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Master's thesis

Gravitational waves from topological defects

Signatures of late-time first-order phase transitions

Nanna Bryne

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Institute of Theoretical Astrophysics, Department of Physics
Faculty of Mathematics and Natural Sciences

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Supervisor:
David Fonseca Mota

Contents

Notation

Constants and units. We use ⌈'natural units'⌋? where $\hbar = c = 1$, where $\hbar$ is the reduced Planck constant and c is the speed of light in vacuum.⌜Planck units? Set $k_B = G_N = 1$?⌟ The Newtonian constant of gravitation G_N is referenced explicitly, and we use Planck units such as the Planck mass $M_{Pl} = (\hbar c / G_N)^{1/2} = G_N^{-1/2} \sim 10^{-8}$ kg.

Tensors. The metric signature $(-,+,+,+)$ is considered, i.e. $\det[g_{\mu\nu}] \equiv |g| < 0$. The Minkowski metric is denoted $\eta_{\mu\nu}$, whereas a general metric is denoted $g_{\mu\nu}$. A four-vector $p^\mu =$

$[\eta_{\mu\nu}] = \mathrm{diag}(-1, 1, 1, 1)$

$f_{,\mu} \equiv \partial_\mu f = \frac{\partial f}{\partial x^\mu}$

Christophel symbols. The Christophel symbols or "connections" are written

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(g_{\mu\sigma,\nu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma} \right) \tag{1}$$

"Lambda tensor." We express the *Lambda tensor*—sometimes called the "projection operator"—that projects onto the TT gauge refer to sec.©0 as

$$\Lambda_{ij.kl}(\boldsymbol{n}) = P_{ik}(\boldsymbol{n}) P_{jl}(\boldsymbol{n}) - \frac{1}{2} P_{ij}(\boldsymbol{n}) P_{kl}(\boldsymbol{n}); \quad P_{ij}(\boldsymbol{n}) = \delta_{ij} - n_i n_j \tag{2}$$

$\forall\, \boldsymbol{n}$ of unit length; $\boldsymbol{n}^2 = n_1^2 + n_2^2 + n_3^2 = 1$. We use a dot ('.') instead of the more conventional comma (',') to distinguish from the Minkowskian partial derivative.

Fourier transforms. We use the following convention for the Fourier transform of $f(x)$, $\tilde{f}(k)$, and its inverse, where x and k are Lorentz four-vectors:

$$\begin{aligned} f(x) &= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x} \tilde{f}(k) \\ \tilde{f}(k) &= \int d^4x\, e^{ik\cdot x} f(x) \end{aligned} \tag{3}$$

Here, $k \cdot x = k_\sigma x^\sigma = g_{\rho\sigma} k^\rho x^\sigma$.

Acronyms

CDM	cold dark matter
CMB	cosmic microwave background (radiation)
DW	domain wall
GR	general relativity
GW	gravitational wave
ΛCDM	Lambda (Greek Λ) cold dark matter model; standard model of cosmology

Nomenclature

In the table below is listed the most frequently used symbols in this paper, for reference.

Table 1: helo

Symbol	Referent	SI-value or definition
Natural constants		
G_{N}	Newtonian constant of gravitation	1.2 kg
k_{B}	Boltzmann's constant	1.2 K
Fiducial quantities		
h_0	Reduced Hubble constant	0.67
Subscripts		
Q_{gw}	Quantity Q related to gravitational wave	
Functions and operators		
$\Theta(\xi)$	Heaviside step function	$\begin{cases} 1 & \xi > 0 \\ 0 & \xi < 0 \end{cases}$
$\mathrm{sgn}(\xi)$	Signum function	$2\Theta(\xi) - 1$
$\delta^{(n)}(\xi)$	Dirac-Delta function of $\xi \in \mathbb{R}^n$, $n \in \mathbb{N}$.	
$\delta^{\mu\nu}$	Kronecker delta.	
Mathematical groups		
Conf($\mathscr{M}$)	The conformal group of the manifold $\mathscr{M}$.	
hh	...	...

DRAFT

Writing Tools

This is a comment.
This needs spelling check.? Perhaps this?
Rephrase this.
Awkward wording.
Needs double-checking.?
This is in need of citation or reference to a section.© (With a comment.)
blah blah [. . .] (Phantom text.)

PHANTOM PARAGRAPH: THIS IS A PHANTOM PARAGRAPH, MAYBE WITH SOME KEYWORDS.

- This is a note.
- This is another note.
  - This is a related note.

This is very important.

This text is highlighted.

Statement. [←That needs to be shown or proven.]■

(chapter) Title Case

Lorem ipsum…

(section) Sentence case

Lorem ipsum…

(subsection) Sentence case

Lorem ipsum…

(paragraph) Sentence case with punctuation. Lorem ipsum…

Below, we describe some phenomena or whatever.

Gravitational waves are called that or GWs, tensor perturbations, …pdsfovnsoz avoszjasvo aoc awvn anvo pewnfvao noav

Stress–energy tensor is called that or SE tensor(?) ⌜OBS: Hilbert SE tensor = HSE tensor?⌟.

Domain walls are called that or walls, never DWs.

General relativity is called that or GR, Einstein's theory of gravity.

Einstein field equation(s) are called that or EFE(s).

Equation(s) of motion are called that or eom(s).

Chapter 1

Introduction

- GOALS:
  - Gather framework about GWs from DWs
  - Remove the need for very expensive N-body simulations with (semi-)analytical predictions
  - Extract as much information as possible from the NANOgrav spectra thingy
- WHY RELEVANT:
  - NANOgrav data wihoo
  - Simulations in this regard are hugely expensive, and will not allow us to constrain the parameters of a model

$$\tilde{h}''_{\circledast} + 2\mathcal{H}\tilde{h}'_{\circledast} + k^2\tilde{h}_{\circledast} = 16\pi G_{\mathrm{N}} a^2 \tilde{\sigma}_{\circledast}; \quad \circledast = +, \times \tag{1.1}$$

$$\left(\tilde{h}^{\mathrm{TT}}\right)_{ij}(\eta, \boldsymbol{k}) = \sum_{\circledast=+,\times} e^{\circledast}_{ij}(\hat{\boldsymbol{k}})\tilde{h}_{\circledast}(\eta, \boldsymbol{k}) \tag{1.2}$$

$$\tilde{h}^{\mathrm{TT}}{}_{ij}(\eta, \boldsymbol{k}) = \sum_{\circledast=+,\times} e^{\circledast}_{ij}(\hat{\boldsymbol{k}})\tilde{h}_{\circledast}(\eta, \boldsymbol{k}) \tag{1.3}$$

1.1 Preliminaries

It is assumed that the reader is familiar with variational calculus and linear perturbation theory.

⌈In the following, we briefly (re)capture some concepts that are important starting points for the rest of the thesis.⌋

- variational calculus/ varying action
- action
- pert. theory?
- line element
- gauge invariance

- FRW cosmology
- classical field theory

$$G_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu} \tag{1.4}$$

1.1.1 Field theory

We formulate a theory in four-dimensional spacetime ⌜Minkowski⌟ in terms of the Lorentz invariant action

$$S = \int d^4x\, \mathcal{L}(\{\phi_i\}, \{\partial_\mu \phi_i\}), \tag{1.5}$$

with $\mathcal{L}$ being the *Lagrangian density* of the theory, a function of the set of fields $\{\phi_i\}$ and its first derivatives. We will refer to $\mathcal{L}$ simply as the Lagrangian, as is customary when working with fields. For a general (i.e. curved) spacetime, blah blah [...] $\partial_\mu \to \nabla_\mu$ blah blah [...] to construct a Lorentz invariant Lagrangian,

$$S = \int d^4x\, \underbrace{\mathcal{L}(\{\phi_i\}, \{\nabla_\mu \phi_i\})}_{\text{not scalar}} = \int \underbrace{d^4x\, \sqrt{-|g|}}_{\text{scalar}}\, \underbrace{\hat{\mathcal{L}}(\{\phi_i\}, \{\nabla_\mu \phi_i\})}_{\text{scalar}}, \tag{1.6}$$

⌜Maybe specify that this is only for scalar fields? Or include other fields?⌟

1.1.2 Expanding universe: standard cosmology

- expansion rate, cosmic time, conformal time
- why is flat assumption OK?
- redshift $z = a_0/a - 1$

The universe expands with the rate $a(t_{\mathrm{ph}})$ at physical or rather *cosmic* time t_{ph}. ⌜Explain cosmic time and $t_{\mathrm{ph}} = 0$.⌟ This work sincerely favours the use of *conformal* time, also known as the comoving horizon. As t is a neat variable name, we shall *not* reserve it for the cosmic time, as is the most common use, but rather let it refer to the conformal time coordinate.

1.1.3 Method of Green's functions

A linear ordinary differential equation (ODE) $\mathrm{L}_x f(x) = g(x)$ assumes a linear differential operator L, a ⌈continuous⌋?, unknown function f, and a right-hand side g that constitutes the inhomogeneous part of the ODE. The *Green's function* G for the ODE (or L) is manifest as any solution to $\mathrm{L}_x G(x, y) = \delta(x - y)$ ⌜check plagiarism (Bringmann)⌟. If L is translation invariant (invariant under $x \mapsto x + a$)—which is equivalent to L having constant coefficients—we can write $G(x, y) = G(x - y)$ and [←show?]

$$f(x) = (G * g)(x) = \int \mathrm{d}y\, G(x - y) g(y) \tag{1.7}$$

solves $\mathrm{L}_x f(x) = g(x)$.

Let $f_i^{(0)}$, $i = 1, 2, 3, \ldots$ be solutions to the homogeneous ODE, i.e. $\mathrm{L}_x f_i^{(0)} = 0$. Then, by the superposition principle, $f(x) + \sum_i c_i f_i^{(0)}$ is also a solution of the original, inhomogeneous equation.

Pulse signal. Consider the very common scenario where the source is a temporary pulse;

$$g(x) = \begin{cases} g(x), & x_0 \leq x \leq x_1, \\ 0, & x \geq x_1. \end{cases} \tag{1.8}$$

1.1.4 Special functions (tmp. name, maybe move (appendix)?)

blah blah [...]

For $\nu = n + {}^1\!/_2, n \in \mathbb{N}$ we have

$$\mathcal{J}_{n+1/2}(x) = \sqrt{\frac{2}{\pi}} x^{n+\frac{1}{2}} \left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{\sin x}{x} \tag{1.9a}$$

and

$$\mathcal{Y}_{n+1/2}(x) = (-1)^{n+1} \mathcal{J}_{-(n+1/2)}(x) \tag{1.9b}$$

for $x \in \mathbb{C}xlxl$

Part I

Background

Chapter 2

Differential Geometry

PHANTOM PARAGRAPH

To develop a classical field theory, we require a handful of mathematical structures? concepts from differential geometry.

A classical field theory consists of the following mathematical structures:

Spacetime . . .

A spacetime is a *smooth manifold* with or without additional mathematical structures.

The aim of this somewhat technical chapter is to provide the necessary mathematical tools to get a sufficient grasp of general relativity as a classical field theory.

2.1 TITLE (Manifolds, tensors, etc.)

The manifolds of primary interest to us as gravity physicists are the *pseudo-Riemannian manifolds*. Such are the very foundation of Einstein's general theory of relativity, and amongst them are Lorentzian and Riemannian manifolds.

We let the spacetime $(\mathscr{M}, g_{\mu\nu})$ be made up of a pseudo-Riemannian manifold $\mathscr{M}$ and a pseudo-Riemannian metric tensor $g_{\mu\nu}$. Then, by definition, $\mathscr{M}$ is differentiable and $g_{\mu\nu}$ is smooth, non-degenerate ⌜(?)⌟ and symmetric.

- Describe:
  - diffeomorphisms
  - maps
  - submanifolds
  - metrics
  - symmetries

2.1.1 Hypersurfaces

- First & second fundamental form

2.2 Riemannian geometry

Riemannian geometry is an important branch of differential geometry and concerns *Riemannian manifolds*. A manifold is Riemannian when it is smooth and admits a *Riemannian metric*. Furthermore, this metric is characterised ⌜by what???⌟

The general theory of relativity is

2.3 Conformal geometry

((Carroll, 2019, App. XXX)) ((Dąbrowski et al., 2009; Feng and Gasperín, 2023))

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⌜

Consider a spacetime $(\mathscr{M}, g_{\mu\nu})$ where $\mathscr{M}$ is a smooth manifold of D dimensions and $g_{\mu\nu}$ is a Lorentzian metric on said manifold. Let $\Upsilon = \Upsilon(x) \in \mathbb{R}^{+}$[1] be a smooth function of spacetime coordinate x^{μ}. Then

$$\tilde{g}_{\mu\nu}(x) = \Upsilon^{2}(x) g_{\mu\nu}(x) \tag{2.1}$$

is a *conformal transformation* and Υ the corresponding *conformal factor*. Such angle-preserving transformations leave the causal structure of the manifold unchanged as they extend or contract the distance between spacetime points. ⌈In this section, a tilde $\tilde{o}$ refers to o in "tilde'd" system.⌋↺

blah blah [...]

It is straight-forward[2] to show that the determinants and inverses of the metrics obey the following relations:

$$\begin{aligned} \sqrt{-\tilde{g}} &= \Upsilon^{D}\sqrt{-g} \\ \tilde{g}^{\mu\nu} &= \Upsilon^{-2} g^{\mu\nu} \end{aligned} \tag{2.2}$$

We apply the transformation Eq. (2.1) to the connection coefficients

blah blah [...]

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + C^{\rho}{}_{\mu\nu} \tag{2.3}$$

$$C^{\rho}{}_{\mu\nu} = \Upsilon^{-1}\left(2\delta^{\rho}{}_{(\mu}\Upsilon_{,\nu)} - g_{\mu\nu} g^{\rho\sigma}\Upsilon_{,\sigma}\right) \tag{2.4}$$

Under Eq. (2.3), the Riemann tensor becomes

$$\tilde{\mathcal{R}}^{\rho}{}_{\sigma\mu\nu} = \mathcal{R}^{\rho}{}_{\sigma\mu\nu} - 2C^{\rho}{}_{[\mu|\sigma|;\nu]} + 2C^{\rho}{}_{[\mu|\lambda}C^{\lambda|}{}_{\nu]\sigma}. \tag{2.5}$$

[1] ⌜$\mathbb{R}^{+} \equiv (0, \infty)$ in notation chapter?⌟

[2] It is, actually, not like when other authors say things like that.

$$\tilde{\mathcal{R}}^{\rho}{}_{\mu\sigma\nu} = \mathcal{R}^{\rho}{}_{\mu\sigma\nu} + \Upsilon^{-1}\Big(\delta^{\rho}{}_{\nu}\Upsilon_{;\mu\sigma} - \delta^{\rho}{}_{\sigma}\Upsilon_{;\mu\nu} + g_{\mu\sigma}\Upsilon^{;\rho}{}_{;\nu} - g_{\mu\nu}\Upsilon^{;\rho}{}_{;\sigma}\Big)$$
$$+2\Upsilon^{-2}\Big(\delta^{\rho}{}_{\sigma}\Upsilon_{,\mu}\Upsilon_{,\nu} - \delta^{\rho}{}_{\nu}\Upsilon_{,\mu}\Upsilon_{,\sigma} + g_{\mu\sigma}\Upsilon^{,\rho}\Upsilon_{,\nu} - g_{\mu\nu}\Upsilon^{,\rho}\Upsilon_{,\sigma}\Big) + \Upsilon^{-2}\Big(\delta^{\rho}{}_{\nu}g_{\mu\sigma} - \delta^{\rho}{}_{\sigma}g_{\mu\nu}\Big)g_{\kappa\tau}\Upsilon^{,\kappa}\Upsilon^{,\tau} \tag{2.6a}$$

$$\tilde{\mathcal{R}}^{\rho}{}_{\mu\sigma\nu} = \mathcal{R}^{\rho}{}_{\mu\sigma\nu} + 2\Upsilon^{-1}\Big(\delta^{\rho}{}_{[\nu}\Upsilon_{;|\sigma|\mu]} + g_{\sigma[\mu}\Upsilon^{;|\rho|}{}_{;\nu]}\Big) + 4\Upsilon^{-2}\Big(\delta^{\rho}{}_{[\mu}\Upsilon_{,|\sigma|}\Upsilon_{,\nu]} + g_{\sigma[\mu}\Upsilon^{,|\rho|}\Upsilon_{,\nu]}\Big)$$
$$+2\Upsilon^{-2}\delta^{\rho}{}_{[\nu}g_{|\sigma|\mu]}g_{\mu\nu}g_{\kappa\tau}\Upsilon^{,\kappa}\Upsilon^{,\tau} \tag{2.7a}$$

$$\tilde{\mathcal{R}}^{\rho}{}_{\mu\sigma\nu} = \mathcal{R}^{\rho}{}_{\mu\sigma\nu} + 2A^{\rho}{}_{[\mu|\sigma|\nu]};$$
$$A^{\rho}{}_{\mu\sigma\nu} = \Upsilon^{-1}\Big(-\delta^{\rho}{}_{\mu}\Upsilon_{;\sigma\nu} + g_{\sigma\mu}\Upsilon^{;\rho}{}_{;\nu}\Big) + 2\Upsilon^{-2}\Big(\delta^{\rho}{}_{\mu}\Upsilon_{,\sigma}\Upsilon_{,\nu} + g_{\sigma\mu}\Upsilon^{,\rho}\Upsilon_{,\nu}\Big) + \Upsilon^{-2}\delta^{\rho}{}_{\nu}g_{\sigma\mu}g_{\mu\nu}g_{\kappa\tau}\Upsilon^{,\kappa}\Upsilon^{,\tau} \tag{2.8a}$$

We

$$\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\phi = \phi_{\tilde{;}\mu\nu} = \nabla_{\mu}\nabla_{\nu}\phi - \Big(\delta^{\kappa}{}_{\mu}\delta^{\tau}{}_{\nu} + \delta^{\tau}{}_{\mu}\delta^{\kappa}{}_{\nu} - g_{\mu\nu}g^{\kappa\tau}\Big)\Upsilon^{-1}(\nabla_{\kappa}\Upsilon)(\nabla_{\tau}\phi) \tag{2.9}$$

$$\tilde{\mathcal{R}} = \Upsilon^{-2}\mathcal{R} - 2(D-1)g^{\kappa\tau}\Upsilon^{-3}(\nabla_{\kappa}\nabla_{\tau}\Upsilon) - (D-1)(D-4)g^{\kappa\tau}\Upsilon^{-4}(\nabla_{\kappa}\Upsilon)(\nabla_{\tau}\Upsilon) \tag{2.10}$$

$$\tilde{\Box}\phi = \Upsilon^{-2}(\ldots) \tag{2.11a}$$
$$\Box\phi = \Upsilon^{2}(\ldots) \tag{2.11b}$$

Conformal invariants. The *Weyl conformal curvature tensor*

$$\mathcal{W}_{\rho\sigma\mu\nu} = \mathcal{R}_{\rho\sigma\mu\nu} + \frac{2}{D-2}\Big(g_{\rho[\nu}\mathcal{R}_{\mu]\sigma} + g_{\sigma[\mu}\mathcal{R}_{\nu]\rho}\Big) + \frac{2}{(D-2)(D-1)}\mathcal{R}g_{\rho[\mu}g_{\nu]\sigma} \tag{2.12}$$

is preserved under conformal transformations, such that

$$\tilde{\mathcal{W}}^{\rho}{}_{\sigma\mu\nu} = \mathcal{W}^{\rho}{}_{\sigma\mu\nu}. \tag{2.13}$$

The *Cotton tensor*

$$C_{\rho\mu\nu} = \mathcal{R}_{\rho[\mu;\nu]} - \frac{1}{2(D-1)}g_{\rho[\mu}\mathcal{R}_{;\nu]} \tag{2.14}$$

is

$C_{\sigma\mu\nu} = \frac{1}{D-3}\mathcal{W}_{\rho\sigma\mu\nu;\rho}$

PHANTOM PARAGRAPH: CONCOMITANT TENSORS

- Conformal flatness: $g_{\mu\nu} = \Upsilon^{-2}(x)\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu} \Rightarrow \tilde{g}_{\mu\nu}(x) = \Upsilon^{2}(x)\eta_{\mu\nu}$
- Conformal trafos of the Hilbert stress–energy tensor

Chapter 3

Classical Field Theory and Gravity

Alongside quantum mechanics, Einstein's theory of gravity—general relativity (GR)—is widely accepted as the most accurate description of our surroundings. GR can be formulated from a geometrical point of view, or it can be viewed as a classical field theory. In the former approach we meet geometrical tools such as the geodesic equation, whereas the latter allows the application of field-theoretical methods. This chapter lays emphasis on the field interpretation of GR.

PHANTOM PARAGRAPH: TWO PERSPECTIVES INSIGHTFUL; BETTER OVERALL UNDERSTANDING OF ASPECTS OF CONCEPTS IN GR

3.1 General relativity

The Einstein–Hilbert action in vacuum is ⌜check Planck mass def.⌟

$$S_{\mathrm{EH}} = {}^{1}\!/_{2} M_{\mathrm{Pl}}^{2} \int d^{4}x \, \sqrt{-|g|}\, \mathcal{R}, \tag{3.1}$$

where $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$. By varying S_{EH} with respect to $g_{\mu\nu}$ one obtains the equation of motion

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} + \frac{1}{2} g_{\mu\nu}\mathcal{R} = 0. \tag{3.2}$$

Thus, we interpret GR as a *classical* field theory where the tensor field $g_{\mu\nu}$ is the gravitational field, ⌈with the particle realisation named "graviton"⌋↻.

- scalar field (ST theories)
- energy momentum tensor

3.2 TITLE (Conformal field theory)

Suppose you have an n-dimensional manifold $\mathscr{M}$ with the associated metric g and coordinate system $\{x\}$. If another spacetime $(\tilde{\mathscr{M}}, \tilde{g})$ of n dimensions is such that $\tilde{g} = \omega(x) g$, we say that said spacetime is *conformal* to the original spacetime $(\mathscr{M}, g)$. This is not just a matter of name-dropping—the situation cause for a number of useful relations. We will see that the expansion of the universe is elegantly handled by conformal transformations. In short, conformality? Is this a word? allows THIS IS WRONG! Same spacetime, different metric

3.2.1 The conformal group

A regular change of the metric tensor under a coordinate transformation $x^\mu \mapsto \tilde{x}^\mu$ looks like

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x). \tag{3.3}$$

A special group of transformations leaves the metric scale invariant (invariant under a local change of scale), $\tilde{g}_{\mu\nu}(\tilde{x}) = \omega^2(x) g_{\mu\nu}(x)$. Such *conformal transformations* make up the conformal group, $\mathsf{Conf}(\mathscr{M})$. We say that $\omega(x)$ is the *conformal factor*.

3.2.2 Friedmann–Lemaitre–Robertson–Walker universe

We consider a four-dimensional expanding universe that is both homogeneous and isotropic with a Lorentzian structure (i.e. metric signature $(-,+,+,+)$). The general metric can be written

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2(\tau)\Big\{-\mathrm{d}\tau^2 + d\Sigma^2\Big\}, \tag{3.4}$$

where τ represents conformal time, $a(\tau)$ is the dimensionless scale factor and Σ is the time-independent three-dimensional space. In polar coordinates, the spatial line element takes the familiar form

$$d\Sigma^2 = \frac{1}{1-kr^2} + r^2 d\Omega^2, \quad k \in \{-1, 0, +1\}. \tag{3.5}$$

However, as we know, it is safe to assume that the universe is flat©(ref to some section) , and we may as well use regular Cartesian coordinates;

$$d\Sigma^2 = \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{3.6}$$

This choice of coordinates implies $g_{\mu\nu} = a^2(x^0)\eta_{\mu\nu}$. Hence, $\mathbb{U} \in \mathsf{Conf}(\mathbb{M})$

- Fourier transform (scale invariance, scalar product preserved)
- Something about the benefit of using $a \propto \tau^\alpha$, and that $\alpha \in \mathbb{Z}$ is a sensible assumption (for completeness, maybe let $\alpha \in \mathbb{R}$?)

Fourier transforms

One very neat consequence of this scale invariance is that in FLRW cosmology we can use the regular, flat-space form of the Fourier transform and its inverse:

$$f(x) = \int \frac{d^4k}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}\eta_{\mu\nu}k^\mu x^\nu} f(k) \qquad = \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{\mathrm{i}\omega\tau} \int \frac{d^3k}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} f(\omega, \boldsymbol{k}) \tag{3.7a}$$

$$f(k) = \int d^4x\, \mathrm{e}^{\mathrm{i}\eta_{\mu\nu}k^\mu x^\nu} f(x) \qquad = \int \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau} \int d^3x\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} f(\tau, \boldsymbol{x}) \tag{3.7b}$$

The four-vectors $[x^\mu] = (\tau, \boldsymbol{x})$ and $[k^\mu] = (\omega, \boldsymbol{k})$ represent the comoving coordinate and wavevector, respectively. **((Maggiore, 2018, Ch. 17.1))**

3.2.3 TITLE (Analytical considerations)

We will encounter several equations of similar forms, for instance $\Box\phi$ = [some source term], whose homogeneous solution satisfies

$$\ddot{\phi} + 2\mathcal{H}\dot{\phi} - \nabla^2\phi = 0. \tag{3.8}$$

This partial differential equation generally depends on initial conditions and expansion history. Transforming the spatial part to Fourier space, ⌈$\nabla \mapsto -\mathrm{i}\boldsymbol{k}$ (check sign!)⌋?, we recognise a wave equation with a damping term $2\mathcal{H}$. The special case for which $a \propto \tau^\alpha$ gives $\mathcal{H} = \dot{a}/a = \alpha/\tau$. A trained eye? will then identify a transformed Bessel's equation ⌜put details in Appendix⌟. We write a somewhat more general (mer hensiktsmessig form) equation for some function $f(x) \to f_k(\tau)$ ⌜fix⌟:

$$\tau^2 \ddot{f}_k + A \cdot \alpha\tau \dot{f}_k + B^2 \cdot k^2\tau^2 f_k = 0; \quad A, B \in \mathbb{C} \tag{3.9}$$

The general solution to this equation is known, and we can use the properties of the Bessel functions to arrive at ⌈nicer⌋↺ expressions for some special cases (define $\nu \equiv n - 1/2, n \equiv A\alpha/2$)

$$f_k(\tau) = \begin{cases} \tau^{-\nu}\{C_k\mathcal{J}_{|\nu|}(Bk\tau) + D_k\mathcal{J}_{-|\nu|}(Bk\tau)\}, & \nu \notin \mathbb{Z} \\ \tau^{1-n}\{C_k j_{|n|}(Bk\tau) + D_k j_{-|n|}(Bk\tau)\}, & \nu \in \mathbb{Z} + \frac{1}{2} \\ \tau^{-\nu}\{C_k\mathcal{J}_{-\nu}(Bk\tau) + D_k\mathcal{Y}_{-\nu}(Bk\tau)\} & \end{cases} \tag{3.10}$$

⌈Carefully check these! I think they are wrong...⌋? ⌜Spoiler alert!⌟ Most physically meaningful scenarios will have $\alpha \in \mathbb{Z}$ (see Table XXX©0) and $A \in \mathbb{Z}$.

⌜Table with matched w_s, α, β etc.⌟

Table 3.1: $w = p/\rho$ blah blah

Constituent	Perfect fluid parameters			Domination	
⌜*FIXME*⌟	w	α	β	a	ρ
cosmological constant, dark energy (Λ)	-1				$\propto a$
non-relativistic matter, dust (m)	0	2	2/3	$\propto \tau^2 \propto t_{\mathrm{ph}}^{2/3}$	$\propto a^{-3}$
relativistic matter, radiation (r)	1/3	1	1/2	$\propto \tau \propto t_{\mathrm{ph}}^{1/2}$	$\propto a^{-4}$

3.3 Perturbation theory

3.4 Classical solitons

Chapter 4

Gravitational Waves

The term "gravitational waves" refers to the tensor perturbations to the background metric$^{©}_{0}$. These "waves" are spacetime distortions whose name comes from the fact that ⌈they obey the wave equation⌋$_{?}$.

4.1 Linearised gravity

The simplest way to find the equation of motion (e.o.m.) for the tensor perturbations to the metric is to *go through* a Minkowski spacetime. First, we establish the law of physics, valid in Minkowski coordinates. We write it in a coordinate-independent form, that is to say write the e.o.m. on tensorial form. ⌈The resulting law (e.o.m.) remains true in any spacetime. This procedure is called the "minimal coupling procedure" and is extremely powerful.⌋$_{?}$

4.1.1 TITLE (Minkowski Spacetime)

For the time being, we take the unperturbed metric to be the flat, static Minkowskian metric such that

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x); \quad \left|h_{\mu\nu}\right| \ll 1 \tag{4.1}$$

is the full, perturbed metric. Note that the $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, whereas $h^{\mu\nu}$ is *not* the inverse of $h_{\mu\nu}$. We aim to find the Einstein tensor $\mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}$.

To zeroth order in $h_{\mu\nu}$, the Einstein tensor is simply zero and the metric Minkowskian. To leading order $\mathcal{O}(h)$, some calculation is required. First, we compute the Christophel symbols

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(2g_{\sigma(\mu,\nu)} - g_{\mu\nu,\sigma}\right) = \frac{1}{2}\eta^{\rho\sigma}\left(2h_{\sigma(\mu,\nu)} - h_{\mu\nu,\sigma}\right) + \mathcal{O}(h^2). \tag{4.2}$$

Next, we find the Riemann tensor to be $\mathcal{R}^{\rho}{}_{\mu\sigma\nu} = 2\Gamma^{\rho}_{\mu[\nu,\sigma]} + \mathcal{O}(h^2)$. The Ricci tensor is then

$$\begin{aligned}\mathcal{R}_{\mu\nu} &= \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h_{\mu}{}^{\rho} - \partial_\rho\partial^{\rho} h_{\mu\nu} - \partial_\mu\partial_\nu h^{\rho}{}_{\rho}\right) + \mathcal{O}(h^2)\\ &= \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h_{\mu}{}^{\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) + \mathcal{O}(h^2),\end{aligned} \tag{4.3}$$

thereby the Ricci scalar $\mathcal{R} = \partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h + \mathcal{O}(h^2)$. The first order Einstein tensor is

$$\mathcal{G}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h_{\mu}{}^{\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h\right) + \mathcal{O}(h^2). \tag{4.4}$$

...

...

...

- Gauge freedom
- Explain perturbation equation
- Separating GWs from background (Maggiore)

4.1.2 TITLE (FLRW Spacetime)

4.1.3 TITLE (TT gauge?)

DRAFT

TITLE (Energy–momentum tensor; Eom.; Scalar field)

The energy–momentum tensor is

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right] \tag{4.5}$$

blah blah [...] Assume no cross terms in $g_{\mu\nu}$. To retrieve the TT-part of $T_{\mu\nu}$ we utilise the Lambda tensor$^{©}_{\text{(repeat or refer?)}}$

$$\begin{aligned} T_{ij}(\eta, \boldsymbol{k}) &= \Lambda_{ij,kl}(\hat{\boldsymbol{k}})(T^{\neg\mathrm{TT}})_{ij}(\eta, \boldsymbol{k}) \\ &\overset{!}{=} \Lambda_{ij,kl}(\hat{\boldsymbol{k}})\left[\partial_i\phi\partial_j\phi\right](\eta, \boldsymbol{k}) \end{aligned} \tag{4.6}$$

$$\begin{aligned} \left[\partial_i\phi\partial_j\phi\right](\eta, \boldsymbol{k}) &= \int d^3x\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left[\partial_i\phi\partial_j\phi\right](\eta, \boldsymbol{x}) \\ &= ???? \end{aligned} \tag{4.7}$$

4.2 Polarisation and decomposition of gravitational waves

4.3 Gravitational waves in vacuum

In an expanding, flat universe, we choose the TT gauge such that $h^i{}_i = 0$ and $h_{ij,j} = 0$. The unsourced equation reads

$$\Box h_\bullet(x) = 0 \Rightarrow \ddot{h}_\bullet(x) + 2\mathcal{H}\dot{h}_\bullet(x) - \nabla^2 h_\bullet(x) = 0. \tag{4.8}$$

Assume $a \propto \tau^\alpha$. We follow Section 3.2.3 to find the general solution for this equation in Fourier space. We specify $\alpha \in \mathbb{Z}$ and get

$$h_\bullet(\tau, \boldsymbol{k}) = \frac{A_k j_{|\alpha|}(k\tau) + B_k j_{-|\alpha|}(k\tau)}{\tau^{\alpha-1}}; \quad k = |\boldsymbol{k}|. \tag{4.9}$$

⌜OR⌟ we solve for the comoving GWs, $\mathsf{h}_\bullet$, defining $p = k\tau$

$$\frac{\partial^2 \mathsf{h}_\bullet}{\partial p^2} + \left[1 - \frac{\alpha(\alpha-1)}{p^2}\right]\mathsf{h}_\bullet = 0 \tag{4.10}$$

4.4 Generation of gravitational waves

- Somehow get to this eq:

$$T^{\mathrm{TT}}_{ij}(\eta, \boldsymbol{k}) = \Lambda_{ij,kl}(\hat{\boldsymbol{k}}) \int \frac{d^3p}{(2\pi)^3}\, p_k p_l \phi(\eta, \boldsymbol{p})\phi(\eta, \boldsymbol{k} - \boldsymbol{p}). \tag{4.11}$$

- Production instead of generation?

4.4.1 General Formalism

DRAFT

⌜

⌜Use conformal trafo! ((Carroll, 2019, p. 467)) ⌟

$$ds^2 = -dt^2 + a^2(t)\left\{\gamma_{ij} + h_{ij}\right\}dx^i dx^j = a^2(\eta)\left\{-d\eta^2 + \left(\gamma_{ij} + h_{ij}\right)dx^i dx^j\right\} \tag{4.12}$$

$h_{ij} = h^{\mathrm{TT}}_{ij}, \gamma_{ij} = \delta_{ij}, (\overline{g}_{\mu\nu} = a^2\eta_{\mu\nu})$
Linearised Einstein equations: ⌜conformal Newtonian gauge⌟

$$\delta G^i{}_j = 8\pi G_{\mathrm{N}} T^i{}_j = \frac{1}{2a^2}\left[\ddot{h}_{ij} + 2\frac{\dot{a}}{a}\dot{h}_{ij} - \nabla^2 h_{ij}\right] \tag{4.13}$$

where a dot (‘ ˙ ’) signifies the *conformal* time derivative. ($T^i{}_j = a^{-2}T_{ij}$) blah blah [...]

$$\ddot{h}_{ij}(\eta, \boldsymbol{k}) + 2\frac{\dot{a}}{a}\dot{h}_{ij}(\eta, \boldsymbol{k}) - k^2 h_{ij}(\eta, \boldsymbol{k}) = 16\pi G_\mathrm{N} T_{ij}(\eta, \boldsymbol{k}) \tag{4.14}$$

Define $\mathfrak{h}_{ij} \equiv a h_{ij}$. By inserting this in Eq. (4.14) and multiplying the equation by a, one finds

$$\ddot{\mathfrak{h}}_{ij}(\eta, \boldsymbol{k}) + \left[k^2 - \frac{\ddot{a}(\eta)}{a(\eta)}\right]\mathfrak{h}_{ij}(\eta, \boldsymbol{k}) = 16\pi G_\mathrm{N} a(\eta) T_{ij}(\eta, \boldsymbol{k}). \tag{4.15}$$

We assume $a(\eta) \propto \eta^\alpha$ and define $\nu \equiv \alpha - \frac{1}{2}$. Letting $\tau = k\eta$, Eq. (4.15) becomes

$$\left[\frac{\partial^2}{\partial\tau^2} + 1 - \frac{4\nu^2 - 1}{4\tau^2}\right]\mathfrak{h}_{ij}(\eta, \boldsymbol{k}) = \frac{16\pi G_\mathrm{N} a(\eta)}{k^2} T_{ij}(\eta, \boldsymbol{k}) \tag{4.16}$$

Now, Eq. (4.16) transforms into a problem of the form $\mathrm{L}_\tau f(\tau) = g(\tau)$; a problem that can be solved using Green's method (see some section$^{©}_{()}$). Kawasaki and Saikawa ((2011)) propose

$$G(\tau, \tau') = \frac{\pi}{2}\Theta(\tau - \tau')[\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')] \tag{4.17}$$

as a solution to $\mathrm{L}_\tau G(\tau, \tau') = \delta(\tau - \tau')$. In some appendix$^{©}_{()}$ we show that this holds for a matter dominated universe where $\nu = 2 - \frac{1}{2} = \frac{3}{2}$.

Now assume the source is active (emits gravitational radiation) between η_ini and η_fi, and followingly? initial conditions $\mathfrak{h}_{ij}(\eta_\mathrm{ini}, \boldsymbol{k}) = \dot{\mathfrak{h}}_{ij}(\eta_\mathrm{ini}, \boldsymbol{k}) = 0$. Thus,

$$\mathfrak{h}_{ij}(\eta \geq \eta_\mathrm{ini}, \boldsymbol{k}) = \frac{8\pi^2 G_\mathrm{N}}{k^2}\int_{k\eta_\mathrm{ini}}^{k\eta} \mathrm{d}\tau' \sqrt{\tau\tau'}[\mathcal{Y}_\nu(\tau)\mathcal{J}_\nu(\tau') - \mathcal{J}_\nu(\tau)\mathcal{Y}_\nu(\tau')]a(\tau')T_{ij}(\tau', \boldsymbol{k}), \tag{4.18}$$

which reduces to

$$\mathfrak{h}_{ij}(\eta \geq \eta_\mathrm{fi}, \boldsymbol{k}) = A_{ij}(\boldsymbol{k})\sqrt{k\eta}\mathcal{J}_\nu(k\eta) + B_{ij}(\boldsymbol{k})\sqrt{k\eta}\mathcal{Y}_\nu(k\eta). \tag{4.19}$$

Combining Eq. (4.18) and Eq. (4.19) at $\eta = \eta_\mathrm{fi}$ gives the coefficients A_{ij} and B_{ij}:

$$\begin{aligned} A_{ij}(\boldsymbol{k}) &= -\frac{8\pi^2 G_\mathrm{N}}{k^2}\int_{k\eta_\mathrm{ini}}^{k\eta_\mathrm{fi}} \mathrm{d}\tau' \sqrt{\tau'} a(\tau')\mathcal{Y}_\nu(\tau')T_{ij}(\tau', \boldsymbol{k}) \\ B_{ij}(\boldsymbol{k}) &= +\frac{8\pi^2 G_\mathrm{N}}{k^2}\int_{k\eta_\mathrm{ini}}^{k\eta_\mathrm{fi}} \mathrm{d}\tau' \sqrt{\tau'} a(\tau')\mathcal{J}_\nu(\tau')T_{ij}(\tau', \boldsymbol{k}) \end{aligned} \tag{4.20}$$

4.4.2 Scalar Field Source (temp. name)

Chapter 5

Lattice-[?] and N-body Simulations

Part II

Project

Chapter 6

Perturbed Domain Walls

PHANTOM PARAGRAPH: MOTIVATION–INSPIRATION(WORK)

⌈ $\mathbb{M} \equiv (\mathbb{R}^4, \eta_{\mu\nu})$, $\mathbb{U} \equiv (\mathbb{R}^4, a^2\eta_{\mu\nu})$ ⌋?

We shall consider a pseudo-Riemannian $(N+1)$-dimensional spacetime $(\mathscr{M}, g_{\mu\nu})$ up until the point where concrete expressions are needed to proceed; here we turn to the conformally flat FLRW universe, for which $\mathscr{M} = \mathbb{U}$, $N = 3$ and $g_{\mu\nu} = a^2\eta_{\mu\nu}$.

We begin in the thin-wall limit, where the width of the domain wall in question is negligible. To go beyond the thin-wall limit eventually turns out to be a simple matter of an additional factor in the final expression. ⌈This is not true, get a grip, Nanna.⌋

$$\Phi(x^\mu - X^\mu) = \frac{1}{\sqrt{2\pi}w_0}\exp\left\{-\frac{(x^\mu - X^\mu)^2}{2w_0^2}\right\} \quad \rightsquigarrow \quad \lim_{w_0\to 0}\Phi(x^\mu - X^\mu) = \delta^{(4)}(x^\mu - X^\mu) \tag{6.1}$$

⌈Needs fixing...⌋

6.1 Dynamics of thin domain walls (general framework)

PHANTOM PARAGRAPH: CONSTANT SURFACE TENSION

We follow Garriga and Vilenkin ((1991)) and Ishibashi and Ishihara ((1999)). The world sheet Σ divides $\mathscr{M}$ into two submanifolds $\mathscr{M}_\pm$ such that $\mathscr{M} = \mathscr{M}_+ \cup \Sigma \cup \mathscr{M}_-$. That is to say, a domain wall holds a world sheet separating two vacua. We take $\mathscr{M}$ to be smooth and $(N+1)$-dimensional, and let Σ be a smooth also and $((N-1)+1)$-dimensional. Consequently, Σ is a timelike hypersurface in $\mathscr{M}$.

- Vary DW action
- Goal: E.O.M. for physically relevant component (epsilon basically)
- Expression for energy–momentum tensor
- Extension to non-thin walls
- Extension to Asymmetron or introduction of energy bias
- What does thin mean? Why is the tension indep. of width?

We invoke a smooth coordinate system $\{x^\mu\}$ ($\mu = 0, 1, \ldots, N$) of the spacetime $(\mathscr{M}, g_{\mu\nu})$ in a neighbourhood of Σ. The embedding of Σ in $\mathscr{M}$ is $x^\mu = x^\mu(y^a)$, where the coordinate system $\{y^a\}$ ($a = 0, 1, \ldots, N-1$) parametrises Σ. The induced metric on Σ is

$$h_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b; \quad e^\mu_a \equiv \frac{\partial x^\mu}{\partial y^a} \tag{6.2}$$

[←argue!]■

The action for a thin domain wall is famously ((e.g. Vachaspati, 2006)) the Nambu-Goto action

$$S_{\rm dw} = -\sigma \int_\Sigma d^N y \sqrt{-h}\,, \tag{6.3}$$

where σ is the wall's energy per unit area, henceforth called "surface tension".

DRAFT

6.1.1 Expanding universe. (my scenario)

6.2 TITLE (Symmetron domain walls)

DRAFT

Not constant surface tension. We now let the surface tension of the domain wall be dependent of time. The variation of the action becomes slightly different, and the resulting equation of motion

$$\ddot{\epsilon} + \left(3\frac{\dot{a}}{a} + \frac{\dot{\sigma}}{\sigma}\right)\dot{\epsilon} + k^2\epsilon = 0. \tag{6.4}$$

A simple way to obtain this equation is to substitute $a \to \sigma^{1/3}a$ in the equations above.

We assume an ideal situation in which SSB occurs at some conformal time η_* in a universe where $a = a_*(\eta/\eta_*)^\alpha$. It is useful to introduce a dimensionless time variable $s \equiv \eta/\eta_*$, s.s. $a = a_* s^\alpha$, as well as a parameter $q \equiv k\eta_*$. The surface tension of a domain wall is computed from the Symmetron effective potential

$$V_{\mathrm{eff}}(\phi) = \frac{\lambda}{4}\phi^4 + \frac{\mu^2}{2}\left(\frac{\rho_{\mathrm{m}}}{\mu^2 M^2} - 1\right)\phi^2 + V_0. \tag{6.5}$$

We define the matter density at SSB to be $\rho_{\mathrm{m}}|_{\eta=\eta_*} = \mu^2 M^2$. Since $\rho_{\mathrm{m}} \propto a^{-3(1+w_{\mathrm{m}})} = a^{-3}$, we get

$$\rho_{\mathrm{m}} = \mu^2 M^2 \left(a_*/a\right)^3 = \mu^2 M^2 \left(\eta_*/\eta\right)^{3\alpha} = \mu^2 M^2 s^{-3\alpha}. \tag{6.6}$$

blah blah [. . .] The surface tension becomes

$$\sigma = \sigma_0 \left(1 - s^{-3\alpha}\right)^{3/2}. \tag{6.7}$$

Finally, the equation of motion for the wall normal coordinate ϵn^μ is

$$\epsilon'' + \left(\frac{3\alpha}{s} + 2\gamma(s)\right)\epsilon' + q^2\epsilon = 0; \quad \gamma(s) = \frac{9\alpha}{4s\left(s^{3\alpha} - 1\right)}. \tag{6.8}$$

6.2.1 Solving the equation of motion in a matter dominated universe

We restrict our discussion to $\alpha = 2$. The generalisation to $\alpha \geq 1/3$ is trivial ⌜in appendix, maybe?⌟. We shall solve Eq. (6.8) in two regimes of s and sow these solutions together. In the following, we neglect the spatial part of ϵ, that which is subject to ⌜explain this⌟

$\epsilon(\eta, x, y) =$

Consider a planar domain wall formed during SSB at spacetime position $X^\mu = (\eta_*, x, y, z_0)$. Assume that such a formation induces a perturbation to the wall that moves the wall normal coordinate from z_0 to $z_0 + \epsilon_*$. This gives the initial condition $\epsilon(1) = \epsilon_*$ to the equation of motion for $\epsilon(s)$ with $\alpha = 2$, namely

$$\epsilon'' + \left(\frac{6}{s} + \frac{9}{s\left(s^6 - 1\right)}\right)\epsilon' + q^2\epsilon = 0. \tag{6.9}$$

The dimensionless variables are restricted $s \geq 1$ and $q \gg 1$ to ensure that SSB has happened and ⌈that the scale of the perturbation is subhorizon.⌋

Shortly after symmetry breaking. We begin by solving the equation of motion for $s \sim 1$. As our equation has a singularity at $s = 1$, the natural way to go is through a Laurent expansion around this point of the damping term in Eq. (6.9). We find

$$\frac{6}{s} + \frac{9}{s\left(s^6 - 1\right)} = \frac{3}{2}(s-1) + \frac{3}{4} + \frac{29}{8}(s-1) - \frac{93}{16}(s-1)^2 + \mathcal{O}((s-1)^3). \tag{6.10}$$

Now $\epsilon(s)$ is also subject to an expansion around $s = 1$;

$$\epsilon(s) = \epsilon_* \cdot \left[1 + c_1(s-1) + c_2(s-1)^2 + c_3(s-1)^3 + \ldots\right]. \tag{6.11}$$

When put together, we get a polynomial in $s-1$ on the left-hand side of Eq. (6.9), for which all coefficients must vanish. We solve the system of equations for $\{c_1, c_2, c_3\}$ and find

$$\epsilon(s) = \epsilon_* \cdot \left[1 - \frac{q^2}{5}(s-1)^2 + \frac{q^2}{35}(s-1)^3\right] + \mathcal{O}((s-1)^4), \quad s \gtrsim 1 \tag{6.12}$$

Later times. For $s \gg 1$, the damping term $\gamma(s) = 9/(2s(s^6-1))$ in the eom for ϵ becomes subdominant ⌜check plag. Julian⌟, and asymptotically the solution is $s^{-2}\{cj_2(qs) + dy_2(qs)\}$. Said damping term is not completely negligible, however, as it causes a *damping envelope* that is considered much like in the case of a damped harmonic oscillator, writing

$$\epsilon(s) \simeq \epsilon_* \cdot w(s) \cdot \exp\left\{-\int^s \mathrm{d}t\, \gamma(t)\right\}. \tag{6.13}$$

Employing this ansatz in the eom gives

$$w'' + \frac{6}{s}w' + \left(q^2 - \theta(s)\right)w = 0; \quad \theta(s) = \gamma'(s) + \gamma^2(s) + \frac{6}{s}\gamma(s) \tag{6.14}$$

whose solution is

$$w(s) \simeq s^{-5/2}\{A\mathcal{J}_{-5/2}(qs) + B\mathcal{Y}_{-5/2}(qs)\} \tag{6.15}$$

when the phase shift introduced by $\theta(s)$ is negligible and A and B are constants.[1] Now, we find that $\exp\left\{-\int^s \mathrm{d}t\, \gamma(t)\right\} = s^{9/2}(s^6-1)^{-3/4} \cdot \text{const.}$, so in redefining the constants A and B, we have

$$\epsilon(s) \simeq \epsilon_* \cdot \frac{A\mathcal{J}_{-5/2}(qs) + B\mathcal{Y}_{-5/2}(qs)}{s^{5/2}} \frac{s^{9/2}}{(s^6-1)^{3/4}}, \quad s \geq s_{\text{sow}} \tag{6.16}$$

Putting it together. We have obtained solutions for $\epsilon(s)$ in two regimes. We say that $\epsilon(s)$ obeys Eq. (6.12) for $s \in [1, s_{\text{sow}}]$ and Eq. (6.13) for $s \in [s_{\text{sow}}, \infty)$, where s_{sow} is close to, but strictly larger than 1. We choose $s_{\text{sow}} = 1 + q^{-1}$ since only ⌈causally conntected modes⌋, for which $q \gg 1$, are of interest to us. We use a computer algebra system, namely *SageMath* ((The Sage Developers, 2023)), to determine A and B in Eq. (6.15) from the system of equations that comes from equating the right-hand sides of Eq. (6.12) and Eq. (6.13) with $s = s_{\text{sow}}$.

- We assume given values of ϵ_* and q
- A and B are constants dep. on choice of s_{sow} (and q)
- Find a place to show the expressions for A and B, maybe an attachment or appendix
- Two free parameters: As we can see, changing the amplitude ϵ_* does not change the motion, but varying the wavenumber q does

$$\epsilon_q(s, x, y) = \epsilon_* \cdot \begin{cases} 1(\dots) & 1 \leq s \leq s_{\text{sow}} \\ 2(\dots) & s_{\text{sow}} \leq s < \infty \quad \mathcal{H} \end{cases} \tag{6.17}$$

[1] In fact, it is possible to show that $\lim_{q\to\infty}\left[\sqrt{q^2 - \theta(1+q^{-1})} - q\right]/q = \sqrt{19}/4 - 1 \approx 0.09$.

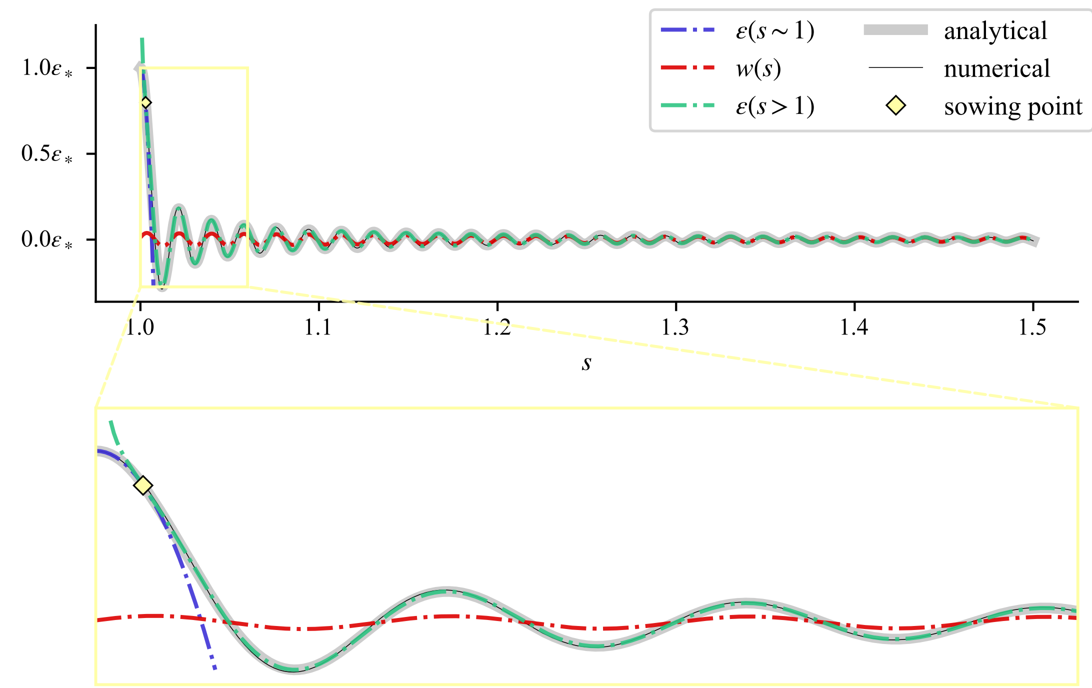

Figure 6.1: Demonstration of ...

6.3 Generation of gravitational waves

⌈ **Neglect back-reaction:** We assume that the topological defect does not change the ⌜un-perturbed⌟ metrics of $\mathscr{M}_\pm$. The domain wall is simply viewed as a sheet separating two domains, and the (un-perturbed) metric $g_{\mu\nu}$ that appears in the covariant derivative, d'Alembertian etc., and raises and lowers indices is unaffected by this.⌋

In the absence of asymmetry, a domain wall will not produce disturbances in the gravitational field. However, perturbations to the wall position, such as ripples or wiggles, can reveal themselves as tensor perturbations to the background metric.

blah blah [...]

6.3.1 Expanding universe: general gramework

⌜Maybe define t to be conformal time? And h to be comoving? Remember conformally flat concept.⌟

$$ds^2 = -\mathrm{d}\check{t}^2 + a^2(\check{t})\Big(\delta_{ij} + \check{h}_{ij}(t,x)\Big)\mathrm{d}x^i\mathrm{d}x^j = a^2(t)\Big(-\mathrm{d}t^2 + \Big(\delta_{ij} + \check{h}_{ij}\Big)\mathrm{d}x^i\mathrm{d}x^j\Big) \tag{6.18}$$

From ref to some section$^{©}_{\text{(GWs chapter)}}$ blah blah [...]

$\mathrm{x} = k\tau$, $\bar{\alpha} = \alpha - \frac{1}{2}$, $\Pi_\circ(\mathrm{x}, \boldsymbol{k}) \triangleq a(\mathrm{x}/k)T^{\mathrm{TT}}_\circ(\mathrm{x}/k, \boldsymbol{k})$

⌜Temporary placeholder definition sign should be used, perhaps $\triangleq$: $\mathrm{x} \triangleq k\tau$ ⌟

$$\mathsf{h}_{\circ}(\tau, \boldsymbol{k}) = \frac{16\pi G_{\mathrm{N}}}{k^2} \int_{\mathrm{x}_{\mathrm{init}}}^{\mathrm{x}} \mathrm{dx}'\, \mathcal{G}_{\bar{\alpha}}(\mathrm{x}, \mathrm{x}')\Pi_{\circ}(\mathrm{x}', \boldsymbol{k}); \quad \circ = +, \times \tag{6.19}$$

((Kawasaki and Saikawa, 2011))

If at some conformal time τ_{fin} switch off the source, we obtain the homogeneous solution for $\tau \geq \tau_{\mathrm{fin}}$,

$$\mathsf{h}_{\circ}(\tau, \boldsymbol{k}) = \sqrt{\mathrm{x}}\{\mathcal{A}_{\circ}(\boldsymbol{k})\mathcal{J}_{\bar{\alpha}}(\mathrm{x}) + \mathcal{B}_{\circ}(\boldsymbol{k})\mathcal{Y}_{\bar{\alpha}}(\mathrm{x})\}. \tag{6.20}$$

The coefficients are determined by sowing together the homogeneous and inhomogeneous solutions at $\tau = \tau_{\mathrm{fin}}$:

$$\begin{aligned}\sqrt{\mathrm{x}_{\mathrm{fin}}}\mathcal{A}_{\circ}(\boldsymbol{k})\mathcal{J}_{\bar{\alpha}}(\mathrm{x}_{\mathrm{fin}}) + \sqrt{\mathrm{x}_{\mathrm{fin}}}\mathcal{B}_{\circ}(\boldsymbol{k})\mathcal{Y}_{\bar{\alpha}}(\mathrm{x}_{\mathrm{fin}})&\\ = \frac{8\pi^2 G_{\mathrm{N}}}{k^2} \int_{\mathrm{x}_{\mathrm{init}}}^{\mathrm{x}_{\mathrm{fin}}} \mathrm{dx}'\, \sqrt{\mathrm{x}\mathrm{x}'}\{\mathcal{Y}_{\bar{\alpha}}(\mathrm{x})\mathcal{J}_{\bar{\alpha}}(\mathrm{x}') - \mathcal{J}_{\bar{\alpha}}(\mathrm{x})\mathcal{Y}_{\bar{\alpha}}(\mathrm{x}')\}\Pi_{\circ}(\mathrm{x}', \boldsymbol{k})&\end{aligned} \tag{6.21}$$

We get that

$$\begin{aligned}\mathcal{A}_{\circ}(\boldsymbol{k}) &= -\frac{8\pi^2 G_{\mathrm{N}}}{k^2} \int_{\mathrm{x}_{\mathrm{init}}}^{\mathrm{x}_{\mathrm{fin}}} \mathrm{dx}'\, \sqrt{\mathrm{x}'}\mathcal{Y}_{\bar{\alpha}}(\mathrm{x}')\Pi_{\circ}(\mathrm{x}', \boldsymbol{k})\\ \mathcal{B}_{\circ}(\boldsymbol{k}) &= +\frac{8\pi^2 G_{\mathrm{N}}}{k^2} \int_{\mathrm{x}_{\mathrm{init}}}^{\mathrm{x}_{\mathrm{fin}}} \mathrm{dx}'\, \sqrt{\mathrm{x}'}\mathcal{J}_{\bar{\alpha}}(\mathrm{x}')\Pi_{\circ}(\mathrm{x}', \boldsymbol{k})\end{aligned} \tag{6.22}$$

DRAFT

See note “gwasevolution parameters”

$$T_{\circ}^{\mathrm{TT}}(t, \boldsymbol{k}) = (\sigma/u^2) \cdot 2\pi^2 W(k_z)\delta\left(\ell_y\right)[\![\ell_x \in \mathbb{Z}]\!] a(t)\mathcal{J}_{\ell_x}(k_z\epsilon_0\varepsilon(ut)) \tag{6.23}$$

¿ Find out if this stress–energy is the same (or how it scales) as the one in $\Box h_{\circ} = 16\pi G_{\mathrm{N}} T_{\circ}^{\mathrm{TT}}$?

Chapter 7

Discussion

Chapter 8

Conclusion and Outlook

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Appendix A

I do not have an appendix