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Master's thesis

Gravitational waves from topological defects

Signatures of late-time first-order phase transitions

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Contents

Notation

Constants and units. We use ⌈'natural units'⌋? where $\hbar = c = 1$, where $\hbar$ is the reduced Planck constant and c is the speed of light in vacuum.⌜Planck units? Set $k_B = G_N = 1$?⌟ The Newtonian constant of gravitation G_N is referenced explicitly, and we use Planck units such as the Planck mass $M_{Pl} = (\hbar c/G_N)^{1/2} = G_N^{-1/2} \sim 10^{-8}$ kg.

Tensors. The metric signature $(-,+,+,+)$ is considered, i.e. $\det[g_{\mu\nu}] \equiv |g| < 0$. The Minkowski metric is denoted $\eta_{\mu\nu}$, whereas a general metric is denoted $g_{\mu\nu}$. A four-vector $p^\mu =$

$[\eta_{\mu\nu}] = \mathrm{diag}(-1, 1, 1, 1)$

$f_{,\mu} \equiv \partial_\mu f = \frac{\partial f}{\partial x^\mu}$

Christophel symbols. The Christophel symbols or "connections" are written

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(g_{\mu\sigma,\nu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}\right) \tag{1}$$

"Lambda tensor." We express the *Lambda tensor*—sometimes called the "projection operator"—that projects onto the TT gauge refer to sec.©0 as

$$\Lambda_{ij.kl}(\boldsymbol{n}) = P_{ik}(\boldsymbol{n})P_{jl}(\boldsymbol{n}) - \frac{1}{2}P_{ij}(\boldsymbol{n})P_{kl}(\boldsymbol{n}); \quad P_{ij}(\boldsymbol{n}) = \delta_{ij} - n_i n_j \tag{2}$$

$\forall\, \boldsymbol{n}$ of unit length; $\boldsymbol{n}^2 = n_1^2 + n_2^2 + n_3^2 = 1$. We use a dot ('.') instead of the more conventional comma (',') to distinguish from the Minkowskian partial derivative.

Fourier transforms. We use the following convention for the Fourier transform of $f(x)$, $\tilde{f}(k)$, and its inverse, where x and k are Lorentz four-vectors:

$$\begin{aligned} f(x) &= \int \frac{d^4k}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}k\cdot x} \tilde{f}(k) \\ \tilde{f}(k) &= \int d^4x\, \mathrm{e}^{\mathrm{i}k\cdot x} f(x) \end{aligned} \tag{3}$$

Here, $k \cdot x = k_\sigma x^\sigma = g_{\rho\sigma} k^\rho x^\sigma$.

Acronyms

CDM	*c*old *d*ark *m*atter
CMB	*c*osmic *m*icrowave *b*ackground (radiation)
DW	*d*omain *w*all
eom	*e*quation *o*f *m*otion
F(L)RW	*F*riedmann–*L*emaître–*R*obertson–*W*alker
GR	*g*eneral *r*elativity
GW	*g*ravitational *w*ave
lhs	*l*eft *h*and *s*ide
rhs	*r*ight *h*and *s*ide
ΛCDM	Lambda (Λ) *c*old *d*ark *m*atter model; standard model of cosmology

Nomenclature

In the table below is listed the most frequently used symbols in this paper, for reference.

Table 1: helo

Symbol	Referent	SI-value or definition
Natural constants		
G_{N}	Newtonian constant of gravitation	1.2 kg
k_{B}	Boltzmann's constant	1.2 K
Fiducial quantities		
h_0	Reduced Hubble constant	0.67
Subscripts		
Q_{gw}	Quantity Q related to gravitational wave	
Functions and operators		
$\Theta\xi$	Heaviside step function	$\begin{cases} 1 & \xi > 0 \\ 0 & \xi < 0 \end{cases}$
$\mathrm{sgn}(\xi)$	Signum function	$2\Theta\xi - 1$
$\delta^{(n)}\xi$	Dirac-Delta function of $\xi \in \mathbb{R}^n$, $n \in \mathbb{N}$.	
$\delta^{\mu\nu}$	Kronecker delta.	
Mathematical groups		
Conf($\mathscr{M}$)	The conformal group of the manifold $\mathscr{M}$.	
hh	...	...

DRAFT

Writing Tools

This is a comment.
This needs spelling check.? Perhaps this?
Rephrase this.
Awkward wording.
Needs double-checking.?
This is in need of citation or reference to a section.© (With a comment.)
blah blah [. . .] (Phantom text.)

PHANTOM PARAGRAPH: THIS IS A PHANTOM PARAGRAPH, MAYBE WITH SOME KEYWORDS.

- This is a note.
- This is another note.
  - This is a related note.

This is very important.

This text is highlighted.

Statement. [←That needs to be shown or proven.]■

(chapter) Title Case

Lorem ipsum...

(section) Sentence case

Lorem ipsum...

(subsection) Sentence case

Lorem ipsum...

(paragraph) Sentence case with punctuation. Lorem ipsum...

Below, we describe some phenomena or whatever.

Gravitational waves are called that or GWs, tensor perturbations, ... pdsfovnsoz avoszjasvo aoc awvn anvo pewnfvao noav

Stress–energy tensor is called that or SE tensor(?) ⌜OBS: Hilbert SE tensor = HSE tensor?⌟.

Domain walls are called that or walls, never DWs.

General relativity is called that or GR, Einstein's theory of gravity.

Einstein field equation(s) are called that or EFE(s).

Equation(s) of motion are called that or eom(s).

Chapter 1

Introduction

- GOALS:
  - Gather framework about GWs from DWs
  - Remove the need for very expensive N-body simulations with (semi-)analytical predictions
  - Extract as much information as possible from the NANOgrav spectra thingy
- WHY RELEVANT:
  - NANOgrav data wihoo
  - Simulations in this regard are hugely expensive, and will not allow us to constrain the parameters of a model

$$\tilde{h}''_{\circledast} + 2\mathcal{H}\tilde{h}'_{\circledast} + k^2\tilde{h}_{\circledast} = 16\pi G_{\mathrm{N}} a^2 \tilde{\sigma}_{\circledast}; \quad \circledast = +, \times \tag{1.1}$$

$$\left(\tilde{h}^{\mathrm{TT}}\right)_{ij}(\eta, \boldsymbol{k}) = \sum_{\circledast=+,\times} e^{\circledast}_{ij}(\hat{\boldsymbol{k}})\tilde{h}_{\circledast}(\eta, \boldsymbol{k}) \tag{1.2}$$

$$\tilde{h}^{\mathrm{TT}}{}_{ij}(\eta, \boldsymbol{k}) = \sum_{\circledast=+,\times} e^{\circledast}_{ij}(\hat{\boldsymbol{k}})\tilde{h}_{\circledast}(\eta, \boldsymbol{k}) \tag{1.3}$$

_
pert_i
î

1.1 Preliminaries

It is assumed that the reader is familiar with variational calculus and linear perturbation theory.

⌈In the following, we briefly (re)capture some concepts that are important starting points for the rest of the thesis.⌋

- variational calculus/ varying action
- action

- pert. theory?
- line element
- gauge invariance
- FRW cosmology
- classical field theory

$$G_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu} \tag{1.4}$$

1.1.1 Field theory

We formulate a theory in four-dimensional spacetime ⌜Minkowski⌟ in terms of the Lorentz invariant action

$$S = \int d^4x\, \mathcal{L}(\{\phi_i\}, \{\partial_\mu \phi_i\}), \tag{1.5}$$

with $\mathcal{L}$ being the *Lagrangian density* of the theory, a function of the set of fields $\{\phi_i\}$ and its first derivatives. We will refer to $\mathcal{L}$ simply as the Lagrangian, as is customary when working with fields. For a general (i.e. curved) spacetime, blah blah [. . .] $\partial_\mu \to \nabla_\mu$ blah blah [. . .] to construct a Lorentz invariant Lagrangian,

$$S = \int d^4x\, \underbrace{\mathcal{L}(\{\phi_i\}, \{\nabla_\mu \phi_i\})}_{\text{not scalar}} = \int \underbrace{d^4x\, \sqrt{-|g|}}_{\text{scalar}}\, \underbrace{\hat{\mathcal{L}}(\{\phi_i\}, \{\nabla_\mu \phi_i\})}_{\text{scalar}}, \tag{1.6}$$

⌜Maybe specify that this is only for scalar fields? Or include other fields?⌟

1.1.2 Expanding universe: standard cosmology

- expansion rate, cosmic time, conformal time
- why is flat assumption OK?
- redshift $\mathfrak{z} = a_0/a - 1$

The universe expands with the rate $a(t_{\mathrm{ph}})$ at physical or rather *cosmic* time t_{ph}. ⌜Explain cosmic time and $t_{\mathrm{ph}} = 0$.⌟ This work sincerely favours the use of *conformal* time, also known as the comoving horizon. As t is a neat variable name, we shall *not* reserve it for the cosmic time, as is the most common use, but rather let it refer to the conformal time coordinate.

1.1.3 Method of Green's functions

A linear ordinary differential equation (ODE) $\mathrm{L}_x f(x) = g(x)$ assumes a linear differential operator L, a ⌈continuous⌋$_?$, unknown function f, and a right-hand side g that constitutes the inhomogeneous part of the ODE. The *Green's function* G for the ODE (or L) is manifest as any solution to $\mathrm{L}_x G(x, y) = \delta(x - y)$ ⌜check plagiarism (Bringmann)⌟. If L is translation invariant (invariant under $x \mapsto x + a$)—which is equivalent to L having constant coefficients—we can write $G(x, y) = G(x - y)$ and [←show?]■

$$f(x) = (G * g)(x) = \int \mathrm{d}y\, G(x - y) g(y) \tag{1.7}$$

solves $\mathrm{L}_x f(x) = g(x)$.

Let $f_i^{(0)}$, $i = 1, 2, 3, \ldots$ be solutions to the homogeneous ODE, i.e. $\mathrm{L}_x f_i^{(0)} = 0$. Then, by the superposition principle, $f(x) + \sum_i c_i f_i^{(0)}$ is also a solution of the original, inhomogeneous equation.

Pulse signal. Consider the very common scenario where the source is a temporary pulse;

$$g(x) = \begin{cases} g(x), & x_0 \leq x \leq x_1, \\ 0, & x \geq x_1. \end{cases} \tag{1.8}$$

1.1.4 Special functions (tmp. name, maybe move (appendix)?)

blah blah [...]

For $\nu = n + {}^1\!/_2, n \in \mathbb{N}$ we have

$$\mathcal{J}_{n+{}^1\!/_2}(x) = \sqrt{\frac{2}{\pi}} x^{n+\frac{1}{2}} \left(-\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x} \right)^n \frac{\sin x}{x} \tag{1.9a}$$

and

$$\mathcal{Y}_{n+{}^1\!/_2}(x) = (-1)^{n+1} \mathcal{J}_{-(n+{}^1\!/_2)}(x) \tag{1.9b}$$

for $x \in \mathbb{C}xlxl$

Part I

Background

Chapter 2

Differential Geometry

PHANTOM PARAGRAPH

To develop a classical field theory, we require a handful of mathematical structures[?] concepts from differential geometry.

A classical field theory consists of the following mathematical structures:

Spacetime ...

A spacetime is a *smooth manifold* with or without additional mathematical structures.

The aim of this somewhat technical chapter is to provide the necessary mathematical tools to get a sufficient grasp of general relativity as a classical field theory.

2.1 TITLE (Manifolds, tensors, etc.)

The manifolds of primary interest to us as gravity physicists are the *pseudo-Riemannian manifolds*. Such are the very foundation of Einstein's general theory of relativity, and amongst them are Lorentzian and Riemannian manifolds.

We let the spacetime $(\mathscr{M}, g_{\mu\nu})$ be made up of a pseudo-Riemannian manifold $\mathscr{M}$ and a pseudo-Riemannian metric tensor $g_{\mu\nu}$. Then, by definition, $\mathscr{M}$ is differentiable and $g_{\mu\nu}$ is smooth, non-degenerate ⌜(?)⌟ and symmetric.

- Describe:
  - diffeomorphisms
  - maps
  - submanifolds
  - metrics
  - symmetries

2.1.1 Hypersurfaces

- First & second fundamental form

2.2 Riemannian geometry

Riemannian geometry is an important branch of differential geometry and concerns *Riemannian manifolds*. A manifold is Riemannian when it is smooth and admits a *Riemannian metric*. Furthermore, this metric is characterised ⌜by what???⌟

The general theory of relativity is

2.3 Conformal geometry

((Carroll, 2019, App. XXX)) ((Dąbrowski et al., 2009; Feng and Gasperín, 2023))

DRAFT

⌜

Consider a spacetime $(\mathscr{M}, g_{\mu\nu})$ where $\mathscr{M}$ is a smooth manifold of D dimensions and $g_{\mu\nu}$ is a Lorentzian metric on said manifold. Let $\Upsilon = \Upsilon(x) \in \mathbb{R}^{+}$[1] be a smooth function of spacetime coordinate x^μ. Then

$$\tilde{g}_{\mu\nu}(x) = \Upsilon^2(x) g_{\mu\nu}(x) \tag{2.1}$$

is a *conformal transformation* and Υ the corresponding *conformal factor*. Such angle-preserving transformations leave the causal structure of the manifold unchanged as they extend or contract the distance between spacetime points. ⌈In this section, a tilde $\tilde{o}$ refers to o in "tilde'd" system.⌋↺

blah blah [...]

It is straight-forward[2] to show that the determinants and inverses of the metrics obey the following relations:

$$\begin{aligned} \sqrt{-\tilde{g}} &= \Upsilon^D \sqrt{-g} \\ \tilde{g}^{\mu\nu} &= \Upsilon^{-2} g^{\mu\nu} \end{aligned} \tag{2.2}$$

We apply the transformation Eq. (2.1) to the connection coefficients

blah blah [...]

$$\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + C^\rho{}_{\mu\nu} \tag{2.3}$$

$$C^\rho{}_{\mu\nu} = \Upsilon^{-1}\left(2\delta^\rho{}_{(\mu}\Upsilon_{,\nu)} - g_{\mu\nu} g^{\rho\sigma}\Upsilon_{,\sigma}\right) \tag{2.4}$$

Under Eq. (2.3), the Riemann tensor becomes

$$\tilde{\mathcal{R}}^\rho{}_{\sigma\mu\nu} = \mathcal{R}^\rho{}_{\sigma\mu\nu} - 2C^\rho{}_{[\mu|\sigma|;\nu]} + 2C^\rho{}_{[\mu|\lambda} C^{\lambda|}{}_{\nu]\sigma}. \tag{2.5}$$

[1] ⌜$\mathbb{R}^+ \equiv (0, \infty)$ in notation chapter?⌟

[2] It is, actually, not like when other authors say things like that.

$$\tilde{\mathcal{R}}^{\rho}{}_{\mu\sigma\nu} = \mathcal{R}^{\rho}{}_{\mu\sigma\nu} + \Upsilon^{-1}\Big(\delta^{\rho}{}_{\nu}\Upsilon_{;\mu\sigma} - \delta^{\rho}{}_{\sigma}\Upsilon_{;\mu\nu} + g_{\mu\sigma}\Upsilon^{;\rho}{}_{;\nu} - g_{\mu\nu}\Upsilon^{;\rho}{}_{;\sigma}\Big)$$
$$+2\Upsilon^{-2}\Big(\delta^{\rho}{}_{\sigma}\Upsilon_{,\mu}\Upsilon_{,\nu} - \delta^{\rho}{}_{\nu}\Upsilon_{,\mu}\Upsilon_{,\sigma} + g_{\mu\sigma}\Upsilon^{,\rho}\Upsilon_{,\nu} - g_{\mu\nu}\Upsilon^{,\rho}\Upsilon_{,\sigma}\Big) + \Upsilon^{-2}\Big(\delta^{\rho}{}_{\nu}g_{\mu\sigma} - \delta^{\rho}{}_{\sigma}g_{\mu\nu}\Big)g_{\kappa\tau}\Upsilon^{,\kappa}\Upsilon^{,\tau} \tag{2.6a}$$

$$\tilde{\mathcal{R}}^{\rho}{}_{\mu\sigma\nu} = \mathcal{R}^{\rho}{}_{\mu\sigma\nu} + 2\Upsilon^{-1}\Big(\delta^{\rho}{}_{[\nu}\Upsilon_{;|\sigma|\mu]} + g_{\sigma[\mu}\Upsilon^{;|\rho|}{}_{;\nu]}\Big) + 4\Upsilon^{-2}\Big(\delta^{\rho}{}_{[\mu}\Upsilon_{,|\sigma|}\Upsilon_{,\nu]} + g_{\sigma[\mu}\Upsilon^{,|\rho|}\Upsilon_{,\nu]}\Big)$$
$$+2\Upsilon^{-2}\delta^{\rho}{}_{[\nu}g_{|\sigma|\mu]}g_{\mu\nu}g_{\kappa\tau}\Upsilon^{,\kappa}\Upsilon^{,\tau} \tag{2.7a}$$

$$\tilde{\mathcal{R}}^{\rho}{}_{\mu\sigma\nu} = \mathcal{R}^{\rho}{}_{\mu\sigma\nu} + 2A^{\rho}{}_{[\mu|\sigma|\nu]};$$
$$A^{\rho}{}_{\mu\sigma\nu} = \Upsilon^{-1}\Big(-\delta^{\rho}{}_{\mu}\Upsilon_{;\sigma\nu} + g_{\sigma\mu}\Upsilon^{;\rho}{}_{;\nu}\Big) + 2\Upsilon^{-2}\Big(\delta^{\rho}{}_{\mu}\Upsilon_{,\sigma}\Upsilon_{,\nu} + g_{\sigma\mu}\Upsilon^{,\rho}\Upsilon_{,\nu}\Big) + \Upsilon^{-2}\delta^{\rho}{}_{\nu}g_{\sigma\mu}g_{\mu\nu}g_{\kappa\tau}\Upsilon^{,\kappa}\Upsilon^{,\tau} \tag{2.8a}$$

We

$$\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\phi = \phi_{\tilde{;}\mu\nu} = \nabla_{\mu}\nabla_{\nu}\phi - \Big(\delta^{\kappa}{}_{\mu}\delta^{\tau}{}_{\nu} + \delta^{\tau}{}_{\mu}\delta^{\kappa}{}_{\nu} - g_{\mu\nu}g^{\kappa\tau}\Big)\Upsilon^{-1}(\nabla_{\kappa}\Upsilon)(\nabla_{\tau}\phi) \tag{2.9}$$

$$\tilde{\mathcal{R}} = \Upsilon^{-2}\mathcal{R} - 2(D-1)g^{\kappa\tau}\Upsilon^{-3}(\nabla_{\kappa}\nabla_{\tau}\Upsilon) - (D-1)(D-4)g^{\kappa\tau}\Upsilon^{-4}(\nabla_{\kappa}\Upsilon)(\nabla_{\tau}\Upsilon) \tag{2.10}$$

$$\tilde{\Box}\phi = \Upsilon^{-2}(\ldots) \tag{2.11a}$$
$$\Box\phi = \Upsilon^{2}(\ldots) \tag{2.11b}$$

Conformal invariants. The *Weyl conformal curvature tensor*

$$\mathcal{W}_{\rho\sigma\mu\nu} = \mathcal{R}_{\rho\sigma\mu\nu} + \frac{2}{D-2}\Big(g_{\rho[\nu}\mathcal{R}_{\mu]\sigma} + g_{\sigma[\mu}\mathcal{R}_{\nu]\rho}\Big) + \frac{2}{(D-2)(D-1)}\mathcal{R}g_{\rho[\mu}g_{\nu]\sigma} \tag{2.12}$$

is preserved under conformal transformations, such that

$$\tilde{\mathcal{W}}^{\rho}{}_{\sigma\mu\nu} = \mathcal{W}^{\rho}{}_{\sigma\mu\nu}. \tag{2.13}$$

The *Cotton tensor*

$$C_{\rho\mu\nu} = \mathcal{R}_{\rho[\mu;\nu]} - \frac{1}{2(D-1)}g_{\rho[\mu}\mathcal{R}_{;\nu]} \tag{2.14}$$

is

$C_{\sigma\mu\nu} = \frac{1}{D-3}\mathcal{W}_{\rho\sigma\mu\nu;\rho}$

PHANTOM PARAGRAPH: CONCOMITANT TENSORS

- Conformal flatness: $g_{\mu\nu} = \Upsilon^{-2}(x)\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu} \Rightarrow \tilde{g}_{\mu\nu}(x) = \Upsilon^{2}(x)\eta_{\mu\nu}$
- Conformal trafos of the Hilbert stress–energy tensor

2.4 Einstein's equation

How does the gravitational field affect how matter behaves, and in what way is matter controlling the gravitational field? Newtonian gravity proposes very good answers to these questions: The acceleration of an object in a gravitational potential Φ is

$$\boldsymbol{a} = -\boldsymbol{\nabla}\Phi, \tag{2.15}$$

and said field is governed by the matter density ρ through the Poisson equation

$$\boldsymbol{\nabla}^2\Phi = 4\pi G_{\mathrm{N}}\,\rho. \tag{2.16}$$

In physics, the answer to a question is highly dependent on *how the question was asked*. A common misconception is that Newtonian gravity was disproven by Einstein. Newton was simply telling a different story; a story about dynamics in non-relativistic systems.[3] Einstein confronted gravitational physics with different but analogous questions, and subsequently more complex answers than Newton. General relativity explains how curvature of spacetime influences matter, manifesting as gravity, and in what way energy and momentum affects spacetime to create curvature. In mathematical terms, these are the geodesic equation

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} = -\Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} \tag{2.17}$$

and Einstein's equation

$$\mathcal{G}_{\mu\nu} = 8\pi G_{\mathrm{N}}\,T_{\mu\nu}. \tag{2.18}$$

These can be obtained by

⌜Not too long chapter, but want to mention the "naive" tankegang from which these can be obtained (minimal coupling etc.).⌟

((Carroll, 2019, Ch. 4))

[3] Which, to be fair, are most common on Earth.

Chapter 3

Classical Field Theory and Gravity

Alongside quantum mechanics, Einstein's theory of gravity—general relativity (GR)—is widely accepted as the most accurate description of our surroundings. GR can be formulated from a geometrical point of view, or it can be viewed as a classical field theory. In the former approach we meet geometrical tools such as the geodesic equation, whereas the latter allows the application of field-theoretical methods. This chapter lays emphasis on the field interpretation of GR.

PHANTOM PARAGRAPH: TWO PERSPECTIVES INSIGHTFUL; BETTER OVERALL UNDERSTANDING OF ASPECTS OF CONCEPTS IN GR

3.1 General relativity

The Einstein–Hilbert action in vacuum is ⌜check Planck mass def.⌟

$$S_{\mathrm{EH}} = {}^1\!/_2 M_{\mathrm{Pl}}^2 \int d^4x \, \sqrt{-|g|}\, \mathcal{R}, \tag{3.1}$$

where $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$. By varying S_{EH} with respect to $g_{\mu\nu}$ one obtains the equation of motion

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} + \frac{1}{2} g_{\mu\nu}\mathcal{R} = 0. \tag{3.2}$$

Thus, we interpret GR as a *classical* field theory where the tensor field $g_{\mu\nu}$ is the gravitational field, ⌈with the particle realisation named "graviton"⌋↻.

- Einstein's field equations
- scalar field (ST theories)
- energy momentum tensor

Einstein's equation for general relativity

$$\boxed{\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}} \tag{3.3}$$

3.1.1 Energy momentum tensor

We address the rhs of Eq. (3.3), also known as the non-geometrical part. We interpret this

Perfect fluid

Lagrangian formulation

3.2 TITLE (Conformal field theory)

Suppose you have an n-dimensional manifold $\mathscr{M}$ with the associated metric g and coordinate system $\{x\}$. If another spacetime $(\tilde{\mathscr{M}}, \tilde{g})$ of n dimensions is such that $\tilde{g} = \omega(x)g$, we say that said spacetime is *conformal* to the original spacetime $(\mathscr{M}, g)$. This is not just a matter of name-dropping—the situation cause for a number of useful relations. We will see that the expansion of the universe is elegantly handled by conformal transformations. In short, conformality? Is this a word? allows THIS IS WRONG! Same spacetime, different metric

3.2.1 The conformal group

A regular change of the metric tensor under a coordinate transformation $x^\mu \mapsto \tilde{x}^\mu$ looks like

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x). \tag{3.4}$$

A special group of transformations leaves the metric scale invariant (invariant under a local change of scale), $\tilde{g}_{\mu\nu}(\tilde{x}) = \omega^2(x) g_{\mu\nu}(x)$. Such *conformal transformations* make up the conformal group, $\mathrm{Conf}(\mathscr{M})$. We say that $\omega(x)$ is the *conformal factor*.

3.2.2 Friedmann–Lemaître–Robertson–Walker universe

We consider a four-dimensional expanding universe that is both homogeneous and isotropic with a Lorentzian structure (i.e. metric signature $(-,+,+,+)$). The general metric can be written

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2(\tau)\left\{-\mathrm{d}\tau^2 + d\Sigma^2\right\}, \tag{3.5}$$

where τ represents conformal time, $a(\tau)$ is the dimensionless scale factor and Σ is the time-independent three-dimensional space. In polar coordinates, the spatial line element takes the familiar form

$$d\Sigma^2 = \frac{1}{1-kr^2} + r^2 d\Omega^2, \quad k \in \{-1, 0, +1\}. \tag{3.6}$$

However, as we know, it is safe to assume that the universe is flat©(ref to some section) , and we may as well use regular Cartesian coordinates;

$$d\Sigma^2 = \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{3.7}$$

This choice of coordinates implies $g_{\mu\nu} = a^2(x^0)\eta_{\mu\nu}$. Hence, $\mathbb{U} \in \mathrm{Conf}(\mathbb{M})$

- Fourier transform (scale invariance, scalar product preserved)
- Something about the benefit of using $a \propto \tau^\alpha$, and that $\alpha \in \mathbb{Z}$ is a sensible assumption (for completeness, maybe let $\alpha \in \mathbb{R}$?)

Fourier transforms

One very neat consequence of this scale invariance is that in FLRW cosmology we can use the regular, flat-space form of the Fourier transform and its inverse:

$$f(x) = \int \frac{d^4k}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}\eta_{\mu\nu}k^\mu x^\nu} f(k) \qquad = \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{\mathrm{i}\omega\tau} \int \frac{d^3k}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} f(\omega, \boldsymbol{k}) \tag{3.8a}$$

$$f(k) = \int d^4x\, \mathrm{e}^{\mathrm{i}\eta_{\mu\nu}k^\mu x^\nu} f(x) \qquad = \int \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau} \int d^3x\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} f(\tau, \boldsymbol{x}) \tag{3.8b}$$

The four-vectors $[x^\mu] = (\tau, \boldsymbol{x})$ and $[k^\mu] = (\omega, \boldsymbol{k})$ represent the comoving coordinate and wavevector, respectively. **((Maggiore, 2018, Ch. 17.1))**

3.2.3 TITLE (Analytical considerations)

We will encounter several equations of similar forms, for instance $\Box\phi = $ [some source term], whose homogeneous solution satisfies

$$\ddot{\phi} + 2\mathcal{H}\dot{\phi} - \nabla^2\phi = 0. \tag{3.9}$$

This partial differential equation generally depends on initial conditions and expansion history. Transforming the spatial part to Fourier space, ⌈$\nabla \mapsto -\mathrm{i}\boldsymbol{k}$ (check sign!)⌋?, we recognise a wave equation with a damping term $2\mathcal{H}$. The special case for which $a \propto \tau^\alpha$ gives $\mathcal{H} = \dot{a}/a = \alpha/\tau$. A trained eye? will then identify a transformed Bessel's equation ⌜put details in Appendix⌟. We write a somewhat more general (mer hensiktsmessig form) equation for some function $f(x) \to f_k(\tau)$ ⌜fix⌟:

$$\tau^2 \ddot{f}_k + A \cdot \alpha\tau \dot{f}_k + B^2 \cdot k^2\tau^2 f_k = 0; \quad A, B \in \mathbb{C} \tag{3.10}$$

The general solution to this equation is known, and we can use the properties of the Bessel functions to arrive at ⌈nicer⌋↺ expressions for some special cases (define $\nu \equiv n - 1/2, n \equiv A\alpha/2$)

$$f_k(\tau) = \begin{cases} \tau^{-\nu}\{C_k \mathcal{J}_{|\nu|}(Bk\tau) + D_k \mathcal{J}_{-|\nu|}(Bk\tau)\}, & \nu \notin \mathbb{Z} \\ \tau^{1-n}\{C_k j_{|n|}(Bk\tau) + D_k j_{-|n|}(Bk\tau)\}, & \nu \in \mathbb{Z} + \frac{1}{2} \\ \tau^{-\nu}\{C_k \mathcal{J}_{-\nu}(Bk\tau) + D_k \mathcal{Y}_{-\nu}(Bk\tau)\} & \end{cases} \tag{3.11}$$

⌈Carefully check these! I think they are wrong...⌋? ⌜Spoiler alert!⌟ Most physically meaningful scenarios will have $\alpha \in \mathbb{Z}$ (see Table XXX©0) and $A \in \mathbb{Z}$.

⌜Table with matched w_s, α, β etc.⌟

Table 3.1: $w = p/\rho$ blah blah

Constituent	Perfect fluid parameters			Domination	
⌜*FIXME*⌟	w	α	β	a	ρ
cosmological constant, dark energy (Λ)	-1				$\propto a$
non-relativistic matter, dust (m)	0	2	$2/3$	$\propto \tau^2 \propto t_{\mathrm{ph}}^{2/3}$	$\propto a^{-3}$
relativistic matter, radiation (r)	$1/3$	1	$1/2$	$\propto \tau \propto t_{\mathrm{ph}}^{1/2}$	$\propto a^{-4}$

3.3 Perturbation theory

3.4 Classical solitons

- topological defects
  - System of kinks: $\phi(x) = \phi_\infty^{1-(N+M)} \prod_{i=1}^{N} \phi_\mathrm{k}(x - k_i) \prod_{j=1}^{M} \bar{\phi}_\mathrm{k}(x - \bar{k}_j)$ ((Vachaspati, 2006))
- basic properties?

3.5 Symmetron model

The simplest of fields is the scalar field; the main ingredient to simple modifications of gravity.

Jordan & Einstein frames. (Some text about these.) Jordan frame (Jf.) and Einstein frame (Ef.)

The origin of the symmetron model is blah blah [. . .]

The Z_2 symmetry $\phi \to -\phi$ is broken by the symmetron potential $V(\phi) = \lambda\phi^4/4 - \mu^2\phi^2/2$. ((Hinterbichler and Khoury, 2010))

The *a*symmetron model is a generalisation of the symmetron in which a cubic term is added to the potential. A nonzero cubic term corresponds to a system where one potential well is deeper than the other. We write the asymmetron ⌈vacuum⌋? potential as

$$\boxed{V(\phi) = \frac{\lambda}{4}\phi^4 - \frac{\kappa}{3}\phi^3 - \frac{\mu^2}{2}\phi^2 + V_0} \tag{3.12}$$

for which the symmetron is retrieved when $\kappa = 0$. In Fig. 3.5 we demonstrate the impact of varying each coupling term separately.

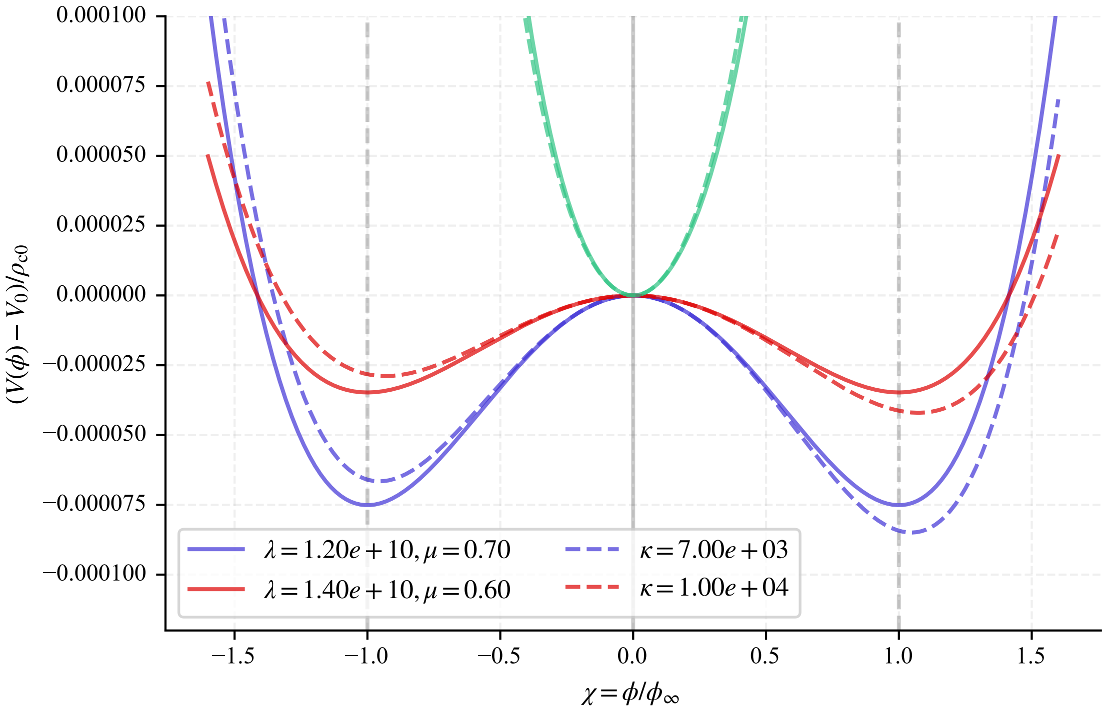


Figure 3.1: Asymmetron potential for different parameter choices.

3.5.1 The symmetron action

We begin with the general chameleon action in the Einstein frame

$$S = S_{\text{SE}} + S_\phi + S_{\text{m}} = \int d^4x \sqrt{-g}\left\{\frac{M_{\text{Pl}}}{2}\mathcal{R} + X(\phi) - V(\phi)\right\} + S_{\text{m}}[\tilde{g}_{\mu\nu}, \psi] \tag{3.13}$$

where $X = -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$ is viewed as the kinetic energy of the chameleon, and S_{m} is the Jordan frame matter action. In truth, the Jf. fields $\tilde{g}_{\mu\nu}$ and ψ are

The matter fields couple to the Jordan frame metric $\tilde{g}_{\mu\nu} = A^2(\phi)g_{\mu\nu}$.

In Jf., the SE tensor $\tilde{T}_{\mu\nu} = -\left(2/\sqrt{-\tilde{g}}\right)\delta\mathcal{L}_{\rm m}/\delta\tilde{g}^{\mu\nu}$ is covariantly conserved; $\tilde{\nabla}_\mu \tilde{T}^\mu{}_\nu = 0$. By varying the Ef. action, we find the symmetron obeys

$$\Box\phi - V_{,\phi} + A^3(\phi)A_{,\phi}\cdot\tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu} = 0. \tag{3.14}$$

We let $A(\phi) = 1+\Delta A = 1+\phi^2/(2M^2)$ be the (a)symmetron conformal factor. The Ef. energy density $\rho = A^3\tilde{\rho}$

Matter density in Einstein frame. The Jf. SE tensor is covariantly conserved; $\tilde{\nabla}_\mu \tilde{T}^\mu{}_\nu = 0$. Several systems, such as galaxies, allow us to assume .... We then find the trace $\tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu} = -\sum_i \tilde{\rho}_i(1-w_i) = -\tilde{\rho}$, and the eom for ϕ becomes

$$\Box\phi = V_{,\phi} + \rho A_{,\phi}. \tag{3.15}$$

It is customary to define the effective potential s.t. $\Box\phi = V_{\text{eff},\phi}$, i.e.

$$V_{\text{eff}}(\phi) = V(\phi) + \rho A(\phi) = \frac{\lambda}{4}\phi^4 - \frac{\kappa}{3}\phi^3 + \frac{\mu^2}{2}\left(\frac{\rho}{\mu^2 M^2} - 1\right)\phi^2 + V_0. \tag{3.16}$$

blah blah [...] This potential becomes unstable when $\rho \leq \mu^2 M^2 \equiv \rho_*$, and the field rolls into either of the two vacua.

Symmetron potential. The (a)symmetron effective potential captures the complete phase transition. Let $\upsilon \equiv \rho_{\rm m}/(\mu^2 M^2)$. By setting $V_{\text{eff},\phi} = 0$, we find the vacuum expectation values

$$\phi_0 = 0 \quad \vee \quad \phi_\pm = \phi_\infty\left(\bar{\kappa} \pm \sqrt{\bar{\kappa}^2 + 1 - \upsilon}\right), \tag{3.17}$$

where we defined $\bar{\kappa} = \kappa/(2\mu\sqrt{\lambda})$ and $\phi_\infty = \mu/\sqrt{\lambda}$. Note that for the symmetron ($\kappa = 0$), since the field is real, VEV is zero before SSB. We determine the stability of these vacua by evaluating $V_{\text{eff},\phi\phi}$ at $\phi = \phi_0, \phi_\pm$ and see that ϕ_0 remains stable until ρ_*.

3.5.2 Fifth-force

blah blah [...]

Chapter 4

Gravitational Waves

The term "gravitational waves" refers to the tensor perturbations to the background metric$^{©}_{()}$. These "waves" are spacetime distortions whose name comes from the fact that ⌈they obey the wave equation⌋$_{?}$.

4.1 Linearised gravity

The simplest way to find the equation of motion (e.o.m.) for the tensor perturbations to the metric is to *go through* a Minkowski spacetime. First, we establish the law of physics, valid in Minkowski coordinates. We write it in a coordinate-independent form, that is to say write the e.o.m. on tensorial form. ⌈The resulting law (e.o.m.) remains true in any spacetime. This procedure is called the "minimal coupling procedure" and is extremely powerful.⌋$_{?}$

4.1.1 TITLE (Minkowski Spacetime)

For the time being, we take the unperturbed metric to be the flat, static Minkowskian metric such that

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x); \qquad \left|h_{\mu\nu}\right| \ll 1 \tag{4.1}$$

is the full, perturbed metric. Note that the $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, whereas $h^{\mu\nu}$ is *not* the inverse of $h_{\mu\nu}$. We aim to find the Einstein tensor $\mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}$.

To zeroth order in $h_{\mu\nu}$, the Einstein tensor is simply zero and the metric Minkowskian. To leading order $\mathscr{O}(h)$, some calculation is required. First, we compute the Christophel symbols

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(2g_{\sigma(\mu,\nu)} - g_{\mu\nu,\sigma}\right) = \frac{1}{2}\eta^{\rho\sigma}\left(2h_{\sigma(\mu,\nu)} - h_{\mu\nu,\sigma}\right) + \mathscr{O}(h^2). \tag{4.2}$$

Next, we find the Riemann tensor to be $\mathcal{R}^{\rho}{}_{\mu\sigma\nu} = 2\Gamma^{\rho}_{\mu[\nu,\sigma]} + \mathscr{O}(h^2)$. The Ricci tensor is then

$$\begin{aligned}\mathcal{R}_{\mu\nu} &= \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h_{\mu}{}^{\rho} - \partial_\rho\partial^\rho h_{\mu\nu} - \partial_\mu\partial_\nu h^{\rho}{}_{\rho}\right) + \mathscr{O}(h^2)\\ &= \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h_{\mu}{}^{\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) + \mathscr{O}(h^2),\end{aligned} \tag{4.3}$$

thereby the Ricci scalar $\mathcal{R} = \partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h + \mathscr{O}(h^2)$. The first order Einstein tensor is

$$\mathcal{G}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h_{\mu}{}^{\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h\right) + \mathscr{O}(h^2). \tag{4.4}$$

...

...

...

- Gauge freedom
- Explain perturbation equation
- Separating GWs from background (Maggiore)

4.1.2 TITLE (FLRW Spacetime)

4.1.3 TITLE (TT gauge?)

DRAFT

4.2 Polarisation and decomposition of gravitational waves

4.3 Gravitational waves in vacuum

In an expanding, flat universe, we choose the TT gauge such that ${h^i}_i = 0$ and $h_{ij,j} = 0$. The unsourced equation reads

$$\Box h_P(x) = 0 \Rightarrow \ddot{h}_P(x) + 2\mathcal{H}\dot{h}_P(x) - \boldsymbol{\nabla}^2 h_P(x) = 0, \tag{4.5}$$

and thus describes a damped harmonic oscillator. We specify $a \propto \tau^\alpha$ and $\alpha \in \mathbb{Z}$ and turn to Fourier space,

$$\ddot{h}_P(\tau, \boldsymbol{k}) + 2\frac{\alpha}{\tau}\dot{h}_P(\tau, \boldsymbol{k}) + k^2 h_P(\tau, \boldsymbol{k}) = 0. \tag{4.6}$$

In defining $\bar{h}_P \equiv a h_P$ and $u \equiv k\tau$ we can recognise the convenient form

$$\frac{\mathrm{d}^2\bar{h}_P}{\mathrm{d}u^2} + \left[1 - \frac{\alpha(\alpha-1)}{u^2}\right]\bar{h}_P = 0 \tag{4.7}$$

whose general solution is a linear combination of the first and second order Riccati-Bessel functions, $\bar{h}_P(\tau, \boldsymbol{k}) = A_P(k)\psi_n(k\tau) + B_P(k)\chi_n(k\tau)$ where $n = \alpha - 1$.

. . . where

$$\begin{aligned} \psi_n(u) &= u j_n(u) = \sqrt{\frac{\pi u}{2}}\mathcal{J}_{n+\frac{1}{2}}(u) \\ \chi_n(u) &= -u y_n(u) = -\sqrt{\frac{\pi u}{2}}\mathcal{Y}_{n+\frac{1}{2}}(u) \end{aligned} \tag{4.8}$$

are the Riccati-Bessel functions.

4.4 Generation of gravitational waves

- Production instead of generation?

Linearised Einstein equations: ⌜conformal Newtonian gauge⌟

$$\delta G^i{}_j = 8\pi G_{\mathrm{N}} T^i{}_j = \frac{1}{2a^2}\left[\ddot{h}_{ij} + 2\frac{\dot{a}}{a}\dot{h}_{ij} - \nabla^2 h_{ij}\right] \tag{4.9}$$

where a dot (' ˙ ') signifies the *conformal* time derivative. ($T^i{}_j = a^{-2}T_{ij}$) blah blah [. . .]

Part II

Methodology

Chapter 5

Thin-Wall Approximation

PHANTOM PARAGRAPH: MOTIVATION–INSPIRATION(WORK)

⌈ $\mathbb{M} \equiv (\mathbb{R}^4, \eta_{\mu\nu})$, $\mathbb{U} \equiv (\mathbb{R}^4, a^2\eta_{\mu\nu})$ ⌋$_?$

We shall consider a pseudo-Riemannian $(N+1)$-dimensional spacetime $(\mathscr{M}, g_{\mu\nu})$ up until the point where concrete expressions are needed to proceed; here we turn to the conformally flat FLRW universe, for which $\mathscr{M} = \mathbb{U}$, $N = 3$ and $g_{\mu\nu} = a^2\eta_{\mu\nu}$.

$$\Phi(x^\mu - X^\mu) = \frac{1}{\sqrt{2\pi}w_0} \exp\left\{-\frac{(x^\mu - X^\mu)^2}{2w_0^2}\right\} \quad \rightsquigarrow \quad \lim_{w_0 \to 0} \Phi(x^\mu - X^\mu) = \delta^{(4)} x^\mu - X^\mu \tag{5.1}$$

⌜Needs fixing...⌟

We will in this chapter consider infinitely thin $((N-1)+1)$-dimensional topological defects as ⌈timelike⌋$_?$ hypersurfaces embedded in $(N+1)$-dimensional spacetime. We will use variational calculus to find the eom for the scalar field that is the wall normal coordinate (ϵ), which is not to be confused with the scalar field that crops up later blah blah [. . .]

5.1 General framework

We follow Garriga and Vilenkin ((1991)) and Ishibashi and Ishihara ((1999)). The world sheet Σ divides $\mathscr{M}$ into two submanifolds $\mathscr{M}_\pm$ such that $\mathscr{M} = \mathscr{M}_+ \cup \Sigma \cup \mathscr{M}_-$. That is to say, a domain wall holds a world sheet separating two vacua. We take $\mathscr{M}$ to be smooth and $(N+1)$-dimensional, and let Σ be a smooth also and $((N-1)+1)$-dimensional. Consequently, Σ is a timelike hypersurface in $\mathscr{M}$.

- Vary DW action
- Goal: E.O.M. for physically relevant component (epsilon basically)
- Expression for energy–momentum tensor
- Extension to non-thin walls
- Extension to Asymmetron or introduction of energy bias
- What does thin mean? Why is the tension indep. of width?

We invoke a smooth coordinate system $\{x^\mu\}$ ($\mu = 0, 1, \ldots, N$) of the spacetime $(\mathscr{M}, g_{\mu\nu})$ in a neighbourhood of Σ. The embedding of Σ in $\mathscr{M}$ is $x^\mu = x^\mu(y^a)$, where the coordinate system $\{y^a\}$ ($a = 0, 1, \ldots, N-1$) parametrises Σ. The induced metric on Σ is

$$q_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b; \qquad e^\mu_a \equiv \frac{\partial x^\mu}{\partial y^a} \tag{5.2}$$

[←argue!]■

We let σ represent the surface energy density of the wall—a quantity we will discuss i much more detail later—and $v_\pm$ the vacuum energy densities of $\mathscr{M}_\pm$. The complete action of the coupled system is

$$S = \underbrace{-\sigma \int_\Sigma d^N y \sqrt{-q}}_{S_{\mathrm{NG}}} - v_+ \int_{\mathscr{M}_+} d^{N+1}x \sqrt{-g} - v_- \int_{\mathscr{M}_-} d^{N+1}x \sqrt{-g} + \underbrace{\frac{M_{\mathrm{Pl}}^2}{2} \int_{\mathscr{M}} d^{N+1}x \sqrt{-g}\,\mathcal{R}}_{S_{\mathrm{EH}}}. \tag{5.3}$$

⌜Comment about Nambu-Goto action. Maybe add matter and ϕ actions?⌟

Under small changes in x^μ on Σ, $x^\mu \to x^\mu + \delta x^\mu$, we obtain the equation

$$\mathrm{D}^a e^\mu_a + \Gamma^\mu_{\kappa\tau} q^{ab} e^\kappa_a e^\tau_b + \frac{v_+ - v_-}{\sigma} n^\mu = 0, \tag{5.4}$$

or equivalently,

$$\mathrm{D}_a \mathrm{D}^a x^\mu + \Gamma^\mu_{\kappa\tau} q^{ab} \frac{\partial x^\kappa}{\partial y^a} \frac{\partial x^\tau}{\partial y^b} + \frac{v_+ - v_-}{\sigma} n^\mu = 0, \tag{5.5}$$

where D_a is the covariant derivative with respect to q_{ab}.

⌈The part of δx^μ that is tangential to Σ are diffeomorphisms on Σ ($y^a \to y^a + \delta y^a$).⌋$_?$ The only physically meaningful component is the transverse one;

$$n_\mu \mathrm{D}^a \mathrm{D}_a x^\mu + n_\mu \Gamma^\mu_{\kappa\tau} q^{ab} e^\kappa_a e^\tau_b + \frac{\Delta v}{\sigma} = 0. \tag{5.6}$$

Without loss of generality, we may align Σ with e.g. the first $N-1$ dimensions of $\mathscr{M}$, i.e. $e^\mu_a = \delta^\mu_a + \delta^\mu_{\nu_*} \epsilon_{,a}$ and $n^\mu = \delta^\mu_{\nu_*}$, with $\nu_* = N$. We let $x^\mu = \delta^\mu_a y^a + \delta^\mu_{\nu_*}(\epsilon(y^a) + \zeta)$ be the embedding function, where ζ is the ν_*-coordinate of Σ in $\mathscr{M}$. Now,

$$\mathrm{D}^2 \epsilon + \Gamma^{\nu_*}_{\kappa\tau} q^{ab} \delta^\kappa{}_a \delta^\tau{}_b + \frac{\Delta v}{\sigma} = 0. \tag{5.7}$$

FIX ME!!!

⌜Will remove several of these equations.⌟

Wall position $X^\mu = X^\mu_0 + \epsilon N^\mu = \delta^\mu_a y^a + \delta^\mu_{\nu_*}(\zeta + ...)$

NB! This is “general” for topological defects, should call σ somthing different.

5.1.1 Energy and momentum

We find the domain wall (hypersurface in (3 + 1) dimensions) Lagrangian as $S_{\mathrm{NG}} = \int d^4x\, \mathcal{L}_{\mathrm{NG}}$, allowing us to compute the associated SE tensor

$$T^{\mu\nu}\Big|_{\mathrm{NG}} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\mathrm{NG}}}{\delta g_{\mu\nu}}. \tag{5.8}$$

We identify the Lagrangian by rewriting the Nambu-Goto action,

$$S_{\mathrm{NG}} = -\sigma \int d^3y \sqrt{-q} = -\sigma \int d^4x \sqrt{-q}\, \delta^{(4)}(x^\mu - X^\mu), \tag{5.9}$$

and perform the variation in Eq.XX$^{\copyright}_{0}$ to find an explicit expression for $T^{\mu\nu}$.

Thickness?

To some extent, we can account for a possibly non-vanishing wall width l by choosing a Gaussian funciton instead of a Dirac-Delta distrubution. In the case of $x^a = y^a$, this looks like

$$\delta(x^{\nu_*} - X^{\nu_*}) \to \Phi_l(x^{\nu_*} - X^{\nu_*}) = \frac{1}{\sqrt{2\pi}l} \exp\left\{-\frac{(x^{\nu_*} - X^{\nu_*})^2}{2l^2}\right\}, \tag{5.10}$$

where $\delta(x^{\nu_*} - X^{\nu_*})$ is retrieved by taking the limit $l \to 0$.

The domain wall Lagrangian is written

$$\mathcal{L}_{\mathrm{NG}} = -\sigma \sqrt{-g}\, \Phi_l(x^{\nu_*} - X^{\nu_*}), \tag{5.11}$$

so that we would have the SE tensor

$$\begin{aligned} T^{\mu\nu}\Big|_{\mathrm{NG}} &= -\frac{2\sigma\, \Phi_l(x^{\nu_*} - X^{\nu_*})}{\sqrt{-g}} \frac{\delta\sqrt{-q}}{\delta g_{\mu\nu}} \\ &= \frac{\sigma\, \Phi_l(x^{\nu_*} - X^{\nu_*})}{\sqrt{-g}\sqrt{-q}} \frac{\delta q}{\delta g_{\mu\nu}}. \end{aligned} \tag{5.12}$$

5.2 Dynamics of planar domain walls in expanding universe

Now the ground work is laid for the scenario that which this project i all about. That is, we consider a conformally flat spacetime with line element

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2(\tau)\left\{-\mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right\}. \tag{5.13}$$

We place a thin domain wall at z-coordinate z_0, represented by a hypersurface Σ, whose induced metric is $q_{ab} = g_{\mu\nu}x^\mu{}_{,a}x^\nu{}_{,b}$. Now, $z = z_0 + \epsilon(y^a)$ and

$$\mathrm{D}^2\epsilon + \Gamma^3_{\kappa\tau}q^{ab}\left(\delta^\kappa{}_a\delta^\tau{}_b + 2\delta^\kappa{}_a\delta^\tau{}_3\epsilon_{,b} + \delta^\kappa{}_3\delta^\tau{}_3\epsilon_{,a}\epsilon_{,b}\right) + \frac{\Delta v}{\sigma}n^\mu = 0 \tag{5.14}$$

blah blah [...] ⌜Set $\Delta v = 0$ etc.⌟

Eventually, we arrive at

$$\ddot{\epsilon} + 3\mathcal{H}\dot{\epsilon} - \left(\partial_x^2 + \partial_y^2\right)\epsilon = 0 \tag{5.15}$$

With the ansatz $\epsilon = \epsilon_p(\tau)\mathscr{E}(x, y)$ we can solve this in terms of eigenvalues $-p^2$,

$$\ddot{\epsilon} + 3\mathcal{H}\dot{\epsilon} + p^2\epsilon = 0, \tag{5.16}$$

where $(\partial_x^2 + \partial_y^2)\mathscr{E}(x, y) = -p^2\mathscr{E}(x, y)$. For a universe with $a \propto \tau^\alpha$, the solution to this equation is of the form $\epsilon_p(\tau) = x^\gamma \mathcal{Z}_\gamma(p\tau)$, $\gamma = (1 - 3\alpha)/2$.

5.2.1 Stress-energy tensor

We perform the variation in Eq. $\mathrm{XX}^{©}_{\text{(of q)}}$. The detailed calculation can be found in appendix $\mathrm{X}^{©}_{\text{(app. with derivations)}}$. The non-vanishing components are

$$\begin{aligned} T_{ab}(\tau, \boldsymbol{x}) &= -a\sigma\, \Phi(z - z_{\mathrm{dw}})\, \eta_{ab} \\ T_{(i3)}(\tau, \boldsymbol{x}) &= -a\sigma\, \Phi(z - z_{\mathrm{dw}})\, \epsilon_{,i} \end{aligned} \tag{5.17}$$

where $a, b = 0, 1, 2$.

5.2.2 Non-constant surface tension

Until now, these equations are model-independent, given constant surface tension ($\dot{\sigma} = 0$) and no energy bias ($v_+ - v_- = 0$). If we allow the surface tension to vary, $\sigma = \sigma(\tau)$, we need to put this inside of the integral in the Nambu-Goto action. We immidiately see that this is equivalent to letting $a^3 \to a^3\sigma$, and with the substitution

$$3\mathcal{H} = \frac{3}{a}\frac{\mathrm{d}a}{\mathrm{d}\tau} \to \frac{3}{a\sigma^{1/3}}\frac{\mathrm{d}a\sigma^{1/3}}{\mathrm{d}\tau} = 3\frac{\dot{a}}{a} + \frac{\dot{\sigma}}{\sigma} = 3\ln\dot{a} + \ln\dot{\sigma} \tag{5.18}$$

in Eq. XXX$^{©}_{0}$, we get

$$\ddot{\epsilon} + (3\dot{a}/a + \dot{\sigma}/\sigma)\dot{\epsilon} + p^2\epsilon = 0. \tag{5.19}$$

This extra term will introduce the model-dependence, seeing as the surface tension in the thin-wall limit is given by

$$\sigma = \int_{\phi_-}^{\phi_+} \mathrm{d}\phi\ \sqrt{2V_{\mathrm{eff}}(\phi) - 2V_{\mathrm{eff}}(\phi_\pm)}, \tag{5.20}$$

where V_{eff} is the effective potential of the theory.

5.3 TITLE (Symmetron domain walls)

We specify our example even further. The symmetron effective potential$^{©}_{\text{(some sec.)}}$ is designed to provoke a phase transition at conformal time τ_*, or scale factor a_*. Inserting this into Eq. XX$^{©}_{0}$, we find that the surface tension of a thin symmetron domain wall is

$$\sigma = \sigma_\infty\left[1 - (a_*/a)^3\right]^{3/2}, \tag{5.21}$$

where $\sigma_\infty = 4\sqrt{2}/3\,\mu^3/\lambda$ is the asymptotic (“true”) surface energy density.

From here it becomes advantagous to introduce some new, dimensionless variables (or dimensions). We will use $s \equiv \tau/\tau_*$ as our time variable, and $u = p\tau_*$ as the eigenvalue, when it comes to that. In a universe with $a = a_* s^\alpha$, the surface tension goes as $\sim (1 - s^{-3\alpha})^{3/2}$, thus

$$\frac{\sigma'}{\sigma} = \frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}s} = \frac{\mathrm{d}\ln\sigma}{\mathrm{d}s} = \frac{9\alpha}{2s(s^{3\alpha} - 1)} \tag{5.22}$$

Finally, the equation of motion for the wall normal coordinate is

$$\epsilon'' + \left(\frac{3\alpha}{s} + 2\gamma(s)\right)\epsilon' + u^2\epsilon = 0; \quad \gamma(s) \equiv \frac{9\alpha}{4s\,(s^{3\alpha} - 1)}. \tag{5.23}$$

From hereon, we express the function in terms of s and u, i.e. $\epsilon_u(s)$.

5.3.1 Solution for $\alpha = 2$

This project only truly experiments in a universe with, and only with, homogeneous matter distribution, i.e. $\alpha = 2$ and $\Omega_{\mathrm{m0}} = 1$. Generalisation of the following to other α should in principle be trivial, but at some point we require $\alpha \geq 1/3$, and so a different analysis would be required for smaller α.

So, restricting our discussion to $\alpha = 2$, we continue with the planar domain wall placed parallel to the xy-plane, spontaneously formed at symmetry break, $s = 1$. Assume an initial perturbation of amplitude ϵ_* was given to the wall, whose spatial part satisfies $(\partial_x^2 + \partial_y^2)\mathscr{E}(x, y) =$

$-p^2\mathscr{E}(x,y)$, giving the "scale" of the perturbation p. With initial conditions $\epsilon_u(s=1)=\epsilon_*$ and $\epsilon_u'(s=1)=0$, we shall solve

$$\epsilon'' + \left(\frac{6}{s} + \frac{9}{2s\,(s^6-1)}\right)\epsilon' + u^2\epsilon = 0 \tag{5.24}$$

analytically in two regimes, and sow these solutions together in the region where they overlap. For notational ease, we write $\epsilon_u(s)=\epsilon_* e(s)$.

Shortly after symmetry breaking. We begin by solving the equation of motion for $s\sim 1$. As our equation has a singularity at $s=1$, the natural way to go is through a Laurent expansion around this point of the damping term in Eq. 5.24. We find

$$\frac{6}{s} + \frac{9}{s\,(s^6-1)} = \frac{3}{2}(s-1) + \frac{3}{4} + \frac{29}{8}(s-1) - \frac{93}{16}(s-1)^2 + \mathscr{O}((s-1)^3). \tag{5.25}$$

Now $e(s)$ is also subject to an expansion around $s=1$;

$$e(s) = \Big[1 + c_1(s-1) + c_2(s-1)^2 + c_3(s-1)^3 + \ldots\Big]. \tag{5.26}$$

When put together, we get a polynomial in $(s-1)$ on the left-hand side of Eq. 5.24, for which all coefficients must vanish. We solve the system of equations for $\{c_1, c_2, c_3\}$ and find

$$e^{(\mathrm{I})}(s) = \left[1 - \frac{u^2}{5}(s-1)^2 + \frac{u^2}{35}(s-1)^3\right] + \mathscr{O}((s-1)^4). \tag{5.27}$$

Adiabatic evolution. The damping term $2\gamma(s) = 9/(2s(s^6-1))$ initially changes extremely rapidly from very large values, before it becomes very small compared to $3a'/a = 6/s$. We expect the solution to quickly approach that of Eq. 5.16 as $s \gg 1$. Said damping term is not completely negligible, however, as it causes a *damping envelope* that is considered much like in the case of a damped harmonic oscillator, writing

$$e^{(\mathrm{II})}(s) \simeq w(s)\cdot \exp\left\{-\int^s \mathrm{d}t\,\gamma(t)\right\}. \tag{5.28}$$

Employing this ansatz in the eom gives

$$w'' + \frac{6}{s}w' + \Big(q^2 - \theta(s)\Big)w = 0; \quad \theta(s) = \gamma'(s) + \gamma^2(s) + \frac{6}{s}\gamma(s), \tag{5.29}$$

whose solution is $w(s) \simeq s^{-5/2}\mathcal{Z}_{-5/2}(us)$ when the phase shift introduced by $\theta(s)$ is negligible. [1] Now, we find that $\exp\left\{-\int^s \mathrm{d}t\,\gamma(t)\right\} = s^{9/2}(s^6-1)^{-3/4}\cdot$ constant. Thus

$$e^{(\mathrm{II})}(s) \simeq \frac{A\mathcal{J}_{-5/2}(us) + B\mathcal{Y}_{-5/2}(us)}{s^{5/2}}\,\frac{s^{9/2}}{(s^6-1)^{3/4}}, \tag{5.30}$$

where A and B are constants to be determined.

[1] In fact, it is possible to show that $\lim_{u\to\infty}\Big[\sqrt{u^2-\theta(1+u^{-1})}-u\Big]/u = \sqrt{19}/4 - 1 \approx 0.09$.

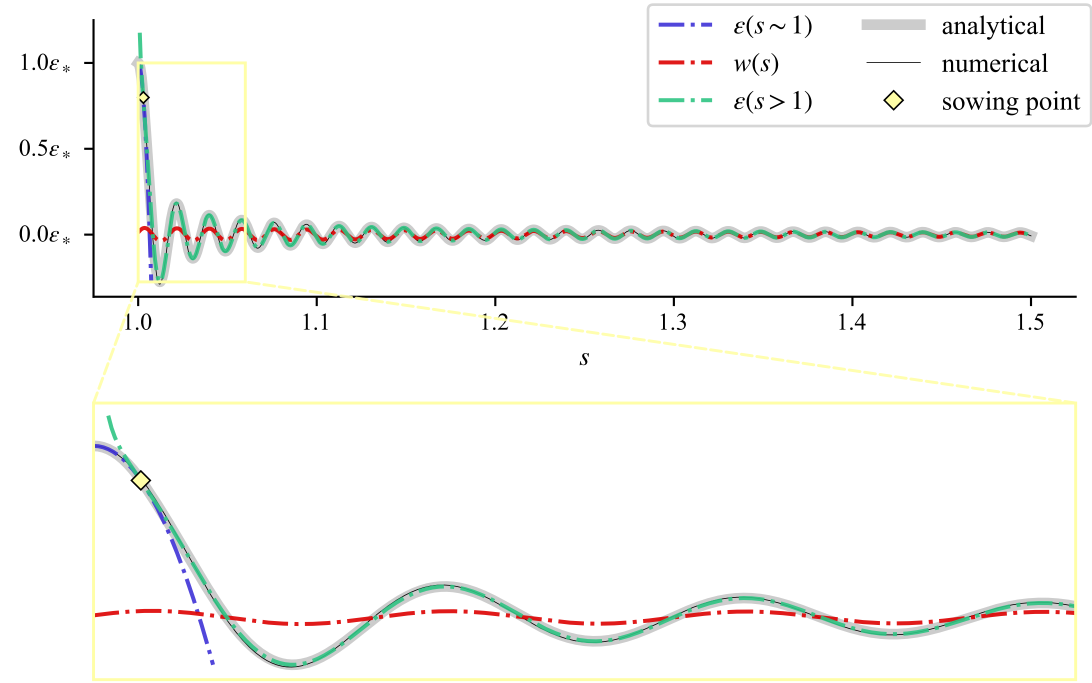


Figure 5.1: Schematic demonstrating how the analytical solution to the eom for $\epsilon_u(s)$.

Complete evolution. We use a computer algebra system, namely *SageMath* ((The Sage Developers, 2023)), to determine A and B from the system of equations

$$\begin{aligned} e^{(\mathrm{I})}(s = s_{\mathrm{sow}}) &= e^{(\mathrm{II})}(s = s_{\mathrm{sow}}) \\ e^{(\mathrm{I})\prime}(s = s_{\mathrm{sow}}) &= e^{(\mathrm{II})\prime}(s = s_{\mathrm{sow}}) \end{aligned} \tag{5.31}$$

⌜maybe put expressions in appendix?⌟ where we choose $s_{\mathrm{sow}} = 1 + u^{-1}$, since only subhorizon modes, $u \gg 1$, are of interest. Now,

$$\epsilon_u(s) = \epsilon_* \cdot \begin{cases} e^{(\mathrm{I})}(s), & s \le s_{\mathrm{sow}}, \\ e^{(\mathrm{II})}(s), & s \ge s_{\mathrm{sow}}. \end{cases} \tag{5.32}$$

In Fig. 5.1 we demonstrate how this solution looks like for arbitrary u, in the different steps described above. ⌜Should include solution in the case where surface tension is constant.⌟

5.3.2 TITLE (Review)

For the time being, let us assume Eq. (5.32) is a good description of the perturbed symmetron wall. We immidiately see that the initial amplitude ϵ_* can be factored out and does not affect the evolution. We keep in mind that this parameter is still important for the validity of the eom that requires $\epsilon_* \ll \tau_*$. The scale parameter $u = p\tau_*$ determines the frequency of oscillations. We observe that this result ⌈is completely independent of τ_*.⌋?

Scaling. It turns out that if plotted over the time variable $t \equiv u(s-1) = p(\tau - \tau_*)$, the solution in Eq. (5.32) is ⌈equal up to normalisation in ϵ_*⌋?

Figure 5.2: Demonstration of the effect of changing scale parameter $u = p\tau_*$.

Spatial part

We need to adress the until now neglected function $\mathscr{E}(x, y)$. It is a solution to

$$\frac{\partial^2 \mathscr{E}}{\partial x^2} + \frac{\partial^2 \mathscr{E}}{\partial y^2} = -p^2 \mathscr{E}, \tag{5.33}$$

and it is natural to demand $\max |\mathscr{E}| = 1$ and $\mathscr{E} \in \mathbb{R}$. The general solution to this equation is $\mathrm{e}^{\mathrm{i}(p_x x + p_y y)}$?

5.3.3 Asymmetron domain walls

Let us have a quick peek at what would happen if introducing asymmetry in the potential. We consider a constant surface tension and vacuum energy densities. In the thin-wall limit, we should be able to use

5.4 Generation of gravitational waves

We have seen that imperfections in what in principle are planar domain walls, do give rise to non-zero coponents $T_{\mu\nu}$. If this does not vanish in the transverse-traceless (TT) gauge, we expect tensor perturbations to the metric, hopefully with a characteristic signature. In this section we present the gravitational-wave calculations blah blah [. . .]. Note that we will only consider conformally flat spacetimes with expansion factor a.

Neglect back-reaction: We assume that the topological defect does not change the un-perturbed metrics of $\mathscr{M}_\pm$. The domain wall is simply viewed as a sheet separating two domains, and the (un-perturbed) metric $g_{\mu\nu}$ that appears in the covariant derivative, d'Alembertian etc., and raises and lowers indices is unaffected by this.

In the absence of asymmetry, a domain wall will not produce disturbances in the gravitational field. However, perturbations to the wall position, such as ripples or wiggles, can reveal themselves as tensor perturbations to the background metric.

blah blah [. . .]

We begin with the perturbed metric $g_{\mu\nu} + \delta g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu})$. ⌈We let $h_{0\mu} = 0$ (why??)⌋?. It is natural to choose the TT gauge, in which $\partial_i h^i{}_j = 0$ and $h^i{}_i = 0$. The perturbed line element noe takes the form

$$ds^2 = a^2(\tau)\Big\{-\mathrm{d}\tau^2 + \Big(\delta_{ij} + h_{ij}(\tau, \boldsymbol{x})\Big)\mathrm{d}x^i\mathrm{d}x^j\Big\}. \tag{5.34}$$

Joke: Are you a wave vector in real space? Because you're impartial! ($k_i \leftrightarrow \mathrm{i}\partial_i$)

5.4.1 TITLE (Dynamics of gravitational waves)

We deduced from the Einstein equation that $\Box h_{\mu\nu} = -16\pi G_{\mathrm{N}} a^2 T_{\mu\nu}{}^{©}_{\text{(some background section)}}$. With the FRLW metric, we get the eom for the tensor perturbation in real space

$$\ddot{h}_{\mu\nu} + 2\mathcal{H}\dot{h}_{\mu\nu} - \boldsymbol{\nabla}^2 h_{\mu\nu} = 16\pi G_{\mathrm{N}} T_{\mu\nu}, \tag{5.35}$$

Suppose $h_{00} = h_{0i} = 0$. We convert to Fourier space $(k_i \leftrightarrow \mathrm{i}\partial_i)^{©}_{\text{(section about this)}}$, and define

$$S_{ij} \equiv 16\pi G_{\mathrm{N}} \Lambda_{ij}{}^{lm} T_{lm}, \tag{5.36}$$

the Fourier image of the TT-part of the SE-tensor multiplied with a prefactor. Now, we recognise the linear polarisation basis for which $S_{ij} = \sum_{P=+,\times} S_P e^P{}_{ij}{}^{©}_{\text{(some prev. section)}}$, and write

$$\ddot{h}_P + 2\mathcal{H}\dot{h}_P + k^2 h_P = S_P. \tag{5.37}$$

Assume $a \propto \tau^\alpha$. The equation can be rewritten in terms of $\bar{h}_P \equiv a h_P$:

$$\ddot{\bar{h}}_P + \left(k^2 - \frac{4\nu^2 - 1}{4\tau^2}\right)\bar{h}_P = aS_P; \quad \nu = \alpha - 1/2 \tag{5.38}$$

$$\ddot{\bar{h}}_P + \left(k^2 - \frac{(\alpha - 1)\alpha}{\tau^2}\right)\bar{h}_P = aS_P \tag{5.39}$$

With the linear differential operator $\mathrm{L}_u = \mathrm{d}^2\big/\mathrm{d}u^2 + 1 - (\nu^2 - \frac{1}{4})/u$ we can write this ODE as $\mathrm{L}_{u=k\tau}\bar{h}_P = k^{-2}aS_P$. Imposing initial conditions $\bar{h}_P(\tau_{\mathrm{init}}) = \dot{\bar{h}}_P(\tau_{\mathrm{init}}) = 0$, we can find the ⌈Green's function $G(u, v) = \Theta(u - u_{\mathrm{init}})G_{\mathrm{r}}(u, v)$ with retarded solution

$$G_{\mathrm{r}}(u, v) = \frac{\pi}{2}\sqrt{uv}\{\mathcal{Y}_\nu(u)\mathcal{J}_\nu(v) - \mathcal{J}_\nu(u)\mathcal{Y}_\nu(v)\}. \tag{5.40}$$

⌋?

$$G_{\mathrm{r}}(u, v) = \chi_n(u)\psi_n(v) - \psi_n(u)\chi_n(v); \quad n = \alpha - 1. \tag{5.41}$$

((Kawasaki and Saikawa, 2011)) We obtain the solution

$$\bar{h}_P(\tau, \boldsymbol{k}) = k^{-2}\int_{\tau_{\mathrm{init}}}^{\tau} \mathrm{d}\hat{\tau}\, G_{\mathrm{r}}(k\tau, k\hat{\tau})a(\hat{\tau})S_P(\hat{\tau}, \boldsymbol{k}) \tag{5.42}$$

⌈Use Boas to argue! (Green's function method, homogeneous initial conditions.)⌋ The complete solution is a long expression, so we decompose $\bar{h}_P = H_P^1 + H_P^2$ where

$$H_P^1(\tau, \boldsymbol{k}) = +\psi_2(k\tau)\int_{\tau_{\mathrm{init}}}^{\tau} \mathrm{d}\theta\, \psi_1(k\theta)\mathcal{T}_P(\theta, \boldsymbol{k}) \tag{5.43}$$

$$H_P^2(\tau, \boldsymbol{k}) = -\psi_1(k\tau)\int_{\tau_{\mathrm{init}}}^{\tau} \mathrm{d}\theta\, \psi_2(k\theta)\mathcal{T}_P(\theta, \boldsymbol{k}) \tag{5.44}$$

$$H_P^1(\tau,\boldsymbol{k}) = +\psi_n(k\tau)\int_{\tau_{\rm init}}^{\tau} \mathrm{d}\theta\, \chi_n(k\theta)\mathscr{T}_P(\theta,\boldsymbol{k}) \tag{5.45}$$

$$H_P^2(\tau,\boldsymbol{k}) = -\chi_n(k\tau)\int_{\tau_{\rm init}}^{\tau} \mathrm{d}\theta\, \psi_n(k\theta)\mathscr{T}_P(\theta,\boldsymbol{k}) \tag{5.46}$$

and $\psi_{\alpha-1}(u) = \mathcal{S}_{\alpha-1}(u)$ with $k^2\mathscr{T}_P(\tau,\boldsymbol{k}) = a(\tau)S_P(\tau,\boldsymbol{k})$.

⌜It is possible since no back-reaction, right? Otherwise, $h_{\mu\nu}$ would contribute on the rhs.⌟

The conformal time derivative becomes

$$\begin{aligned}\dot H_P^1(\tau,\boldsymbol{k}) &= +k\left[\psi_\alpha(k\tau) - nj_n(k\tau)\right]\int_{\tau_{\rm init}}^{\tau} \mathrm{d}\theta\, \chi_n(k\theta)\mathscr{T}_P(\theta,\boldsymbol{k})\\ \dot H_P^2(\tau,\boldsymbol{k}) &= -k\left[\chi_\alpha(k\tau) - ny_n(k\tau)\right]\int_{\tau_{\rm init}}^{\tau} \mathrm{d}\theta\, \psi_n(k\theta)\mathscr{T}_P(\theta,\boldsymbol{k})\end{aligned} \tag{5.47}$$

Free waves. If at some point in time $\tau_{\rm fin}$ the source is gone blah blah [...] , and so the waves propagates freely in the universe (vacuum).

5.4.2 Fourier space stress-energy tensor

- Fourier space SE tensor
- TT gauge

From section above$^{©}_{0}$ we found that the SE tensor of a thin domain wall in an expanding universe looks like this:

$$\begin{aligned}T_{ab}(\tau,\boldsymbol{x}) &= -a(\tau)\sigma(\tau)\Phi_l(z - z_{\rm dw})\,\eta_{ab}\\ T_{(iz)}(\tau,\boldsymbol{x}) &= -a(\tau)\sigma(\tau)\Phi_l(z - z_{\rm dw})\,\epsilon_{,i}\end{aligned} \tag{5.48}$$

where $z_{\rm dw} = z_0 + \epsilon(x^a)$ and $\Phi_l(z - z_{\rm dw}) = (2\pi l^2)^{-1/2}\exp\left\{-(z - z_{\rm dw})^2/(2l^2)\right\}$. We go further and look at this quantity in Fourier space:

$$\begin{aligned}T_{ab}(\tau,\boldsymbol{k}) &= -a(\tau)\sigma(\tau)\,\eta_{ab}\,\mathrm{e}^{-k_z^2l^2/2}\mathrm{e}^{-\mathrm{i}k_z z_0}\int d^2x\,\mathrm{e}^{-\mathrm{i}k_z\epsilon(\tau,x,y)}\mathrm{e}^{\mathrm{i}k_x x}\mathrm{e}^{\mathrm{i}k_y y}\\ T_{(iz)}(\tau,\boldsymbol{k}) &= -a(\tau)\sigma(\tau)\,\mathrm{e}^{-k_z^2l^2/2}\mathrm{e}^{-\mathrm{i}k_z z_0}\int d^2x\,\partial_i\epsilon(\tau,x,y)\mathrm{e}^{-\mathrm{i}k_z\epsilon(\tau,x,y)}\mathrm{e}^{\mathrm{i}k_x x}\mathrm{e}^{\mathrm{i}k_y y}\end{aligned} \tag{5.49}$$

Let us now say $\epsilon(x^a) = \epsilon_p(\tau)\mathscr{E}(x,y)$ where $\partial_a\mathscr{E} = -\mathrm{i}p_a\mathscr{E}$ blah blah [...] . In addition, we can put p_a along the y-axis. Now

$$\underbrace{\int \mathrm{d}y\,\mathrm{e}^{-\mathrm{i}k_z\epsilon_p(\tau)\cdot\mathscr{E}(y)}\mathrm{e}^{\mathrm{i}k_y y}}_{\mathscr{I}_1} \quad\text{and}\quad \underbrace{\epsilon_p(\tau)\int \mathrm{d}y\,\partial_y\mathscr{E}\,\mathrm{e}^{-\mathrm{i}k_z\epsilon_p(\tau)\cdot\mathscr{E}(y)}\mathrm{e}^{\mathrm{i}k_y y}}_{\mathscr{I}_2} \tag{5.50}$$

are all we need to solve to have a completely analytic expression for $T_{ij}(\tau,\boldsymbol{k})$. We write $T_{xx}(\tau,\boldsymbol{k}) = \mathcal{T}(\tau,k_x,k_z)\mathscr{I}_1(\tau,k_y,k_z)$ and $T_{yz}(\tau,\boldsymbol{k}) = \mathcal{T}(\tau,k_x,k_z)\mathscr{I}_2(\tau,k_y,k_z)$ where

$$\mathcal{T}(\tau,k_x,k_z) = -2\pi a(\tau)\sigma(\tau)\delta(k_x)\mathrm{e}^{-k_z^2l^2/2}\mathrm{e}^{-\mathrm{i}k_z z_0} \tag{5.51}$$

is considered dimensionless. ⌜Comment about cylindrical coordinates?⌟

Choice of spatial part

It is not obvious what to choose for $\mathscr{E}(y)$. In this project, we started out with $\mathscr{E}(y) = \sin py$, which luckily worked out (though not easily).

The advantage of reducing the problem to spatial dimensions y and z becomes very clear when converting to a linear polarisation basis. We show in appendix $\mathrm{X}_0^{©}$ that

$$e^{+}_{ij}(\boldsymbol{k}) = \frac{1}{k^2}\begin{pmatrix} k^2 & 0 & 0 \\ 0 & -k_z^2 & k_y k_z \\ 0 & k_y k_z & -k_y^2 \end{pmatrix}_{ij} \quad \wedge \quad e^{\times}_{ij}(\boldsymbol{k}) = \frac{1}{k}\begin{pmatrix} 0 & k_z & -k_y \\ -k_z & 0 & 0 \\ k_y & 0 & 0 \end{pmatrix}_{ij} \tag{5.52}$$

holds for $\boldsymbol{k} = (0, k_y, k_z)$, which is enforced by the Fourier transform of unity in $T_{ij}(\tau, \boldsymbol{x})$. We observe that $T_{ix} = \delta_{ix} T_{xx}$, so $T_\times = 0$.

blah blah [. . .] (appendix)

$$\begin{aligned} \mathscr{I}_1 &= 2\pi \sum_{n\in\mathbb{Z}} \delta(k_y + np) \cdot \mathcal{J}_n(k_z \epsilon_p) \\ \mathscr{I}_2 &= 2\pi \sum_{n\in\mathbb{Z}} \delta(k_y + np) \cdot \mathcal{J}_n(k_z \epsilon_p) \cdot \frac{np}{k_z} \end{aligned} \tag{5.53}$$

We explore the general behaviour of the non-vanishing modes.

Transverse-traceless gauge

We extract the transverse and traceless part of the SE tensor by use of the projection $\text{operator}^{©}_{\text{(early chap.)}}$.

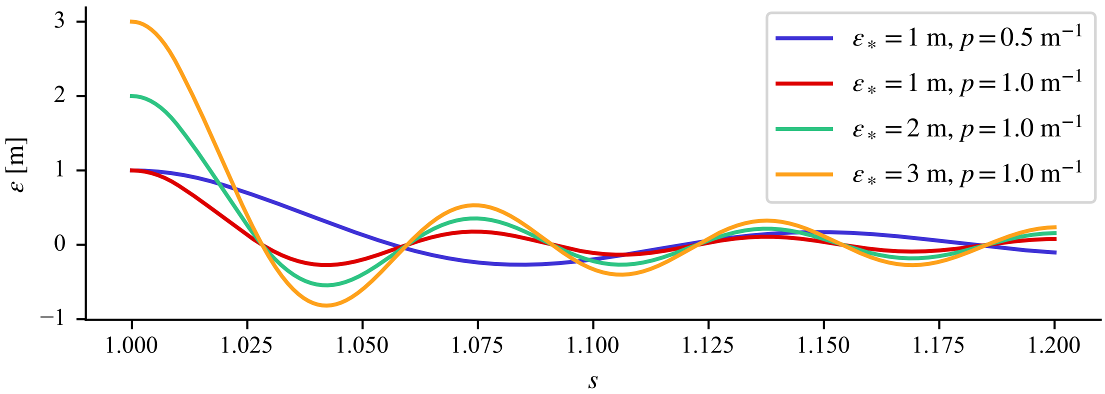
$$T_+^{\mathrm{TT}} = T_{xx}^{\mathrm{TT}} = \frac{1}{2k^2}\Big[k_y^2 T_{xx} + 2k_y k_z T_{xy}\Big] \tag{5.54}$$

$$T_+^{\mathrm{TT}} = \frac{1}{2k^2}\,\mathcal{T}(\tau, k_x, k_z)\Big[k_y^2 \mathscr{I}_1 + 2k_y k_z \mathscr{I}_2\Big] = -\frac{k_y^2}{2k^2}\,\mathcal{T}(\tau, k_x, k_z)\mathscr{I}_1(\tau, k_y, k_z) \tag{5.55}$$

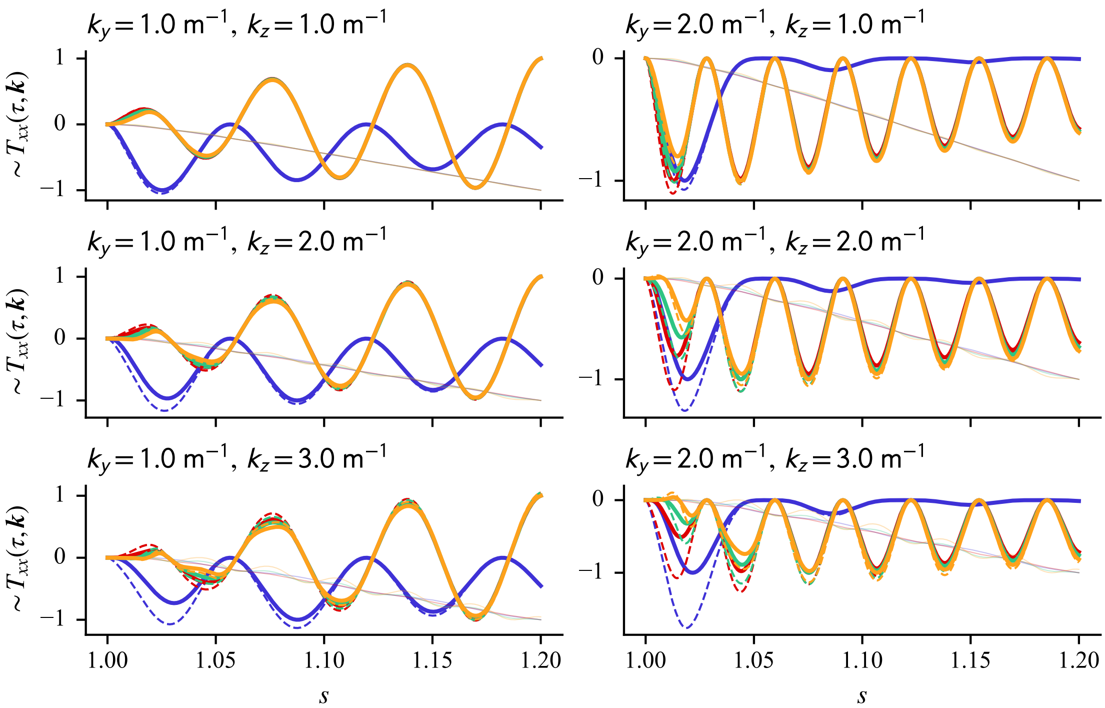
5.4.3 Examples

How will the domain wall manifest in the gravitational waves, given our equations? Let us have a look at some examples. When $z_0 = 0$, $h_+ = \Re\mathfrak{e}\{h_+\}$.

Both the perturbation scale parameter u *and* the initial amplitude ϵ_* contribute to the GW signature.

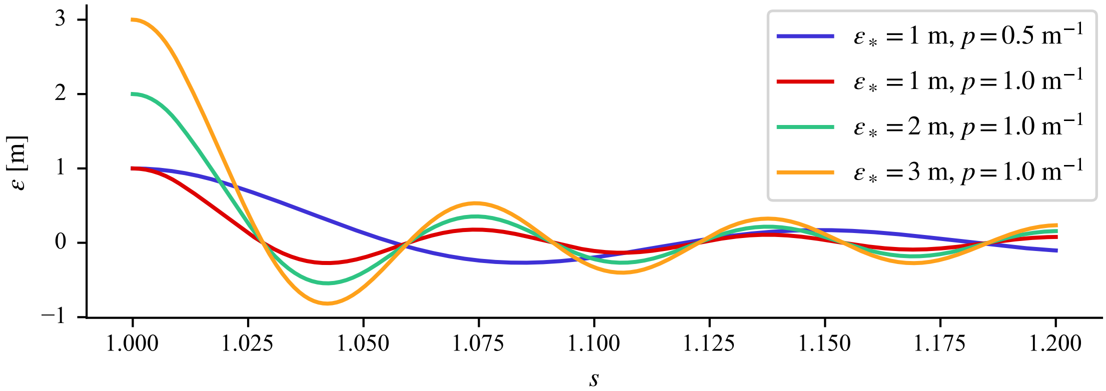


(a) X.

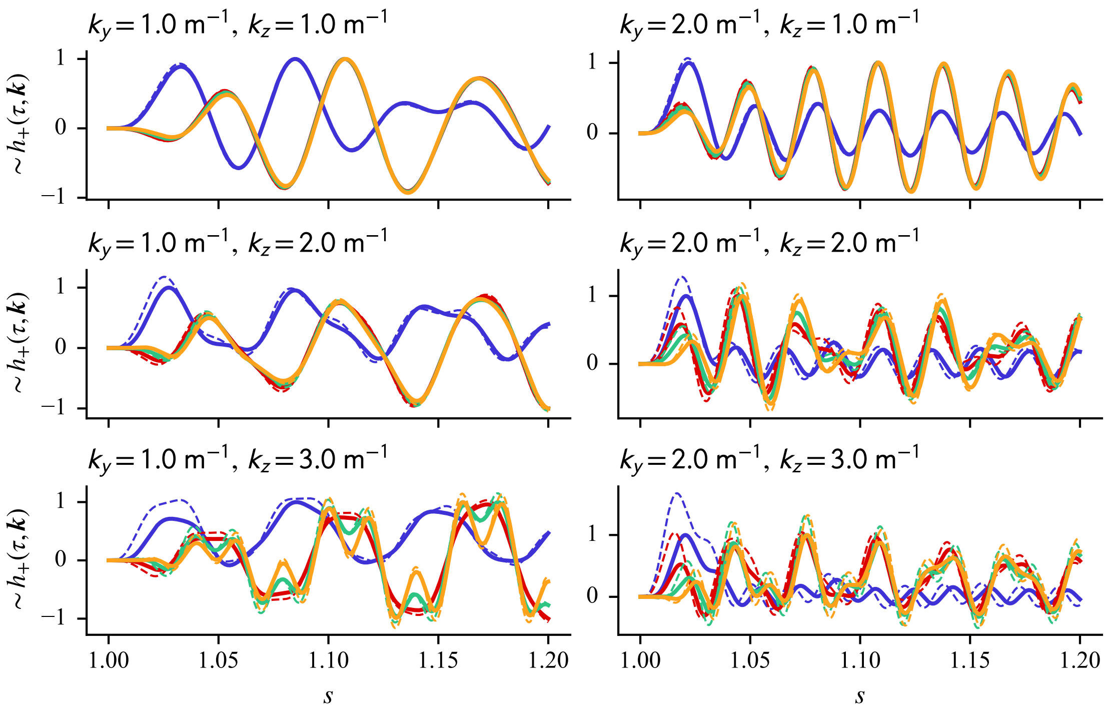


(b) Gravitational waves from ....

Figure 5.3: In units where m $\equiv \tau_*/100$.



(a) X.



(b) Gravitational waves from ....

Figure 5.4: In units where $\mathrm{m} \equiv \tau_*/100$.

Chapter 6

TITLE (Kinks in Cosmology)

- Testing the framework
- Different theory
- First, go through the field description (modified gravity, Symmetron)

A variety of challenges present themselves when simulating cosmological scenarios.

6.1 Symmetron domain walls

From Sect. 3.5 we have the (a)symmetron effective potential and . . .

Using the symmetron model to represent a domain wall in an FRLW universe, $\Box\phi = V_{\text{eff},\phi}$ becomes

$$-a^{-2}\Big[\ddot{\phi} + 2\mathcal{H}\dot{\phi} - \nabla^2\phi\Big] = \lambda\phi^3 + \mu^2(\upsilon - 1)\phi. \tag{6.1}$$

From here, we will use $\chi = \phi/\phi_\infty$ and $\chi_\pm = \sqrt{1-\upsilon}$. We recall that $\upsilon = \rho_{\text{m}}/\rho_{\text{m}*} = (a_*/a)^3$.

Prior to SSB, the scalar field is trivial, and so we move on to consider χ from this critical point where the quartic term turns negative and the $\mathbb{Z}_2$ symmetry is spontaneously broken.

6.1.1 Quasistatic limit

We can solve

$$\nabla^2\chi \simeq +\big(\mu^3/\sqrt{\lambda}\big)\cdot a^2\big[\chi^2 - (1-\upsilon)\big]\chi \tag{6.2}$$

to obtain the solution in the limit where spatial gradient plays a much larger role that time derivatives. We let $\chi = \chi(a, z)$ and use the well-established ((see e.g. Llinares and Pogosian, 2014)) expression for a domain wall when $a \propto \tau^\alpha$

$$\chi(a, z) = \sqrt{1-\upsilon}\tanh\left(\frac{a(z - z_{\text{dw}})}{2L_{\text{C}}}\sqrt{1-\upsilon}\right). \tag{6.3}$$

$L_{\text{C}} = (\sqrt{2}\mu)^{-1}$ is known as the Compton wavelength of the symmetron, a measurement that will become significant later when blah blah [. . .]. We write $a = a_*(\tau/\tau_*)^\alpha$, which gives $\upsilon = (a_*/a)^3 = (\tau_*/\tau)^{3\alpha}$.

PHANTOM PARAGRAPH: SÅRBARHETER ETC., THICKNESS

Figure 6.1: Quasistatic evolution of the domain wall represented by χ.

Basic properties. Vilenkin and Shellard ((1994)) define the wall thickness as the $(z - z_{\rm dw})$-coordinate at which the argument in the tanh-function is $1/\sqrt{2}$, which is $\delta_\infty \equiv \mu^{-1} = \sqrt{2}L_{\rm C}$. The surface tension of such a conventional wall is estimated $\sigma_\infty \equiv \frac{2\sqrt{2}}{3}\mu^3/\lambda^{©}_{\text{(maybe background section?)}}$. Extrapolated to expanding spacetime, we get

$$a\delta_{\rm dw} = \delta_\infty(1-\upsilon)^{-1/2} \quad \text{and} \quad \sigma_{\rm dw} = \sigma_\infty(1-\upsilon)^{3/2} \tag{6.4}$$

as expressions for the comoving thickness $\delta_{\rm dw}$ and the surface energy density $\sigma_{\rm dw}$. The system of N kinks and M antikinks is represented by the field $\phi = \phi_\infty\chi$ with

$$\chi = \prod_i^N \chi_{\rm k}(z - k_i)\prod_j^M \bar{\chi}_{\rm k}(z - \bar{k}_j), \tag{6.5}$$

where

$$\chi_{\rm k}(z - k_i) = \sqrt{1-\upsilon}\tanh\left(\frac{a(z-k_i)}{\sqrt{2}\delta_{{\rm k},i}}\right) \tag{6.6}$$

and $\bar{\chi}_{\rm k} = -\chi_{\rm k}$.

6.1.2 Asymptotic limit

We let $\check{\chi}$ denote the field value far away from the wall, well inside the domain. Here,

$$\ddot{\check{\chi}} + 2\mathcal{H}\dot{\check{\chi}} = -\left(\mu^3/\sqrt{\lambda}\right)\cdot a^2\left[\check{\chi}^2 + \upsilon - 1\right]\check{\chi} \tag{6.7}$$

controls the evolution of the field strength. The trivial solution becomes unstable after the phase transition, and the field may fall into any of the two vacua. We take a look at one of the minima. The positive minimum, which was zero at PT, goes as $\chi_+ = \sqrt{1-\upsilon}$. Now, the rate at which χ_+ moves from its initial value, blows up at the phase transition, but decays rapidly when approaching the limit value. We solve this issue by parting the equation in two regimes; **(I)** a quick, non-adiabatic regime in which V_{eff} changes faster than what χ can possibly follow and **(II)** an adiabatic regime where V_{eff} changes much slower than χ. What happens is that the field rolls towards the minimum and begins to oscillate around it whilst following its slow drift. ⌈Plagiarism? (Julian's notes)⌋? The oscillation amplitude is decided by the initial conditions of the field.

PHANTOM PARAGRAPH: WHY DO WE NOT WANT OSCILLATIONS?

We can rewrite Eq. 6.7 in terms of the time coordinate $\chi_+ = \sqrt{1-\upsilon}$,

$$\frac{\mathrm{d}^2\breve{\chi}}{\mathrm{d}\chi_+^2} - \frac{1}{\chi_+\left(1-\chi_+^2\right)}\frac{\mathrm{d}\breve{\chi}}{\mathrm{d}\chi_+} + m^2\frac{\chi_+^2\left(\breve{\chi}^2-\chi_+^2\right)}{\left(1-\chi_+^2\right)^3}\breve{\chi} = 0, \tag{6.8}$$

where

$$m = \frac{2\mu}{3\mathcal{H}_*(1+\mathfrak{z}_*)} = \frac{\sqrt{2}a_*^{3/2}}{3\xi_*}. \tag{6.9}$$

The idea is to use this solution as boundary conditions for χ:

$$\chi|_{z\to\pm\infty} = \pm\breve{\chi} \quad \wedge \quad \dot{\chi}|_{z\to\pm\infty} = \pm\dot{\breve{\chi}} \tag{6.10}$$

We solve in two regimes, each solution expanded around **(I)** $\chi_+ = 0$ and **(II)** $\chi_+ = 1$:

$$\breve{\chi}^{\textbf{(I)}} = \chi_* + \frac{C}{2}\chi_+^2 + \frac{C-\chi_*^3 m^2}{8}\chi_+^4 \tag{6.11a}$$

$$\breve{\chi}^{\textbf{(II)}} = \chi_+ + \frac{8(3-m^2)}{m^4}(\chi_+-1)^3 + \frac{1440-636m^2+41m^4}{2m^6}(\chi_+-1)^4 \tag{6.11b}$$

6.1.3 Examples

We employ the

We solve 6.7 numerically for various sets of initial conditions to demonstrate blah blah [. . .]

6.1.4 Compton wavelength

Much like GR, this thesis is highly non-linear. It is near impossible to preserve a causal structure, to write it "chronologically," if one also aims to divide it into subjects for readability.

6.2 Dynamic modelling

When translating a theory consisting of nice, continous functions with theoretical limits to a discrete, numerical system, there are blah blah [. . .]. When said theory also includes a phase transition, it does not become any easier. Phase transitions are manifestly computational headaches, and we are limited by blah blah [. . .]. The discontinuity introdused by the phase transition is even more complicated to replicate in simulations.

PHANTOM PARAGRAPH: THE CODE (GEVOLUTION ETC.) (MAYBE OWN SECTION?)

PHANTOM PARAGRAPH: BCS

We will have a look some of the most pressing matters when aiming to test the theory from Ch. 5 in simulations.

Figure 6.2: Evolution of the minima $\check{\chi}$.

6.2.1 Spatial and temporal resolution

6.2.2 Memory

6.3 `gwasevolution` (On the lattice?)

The code `gwasevolution`[1] is an extended version of the massively parallelised N-body relativistic code `gevolution` ((Adamek et al., 2016)).

The general setup is a three-dimensional box of equal side lengths L on a lattice of N^3 points, giving a spatial resolution of $\Delta x = L/N$.

The mapping from code configuration space to comoving coordinates is:

$$\begin{aligned} \boldsymbol{x} &= \mathtt{x}_{\mathtt{i},\mathtt{j},\mathtt{k}} \cdot \Delta x \\ (x, y, z) &= (\mathtt{i}, \mathtt{j}, \mathtt{k}) \cdot \Delta x \end{aligned} \tag{6.12}$$

Meanwhile, the corresponding mapping in Fourier space is given in terms of the fundamental frequency $k_\mathrm{f} = 2\pi/L$:

$$\begin{aligned} \boldsymbol{k} &= \mathtt{k}_{\mathtt{u},\mathtt{v},\mathtt{w}} \cdot k_\mathrm{f} \\ (k_x, k_y, k_z) &= (\mathtt{u}, \mathtt{v}, \mathtt{w}) \cdot k_\mathrm{f} \end{aligned} \tag{6.13}$$

⌜comment about hermitian symmetry?⌟

6.3.1 Initial configuration

For the purpose of exploring the validity of our equations, the simulation setup needs be controlled ⌜no stochasticity⌟. This is why we cannot initialise a cosmological scenario where

[1] Written by Christiansen et al..

a phase transition is ⌈in the cards (on the horizon)⌋; we would never achieve a domain wall lying perfectly at $z = z_0 + \epsilon_* \sin py$.

Ignore the perturbation for now. The 3D simulation box of side lengths L has in total N^3 lattice points.[2] To preserve the periodic boundary conditions, we need at least two walls. We place one topological defect at $z = L/2$ and its counterpart (the antikink) at $z = 0$. The simplest way to achieve this is by initialising the scalar field χ (`achi` in the code) with

$$\chi_+ \tanh\left(\frac{a(z - z_{\mathrm{dw}})}{2L_{\mathrm{C}}}\chi_+\right). \tag{6.14}$$

6.3.2 Discrete Fourier space

blah blah [...]

$h_{ij}(\eta, \boldsymbol{k}) = \Lambda_{ijkl}(\boldsymbol{k}) h_{kl}(\eta, \boldsymbol{k})$ where $\boldsymbol{k}$ is *not* defined $\boldsymbol{k} = 2\pi(\mathtt{u}, \mathtt{v}, \mathtt{w})/L$

6.3.3 Options/parameters

We now turn to the details of the simulations we are to perform. For starters, we provide options and compiler flags to make certain that fields χ and h_{ij} are computed.

Numerics. There are choices to be made concerning temporal and spatial resolution, and what integration method to use. Of solvers, there are a handful to choose from, for instance Leap-Frog or fourth order Runge-Kutta. The spatial resolution $\Delta x = L/N$ is set by the users choice of box side length L in Mpc$/h_0$ and the number of grid points N in each direction. The temporal resolutions vary for some field updates. They are controlled by the *Courant factors*

$$C_{\mathrm{f}} = v_{\mathrm{g}} \Delta\tau / \Delta x \tag{6.15}$$

where blah blah [...]

⌜L and $\mathfrak{z}_{\mathrm{init}}$ difficult to place in category.⌟

Physics. More exciting are the parameters that directly relates to our theoretical setup and cosmological scenarios. We can set the simulation box size, L,

Asymmetron parameters. The code takes the phenomenological parameters $\mathrm{As}_* = \{a_*, \xi_*, \beta_+, \beta_-\}$ and maps them to Lagrangian parameters $\mathrm{As}_{\mathcal{L}} = \{M, \mu, \lambda, \kappa\}$ as described in some section or appendix$^{©}_{0}$.

Perturbation parameters. We added some options to the initialisation-part of the code so to easily create the idealised scenario we want. It has options for this kind of perturbation in the z-direction:

$$\epsilon = \epsilon_p(\tau)\,\mathrm{tri}\left\{p_x x + p_y y\right\}; \quad \mathrm{tri} \in \{\sin, \cos\}, \tag{6.16}$$

for which the user can provide initial amplitude ϵ_* and perturbation scale in terms of scaled, integer wavenumbers $m_{\mathtt{i},\mathtt{j}} = p_{x,y}/k_{\mathrm{f}}$.

We need to be careful blah blah [...]

⌜Surface tension "tension": Sensitivity to this.⌟

[2]Two dimensions would suffice for this problem, but `gevolution` only takes cubic boxes.

6.4 Simulation setups

The simulations that were performed in this project and are referenced in the coming discussions are listed in table XXX for reference.

type-0 Use Eq. 6.14 for χ and its conformal time derivative for $\dot{\chi}$ at some time shortly after $\mathfrak{z}_*$

type-$\star$ Use tweaked initial conditions for field,

6.4.1 Catalogue

Every simulation used a 4th order Runge-Kutta solver blah blah [. . .]

Part III

Findings

Chapter 7

TITLE (Results)

This chapter both presents and discusses the results concerning the wall normal coordinate ϵ from some section$^{©}_{\text{(about analytic work)}}$ and the domain wall scalar field χ. We compare results from analytical calculations and numerical simulations.

In order to keep track of the various results, or rather how they were obtained, we create a syntax that should make this task simple. When relevant, we will use the simulation labels from XXX$^{©}_{()}$. In this section we make use of keys **A**, **N** and **S**, described below.

A Completely analytical solution.

N Numerical solution (`odeint`).

S Simulated result.

In general, the complete label of one result (perhaps a graph) might be **3)**ϵ**:N1.b**, referring to setup 3, numerical solution (variant 1) with initial conditions b. It will become clear what this means in particular in the following sections.

7.1 Symmetron field

The symmetron field χ (`achi` in the code) will at SSB roll into either minima, depending on the sign of the field right before it happens. The strength of the oscillations around the true minima depend on both the initial field value and its time derivative.

As we saw in section XX$^{©}_{()}$, the solution for

7.1.1 Technical note

We have the equations

$$\ddot{\phi} + 2\dot{a}/a\,\dot{\phi} - \nabla^2\phi = -a^2\,\mathrm{d}V_{\text{eff}}/\mathrm{d}\phi \tag{7.1a}$$

and

$$\ddot{\mathring{\phi}} + 2\dot{a}/a\,\dot{\mathring{\phi}} = -a^2\,\mathrm{d}V_{\text{eff}}/\mathrm{d}\phi \tag{7.1b}$$

The latter is in focus first. We compare solutions obtained with different methods (**A**/**N**/**S**), variants (**–**/**0**/**1**/**X**) and initial conditions (**a**/**b**/**c**). Subtype:

0 Minima $\chi_{\pm} = \pm\sqrt{1-\upsilon}$

1

T Equation

X χ evaluated at $z = 0.8L$

B $\max\chi$ ⌜or $\sqrt{\chi_{\text{avg}}}$⌟

Initial conditions $(\tau_0, \check{\chi}(\tau_0), \dot{\check{\chi}}(\tau_0))$:

t $(\tau_*, \chi_*, 3C)$

b

7.1.2 Some title

The solution to the

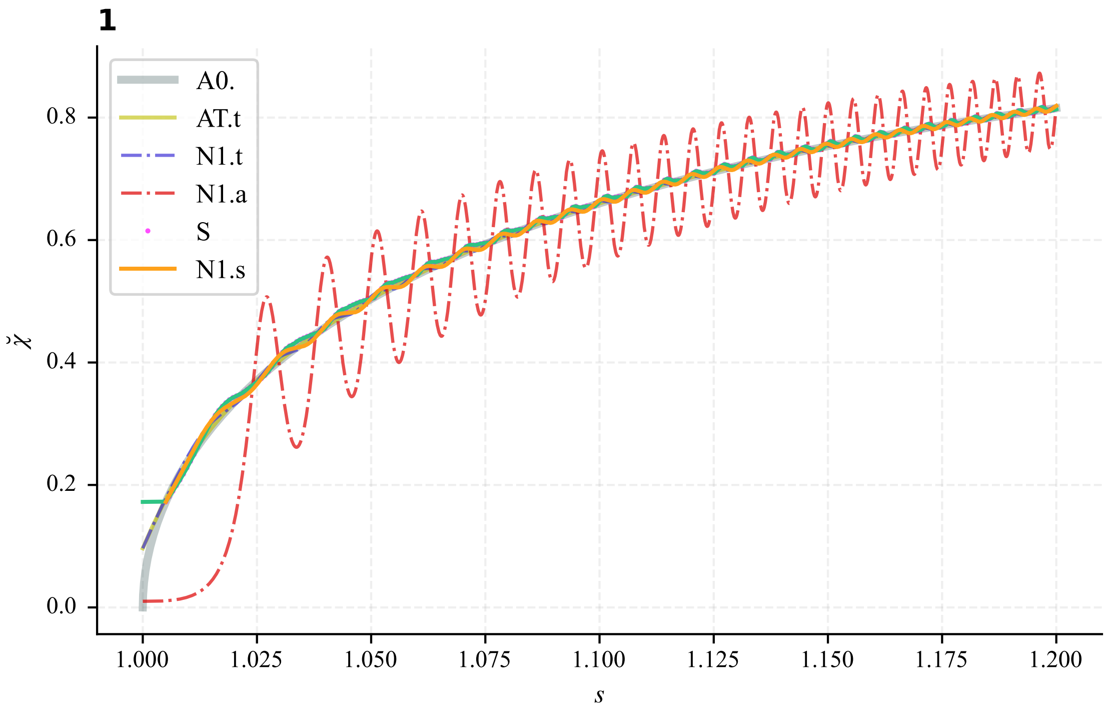


Figure 7.1: $\check{\chi}$ as function of conformal time $s = \tau/\tau_*$.

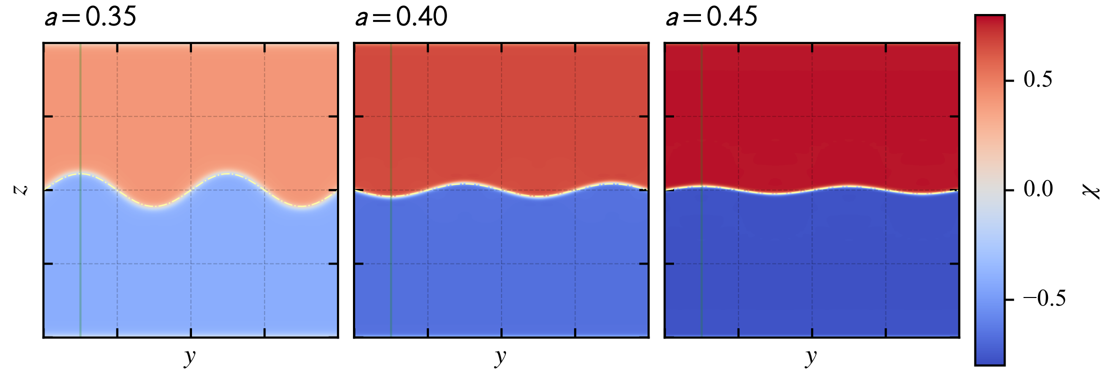
These results, and simply the fact that they are blah blah [...]
The dimensionless time variable $t_p = (\tau - \tau_{\text{init}}) \cdot p$ or $t_p = (\tau - \tau_*) \cdot p = (s-1)u$

7.2 Domain wall dynamics

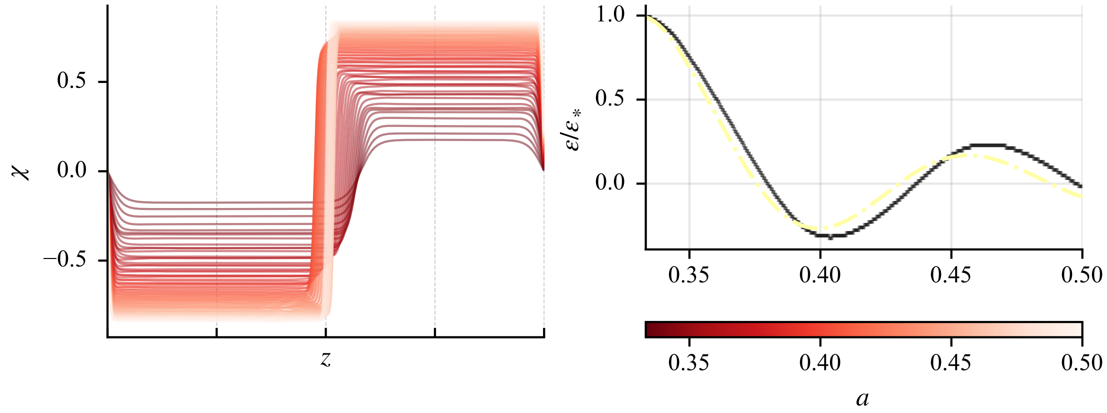
Simulation-wise, the (middle) wall position is tracked by the minimum value of $|\chi|$, i.e. the z-coordinate at which the field is closest to zero.

We reduce the problem from three dimensions to two with cylindrical symmetry, and then again to one dimension by considering a suitable slice in the y-direction and taking the coordinate of the minimal absolute value of the scalar field χ. An example of the two-dimensional perspective is shown in Fig. 7.2a. This picture is more or less the same for all simulations, at least when comparing by-eye.

We see that the quasi-static tanh-solution varies in applicability as it occurs "bumps" around each wall after some time.



(a) The domain wall evolution in two dimensions. We indicate the aforementioned slice with a green vertical line.



(b) *Left panel:* The scalar field value along y-coordinate XXX at different redshifts. *Right panel:* The wall coordinate as function of time. Note that the colour bar share the same axis as the blah blah [. . .] .

Figure 7.2: Demonstration of results from simulation 1.

–: constant surface tension . . . 0: surface tension $\sim (1-\upsilon)^{3/2}$ . . . 1: surface tension different? . . . X: ?

Labels: **Tn.i** = Type T, equation/method n, initial conditions i

7.2.1 Technical note

This section's focus is on the solutions to

$$\ddot{\epsilon}_p + (3\dot{a}/a + \dot{\sigma}/\sigma)\,\dot{\epsilon}_p + p^2\epsilon_p = 0. \tag{7.2}$$

That is, we compare solutions obtained with different methods (**A**/**N**/**S**), variants (**–**/**0**/**1**/**X**) and initial conditions (**a**/**b**/**c**) Types:

A Completely analytical solution to eom.

N Numerical solution (`odeint`) to eom.

S Simulated result.

Subtype:

– Eom for $\epsilon_p(\tau)$ with $\sigma = \sigma_\infty$.

0 Eom for $\epsilon_p(\tau)$ with $\sigma = \sigma_\infty(1-\upsilon)^{3/2}$.

1 Eom for $\epsilon_p(\tau)$ with $\sigma = \sigma_\infty/2 \cdot \big(3(1-\upsilon) - \breve{\chi}^2\big)\breve{\chi}$.

X Where $|\chi|$ takes its minimum value.

Initial conditions $(\tau_0, \epsilon_p(\tau_0), \dot{\epsilon}_p(\tau_0))$:

a $(\tau_*, \epsilon_*, 0)$

b $(\tau_{\text{init}}, \epsilon_p(\tau_{\text{init}})$ from **A0.a**, $\dot{\epsilon}_p(\tau_{\text{init}})$ from **A0.a**)

7.2.2 Idk

- The initial amplitude affects the phase of the simulated wall evolution, according to simulations.
- In any case, the wall pos. evolution is quite consistent for different levels of oscillations
- $\epsilon_* > 1/p$: We will see great impact of changing ϵ_*
- Figure showing difference in position graphs

The simulated wall position graph is not perfectly overlapping with the analytical one. Simulated walls have tendency to change slower, at least initially, manifesting in a phase difference between $\epsilon_p(\tau)$ from simulation and thin-wall approximation. If not due to numerical error, this is necessarily either a consequence of the field-like description or possibly another damping term in the eom for ϵ_p. In the latter scenario, one could guess that the expression for the surface tension is not flawless (something else would insinuate that the expansion term is wrong, which is not the case.) With better spatial resolution, there was no improvement for this part. Initialising simulations even closer to symmetry break enhanced oscillations and increased the phase difference. Increasing the box size—and scaling all parameters thereafter—did not have any effect in this matter.

We saw that initial amplitude actually did matter in simulations, cf. simulation 1 vs. 2. The thin-wall approximation does not say this, however, in fact it says the opposite; ⌈the eom is scale invariant, and thus unchanged by translations.⌋? It is therefore hard to argue that this motion is possible to reproduce by adjusting terms in the eom.

⌜Possible explanation: Field evolution not independent in y-direction, at least when ϵ_* is comparable to $1/p$⌟

The Wiener process is scale-invariant…

We take a closer look at the wall evolution in one particular simulation.

Why not?

In this section, we want to provide answers to the most likely questions the reader might have. Near on any question starting with “why did you not ...” may be answered “because of temporal and computational limitations.”

Perturbation amplitude. Why was it not increased to better resolve the motion in space? Recall that there needs to be *two* walls present, and the kink profile should really not affect the antikink profile. In the quasi-static limit, the wall's thickness goes as $\sim (a\chi_+)^{-1}$, i.e. from infinitely large at symmetry break.

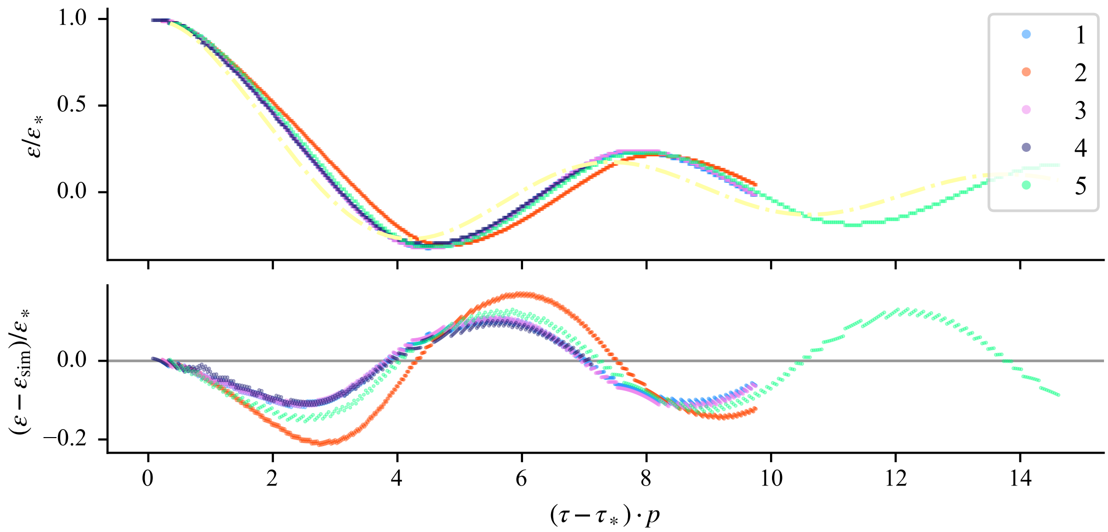


Figure 7.3: The absolute difference between the wall position from calculations and simulations as functions of time.

7.3 Gravitational waves from perturbed symmetron domain walls

The tensor field is in principle more complex (literally) than the previously discussed scalar field. It is therefore advantageous that our proposed field only has one degree of freedom since—in theory—$h_\times = 0$.

As we saw in the previous section, the wall evolution differs somewhat in the two models. In this section, we will investigate if and how this difference affects the gravitational wave modes, and we will blah blah [. . .]

About the output from the code ... We quickly see from calculations that $h_{ij}(\tau, \boldsymbol{k}) \in \mathbb{R}$. The code will have it differently, however, and consistently produces non-negligible imaginary components. Likely, this has to do with the different Fourier conventions used by hand and by code. We have not been able to resolve this completely (i.e. find a suitable mapping), and so we present only the magnitude of the strain.

Computing the semi-analytical expression ... To find $H_{1,2}(\tau, \boldsymbol{k})$ we need to use a numerical solver, and for this we chose *Numpy*'s `cumtrapz`; a method for integrating cumulatively with the trapezoidal rule.

We extract the relevant output from `gwasevolution` to compare with the analytical calculations. Nothing is assumed about the temporal part of the wall normal coordinate, so we may insert any function as $\epsilon_q(\tau)$ into Eq. XXX$^{©}_{\text{(main expr.)}}$. This is a huge advantage since the results from Ch. 7 are not perfect.

An even bigger advantage would be to have the code output the wall position as a near-continuous function of time, but we only have the profile extracted from `achi` animation outputs, giving it a function with blah blah [. . .]

A few take-aways. There are some results that need be mentioned, but not necessarily presented plots.

Figure 7.4: The wall position as depicted by . . . .

Periodicity in y-mode. The outputted `hijFT` shows significantly smaller strains for $K_Y \neq nm_Y$ than for $K_Y = nm_Y$, something we interpret as a corroboration to the Dirac–delta factor in Eq. XXX$^{©}_{\text{(main expr.)}}$. It being non-zero may be a result of numerical error, but it is likely also related to the issue with the wavenumber ambiguity$^{©}_{()}$.

$K_Y = 0$ is non-zero. For whatever reason, the code insists there are significant tensor perturbation propagating in the Z-direction. This is not what we expected from calculations, where $\boldsymbol{k} = (0, 0, k_z)$ corresponds to zero strain.

⌜Looks like $k_z < k_y$ does not carry much information, but I believe that makes sense. They should not be "kinky" here, but free.⌟ ⌜Real component looks generally more messy, might be due to not perfect spatial part?⌟

7.3.1 Technical note

7.3.2 Idk

- Adding ingredients to expression for h_{ij} $\rightsquigarrow$ Separate effects
  - Changing ϵ: Changes the "phase"
  - Changing σ: Introduces more small oscillations
  - Changing l (thickness): Even more of the small oscillations

Chapter 8

TITLE (Ifs, buts and maybes)

⌜Maybe this chapter should be post-simulation work? I.e. implications? ⌟

So the framework is not rock solid, but it is definitely *something* true about the equations. We reserve this chapter for the ifs, buts and maybes. To keep the discussion at bay, we focus only on the +-polarised wave of the tensor perturbation.

- insert sensible parameters

8.1 Limits of the framework

Let us review the equations this thin-wall approximation is built on. We want everything up to the Fourier SE tensor to be analytically solvable, at least to some level that resembles the actual situation, like when using $\sigma \propto (1-\upsilon)^{3/2}$ in this project. ⌜The behaviour is recognisable at this stage.⌟ Okay, so far, so good. We found that for a two-dimensional topological defect in a conformally flat spacetime, we have the SE tensor

$$T_{\mu\nu} = \tag{8.1}$$

(z-plane)

8.2 Superpositions

- Adding propagating waves on torus
- What would happen if there were two such perturbations? or several pert. walls?

Chapter 9

Conclusion and Outlook

- Surface-tension tension: There is a strong dependence on σ.
- what I would do if time
- actual observables
- concluding remarks

9.1 Applications

9.2 Future work

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Appendix A

I do not have an appendix