UNIVERSITY
OF OSLO

Master's thesis

Gravitational waves from topological defects

Any short subtitle

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CS: Astrophysics
60 ECTS study points

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Spring 2024

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Contents

Notation

Constants and units. We use ⌈'natural units'⌋? where $\hbar = c = 1$, where $\hbar$ is the reduced Planck constant and c is the speed of light in vacuum.⌜Planck units? Set $k_\mathrm{B} = G_\mathrm{N} = 1$?⌟ The Newtonian constant of gravitation G_N is referenced explicitly, and we use Planck units such as the Planck mass $M_\mathrm{Pl} = (\hbar c/G_\mathrm{N})^{1/2} = G_\mathrm{N}^{-1/2} \sim 10^{-8}$ kg.

Tensors. The metric signature $(-,+,+,+)$ is considered, i.e. $\det[g_{\mu\nu}] \equiv |g| < 0$. The Minkowski metric is denoted $\eta_{\mu\nu}$, whereas a general metric is denoted $g_{\mu\nu}$. A four-vector $p^\mu =$

$[\eta_{\mu\nu}] = \mathrm{diag}(-1, 1, 1, 1)$

$f_{,\mu} \equiv \partial_\mu f = \frac{\partial f}{\partial x^\mu}$

Christophel symbols. The Christophel symbols or "connections" are written

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(g_{\mu\sigma,\nu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma} \right) \tag{1}$$

"Lambda tensor." We express the *Lambda tensor*—sometimes called the "projection operator"—that projects onto the TT gauge refer to sec.©0 as

$$\Lambda_{ij.kl}(\boldsymbol{n}) = P_{ik}(\boldsymbol{n}) P_{jl}(\boldsymbol{n}) - \frac{1}{2} P_{ij}(\boldsymbol{n}) P_{kl}(\boldsymbol{n}); \quad P_{ij}(\boldsymbol{n}) = \delta_{ij} - n_i n_j \tag{2}$$

$\forall\, \boldsymbol{n}$ of unit length; $\boldsymbol{n}^2 = n_1^2 + n_2^2 + n_3^2 = 1$. We use a dot ('.') instead of the more conventional comma (',') to distinguish from the Minkowskian partial derivative.

Fourier transforms. We use the following convention for the Fourier transform of $f(x)$, $\tilde{f}(k)$, and its inverse, where x and k are Lorentz four-vectors:

$$\begin{aligned} f(x) &= \int \frac{d^4k}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}k\cdot x} \tilde{f}(k) \\ \tilde{f}(k) &= \int d^4x\, \mathrm{e}^{\mathrm{i}k\cdot x} f(x) \end{aligned} \tag{3}$$

Here, $k \cdot x = k_\sigma x^\sigma = g_{\rho\sigma} k^\rho x^\sigma$.

Acronyms

CDM	cold dark matter
CMB	cosmic microwave background (radiation)
DW	domain wall
GR	general relativity
GW	gravitational wave
ΛCDM	Lambda (Greek Λ) cold dark matter model; standard model of cosmology

Nomenclature

In the table below is listed the most frequently used symbols in this paper, for reference.

Table 1: helo

Symbol	Referent	SI-value or definition
Natural constants		
G_N	Newtonian constant of gravitation	1.2 kg
k_B	Boltzmann's constant	1.2 K
Fiducial quantities		
h_0	Reduced Hubble constant	0.67
Subscripts		
Q_gw	Quantity Q related to gravitational wave	
Functions and operators		
$\Theta(\xi)$	Heaviside step function	$\begin{cases} 1 & \xi > 0 \\ 0 & \xi < 0 \end{cases}$
$\mathrm{sgn}(\xi)$	Signum function	$2\Theta(\xi) - 1$
$\delta^{(n)}(\xi)$	Dirac-Delta function of $\xi \in \mathbb{R}^n$, $n \in \mathbb{N}$.	
$\delta^{\mu\nu}$	Kronecker delta.	

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Writing Tools

This is a comment.
This needs spelling check.? Perhaps this?
Rephrase this.
Awkward wording.
Needs double-checking.?
This is in need of citation or reference to a section.© (With a comment.)
blah blah [...] (Phantom text.)

PHANTOM PARAGRAPH: THIS IS A PHANTOM PARAGRAPH, MAYBE WITH SOME KEYWORDS.

- This is a note.
- This is another note.
  - This is a related note.

This is very important.
This text is highlighted.
Statement. [←That needs to be shown or proven.]

Chapter 1

Introduction

- GOALS:
  - Gather framework about GWs from DWs
  - Remove the need for very expensive N-body simulations with (semi-)analytical predictions
  - Extract as much information as possible from the NANOgrav spectra thingy
- WHY RELEVANT:
  - NANOgrav data wihoo
  - Simulations in this regard are hugely expensive, and will not allow us to constrain the parameters of a model

$$\tilde{h}''_{\circledast} + 2\mathcal{H}\tilde{h}'_{\circledast} + k^2\tilde{h}_{\circledast} = 16\pi G_{\mathrm{N}} a^2 \tilde{\sigma}_{\circledast}; \quad \circledast = +, \times \tag{1.1}$$

$$\left(\tilde{h}^{\mathrm{TT}}\right)_{ij}(\eta, \boldsymbol{k}) = \sum_{\circledast=+,\times} e^{\circledast}_{ij}(\hat{\boldsymbol{k}})\tilde{h}_{\circledast}(\eta, \boldsymbol{k}) \tag{1.2}$$

$$\tilde{h}^{\mathrm{TT}}{}_{ij}(\eta, \boldsymbol{k}) = \sum_{\circledast=+,\times} e^{\circledast}_{ij}(\hat{\boldsymbol{k}})\tilde{h}_{\circledast}(\eta, \boldsymbol{k}) \tag{1.3}$$

1.1 Preliminaries

It is assumed that the reader is familiar with variational calculus and linear perturbation theory.

In the following, we briefly (re)capture some concepts that are important starting points for the rest of the thesis.

- variational calculus/ varying action
- action
- pert. theory?
- line element
- gauge invariance

- FRW cosmology
- classical field theory

$$G_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu} \tag{1.4}$$

1.1.1 Field theory

We formulate a theory in four-dimensional spacetime ⌜Minkowski⌟ in terms of the Lorentz invariant action

$$S = \int d^4x\, \mathcal{L}(\{\phi_i\}, \{\partial_\mu \phi_i\}), \tag{1.5}$$

with $\mathcal{L}$ being the *Lagrangian density* of the theory, a function of the set of fields $\{\phi_i\}$ and its first derivatives. We will refer to $\mathcal{L}$ simply as the Lagrangian, as is customary when working with fields. For a general (i.e. curved) spacetime, blah blah [...] $\partial_\mu \to \nabla_\mu$ blah blah [...] to construct a Lorentz invariant Lagrangian,

$$S = \int d^4x\, \underbrace{\mathcal{L}(\{\phi_i\}, \{\nabla_\mu \phi_i\})}_{\text{not scalar}} = \int \underbrace{d^4x\, \sqrt{-|g|}}_{\text{scalar}}\, \underbrace{\hat{\mathcal{L}}(\{\phi_i\}, \{\nabla_\mu \phi_i\})}_{\text{scalar}}, \tag{1.6}$$

⌜Maybe specify that this is only for scalar fields? Or include other fields?⌟

1.1.2 Expanding universe: FRW cosmology

The universe expands with the rate $a(t)$ at cosmic time t.

- expansion rate, cosmic time, conformal time
- why is flat assumption OK?

1.1.3 Method of Green's Functions

A linear ordinary differential equation (ODE) $\mathrm{L}_x f(x) = g(x)$ assumes a linear differential operator L, a ⌈continuous⌋? , unknown function f, and a right-hand side g that constitutes the inhomogeneous part of the ODE. The *Green's function* G for the ODE (or L) is manifest as any solution to $\mathrm{L}_x G(x, y) = \delta(x - y)$ ⌜check plagiarism (Bringmann)⌟. If L is translation invariant (invariant under $x \mapsto x + a$)—which is equivalent to L having constant coefficients—we can write $G(x, y) = G(x - y)$ and [←show?]■

$$f(x) = (G * g)(x) = \int \mathrm{d}y\, G(x - y) g(y) \tag{1.7}$$

solves $\mathrm{L}_x f(x) = g(x)$.

Let $f_i^{(0)}$, $i = 1, 2, 3, \ldots$ be solutions to the homogeneous ODE, i.e. $\mathrm{L}_x f_i^{(0)} = 0$. Then, by the superposition principle, $f(x) + \sum_i c_i f_i^{(0)}$ is also a solution of the original, inhomogeneous equation.

Pulse signal. Consider the very common scenario where the source is a temporary pulse;

$$g(x) = \begin{cases} g(x), & x_0 \leq x \leq x_1, \\ 0, & x \geq x_1. \end{cases} \tag{1.8}$$

1.1.4 Special functions (tmp. name, maybe move (appendix)?)

blah blah [...]

For $\nu = n + {}^1\!/_2, n \in \mathbb{N}$ we have

$$\mathcal{J}_{n+1/2}(x) = \sqrt{\frac{2}{\pi}} x^{n+\frac{1}{2}} \left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{\sin x}{x} \tag{1.9a}$$

and

$$\mathcal{N}_{n+1/2}(x) = (-1)^{n+1} \mathcal{J}_{-(n+1/2)}(x) \tag{1.9b}$$

for $x \in \mathbb{C}xlxl$

Part I

Background

Chapter 2

Classical Field Theory and Gravity

Alongside quantum mechanics, Einstein's theory of gravity—general relativity (GR)—is widely accepted as the most accurate description of our surroundings. GR can be formulated from a geometrical point of view, or it can be viewed as a classical field theory. In the former approach we meet geometrical tools such as the geodesic equation, whereas the latter allows the application of field-theoretical methods. This chapter lays emphasis on the field interpretation of GR.

PHANTOM PARAGRAPH: TWO PERSPECTIVES INSIGHTFUL; BETTER OVERALL UNDERSTANDING OF ASPECTS OF CONCEPTS IN GR

2.1 General Relativity

The Einstein–Hilbert action in vacuum is check Planck mass def.

$$S_{\mathrm{EH}} = {}^{1}\!/_{2} M_{\mathrm{Pl}}^{2} \int d^4x \, \sqrt{-|g|}\, \mathcal{R}, \tag{2.1}$$

where $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$. By varying S_{EH} with respect to $g_{\mu\nu}$ one obtains the equation of motion

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} + \frac{1}{2} g_{\mu\nu}\mathcal{R} = 0. \tag{2.2}$$

Thus, we interpret GR as a *classical* field theory where the tensor field $g_{\mu\nu}$ is the gravitational field, with the particle realisation named "graviton".

- scalar field (ST theories)
- energy momentum tensor

2.2 TITLE (Conformal Transformations)

2.3 TITLE (Scalar–Tensor Theories (maybe subsec. of prev.))

2.4 Perturbation Theory

2.5 Classical Solitons

Chapter 3

Gravitational Waves

The term “gravitational waves” refers to the tensor perturbations to the background metric©() . These “waves” are spacetime distortions whose name comes from the fact that ⌈they obey the wave equation⌋?.

3.1 Linearised Gravity

⌜CONFORMAL TRAFOS!!!! Carroll ((2019, p. 467))⌟

If we let $\overline{g}_{\mu\nu}$ be the background metric, the perturbed metric is given by

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}; \quad \left|h_{\mu\nu}\right| \ll 1, \tag{3.1}$$

where $h_{\mu\nu}$ describes a tensor field propagating on said background. Now, $g^{\mu\nu} = \overline{g}^{\mu\nu} - h^{\mu\nu}$ is the inverse metric, however $h^{\mu\nu}h_{\nu\lambda} = \overline{g}^{\mu\rho}\overline{g}^{\nu\sigma}h_{\rho\sigma}h_{\nu\lambda} \neq \delta^{\mu}{}_{\lambda}$.

PHANTOM PARAGRAPH: FIND THE LINEARISED EINSTEIN EQS – USING COMMA NOTATION – COMMENT ABOUT NO BACKREACTION

The Christophel symbols are now

$$\begin{aligned}
\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}g^{\rho\sigma}\left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}\right) \\
&= \overline{\Gamma}^{\rho}_{\mu\nu} + \frac{1}{2}\overline{g}^{\rho\sigma}\left(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma}\right) - \frac{1}{2}h^{\rho\sigma}\underbrace{\left(\overline{g}_{\mu\sigma,\nu} + \overline{g}_{\nu\sigma,\mu} - \overline{g}_{\mu\nu,\sigma}\right)}_{=2\overline{g}_{\sigma\lambda}\overline{\Gamma}^{\lambda}_{\mu\nu}} + \mathcal{O}(h^2) \\
&= \overline{\Gamma}^{\rho}_{\mu\nu} + \frac{1}{2}\overline{g}^{\rho\sigma}\left(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma}\right) - h^{\rho}{}_{\lambda}\overline{\Gamma}^{\lambda}_{\mu\nu} + \mathcal{O}(h^2) \\
&= \overline{\Gamma}^{\rho}_{\mu\nu} + \frac{1}{2}\overline{g}^{\rho\sigma}\left(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma} - 2h_{\sigma\lambda}\overline{\Gamma}^{\lambda}_{\mu\nu}\right) + \mathcal{O}(h^2) \\
&= \overline{\Gamma}^{\rho}_{\mu\nu} + \frac{1}{2}\overline{g}^{\rho\sigma}\left(h_{\mu\sigma;\nu} + h_{\nu\sigma;\mu} - h_{\mu\nu;\sigma}\right) + \mathcal{O}(h^2),
\end{aligned} \tag{3.2}$$

where $h_{\mu\nu;\sigma} = \overline{\nabla}_{\sigma}h_{\mu\nu}$ ⌜$+\mathcal{O}(h^2)$??⌟. [←Prove this last line.]■

⌜Maybe have such things in an appendix?⌟

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3.1.1 TITLE (Energy–momentum tensor; Eom.; Scalar field)

The energy–momentum tensor is

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right] \tag{3.3}$$

blah blah [...] Assume no cross terms in $g_{\mu\nu}$. To retrieve the TT-part of $T_{\mu\nu}$ we utilise the Lambda tensor©(repeat or refer?)

$$\begin{aligned} T_{ij}(\eta, \boldsymbol{k}) &= \Lambda_{ij,kl}(\hat{\boldsymbol{k}})\,(T^{\neg\mathrm{TT}})_{ij}\,(\eta, \boldsymbol{k}) \\ &\stackrel{!}{=} \Lambda_{ij,kl}(\hat{\boldsymbol{k}})\left[\partial_i\phi\partial_j\phi\right](\eta, \boldsymbol{k}) \end{aligned} \tag{3.4}$$

$$\begin{aligned} \left[\partial_i\phi\partial_j\phi\right](\eta, \boldsymbol{k}) &= \int d^3x\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left[\partial_i\phi\partial_j\phi\right](\eta, \boldsymbol{x}) \\ &= ???? \end{aligned} \tag{3.5}$$

3.2 Generation of Gravitational Waves

- Somehow get to this eq:

$$T^{\mathrm{TT}}_{ij}(\eta, \boldsymbol{k}) = \Lambda_{ij,kl}(\hat{\boldsymbol{k}})\int \frac{d^3p}{(2\pi)^3}\, p_k p_l \phi(\eta, \boldsymbol{p})\phi(\eta, \boldsymbol{k}-\boldsymbol{p}). \tag{3.6}$$

- Production instead of generation?

3.2.1 General Formalism

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⌜Use conformal trafo! ((Carroll, 2019, p. 467)) ⌟

$$ds^2 = -\mathrm{d}t^2 + a^2(t)\left\{\gamma_{ij} + h_{ij}\right\}\mathrm{d}x^i\mathrm{d}x^j = a^2(\eta)\left\{-\mathrm{d}\eta^2 + \left(\gamma_{ij} + h_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j\right\} \tag{3.7}$$

$h_{ij} = h_{ij}^{\mathrm{TT}}, \gamma_{ij} = \delta_{ij}, (\overline{g}_{\mu\nu} = a^2\eta_{\mu\nu})$
Linearised Einstein equations: ⌜conformal Newtonian gauge⌟

$$\delta G^i{}_j = 8\pi G_{\mathrm{N}} T^i{}_j = \frac{1}{2a^2}\left[\ddot{h}_{ij} + 2\frac{\dot{a}}{a}\dot{h}_{ij} - \boldsymbol{\nabla}^2 h_{ij}\right] \tag{3.8}$$

where a dot ('$\dot{}$') signifies the *conformal* time derivative. ($T^i{}_j = a^{-2}T_{ij}$) blah blah [...]

$$\ddot{h}_{ij}(\eta, \boldsymbol{k}) + 2\frac{\dot{a}}{a}\dot{h}_{ij}(\eta, \boldsymbol{k}) - k^2 h_{ij}(\eta, \boldsymbol{k}) = 16\pi G_{\mathrm{N}} T_{ij}(\eta, \boldsymbol{k}) \tag{3.9}$$

Define $\mathfrak{h}_{ij} \equiv a h_{ij}$. By inserting this in Eq. (3.9) and multiplying the equation by a, one finds

$$\ddot{\mathfrak{h}}_{ij}(\eta, \boldsymbol{k}) + \left[k^2 - \frac{\ddot{a}(\eta)}{a(\eta)}\right]\mathfrak{h}_{ij}(\eta, \boldsymbol{k}) = 16\pi G_{\mathrm{N}} a(\eta) T_{ij}(\eta, \boldsymbol{k}). \tag{3.10}$$

We assume $a(\eta) \propto \eta^\alpha$ and define $\nu \equiv \alpha - \frac{1}{2}$. Letting $\tau = k\eta$, Eq. (3.10) becomes

$$\left[\frac{\partial^2}{\partial\tau^2} + 1 - \frac{4\nu^2 - 1}{4\tau^2}\right]\mathfrak{h}_{ij}(\eta, \boldsymbol{k}) = \frac{16\pi G_{\mathrm{N}} a(\eta)}{k^2} T_{ij}(\eta, \boldsymbol{k}) \tag{3.11}$$

Now, Eq. (3.11) transforms into a problem of the form $\mathrm{L}_\tau f(\tau) = g(\tau)$; a problem that can be solved using Green's method (see some section©₀). Kawasaki and Saikawa ((2011)) propose

$$G(\tau, \tau') = \frac{\pi}{2}\Theta(\tau - \tau')\left[\mathcal{N}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{N}_\nu(\tau')\right] \tag{3.12}$$

as a solution to $\mathrm{L}_\tau G(\tau, \tau') = \delta(\tau - \tau')$. In some appendix©₀ we show that this holds for a matter dominated universe where $\nu = 2 - \frac{1}{2} = \frac{3}{2}$.

Now assume the source is active (emits gravitational radiation) between η_{ini} and η_{fi}, and followingly? initial conditions $\mathfrak{h}_{ij}(\eta_{\mathrm{ini}}, \boldsymbol{k}) = \dot{\mathfrak{h}}_{ij}(\eta_{\mathrm{ini}}, \boldsymbol{k}) = 0$. Thus,

$$\mathfrak{h}_{ij}(\eta \geq \eta_{\mathrm{ini}}, \boldsymbol{k}) = \frac{8\pi^2 G_{\mathrm{N}}}{k^2}\int_{k\eta_{\mathrm{ini}}}^{k\eta}\mathrm{d}\tau'\ \sqrt{\tau\tau'}\left[\mathcal{N}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{N}_\nu(\tau')\right] a(\tau')T_{ij}(\tau', \boldsymbol{k}), \tag{3.13}$$

which reduces to

$$\mathfrak{h}_{ij}(\eta \geq \eta_{\mathrm{fi}}, \boldsymbol{k}) = A_{ij}(\boldsymbol{k})\sqrt{k\eta}\mathcal{J}_\nu(k\eta) + B_{ij}(\boldsymbol{k})\sqrt{k\eta}\mathcal{N}_\nu(k\eta). \tag{3.14}$$

Combining Eq. (3.13) and Eq. (3.14) at $\eta = \eta_{\mathrm{fi}}$ gives the coefficients A_{ij} and B_{ij}:

$$\begin{aligned} A_{ij}(\boldsymbol{k}) &= -\frac{8\pi^2 G_{\mathrm{N}}}{k^2}\int_{k\eta_{\mathrm{ini}}}^{k\eta_{\mathrm{fi}}}\mathrm{d}\tau'\ \sqrt{\tau'}a(\tau')\mathcal{N}_\nu(\tau')T_{ij}(\tau', \boldsymbol{k}) \\ B_{ij}(\boldsymbol{k}) &= +\frac{8\pi^2 G_{\mathrm{N}}}{k^2}\int_{k\eta_{\mathrm{ini}}}^{k\eta_{\mathrm{fi}}}\mathrm{d}\tau'\ \sqrt{\tau'}a(\tau')\mathcal{J}_\nu(\tau')T_{ij}(\tau', \boldsymbol{k}) \end{aligned} \tag{3.15}$$

3.2.2 Scalar Field Source (temp. name)

Chapter 4

Lattice-? and N-body simulations

Part II

Project

Chapter 5

Calculating Gravitational Waves from Domain Walls

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⌜

Consider a planar domain wall in the xy-plane in a flat FRW universe, represented by a scalar field $\phi(\eta, \boldsymbol{x})$ and a potential $V(\phi)$. The action of this theory is

$$S = \int d^4x \sqrt{-g} \left\{ 16\pi G_{\mathrm{N}} \mathcal{R} - \frac{1}{2} \phi^{;\mu} \phi_{;\mu} + V(\phi) \right\}. \tag{5.1}$$

The background metric is

$$d\bar{s}^2 = \bar{g}_{\mu\nu} \mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu = -\mathrm{d}t^2 + a^2(t) \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j = a^2(\eta) \left\{ -\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \right\}. \tag{5.2}$$

The solution to $\bar{\Box}\phi = \mathrm{d}V/\mathrm{d}\phi$ is denoted $\bar{\phi}(\eta, z)$. We let indices $a, b, c = 1, 2$ and $i, j, k, l, \ldots = 1, 2, 3$. Now we add a linear perturbation $\zeta(\eta, x^a)$ to the wall such that

$$\phi(\eta, \boldsymbol{x}) = \bar{\phi}(\eta, z; \zeta(\eta, x^a)) = \bar{\phi}(\eta, z; 0) + \zeta(\eta, x^a) \left. \frac{\partial \bar{\phi}}{\partial z} \right|_{\zeta = 0} + \mathcal{O}(\zeta^2). \tag{5.3}$$

⌜Remember eqs for ζ!⌟ Furthermore, Fourier transforming [←show this!]■ the spatial components gives

$$\phi(\eta, \boldsymbol{k}) = \int d^3x \, \mathrm{e}^{\mathrm{i}k_i x^i} \phi(\eta, \boldsymbol{x}) = \left[(2\pi)^2 \delta^{(2)}(k_a) - \mathrm{i}k_3 \zeta(\eta, k_a) \right] \bar{\phi}(\eta, k_3; 0) + O\left(\zeta^2\right). \tag{5.4}$$

The TT-part of the energy-momentum tensor is [←refer to some section]■ ⌜NB: g cannot have cross terms!!⌟

$$T_{ij}^{\mathrm{TT}}(\eta, \boldsymbol{k}) = \Lambda_{ij.kl}(\hat{\boldsymbol{k}}) \int \frac{d^3p}{(2\pi)^3} \, p_k p_l \phi(\eta, \boldsymbol{p}) \phi(\eta, \boldsymbol{k} - \boldsymbol{p}). \tag{5.5}$$

We define a quantity t_{kl} by

$$T_{ij}^{\mathrm{TT}}(\eta, \boldsymbol{k}) = \Lambda_{ij.kl}(\hat{\boldsymbol{k}}) \left(\frac{1}{2\pi} \cdot t_{kl}(\eta, \boldsymbol{k}) + \mathcal{O}(\zeta^2) \right), \tag{5.6}$$

and the additional function

$$\Im_n(\eta, q_0) = \int_{\mathbb{R}} \mathrm{d}q \, q^n \bar{\phi}(\eta, q; 0) \bar{\phi}(\eta, q_0 - q; 0). \tag{5.7}$$

After some manipulation [←show this!]■, we get the following:

$$t_{ab}(\eta, \boldsymbol{k}) = k_a k_b \left[-\mathrm{i}\zeta(\eta, k_c) \right] \Im_1(\eta, k_3) \tag{5.8a}$$

$$t_{a3}(\eta, \boldsymbol{k}) = k_a \left[-\mathrm{i}\zeta(\eta, k_c) \right] \Im_2(\eta, k_3) \tag{5.8b}$$

$$t_{33}(\eta, \boldsymbol{k}) = k_3 \left[-\mathrm{i}\zeta(\eta, k_c) \right] \Im_2(\eta, k_3) + (2\pi)^2 \delta^{(2)}(k_a) \, \Im_2(\eta, k_3) \tag{5.8c}$$

⌜There are some *small* constraint on the perturbation from this. Need to be commented!⌟

Gravitational waves sourced by this field is – to first order in ζ – given by

$$
\begin{aligned}
ah_{ij}(\eta,\boldsymbol{k}) &= \frac{16\pi G_{\mathrm{N}}}{k}\int_{\eta_{\mathrm{i}}}^{\eta}\mathrm{d}\eta'\ \sin\left(k\left[\eta-\eta'\right]\right)a(\eta')T_{ij}^{\mathrm{TT}}(\eta',\boldsymbol{k}) \\
&= \frac{8G_{\mathrm{N}}}{k}\Lambda_{ij.kl}(\hat{\boldsymbol{k}})\int_{\eta_{\mathrm{i}}}^{\eta}\mathrm{d}\eta'\ \sin\left(k\left[\eta-\eta'\right]\right)a(\eta')t_{kl}(\eta',\boldsymbol{k})+\mathscr{O}(\zeta^2).
\end{aligned}
\tag{5.9}
$$

Remaining are the $\Lambda_{ij.kl}t_{kl}$-elements, which in total are ⌈6⌋? terms per ij, due to symmetry in t_{kl}:

$$
\begin{aligned}
\Lambda_{ij.kl}(\hat{\boldsymbol{k}})t_{kl}(\eta,\boldsymbol{k}) = &\left\{\left(\Lambda_{ij.12}+\Lambda_{ij.21}\right)t_{12}+\left(\Lambda_{ij.13}+\Lambda_{ij.31}\right)t_{13}+\left(\Lambda_{ij.23}+\Lambda_{ij.32}\right)t_{23}\right\}(\eta,k\hat{\boldsymbol{k}}) \\
&+\left\{\Lambda_{ij.11}t_{11}+\Lambda_{ij.22}t_{22}+\Lambda_{ij.33}t_{33}\right\}(\eta,k\hat{\boldsymbol{k}})
\end{aligned}
\tag{5.10}
$$

All of these are on the form

$$
-\mathrm{i}\zeta(\eta,k_a)\times\left\{k^2k^2k^2\Im_1(\eta,k_3)A_{ij}(\hat{\boldsymbol{k}})+k\Im_2(\eta,k_3)B_{ij}(\hat{\boldsymbol{k}})\right\},
\tag{5.11}
$$

leaving

$$
ah_{ij}(\eta,\boldsymbol{k}) = 8G_{\mathrm{N}}\left[kA_{ij}(\hat{\boldsymbol{k}})\mathcal{I}_1(\eta,\boldsymbol{k};\eta_{\mathrm{i}})+B_{ij}(\hat{\boldsymbol{k}})\mathcal{I}_2(\eta,\boldsymbol{k};\eta_{\mathrm{i}})\right]
\tag{5.12}
$$

where

$$
\mathcal{I}_n(\eta,\boldsymbol{k};\eta_{\mathrm{i}}) = -\mathrm{i}\int_{\eta_{\mathrm{i}}}^{\eta}\mathrm{d}\eta'\ a(\eta')\sin\left(k\left(\eta-\eta'\right)\right)\times\zeta(\eta',k_a)\Im_n(\eta',k_3).
\tag{5.13}
$$

Furthermore, we can show [←proof!]■ that $A_{ij}(\boldsymbol{n}) = -n_3B_{ij}(\boldsymbol{n}) \equiv +2n_3C_{ij}(\boldsymbol{n})$ for $|\boldsymbol{n}|^2 = n_1^2+n_2^2+n_3^2 = 1$, allowing for the slightly simpler expression

$$
ah_{ij}(\eta,\boldsymbol{k}) = 4G_{\mathrm{N}}C_{ij}(\hat{\boldsymbol{k}})\left[k_3\mathcal{I}_1(\eta,\boldsymbol{k};\eta_{\mathrm{i}})-\mathcal{I}_2(\eta,\boldsymbol{k};\eta_{\mathrm{i}})\right],
\tag{5.14}
$$

where⌈:

$$
\begin{aligned}
C_{ab}(\boldsymbol{n}) &= n_3\left[n_an_b\left(n_3^2+1\right)-\delta_{ab}\left(1-n_3^2\right)\right] \\
C_{a3}(\boldsymbol{n}) &= -n_an_3^2\left(1-n_3^2\right) \\
C_{33}(\boldsymbol{n}) &= n_3^2\left(1-n_3^2\right)^2
\end{aligned}
\tag{5.15}
$$

⌋?

Redshift $\mathfrak{z}_* = 2 \therefore a(\eta_{\mathrm{i}}) = (1+\mathfrak{z}_*)^{-1} = 1/3$
$ds^2 = a^2(\eta)\left(\delta_{\mu\nu}+h_{\mu\nu}\right)\mathrm{d}x^\mu\mathrm{d}x^\nu,\ x^0=\eta$
$u_{\mathfrak{a}}x^{\mathfrak{a}},\ \mathfrak{a}=0,1,2$
$u_\iota x^\iota,\ \iota = 0,1,2\mathfrak{b}\mathfrak{b}$

Important references: ((Vachaspati, 2006, p. 145)), ((Vilenkin, 1985, p. 291)), ((Vilenkin and Shellard, 1994, p. 375))

5.1 General Formalism

Chapter 6

Simulating Gravitational Waves from Domain Walls

Chapter 7

Studying Gravitational Waves from Domain Walls

Chapter 8

Discussion

Chapter 9

Conclusion and Outlook

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Appendix A

I do not have an appendix