UNIVERSITY
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Master's thesis

Gravitational waves from topological defects

Any short subtitle

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Contents

Notation

Constants and units. We use ⌈'natural units'⌋? where $\hbar = c = 1$, where $\hbar$ is the reduced Planck constant and c is the speed of light in vacuum.⌜Planck units? Set $k_B = G_N = 1$?⌟ The Newtonian constant of gravitation G_N is referenced explicitly, and we use Planck units such as the Planck mass $M_{Pl} = (\hbar c/G_N)^{1/2} = G_N^{-1/2} \sim 10^{-8}$ kg.

Tensors. The metric signature $(-,+,+,+)$ is considered, i.e. $\det[g_{\mu\nu}] \equiv |g| < 0$. The Minkowski metric is denoted $\eta_{\mu\nu}$, whereas a general metric is denoted $g_{\mu\nu}$. A four-vector $p^\mu =$
$[\eta_{\mu\nu}] = \mathrm{diag}(-1,1,1,1)$

Christophel symbols:

$$\Gamma^\rho_{\mu\nu} = \ldots \tag{1}$$

"Lambda tensor":

$$\Lambda_{ij,kl} = \ldots \tag{2}$$

Fourier transforms. We use the following convention for the Fourier transform of $f(x)$, $\tilde{f}(k)$, and its inverse, where x and k are Lorentz four-vectors:

$$\begin{aligned} f(x) &= \int \frac{d^4k}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}k\cdot x} \tilde{f}(k) \\ \tilde{f}(k) &= \int d^4x\, \mathrm{e}^{\mathrm{i}k\cdot x} f(x) \end{aligned} \tag{3}$$

Here, $k\cdot x = k_\sigma x^\sigma = g_{\rho\sigma} k^\rho x^\sigma$.

Acronyms

CDM	cold dark matter
CMB	cosmic microwave background (radiation)
DW	domain wall
GR	general relativity
GW	gravitational wave
ΛCDM	Lambda (Greek Λ) cold dark matter model; Standard cosmological model

Nomenclature

In the table below is listed the most frequently used symbols in this paper, for reference.

Table 1: helo

Symbol	Referent	SI-value or definition
Natural constants		
G_{N}	Newtonian constant of gravitation	$1.2\ \mathrm{kg}$
k_{B}	Boltzmann's constant	$1.2\ \mathrm{K}$
Fiducial quantities		
h_0	Reduced Hubble constant	0.67
Subscripts		
Q_{gw}	Quantity Q related to gravitational wave	
Functions and operators		
$\Theta(\xi)$	Heaviside step function	$\begin{cases} 1 & \xi > 0 \\ 0 & \xi < 0 \end{cases}$
$\mathrm{sgn}(\xi)$	Signum function	$2\Theta(\xi) - 1$
$\delta^{(n)}(\xi)$	Dirac-Delta function of $\xi \in \mathbb{R}^n$, $n \in \mathbb{N}$.	
$\delta^{\mu\nu}$	Kronecker delta.	

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Writing Tools

This is a comment.
This needs spelling check.? Perhaps this?
Rephrase this.
Awkward wording.
Needs double-checking.?
This is in need of citation or reference to a section.©
blah blah [. . .](Phantom text.)

PHANTOM PARAGRAPH: THIS IS A PHANTOM PARAGRAPH, MAYBE WITH SOME KEYWORDS.

- This is a note.
- This is another note.
  - This is a related note.

This is very important.

This text is highlighted.

Statement. [←That needs to be shown or proven.]

Chapter 1

Introduction

- GOALS:
  - Gather framework about GWs from DWs

$$\tilde{h}''_{\circledast} + 2\mathcal{H}\tilde{h}'_{\circledast} + k^2\tilde{h}_{\circledast} = 16\pi G_{\mathrm{N}} a^2 \tilde{\sigma}_{\circledast}; \quad \circledast = +, \times \tag{1.1}$$

$$\left(\tilde{h}^{\mathrm{TT}}\right)_{ij}(\eta, \boldsymbol{k}) = \sum_{\circledast=+,\times} e^{\circledast}_{ij}(\hat{\boldsymbol{k}})\tilde{h}_{\circledast}(\eta, \boldsymbol{k}) \tag{1.2}$$

$$\tilde{h}^{\mathrm{TT}}{}_{ij}(\eta, \boldsymbol{k}) = \sum_{\circledast=+,\times} e^{\circledast}_{ij}(\hat{\boldsymbol{k}})\tilde{h}_{\circledast}(\eta, \boldsymbol{k}) \tag{1.3}$$

1.1 Preliminaries

It is assumed that the reader is familiar with variational calculus and linear perturbation theory.

⌈In the following, we briefly (re)capture some concepts that are important starting points for the rest of the thesis.⌋

- variational calculus/ varying action
- action
- pert. theory?
- line element
- gauge invariance
- FRW cosmology
- classical field theory

$$G_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu} \tag{1.4}$$

1.1.1 Field theory

We formulate a theory in four-dimensional spacetime ⌜Minkowski⌟ in terms of the Lorentz invariant action

$$S = \int d^4x\, \mathcal{L}(\{\phi_i\}, \{\partial_\mu \phi_i\}), \tag{1.5}$$

with $\mathcal{L}$ being the *Lagrangian density* of the theory, a function of the set of fields $\{\phi_i\}$ and its first derivatives. We will refer to $\mathcal{L}$ simply as the Lagrangian, as is customary when working with fields. For a general (i.e. curved) spacetime, blah blah [...]$\partial_\mu \to \nabla_\mu$ blah blah [...]to construct a Lorentz invariant Lagrangian,

$$S = \int d^4x\, \underbrace{\mathcal{L}(\{\phi_i\}, \{\nabla_\mu \phi_i\})}_{\text{not scalar}} = \int \underbrace{d^4x\, \sqrt{-|g|}}_{\text{scalar}}\, \underbrace{\hat{\mathcal{L}}(\{\phi_i\}, \{\nabla_\mu \phi_i\})}_{\text{scalar}}, \tag{1.6}$$

⌜Maybe specify that this is only for scalar fields? Or include other fields?⌟

1.1.2 Expanding universe: FRW cosmology

The universe expands with the rate $a(t)$ at cosmic time t.

- expansion rate, cosmic time, conformal time
- why is flat assumption OK?

Part I

Background

Chapter 2

Classical Field Theory and Gravity

Alongside quantum mechanics, Einstein's theory of gravity—general relativity (GR)—is widely accepted as the most accurate description of our surroundings. GR can be formulated from a geometrical point of view, or it can be viewed as a classical field theory. In the former approach we meet geometrical tools such as the geodesic equation, whereas the latter allows the application of field-theoretical methods. This chapter lays emphasis on the field interpretation of GR.

PHANTOM PARAGRAPH: TWO PERSPECTIVES INSIGHTFUL; BETTER OVERALL UNDERSTANDING OF ASPECTS OF CONCEPTS IN GR

2.1 General Relativity

The Einstein–Hilbert action in vacuum is check Planck mass def.

$$S_{\mathrm{EH}} = {}^{1}\!/_{2} M_{\mathrm{Pl}}^{2} \int d^4x \, \sqrt{-|g|}\, \mathcal{R}, \tag{2.1}$$

where $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$. By varying S_{EH} with respect to $g_{\mu\nu}$ one obtains the equation of motion

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} + \frac{1}{2} g_{\mu\nu}\mathcal{R} = 0. \tag{2.2}$$

Thus, we interpret GR as a *classical* field theory where the tensor field $g_{\mu\nu}$ is the gravitational field, with the particle realisation named "graviton".

- scalar field (ST theories)

2.2 Scalar-Tensor Theories?

2.3 Perturbation Theory

2.4 Classical Solitons

Chapter 3

Gravitational Waves

The term "gravitational waves" refers to the tensor perturbations to the background metric© . These "waves" are spacetime distortions whose name comes from the fact that ⌈they obey the wave equation⌋?.

3.1 Linearised Gravity

If we let $\bar{g}_{\mu\nu}$ be the background metric, the perturbed metric is given by

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}; \quad \left|h_{\mu\nu}\right| \ll 1, \tag{3.1}$$

where $h_{\mu\nu}$ describes a tensor field propagating on said background. Now, $g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu}$ is the inverse metric, however $h^{\mu\nu}h_{\nu\lambda} = \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}h_{\rho\sigma}h_{\nu\lambda} \neq \delta^{\mu}_{\lambda}$.

PHANTOM PARAGRAPH: FIND THE LINEARISED EINSTEIN EQS – USING COMMA NOTATION – COMMENT ABOUT NO BACKREACTION

The Christophel symbols are now

$$\begin{aligned}
\Gamma^{\rho}{}_{\mu\nu} &= \frac{1}{2}g^{\rho\sigma}\left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}\right) \\
&= \bar{\Gamma}^{\rho}{}_{\mu\nu} + \frac{1}{2}\bar{g}^{\rho\sigma}\left(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma}\right) - \frac{1}{2}h^{\rho\sigma}\underbrace{\left(\bar{g}_{\mu\sigma,\nu} + \bar{g}_{\nu\sigma,\mu} - \bar{g}_{\mu\nu,\sigma}\right)}_{=2\bar{g}_{\sigma\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\nu}} + O(h^2) \\
&= \bar{\Gamma}^{\rho}{}_{\mu\nu} + \frac{1}{2}\bar{g}^{\rho\sigma}\left(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma}\right) - h^{\rho}{}_{\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\nu} + O(h^2) \\
&= \bar{\Gamma}^{\rho}{}_{\mu\nu} + \frac{1}{2}\bar{g}^{\rho\sigma}\left(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma} - 2h_{\sigma\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\nu}\right) + O(h^2) \\
&= \bar{\Gamma}^{\rho}{}_{\mu\nu} + \frac{1}{2}\bar{g}^{\rho\sigma}\left(h_{\mu\sigma;\nu} + h_{\nu\sigma;\mu} - h_{\mu\nu;\sigma}\right) + O(h^2),
\end{aligned} \tag{3.2}$$

where $h_{\mu\nu;\sigma} = \bar{\nabla}_{\sigma}h_{\mu\nu}$ ⌜$+O(h^2)$??⌟. [←Prove this last line.]■

3.2 Generation of Gravitational Waves

- Somehow get to this eq:

$$T^{\mathrm{TT}}_{ij}(\eta, \boldsymbol{k}) = \Lambda_{ij,kl}(\hat{\boldsymbol{k}}) \int \frac{d^3p}{(2\pi)^3}\, p_k p_l \phi(\eta, \boldsymbol{p})\phi(\eta, \boldsymbol{k} - \boldsymbol{p}). \tag{3.3}$$

- Production instead of generation?

3.2.1 General Formalism

3.2.2 Scalar Field Source (temp. name)

Chapter 4

Lattice-? and N-body simulations

Part II

Project

Chapter 5

Calculating Gravitational Waves from Domain Walls

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Consider a planar domain wall in the xy-plane in a flat FRW universe, represented by a scalar field $\phi(\eta, \boldsymbol{x})$ and a potential $V(\phi)$. The background metric is

$$d\bar{s}^2 = \bar{g}_{\mu\nu}\mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j = a^2(\eta)\left\{-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right\}. \tag{5.1}$$

The solution to $\bar{\Box}\phi = \mathrm{d}V/\mathrm{d}\phi$ is denoted $\bar{\phi}(\eta, z)$. We let indices $a, b, c = 1, 2$ and $i, j, k, l, \ldots = 1, 2, 3$. Now we add a linear perturbation $\zeta(\eta, x^a)$ to the wall such that

$$\phi(\eta, \boldsymbol{x}) = \bar{\phi}(\eta, z; \zeta(\eta, x^a)) = \bar{\phi}(\eta, z; 0) + \zeta(\eta, x^a)\left.\frac{\partial\bar{\phi}}{\partial z}\right|_{\zeta=0} + O(\zeta^2). \tag{5.2}$$

Furthermore, Fourier transforming [←show this!]■ the spatial components gives

$$\phi(\eta, \boldsymbol{k}) = \int d^3x\, \mathrm{e}^{\mathrm{i}k_i x^i}\phi(\eta, \boldsymbol{x}) = \left[(2\pi)^2\delta^{(2)}(k_a) - \mathrm{i}k_3\zeta(\eta, k_a)\right]\bar{\phi}(\eta, k_3; 0) + O\!\left(\zeta^2\right). \tag{5.3}$$

The TT-part of the energy-momentum tensor is [←refer to some section]■

$$T^{\mathrm{TT}}_{ij}(\eta, \boldsymbol{k}) = \Lambda_{ij,kl}(\hat{\boldsymbol{k}})\int \frac{d^3p}{(2\pi)^3}\, p_k p_l \phi(\eta, \boldsymbol{p})\phi(\eta, \boldsymbol{k} - \boldsymbol{p}). \tag{5.4}$$

We define a quantity t_{kl} by

$$T^{\mathrm{TT}}_{ij}(\eta, \boldsymbol{k}) = \Lambda_{ij,kl}(\hat{\boldsymbol{k}})\left(\frac{1}{2\pi}\cdot t_{kl}(\eta, \boldsymbol{k}) + O(\zeta^2)\right), \tag{5.5}$$

and the additional function

$$\mathfrak{I}_n(\eta, q_0) = \int_{\mathbb{R}} \mathrm{d}q\, q^n \bar{\phi}(\eta, q; 0)\bar{\phi}(\eta, q_0 - q; 0). \tag{5.6}$$

After some manipulation [←show this!]■, we get the following:

$$t_{ab}(\eta, \boldsymbol{k}) = k_a k_b\left[-\mathrm{i}\zeta(\eta, k_c)\right]\mathfrak{I}_1(\eta, k_3) \tag{5.7a}$$
$$t_{a3}(\eta, \boldsymbol{k}) = k_a\left[-\mathrm{i}\zeta(\eta, k_c)\right]\mathfrak{I}_2(\eta, k_3) \tag{5.7b}$$
$$t_{33}(\eta, \boldsymbol{k}) = k_3\left[-\mathrm{i}\zeta(\eta, k_c)\right]\mathfrak{I}_2(\eta, k_3) + (2\pi)^2\delta^{(2)}(k_a)\mathfrak{I}_2(\eta, k_3) \tag{5.7c}$$

There are some *small* constraint on the perturbation from this. Need to be commented!

Gravitational waves sourced by this field is – to first order in ζ – given by

$$\begin{aligned} ah_{ij}(\eta, \boldsymbol{k}) &= \frac{16\pi G_{\mathrm{N}}}{k}\int_{\eta_{\mathrm{i}}}^{\eta}\mathrm{d}\eta'\, \sin\left(k\left[\eta - \eta'\right]\right)a(\eta')T^{\mathrm{TT}}_{ij}(\eta', \boldsymbol{k}) \\ &= \frac{8G_{\mathrm{N}}}{k}\Lambda_{ij,kl}(\hat{\boldsymbol{k}})\int_{\eta_{\mathrm{i}}}^{\eta}\mathrm{d}\eta'\, \sin\left(k\left[\eta - \eta'\right]\right)a(\eta')t_{kl}(\eta', \boldsymbol{k}) + O(\zeta^2). \end{aligned} \tag{5.8}$$

Remaining are the $\Lambda_{ij,kl}t_{kl}$-elements, which in total are ⌈6⌋? terms per ij, due to symmetry in t_{kl}:

$$\begin{aligned}\Lambda_{ij,kl}(\hat{\boldsymbol{k}})t_{kl}(\eta,\boldsymbol{k}) = &\left\{\left(\Lambda_{ij,12}+\Lambda_{ij,21}\right)t_{12}+\left(\Lambda_{ij,13}+\Lambda_{ij,31}\right)t_{13}+\left(\Lambda_{ij,23}+\Lambda_{ij,32}\right)t_{23}\right\}(\eta,k\hat{\boldsymbol{k}})\\ &+\left\{\Lambda_{ij,11}t_{11}+\Lambda_{ij,22}t_{22}+\Lambda_{ij,33}t_{33}\right\}(\eta,k\hat{\boldsymbol{k}})\end{aligned} \tag{5.9}$$

All of these are on the form

$$-\mathrm{i}\zeta(\eta,k_a)\times\left\{k^2\Im_1(\eta,k_3)A_{ij}(\hat{\boldsymbol{k}})+k\Im_2(\eta,k_3)B_{ij}(\hat{\boldsymbol{k}})\right\}, \tag{5.10}$$

leaving

$$ah_{ij}(\eta,\boldsymbol{k}) = 8G_{\mathrm{N}}\left[kA_{ij}(\hat{\boldsymbol{k}})\mathcal{I}_1(\eta,\boldsymbol{k};\eta_{\mathrm{i}})+B_{ij}(\hat{\boldsymbol{k}})\mathcal{I}_2(\eta,\boldsymbol{k};\eta_{\mathrm{i}})\right] \tag{5.11}$$

where

$$\mathcal{I}_n(\eta,\boldsymbol{k};\eta_{\mathrm{i}}) = -\mathrm{i}\int_{\eta_{\mathrm{i}}}^{\eta}\mathrm{d}\eta'\,a(\eta')\sin\left(k\left(\eta-\eta'\right)\right)\times\zeta(\eta',k_a)\Im_n(\eta',k_3). \tag{5.12}$$

Furthermore, we can show [←proof!]■ that $A_{ij}(\boldsymbol{n}) = -n_3B_{ij}(\boldsymbol{n}) \equiv +2n_3C_{ij}(\boldsymbol{n})$ for $|\boldsymbol{n}|^2 = n_1^2+n_2^2+n_3^2 = 1$, allowing for the slightly simpler expression

$$ah_{ij}(\eta,\boldsymbol{k}) = 4G_{\mathrm{N}}C_{ij}(\hat{\boldsymbol{k}})\left[k_3\mathcal{I}_1(\eta,\boldsymbol{k};\eta_{\mathrm{i}})-\mathcal{I}_2(\eta,\boldsymbol{k};\eta_{\mathrm{i}})\right], \tag{5.13}$$

where⌈:

$$\begin{aligned}C_{ab}(\boldsymbol{n}) &= n_3\left[n_an_b\left(n_3^2+1\right)-\delta_{ab}\left(1-n_3^2\right)\right]\\ C_{a3}(\boldsymbol{n}) &= -n_an_3^2\left(1-n_3^2\right)\\ C_{33}(\boldsymbol{n}) &= n_3^2\left(1-n_3^2\right)^2\end{aligned} \tag{5.14}$$

⌋?

5.1 General Formalism

Chapter 6

Simulating Gravitational Waves from Domain Walls

Chapter 7

Studying Gravitational Waves from Domain Walls

Chapter 8

Discussion

Chapter 9

Conclusion and Outlook

Bibliography

[1] Majdi Amr. Particle production by gravitational perturbations in domain walls. *Nuclear Physics B*, 945:114648, August 2019.

[2] Aleksandr Azatov, Miguel Vanvlasselaer, and Wen Yin. Dark Matter production from relativistic bubble walls. *Journal of High Energy Physics*, 2021(3):288, March 2021.

[3] Tanmay Vachaspati. *Kinks and Domain Walls: An Introduction to Classical and Quantum Solitons*. Cambridge University Press, Cambridge, 2006.

Appendix A

I do not have an appendix