4x\, \sqrt{-|g|} \left\{ \frac{M_{\mathrm{Pl}}^2}{2} R - \frac{1}{2}\nabla_{\rho}\phi\nabla^{\rho}\phi - V(\phi) \right\} + S_{\mathrm{m}} \tag{1.5}$$

GR as we know it is reconstructed when varying S_{E} with respect to the metric $g_{\mu\nu}(x)$

Part I

Background

Chapter 2

Classical Field Theory and Gravity

The action and blah blah

2.1 General Relativity

The Einstein–Hilbert action ⌈in vacuum⌋? is ⌜check Planck mass def.⌟

$$S_{\mathrm{EH}} = {}^1\!/\!_2 M_{\mathrm{Pl}}^2 \int d^4x\, \mathcal{R}, \tag{2.1}$$

where $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$. By varying S_{EH} with respect to $g_{\mu\nu}$ one obtains the equation of motion

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} + \frac{1}{2} g_{\mu\nu}\mathcal{R} = 0. \tag{2.2}$$

Thus, we interpret GR *classical* field theory where the tensor field $g_{\mu\nu}$ is the gravitational field, ⌈with the particle realisation? z? named "graviton"⌋↻.

- scalar field (ST theories)

2.2 Scalar-Tensor Theories?

2.3 Perturbation Theory

2.4 Classical Solitons

Chapter 3

Gravitational Waves

Chapter 4

Lattice- and N-body simulations

Part II

Project

Chapter 5

Calculating Gravitational Waves from Domain Walls

DRAFT

Consider a planar domain wall in the xy-plane in a flat FRW universe, represented by a scalar field $\phi(\eta, \boldsymbol{x})$ and a potential $V(\phi)$. The background metric is

$$d\bar{s}^2 = \bar{g}_{\mu\nu}\mathrm{d}\bar{x}^\mu \bar{x}^\nu = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j = a^2(\eta)\left\{-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right\}. \tag{5.1}$$

The solution to $\bar{\Box}\phi = \mathrm{d}V/\mathrm{d}\phi$ is denoted $\bar{\phi}(\eta, z)$. We let indices $a, b, c = 1, 2$ and $i, j, k, l, \ldots = 1, 2, 3$. Now we add a linear perturbation $\zeta(\eta, x^a)$ to the wall such that

$$\phi(\eta, \boldsymbol{x}) = \bar{\phi}(\eta, z; \zeta(\eta, x^a)) = \bar{\phi}(\eta, z; 0) + \zeta(\eta, x^a)\frac{\partial\bar{\phi}}{\partial z}\bigg|_{\zeta=0} + O(\zeta^2). \tag{5.2}$$

Furthermore, Fourier transforming [←show this!]■ the spatial components gives

$$\phi(\eta, \boldsymbol{k}) = \int d^3x\, \mathrm{e}^{\mathrm{i}k_i x^i}\phi(\eta, \boldsymbol{x}) = \left[(2\pi)^2\delta^{(2)}(k_a) - \mathrm{i}k_3\zeta(\eta, k_a)\right]\bar{\phi}(\eta, k_3; 0) + O\!\left(\zeta^2\right). \tag{5.3}$$

The TT-part of the energy-momentum tensor is [←refer to some section]■

$$T^{\mathrm{TT}}_{ij}(\eta, \boldsymbol{k}) = \Lambda_{ij,kl}(\hat{\boldsymbol{k}})\int \frac{d^3p}{(2\pi)^3}p_k p_l \phi(\eta, \boldsymbol{p})\phi(\eta, \boldsymbol{k}-\boldsymbol{p}). \tag{5.4}$$

We define a quantity t_{kl} by

$$T^{\mathrm{TT}}_{ij}(\eta, \boldsymbol{k}) = \Lambda_{ij,kl}(\hat{\boldsymbol{k}})\left(\frac{1}{2\pi}\cdot t_{kl}(\eta, \boldsymbol{k}) + O(\zeta^2)\right), \tag{5.5}$$

and the additional function

$$\mathfrak{I}_n(\eta, q_0) = \int_{\mathbb{R}} \mathrm{d}q\, q^n\bar{\phi}(\eta, q; 0)\bar{\phi}(\eta, q_0 - q; 0). \tag{5.6}$$

After some manipulation [←show this!]■, we get the following:

$$t_{ab}(\eta, \boldsymbol{k}) = k_a k_b\left[-\mathrm{i}\zeta(\eta, k_c)\right]\mathfrak{I}_1(\eta, k_3) \tag{5.7a}$$
$$t_{a3}(\eta, \boldsymbol{k}) = k_a\left[-\mathrm{i}\zeta(\eta, k_c)\right]\mathfrak{I}_2(\eta, k_3) \tag{5.7b}$$
$$t_{33}(\eta, \boldsymbol{k}) = k_3\left[-\mathrm{i}\zeta(\eta, k_c)\right]\mathfrak{I}_2(\eta, k_3) + (2\pi)^2\delta^{(2)}(k_a)\mathfrak{I}_2(\eta, k_3) \tag{5.7c}$$

There are some *small* constraint on the perturbation from this. Need to be commented!

Gravitational waves sourced by this field is – to first order in ζ – given by

$$\begin{aligned} ah_{ij}(\eta, \boldsymbol{k}) &= \frac{16\pi G_\mathrm{N}}{k}\int_{\eta_\mathrm{i}}^{\eta}\mathrm{d}\eta'\ \sin\left(k\left[\eta-\eta'\right]\right)a(\eta')T^{\mathrm{TT}}_{ij}(\eta', \boldsymbol{k}) \\ &= \frac{8G_\mathrm{N}}{k}\Lambda_{ij,kl}(\hat{\boldsymbol{k}})\int_{\eta_\mathrm{i}}^{\eta}\mathrm{d}\eta'\ \sin\left(k\left[\eta-\eta'\right]\right)a(\eta')t_{kl}(\eta', \boldsymbol{k}) + O(\zeta^2). \end{aligned} \tag{5.8}$$

Remaining is the $\Lambda_{ij,kl}t_{kl}$-elements, which in total are ⌈6⌋? terms per ij, due to symmetry in t_{kl}:

$$\begin{aligned}\Lambda_{ij,kl}(\hat{\boldsymbol{k}})t_{kl}(\eta,\boldsymbol{k}) &= 2\left\{\Lambda_{ij,12}t_{12}+\Lambda_{ij,13}t_{13}+\Lambda_{ij,23}t_{23}\right\}(\eta,k\hat{\boldsymbol{k}}) \\ &\quad+\left\{\Lambda_{ij,11}t_{11}+\Lambda_{ij,22}t_{22}+\Lambda_{ij,33}t_{33}\right\}(\eta,k\hat{\boldsymbol{k}})\end{aligned} \tag{5.9}$$

5.1 General Formalism

Chapter 6

Simulating Gravitational Waves from Domain Walls

Chapter 7

Studying Gravitational Waves from Domain Walls

Chapter 8

Discussion

Chapter 9

Conclusion and Outlook

Bibliography

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Appendix A

I do not have an appendix