# 1 Thin Wall Approximation

24th November 2023

## 1.1 Dynamics of Domain Walls in The Thin Wall Approximation

Preceding Julian's notes ('Dynamics of Domain Walls in the Thin Wall approximation'). I could not get a symmetric stress-energy tensor from equations (18) and (4) in said notes. I then calculated the determinant ($g^{(3)}$) for myself, and by using that expression (Eq. (1.3) and Eq. (1.4) below) the functional derivative $\delta g^{(3)}/\delta g_{\rho\sigma}$ becomes symmetric, hence $T^{\rho\sigma}$ is symmetric.

**Covariant action.** Consider symmetron potential, thin wall limit. Surface tension is

$$\sigma \simeq \int_{\phi_-}^{\phi_+} \mathrm{d}\phi \ \sqrt{2V_{\mathrm{eff}}(\phi) - 2V_{\mathrm{eff}}(\phi_\pm)}, \tag{1.1}$$

where $\phi_\pm = \phi(z \to \pm\infty)$. We write the covariant action as $S_{\mathrm{dw}} = -\sigma \int d^3\xi \sqrt{-g^{(3)}}$. The induced metric is

$$g^{(3)}_{AB} = g_{\mu\nu} \frac{\mathrm{d}x_{\mathrm{dw}}{}^{\mu}}{\mathrm{d}\xi^A} \frac{\mathrm{d}x_{\mathrm{dw}}{}^{\nu}}{\mathrm{d}\xi^B}; \quad A, B = 0, 1, 2, \tag{1.2}$$

where $x_{\mathrm{dw}}{}^{\mu}(\xi^A)$ is the embedding function. The determinant of the world volume metric is

$$g^{(3)} = \tilde{\varepsilon}_{ABC} g^{(3)}_{0A} g^{(3)}_{1B} g^{(3)}_{2C} = g_{\mu\nu} g_{\kappa\lambda} g_{\alpha\beta} \mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta}, \tag{1.3}$$

where $\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} = \tilde{\varepsilon}_{ABC} \Delta^{\mu}{}_{0} \Delta^{\nu}{}_{A} \Delta^{\kappa}{}_{1} \Delta^{\lambda}{}_{B} \Delta^{\alpha}{}_{2} \Delta^{\beta}{}_{C}$; $\Delta^{\mu}{}_{A} \equiv \mathrm{d}x_{\mathrm{dw}}{}^{\mu} \big/ \mathrm{d}\xi^A$. In particular,

$$\begin{aligned}\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} = {} & \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{0}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{1}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{2} + \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{1}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{2}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{0} + \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{2}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{0}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{1} \\ & - \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{0}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{2}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{1} - \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{1}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{0}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{2} - \Delta^{\mu}{}_{0}\Delta^{\nu}{}_{2}\Delta^{\kappa}{}_{1}\Delta^{\lambda}{}_{1}\Delta^{\alpha}{}_{2}\Delta^{\beta}{}_{0}.\end{aligned} \tag{1.4}$$

**Stress-energy tensor.** We consider a planar wall lying in the $xy$-plane with a small perturbation in the $z$-direction. The stress-energy tensor is given by

$$T^{\rho\sigma} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g_{\rho\sigma}} = \frac{\sigma \delta(z - z_{\mathrm{dw}})}{\sqrt{-g}\sqrt{-g^{(3)}}} \frac{\delta g^{(3)}}{\delta g_{\rho\sigma}}. \tag{1.5}$$

We need the functional derivative of $g^{(3)}$ and the quantity $\sigma\delta(z - z_{\mathrm{dw}})$.

### 1.1.1 My Calculation

We vary $g^{(3)}$ w.r.t. $g_{\rho\sigma}$, ignoring $\mathcal{O}((\delta g_{\rho\sigma})^2)$-terms:

$$\begin{aligned}
g^{(3)} + \delta g^{(3)} &= \left(g_{\mu\nu} + \delta g_{\mu\nu}\right)\left(g_{\kappa\lambda} + \delta g_{\kappa\lambda}\right)\left(g_{\alpha\beta} + \delta g_{\alpha\beta}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} \\
&= g^{(3)} + \left(\delta g_{\mu\nu} g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu}\delta g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu} g_{\kappa\lambda}\delta g_{\alpha\beta}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} \\
&= g^{(3)} + \left(\frac{\partial g_{\mu\nu}}{\partial g_{\rho\sigma}}\delta g_{\rho\sigma} g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu}\frac{\partial g_{\kappa\lambda}}{\partial g_{\rho\sigma}}\delta g_{\rho\sigma} g_{\alpha\beta} + g_{\mu\nu} g_{\kappa\lambda}\frac{\partial g_{\alpha\beta}}{\partial g_{\rho\sigma}}\delta g_{\rho\sigma}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta} \\
&= g^{(3)} + \left(\delta^{\rho}{}_{\mu}\delta^{\sigma}{}_{\nu} g_{\kappa\lambda} g_{\alpha\beta} + g_{\mu\nu}\delta^{\rho}{}_{\kappa}\delta^{\sigma}{}_{\lambda} g_{\alpha\beta} + g_{\mu\nu} g_{\kappa\lambda}\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}\right)\mathbf{Q}^{\mu\nu\kappa\lambda\alpha\beta}\cdot\delta g_{\rho\sigma} \\
&= g^{(3)} + \left(g_{\kappa\lambda} g_{\alpha\beta}\mathbf{Q}^{\rho\sigma\kappa\lambda\alpha\beta} + g_{\mu\nu} g_{\alpha\beta}\mathbf{Q}^{\mu\nu\rho\sigma\alpha\beta} + + g_{\mu\nu} g_{\kappa\lambda}\mathbf{Q}^{\mu\nu\kappa\lambda\rho\sigma}\right)\cdot\delta g_{\rho\sigma}
\end{aligned} \tag{1.6}$$

Thus,

$$\frac{\delta g^{(3)}}{\delta g_{\rho\sigma}} = g_{\kappa\lambda} g_{\alpha\beta}\left\{\mathbf{Q}^{\rho\sigma\kappa\lambda\alpha\beta} + \mathbf{Q}^{\alpha\beta\rho\sigma\kappa\lambda} + \mathbf{Q}^{\kappa\lambda\alpha\beta\rho\sigma}\right\}. \tag{1.7}$$

**Flat FRW universe.** With $g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ and ${x_{\mathrm{dw}}}^A = \xi^A, {x_{\mathrm{dw}}}^3 = z_{\mathrm{dw}} = \epsilon(\xi^A)$, we may insert

$$\Delta^{\mu}{}_{A} = \begin{cases} \delta^{\mu}{}_{A}, & \mu \neq 3 \\ \partial\epsilon\big/\partial\xi^A\,, & \mu = 3 \end{cases} \tag{1.8}$$

into Eq. (1.4) to compute $g^{(3)}$ and $\delta g^{(3)}/\delta g_{\rho\sigma}$. The result of the latter is a symmetric tensor of type (2,0);

$$\left[\frac{\delta g^{(3)}}{\delta g_{\rho\sigma}}\right] = \\
\begin{pmatrix}
\left({\iota_1}^2 + {\iota_2}^2 + 1\right)a^4 & -\iota_0\iota_1 a^4 & -\iota_0\iota_2 a^4 & \iota_0 a^4 \\
-\iota_0\iota_1 a^4 & {\iota_0}^2 a^4 - \left({\iota_2}^2 + 1\right)a^2 & \iota_1\iota_2 a^2 & -\iota_1 a^2 \\
-\iota_0\iota_2 a^4 & \iota_1\iota_2 a^2 & {\iota_0}^2 a^4 - \left({\iota_1}^2 + 1\right)a^2 & -\iota_2 a^2 \\
\iota_0 a^4 & -\iota_1 a^2 & -\iota_2 a^2 & {\iota_0}^2 a^4 - \left({\iota_1}^2 + {\iota_2}^2\right)a^2
\end{pmatrix}, \tag{1.9}$$

where we defined $\iota_A \equiv \partial\epsilon\big/\partial\xi^A$.

**Symmetron potential.** We let $V_{\mathrm{eff}}(\phi) = V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2$. As such, $\phi_\pm = \pm\phi_0$ and $V(\phi_\pm) = 0$. Now,

$$\begin{aligned}
\sigma\delta(z - z_{\mathrm{dw}}) &= \int_{-\infty}^{\infty}\mathrm{d}z\,\frac{\mathrm{d}\phi}{\mathrm{d}z}\sqrt{2V_{\mathrm{eff}}(\phi(z)) - 2V_{\mathrm{eff}}(\phi_\pm)}\,\delta(z - z_{\mathrm{dw}}) \\
&= \sqrt{2}\int_{-\infty}^{\infty}\mathrm{d}z\,\frac{\mathrm{d}\phi}{\mathrm{d}z}\sqrt{V_{\mathrm{eff}}(\phi(z))}\,\delta(z - z_{\mathrm{dw}}) \\
&= \sqrt{2}\sqrt{V_{\mathrm{eff}}(\phi(z_{\mathrm{dw}}))}\left.\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|_{z=z_{\mathrm{dw}}} \\
&= \sqrt{\frac{\lambda}{2}}\left(\phi(z_{\mathrm{dw}})^2 - \phi_0^2\right)\left.\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|_{z=z_{\mathrm{dw}}}.
\end{aligned} \tag{1.10}$$

**Have I completely misunderstood something here?**

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## 1.2 Dynamics of Domain Walls in The Thin Wall Approximation, cont.

It is safe to assume that the wall thickness is much smaller than the horizon. The adiabatically static solution to the e.o.m. for $\phi$ is $\phi(t,z) = \phi_0 \tanh\left\{\frac{a(t)}{w_0} z\right\}$, where $w_0 = \phi_0^{-1}\sqrt{2/\lambda}$ is the wall thickness ((Press, Ryden, and Spergel, 1989)). Dismissing Eq. (1.10), we find the following:

$$\begin{aligned}\sigma\delta(z-z_{\mathrm{dw}}) &= \int_{-\infty}^{\infty} \mathrm{d}z' \,\frac{\mathrm{d}\phi}{\mathrm{d}z'}\sqrt{2V_{\mathrm{eff}}(\phi)-2V_{\mathrm{eff}}(\phi_\pm)} \times \delta(z-z_{\mathrm{dw}}) \\ &= \delta(z-z_{\mathrm{dw}})\sqrt{\frac{\lambda}{2}}\int_{\mathbb{R}} \mathrm{d}z' \,\frac{\mathrm{d}\phi}{\mathrm{d}z'}\left(\phi^2-\phi_0^2\right) \\ &= \delta(z-z_{\mathrm{dw}})\sqrt{\frac{\lambda}{2}}\frac{a\phi_0^3}{w_0}\int_{\mathbb{R}} \mathrm{d}z' \,\cosh^{-2}\left\{\frac{az'}{w_0}\right\}\left(\tanh^2\left\{\frac{az'}{w_0}\right\}-1\right) \\ &= \delta(z-z_{\mathrm{dw}})\sqrt{\frac{\lambda}{2}}\frac{a\phi_0^3}{w_0}\int_{\mathbb{R}} \mathrm{d}z' \,\cosh^{-4}\left\{\frac{az'}{w_0}\right\}\cdot(-1) \\ &= \ldots \\ &= (-1)\times\frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3 \times \delta(z-z_{\mathrm{dw}})\end{aligned} \tag{1.11}$$

⌜There might be a sign error somewhere as I believe the tension should be positive.⌟ Gathering it all, we have

$$T^{\mu\nu} = (\pm)\,\delta(z-\epsilon)\cdot\frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3\cdot a^{-6}\left[\left(\frac{\partial\epsilon}{\partial\xi^0}\right)^2 - a^{-2}\left(\left(\frac{\partial\epsilon}{\partial\xi^1}\right)^2+\left(\frac{\partial\epsilon}{\partial\xi^2}\right)^2+1\right)\right]^{-1/2}\cdot\frac{\delta g^{(3)}}{\delta g_{\mu\nu}}, \tag{1.12}$$

where the last factor is found from Eq. (1.9).

**Perturbation.** The small perturbation $\epsilon$ obeys

$$\ddot{\epsilon} + 4\frac{\dot{a}}{a}\dot{\epsilon} - \frac{1}{a^2}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\epsilon = 0; \tag{1.13}$$

a linear PDE with solutions of the form $\epsilon(\xi^A) = \tilde{\epsilon}(\xi^0)e^{i(u_1\xi^1+u_2\xi^2)}$, where $\tilde{\epsilon}$ satisfies

$$\ddot{\tilde{\epsilon}} + 4\frac{\dot{a}}{a}\dot{\tilde{\epsilon}} + \frac{u^2}{a^2}\tilde{\epsilon} = 0;\ u^2 = u_1^2+u_2^2. \tag{1.14}$$

**Do you have any suggestions as to how to solve this?**

We can rewrite Eq. (1.12):

$$\begin{aligned}T^{\mu\nu} &= (\pm)\,\delta(z-\epsilon)\cdot\frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3\cdot a^{-5}\left[a^2\dot{\epsilon}^2+u^2\epsilon-1\right]^{-1/2}\cdot\frac{\delta g^{(3)}}{\delta g_{\mu\nu}} \\ &= (\pm)\,\delta(z-\epsilon)\cdot\frac{4}{3}\sqrt{\frac{\lambda}{2}}\phi_0^3\cdot a^{-5}\left[a^2\dot{\tilde{\epsilon}}^2e^{2i(u_1x+u_2y)}+u^2\tilde{\epsilon}e^{i(u_1x+u_2y)}-1\right]^{-1/2}\cdot\frac{\delta g^{(3)}}{\delta g_{\mu\nu}}\end{aligned} \tag{1.15}$$

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## 1.3 Perturbation: Solving the e.o.m.

We have

$$T^{\mu\nu} = \frac{(\pm)2\,\sigma\delta(z-\epsilon)}{a^5\sqrt{a^2\dot{\epsilon}^2+u^2\epsilon^2-1}}\frac{\delta g^{(3)}}{\delta g_{\mu\nu}}. \tag{1.16}$$

Eq. (1.14) can be solved analytically for $a \propto t^\beta$. We let $\eta$ denote conformal time s.t. $\mathrm{d}\eta = a^{-1}\mathrm{d}t$ and $a \propto \eta^\alpha$, where $\alpha = \beta/(1-\beta)$. We let $\epsilon(t,x,y) = \epsilon(t)f(x,y)$, and ignore the tilde from before. For a matter dominated universe, $\beta = 2/3$ and $\alpha = 2$.

We make use of the transformed Bessel's equation of the form

$$x^2y'' + (1-2\mathfrak{a})\,xy' + \left(\mathfrak{b}^2\mathfrak{c}^2x^{2\mathfrak{c}} + \mathfrak{a}^2 - \ell^2\mathfrak{c}^2\right)y = 0; \quad \mathfrak{a},\mathfrak{b},\mathfrak{c},\ell \in \mathbb{C}, \tag{1.17}$$

whose general solution is $y(x) = x^{\mathfrak{a}}\left\{c_1\mathcal{J}_\ell(\mathfrak{b}x^{\mathfrak{c}}) + c_2\mathcal{N}_\ell(\mathfrak{b}x^{\mathfrak{c}})\right\}$ ((see Bowman, 1958, p.117–118)). The properties of the Bessel functions of the first and second kind gives

$$y(x) = \begin{cases} x^{\mathfrak{a}}\left\{c_1\mathcal{J}_\ell(\mathfrak{b}x^{\mathfrak{c}}) + c_2\mathcal{J}_{-\ell}(\mathfrak{b}x^{\mathfrak{c}})\right\}, & \ell \notin \mathbb{Z} \\ x^{\mathfrak{a}}\left\{c_1\mathcal{J}_\ell(\mathfrak{b}x^{\mathfrak{c}}) + c_2\mathcal{N}_\ell(\mathfrak{b}x^{\mathfrak{c}})\right\}, & \ell \in \mathbb{Z} \end{cases}. \tag{1.18}$$

### 1.3.1 Conformal Time Frame

For $\epsilon(t) \to \epsilon(\eta)$, Eq. (1.14) reads

$$\eta^2\epsilon'' + 3\alpha\eta\epsilon' + u^2\eta^2\epsilon = 0, \tag{1.19}$$

where primed means conformal time derivative. The solution to this equation is

$$\epsilon(\eta) = \eta^\ell\left\{c_1\mathcal{J}_\ell(u\eta) + c_2\mathcal{N}_\ell(u\eta)\right\}; \quad \ell = \frac{1-3\alpha}{2}. \tag{1.20}$$

**Matter domination.** With $\alpha = 2$, $\ell = -5/2$ and thus

$$\epsilon(\eta) = c\cdot\frac{\mathcal{J}_{5/2}(u\eta)}{\eta^{5/2}} \tag{1.21}$$

is a solution.

### 1.3.2 Cosmic Time Frame

When $a(t) = Kt^\beta$, Eq. (1.14) is simply

$$t^2\ddot{\epsilon} + 4\beta t\dot{\epsilon} + (u/K)^2t^{2(1-\beta)}\epsilon = 0, \tag{1.22}$$

with solution

$$\epsilon(t) = t^{\ell\gamma}\left\{c_1\mathcal{J}_\ell\left(\frac{u}{K\gamma}t^\gamma\right) + c_2\mathcal{N}_\ell\left(\frac{u}{K\gamma}t^\gamma\right)\right\}; \quad \ell = \frac{1-4\beta}{2\gamma}, \gamma = 1-\beta. \tag{1.23}$$

**Matter domination.** With $\beta = 2/3$, $\ell = -5/2$ and a solution is

$$\epsilon(t) = c\cdot\frac{\mathcal{J}_{5/2}\left(\frac{3u}{K}t^{1/3}\right)}{t^{5/6}}. \tag{1.24}$$

### 1.3.3 Stress-Energy Tensor

We once again turn our attention to the stress-energy tensor, writing it out for $\epsilon(t,x,y) = \epsilon(t)f(x,y)$, where $f(x,y) = \mathrm{e}^{\mathrm{i}(u_1 x + u_2 y)}$:

$$\begin{aligned} T^{\mu\nu} &= (\pm)\frac{2\,\sigma\delta(z - f\epsilon)}{a^5\sqrt{a^2 f^2\dot{\epsilon}^2 + u^2 f^2\epsilon^2 - 1}}\frac{\delta g^{(3)}}{\delta g_{\mu\nu}} \\ &= (\pm)\frac{2\sigma\delta(z - f\epsilon)}{a^5\sqrt{f^2\left[\epsilon'^2 + u^2\epsilon^2\right] - 1}}\frac{\delta g^{(3)}}{\delta g_{\mu\nu}} \end{aligned} \tag{1.25}$$

The square root in the denominator is straight-forwardly computed when using e.g. Eq. (1.21).

## 1.4 Wall Profile

We still need to replace $\sigma\delta(z - \epsilon(t,x,y))$ in the expression for $T^{\mu\nu}$. **I have still not figured out how to work this out.**

# 2 Thin Wall Approximation; Vol. 2

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## 2.1 Dynamics of Domain Walls in The Thin Wall Approximation

We have the spacetime metric $g_{\mu\nu}$ and the induced metric

$$\gamma_{ab} = g_{\mu\nu}\frac{\mathrm{d}x^{\mu}_{\mathrm{dw}}}{\mathrm{d}\xi^{a}}\frac{\mathrm{d}x^{\nu}_{\mathrm{dw}}}{\mathrm{d}\xi^{b}}; \quad \left[x^{\mu}_{\mathrm{dw}}\right] = (\xi^0, \xi^1, \xi^2, \epsilon(\xi^a)), \tag{2.1}$$

where we let $a, b = 0, 1, 2$. Consider $g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$. We define $\iota_a \equiv \partial\epsilon/\partial\xi^a$ and $\iota_3 \equiv -1$ for notational ease. The determinant of the induced metric is

$$\gamma = -a^4[-(a\iota_0)^2 + \underbrace{\iota_1{}^2 + \iota_2{}^2 + \iota_3{}^2}_{\equiv I}]. \tag{2.2}$$

In the thin wall approximation, the surface tension

$$\sigma = \int_{-\infty}^{+\infty}\mathrm{d}z\, T_{00} \simeq - \int_{\phi_-}^{\phi_+}\mathrm{d}\phi\,\sqrt{2V_{\mathrm{eff}}(\phi) - 2V_{\mathrm{eff}}(\phi_{\pm})}. \tag{2.3}$$

The covariant action

$$S_{\mathrm{dw}} = \int d^4x\,\mathcal{L}_{\mathrm{dw}} = -\sigma\int d^3\xi\,\sqrt{-\gamma} = -\sigma\int d^4x\,\sqrt{-\gamma}\,\delta(z - z_{\mathrm{dw}}) \tag{2.4}$$

and thus, the stress–energy tensor

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta\mathcal{L}_{\mathrm{dw}}}{\delta g_{\mu\nu}} = \frac{2\sigma\delta(z-\epsilon)}{\sqrt{-g}\,\sqrt{-\gamma}}\frac{\delta\gamma}{\delta g_{\mu\nu}} = \frac{2\sigma\delta(z-\epsilon)}{a^5\,\sqrt{I-(a\iota_0)^2}}\frac{\delta\gamma}{\delta g_{\mu\nu}}. \tag{2.5}$$

We have calculated the functional derivative $\delta\gamma/\delta g_{\mu\nu}$ before:

$$\frac{\delta\gamma}{\delta g_{00}} = a^4 I \qquad \frac{\delta\gamma}{\delta g_{0i}} = -a^4\iota_0\iota_i \qquad \frac{\delta\gamma}{\delta g_{ij}} = a^2\left[\iota_i\iota_j + \delta_{ij}\left((a\iota_0)^2 - I\right)\right] \tag{2.6}$$

With the ansatz $\epsilon(\xi^a) = \epsilon_t(\xi^0)\mathrm{e}^{\mathrm{i}u_1\xi^1}\mathrm{e}^{\mathrm{i}u_2\xi^2}$, solutions for the equations of motion for $\epsilon_t$ are known for $a \propto t^{\beta}$. In that case, $\iota_0 = \dot{\epsilon}$, $\iota_1 = \mathrm{i}u_1\epsilon$, $\iota_2 = \mathrm{i}u_2\epsilon$ and, of course, $\iota_3 = -1$.

### 2.1.1 The Symmetron Potential

It is easily shown that for $V_{\mathrm{eff}}(\phi) = V_{\mathrm{Sym}}(\phi) \equiv \frac{\lambda}{4}\left(\phi^2 - {\phi_0}^2\right)^2$, the surface tension reduces to $\sigma = \sigma_0 \equiv \frac{4}{3}{\phi_0}^3\sqrt{\lambda/2}$. We consider matter domination ($a \propto t^{2/3} \propto \eta^2$) and assume a solution $\epsilon_\eta = \epsilon_0\eta^{-5/2}\mathcal{J}_{5/2}(u\eta)$, where $\eta$ is conformal time. Note that $\epsilon' = a\dot{\epsilon}$. We have

$$T^{\mu\nu}(\eta,\boldsymbol{x}) = \frac{2\sigma_0\delta(z-\epsilon)}{a^5\sqrt{I-(a\dot{\epsilon})^2}}\frac{\delta\gamma}{\delta g_{\mu\nu}} = \frac{2\sigma_0\delta(z-\epsilon)}{a^5\sqrt{1-(u\epsilon)^2-\epsilon'^2}}\frac{\delta\gamma}{\delta g_{\mu\nu}}. \tag{2.7}$$

Neglecting all $\mathscr{O}(\epsilon^2)$-terms, we get that the only non-vanishing contributions are:

$$\delta\gamma\big/\delta g_{00} = a^4 \qquad \delta\gamma\big/\delta g_{11} = -a^2 \qquad \delta\gamma\big/\delta g_{22} = -a^2 \tag{2.8}$$
$$\delta\gamma\big/\delta g_{03} = a^3\epsilon' \qquad \delta\gamma\big/\delta g_{13} = -\mathrm{i}u_1a^2\epsilon \qquad \delta\gamma\big/\delta g_{23} = -\mathrm{i}u_2a^2\epsilon \tag{2.9}$$

$$[T^{\mu\nu}](\eta,\boldsymbol{k}) = \frac{8\sigma_0\pi^2}{a^3}\begin{pmatrix} a^2A & 0 & 0 & a\epsilon'_\eta B \\ 0 & -A & 0 & -\mathrm{i}u_1\epsilon_\eta B \\ 0 & 0 & -A & -\mathrm{i}u_2\epsilon_\eta B \\ a\epsilon'_\eta B & -\mathrm{i}u_1\epsilon_\eta B & -\mathrm{i}u_2\epsilon_\eta B & 0 \end{pmatrix};$$
$$A = \delta\big(k_1\big)\,\delta\big(k_2\big) + \mathrm{i}k_3\epsilon_\eta B; \quad B = \delta\big(k_1+u_1\big)\,\delta\big(k_2+u_2\big) \tag{2.10}$$

⌜Maybe not the best representation of this ...⌟

**Gravitational Waves.** The transverse, traceless tensor perturbation $h_{ij}$, cropping up in the perturbed line element $ds^2 = a^2\left\{-\mathrm{d}\eta^2 + (\delta_{ij}+h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right\}$, has the e.o.m.

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + 2\frac{a'}{a}\frac{\mathrm{d}}{\mathrm{d}\eta} + k^2\right]h_{ij}(\eta,\boldsymbol{k}) = 16\pi G_{\mathrm{N}}\Lambda_{ij,kl}(\boldsymbol{n})T_{kl}(\eta,\boldsymbol{k}); \quad \boldsymbol{k} = k\boldsymbol{n}, |\boldsymbol{n}| = 1. \tag{2.11}$$

We extracted the transverse, traceless (TT) part of the symmetric stress–energy tensor by use of the “Lambda tensor” $\Lambda_{ij,kl}$.

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