# Equations of Order One

## Homogeneous Equations

### Example (1)

(1)

$$(x^2+y^2)dx+(x^2-xy)dy=0$$

we use $y=vx$

$$dy = vdx + xdv$$

Step (1)
test of homogeinity:

$$((tx)^2+(ty)^2)dx+((tx)^2-(tx\cdot ty)^2)dy=0$$

$$t^2(x^2+y^2)dx+t^2(x^2-xy)dy=0$$

$$=t^\alpha f(x,y)$$

∴ this is a HDE

Step (2) Substitute $y=vx$

$$(x^2+(vx)^2)dx+(x^2-x(vx))dy=0$$

Note that $dy=vdx+xdv$

$$(x^2+(vx)^2)dx+(x^2-x(vx))[vdx+xdv]=0$$

$$(x^2+v^2x^2)dx+(x^2-x^2v)[vdx+xdv]=0$$

eliminate $x^2$

$$(1+v^2)dx+(1-v)[vdx+xdv]=0$$

(separable DE?)

$$(1+v^2)dx+v(1-v)dx+x(1-v)dv=0$$

$$(1+v^2)dx+vdx-v^2dx+xdv-xvdv=0$$

$$dv\cancel{+v^2dx}+vdx\cancel{-v^2dx}+xdv-xvdv=0$$

$$dv+vdx+xdv-xvdv=0$$

$$(1+v)dx+x(1-v)dv=0$$

$$\frac{dx}{x}+\frac{(1-v)}{(v+1)}dv=0$$

$$\frac{dx}{x}-\frac{(v-1)}{v+1}dv=0$$

Note that:

$$\frac{v-1}{v+1}=1-\frac{2}{v+1}$$

(2)

$$\frac{dx}{x}-\left[1-\frac{2}{v+1}\right]dv=0$$

$$\frac{dx}{x}-dv+\frac{2dv}{v+1}=0$$

Step (3) Integrate & Simplify

$$\int\frac{dx}{x}-\int dv+2\int\frac{dv}{v+1}=\int 0$$

$$\ln x+\ln c_1-v+\ln c_2+2\ln|v+1|+\ln c_3=\ln c_4$$

$$\ln x-v+2\ln|v+1|=\ln c_4-\ln c_1-\ln c_2-\ln c_3$$

let $\ln c_4-\ln c_1-\ln c_2-\ln c_3=\ln c_5$

$$\ln x-v+2\ln|v+1|=\ln c_5$$

$$\ln x+\ln|v+1|^2-\ln c_5=v$$

$$\ln\left|\frac{x(v+1)^2}{c_5}\right|=v$$

Recall: $v=\frac{y}{x}$

$$\ln\left|\frac{x(\frac{y}{x}+1)^2}{c_5}\right|=\frac{y}{x}$$

$$\ln\left|\frac{x(\frac{y+x}{x})^2}{c_5}\right|=\frac{y}{x}$$

$$\ln\left|x\frac{(y+x)^2}{x^2c_5}\right|=\frac{y}{x}$$

$$e^{\ln\left|\frac{(y+x)^2}{c_5x}\right|}=e^{\frac{y}{x}}$$

$$(x+y)^2=c_5xe^{\frac{y}{x}}$$

~~let $c_5=c$~~

$$\boxed{(x+y)^2=cxe^{\frac{y}{x}}}$$

# Equations of Order One

## Homogeneous Equations (Dennis Zill book)

– substitution method

(HDE) — Homogeneous DE

fail ◇ pass → homogeneity test ✓

## Exercise

① $\boxed{(x-y)\,dx + x\,dy = 0}$ Example: ① $x^2+$

Ans. $\boxed{xe^{\frac{y}{x}} = C}$

①

$(x^2$

$(x^2+$

$\cancel{x^2}(1$

$(1+$

$(1+$

$dx-$

(2)

$$\frac{dx}{x} - \left(1 - \frac{2}{v+1}\right) dv = 0$$

$$\frac{dx}{x} - dv + \frac{2\,dv}{v+1} = 0$$

$$\int \frac{dx}{x} - \int dv + 2\int \frac{dv}{v+1} = \int 0$$

$(c) = \int 0$

$$\ln|x| + \ln C_1 - v + \ln C_2 + 2\ln|v+1| + \ln C_3 = \ln C_4$$

$$\ln x - v + 2\ln|v+1| = \ln C_4 - \ln C_1 - \ln C_2 - \ln C_3$$

let $= \ln C_5$

$$\ln x - v + 2\ln|v+1| = \ln C_5$$

$$\ln x + \ln|v+1|^2 - \ln C_5 = v$$

$$\ln \frac{x(v+1)^2}{C_5} = v$$

SDE

(3)

$$e^{\ln\left|\frac{x(v+1)^2}{C_5}\right|} = e^{v}$$

$$\frac{x(v+1)^2}{C_5} = e^{v}$$

$y = vx$

$v = \frac{y}{x}$

$$x(v+1)^2 = C_5 e^{\frac{y}{x}}$$

$$x\left(\frac{y}{x}+1\right)^2 = C_5 e^{\frac{y}{x}}$$

$$x\left(\frac{y+x}{x}\right)^2 = C_5 e^{\frac{y}{x}}$$

$$x\,\frac{(x+y)^2}{x^2} = C_5 e^{\frac{y}{x}}$$

$$\frac{(x+y)^2}{x} = C_5 e^{\frac{y}{x}}$$

$$(x+y)^2 = C_5 x e^{\frac{y}{x}}$$

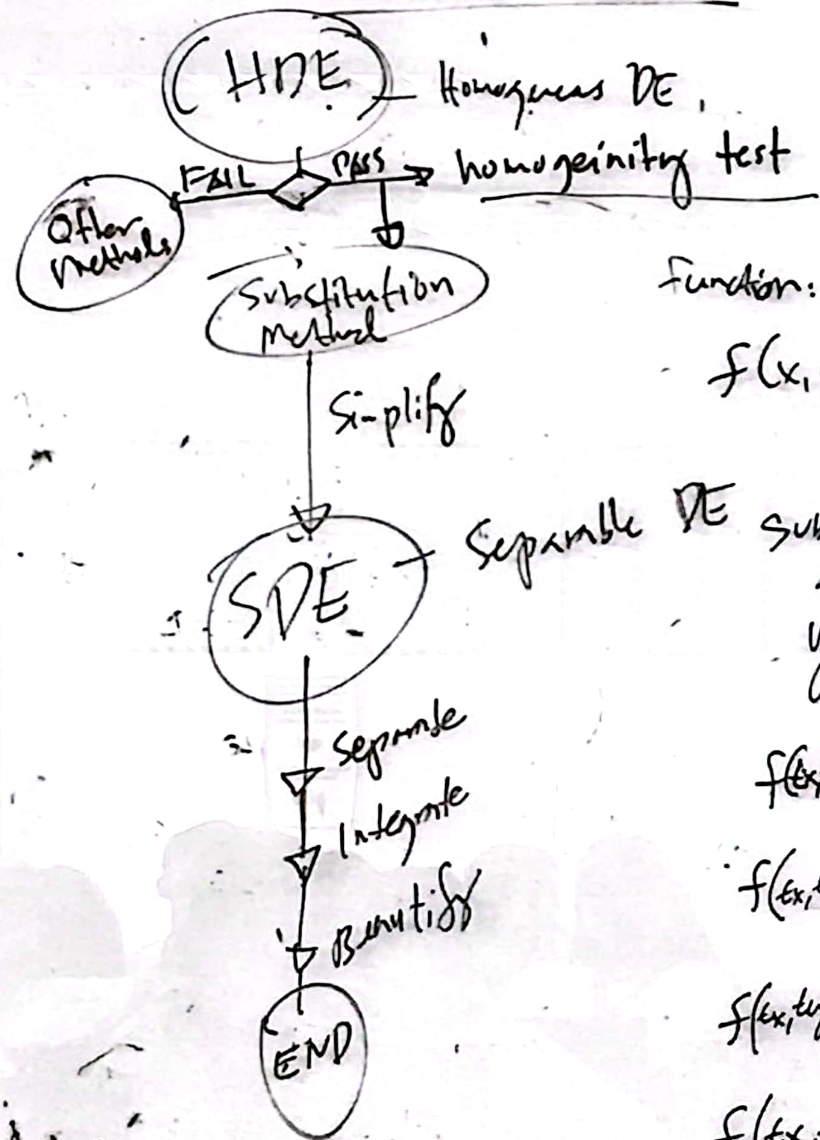
$C_5 = C$

$$\boxed{(x+y)^2 = Cxe^{\frac{y}{x}}}$$

# Equations of Order One

## Homogeneous Equations (Dennis Zill book)

- substitution method




function: $t^{\alpha} f(x,y) \stackrel{?}{=} f(tx, ty)$

$f(x,y) = x^3 + y^3 \rightarrow$ homogeneous of degree 3

$\alpha = 3$

Substitution

$x \rightarrow tx$

$y \rightarrow ty$

$f(tx,ty) = (tx)^3 + (ty)^3$

$f(tx,ty) = t^3x^3 + t^3y^3$

$f(tx,ty) = t^3(x^3+y^3)$

$f(tx,ty) = t^3 f(x,y)$

$\alpha = 3$

Example: ① $(x^2+y^2)dx + (x^2 - xy)dy = 0$

① $y = vx \,, \; dy = vdx + xdv$

$(x^2 + v^2x^2)dx + (x^2 - vx^2)dy = 0$

$(x^2 + v^2x^2)dx + (x^2 - vx^2)(vdx + xdv) = 0$

$\frac{\cancel{x^2}(1+v^2)dx + \cancel{x^2}(1-v)(vdx + xdv)}{\cancel{x^2}} = \frac{0}{x^2}$

$(1+v^2)dx + (1-v)(vdx + xdv) = 0$

$(1+v^2)dx + v(1-v)dx + x(1-v)dv = 0$

$dx + \cancel{v^2dx} + vdx \cancel{- v^2dx} + xdv - xvdv = 0$

$dx + vdx + xdv - xvdv = 0$

$(1+v)dx + (x - xv)dv = 0$

$\frac{\cancel{(1+v)}dx}{x\cancel{(1+v)}} + \frac{\cancel{x}(1-v)dv}{\cancel{x}(1+v)} = 0$

$\boxed{SDE}$ $\frac{dx}{x} + \frac{(-v+1)}{(v+1)}dv = 0$

$\frac{dx}{x} - \frac{(v-1)}{v+1}dv = 0$

$(c) = \int 0$

$v+1 \overline{\big) \, v-1}$ with quotient $1$; subtracting $v+1$ gives $0 - 2$

$\boxed{\frac{v-1}{v+1} = 1 - \frac{2}{v+1}}$

$) \, dv = 0$

$dv \quad = 0$

$+1$

$\int \frac{dv}{v+1} = \int 0$

$nC_2 + 2\ln|v+1| + \ln C_3 = \ln C_4$

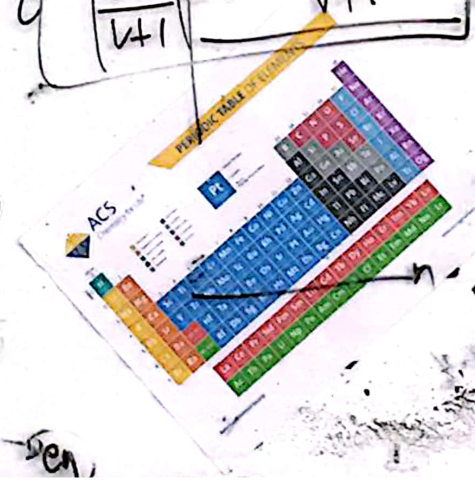
$| = \ln C_4 - \ln C_1 - \ln C_2 - \ln C_3$

let $= \ln C_5$

$nC_5$

$= v$

b. ____



* Verifying the solution of a DE

(a) $\frac{dy}{dx} = xy^{1/2}$, $y = \frac{1}{16}x^4$

$y'$, $y' = \frac{4x^3}{16} = \frac{x^3}{4}$

$y$

$\frac{x^3}{4} = x\left(\frac{1}{16}x^4\right)^{\frac{1}{2}}$

$\frac{x^3}{4} = x\left(\frac{1}{\sqrt{16}}x^2\right)$

$\frac{x^3}{4} = \frac{x^3}{4}$

$\therefore y = \frac{1}{16}x^4$ is a solution to the given DE.

(b) $y'' - 2y' + y = 0$ ③ ; $y = xe^x$

$y' = (xe^x)'$

product rule

Recall: $d(uv) = udv + vdu$

| | |
|---|---|
| $u$ | $= x$ |
| $du$ | $1$ |
| $v$ | $e^x$ |
| $dv$ | $e^x$ |

Recall: $de^u = e^u du$

$u = x$, $du = 1$

$de^x = e^x(1) = e^x$

$y' = (x)e^x + e^x(1)$

$y' = xe^x + e^x$ — ①

$y'' = (xe^x + e^x)'$

$= (xe^x)' + (e^x)'$

$= (xe^x + e^x) + (e^x)$

$= xe^x + e^x + e^x$

$y'' = xe^x + 2e^x$ — ②

$y'' - 2y' + y = 0$

substitute eq ① and ② to eq ③

$(xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0$

$xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0$

$\cancel{2xe^x} + \cancel{2e^x} - \cancel{2xe^x} - \cancel{2e^x} = 0 \rightarrow 0 = 0$ ✓

$\therefore y = xe^x$ is a sol'n to the given equation ✓

## DIFFERENTIAL EQUATIONS

Date:

- an equation containing of one or more unknown functions (or variables) with respect to one or more independent variables.

### CLASSIFICATIONS OF DE

① TYPE

a. Partial DE

b. Ordinary DE

② ORDER - highest order of the derivative of a DE.

$$\left(\frac{d^2y}{dx^2}\right) + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

$\frac{d^2y}{dx^2}$ → 2nd order; $\frac{dy}{dx}$ → (1st order)

③ linearity

a. linear

b. non-linear

$$\left(\frac{d^2y}{dx^2}\right) + 5\left(\frac{dy}{dx}\right)^3 \rightarrow \text{non-linear}$$

$$\left(\frac{dy}{dx}\right) + 5y = e^x \rightarrow \text{linear}$$

④ DEGREE - exponent of your highest derivative.

$$\left(\frac{dy}{dx}\right) + 5y = e^x$$

1st degree

EXAMPLE:

① $\left(\frac{dy}{dx}\right)^2 + 2x + 4 = 0$ Order = 1

③ $\left(\frac{d^2y}{dx^2}\right) + 7\frac{dy}{dx} + 2 = 0$ Degree = 1, Order = 2

② $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 6 = 0$ Order: 2

④ $\left(\frac{d^2y}{dx^3}\right)^3 + 6\frac{dy}{dx} + 4 = 0$ Degree = 3, Order = 2

⑤ $\frac{dy}{dx} = 5$ Linear

⑥ $\left(\frac{d^2y}{dx^2}\right)^1 + y = 0$ Linear

⑦ $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0$ degree = 2, non-linear

⑧ $\frac{dy}{dx}\left(x\frac{dy}{dx} + \frac{3}{\frac{dy}{dx}}\right) = y^2 \cdot \frac{dy}{dx}$

$= x\left(\frac{dy}{dx}\right)^2 + 3 = y^2 \cdot \frac{dy}{dx}$ Order : 1

Degree = 2

$= x\left(\frac{dy}{dx}\right)^2 - y^2 \cdot \frac{dy}{dx} + 3 = 0$ non-linear

Date:

$$\left(\frac{dy}{dx}\right) + 5y = e^x$$ Order = 1, Degree = 1, linear

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$$ Order = 2, linear

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$ Order = 1, nonlinear

$x = f(t)$ $y = f(t)$

ORDINARY DE (ODE's)

$y = f(x)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}$$

2nd order PDE, 2nd order PDE

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ 1st order PDE (nonlinear)

PARTIAL DE (PDE's)

$u = f(x, y)$

→ unknown function/dependent variable

$$\frac{d^2x}{dt^2} + 16x = 0 \rightarrow \text{order} = 2 \text{ ODE}$$

↳ independent variable

$\frac{dy}{dx}$ or $\frac{dy}{dt}$

$a(n)y^n + a(n-1)y^{n-1} + \ldots + y + a(0)y = g(x)$

linear DE $$a(n)\frac{d^ny}{dx^n} + a(n-1)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + \frac{dy}{dx} + a(0)y = g(x)$$

*** VERIFYING THE SOLUTION OF DE**

(a) $\frac{dy}{dx} = xy^{1/2}$, $y = \frac{1}{16}x^4$

$y' =$ 

$y = \frac{4x^3}{16} = \frac{x^3}{4}$

$y' =$

$$\frac{x^3}{4} = x\left(\frac{1}{16}x^4\right)^{1/2}$$

$$\frac{x^3}{4} = x\left(\frac{1}{\sqrt{16}}x^2\right)$$

$$\frac{x^3}{4} = \frac{x^3}{4}$$

$y = \frac{1}{16}x^4$ is a solution to the given DE.

(b) $y'' - 2y' + y = 0$

$y = xe^x$

$y' = (xe^x)'$

$y' = (x)e^x + e^x(1)$

$y' = xe^x + e^x$

$y'' = (xe^x + e^x)$

$= (xe^x) + (e^x)$

$= (xe^x + e^x) + (e^x)'$

$= xe^x + e^x + e^x$

$y'' = xe^x + 2e^x$

Recall: $d(uv) = udv + vdu$

| | |
|---|---|
| u | = x |
| du | 1 |
| v | ex |
| dv | ex |

Recall: $de^u = e^u du$

$u = x, du = 1$

$de^x = e^x(1)e^x$

Date:

$y'' - 2y' + y = 0$

SUBSTITUTE EQ. 1 & 2 to EQ. 3

$(xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0$

$xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0$

$\cancel{2xe^x} + \cancel{2e^x} - \cancel{2xe^x} - \cancel{2e^x} = 0 \to 0 \overset{\checkmark}{=} 0$

$y = xe^x$ is a solution to the given equation.

## • SEPARABLE EQUATIONS •

$\frac{dy}{dx} = g(x)h(y)$

$\int g(x)\,dx = \int g(y)\,dy$

ex. $\cancel{(dx)}\frac{dy}{\cancel{\cos y}\,\cancel{dx}} = \frac{x^2 \cancel{\cos y}}{\cancel{\cos y}}(dx) = \int \frac{dy}{\cos y} = \int x^2 dx \to \int \sec y\, dy = \int x^2 dx$

$\left(\frac{1}{\cos y}\right)dy$ Recall: trigo $\frac{1}{\cos y} = \sec y$

$\ln|\sec y + \tan y| + C_1 = \frac{x^3}{3} + C_2$

$\ln|\sec y + \tan y| = \frac{x^3}{3} + C_2 - C_1$

$\boxed{\ln|\sec y + \tan y| = \frac{x^3}{3} + C}$

let $C_2 - C_1 = C$

$e^{\ln|\sec y + \tan y|} = e^{\frac{x^3}{3}}$

$\sec y + \tan y = e^{\frac{x^3}{3} + C}$

$\sec y + \tan y = e^{\frac{x^3}{3}} e^C$

let $e^C = C_3$

$\boxed{\sec y + \tan y = C_3 e^{\frac{x^3}{3}}}$

$(x+y)\,dx = (x-y)\,dy$

$x\,dx + y\,dx =$

② $\frac{dy}{dx} = y^2 - 4$

$1 = Ax - A + Bx + B$

$1 = A(x-1) + B(x+1)$

$x \quad \frac{1}{2} = (AB)$

$x_2 \quad A = \frac{1}{2}$

$\frac{1}{(x^2-1)} = \frac{-\frac{1}{2}}{(x+1)} + \frac{\frac{1}{2}}{(x-1)}$

$C_1 \quad 1 = -A + B$ ①

$x, \quad 0 = A + B$ ②

$A = -B$ $\boxed{2\cdot 1}$ subs eq. 2·1 to eq 1

$(x+y)\,dx = (x-y)\,dy$

$x\,dx + y\,dx =$

$1 = -A + B$

$1 = (-B) + B$

$1 = -B + B$

$\frac{1}{2} = \frac{2B}{2}$

$B = \frac{1}{2} \to$ eq. ③

subs. eq ③ to ①

$1 = -A + B$

$1 = -A + \frac{1}{2}$

$1 - \frac{1}{2} = -A$

$(-1)\frac{1}{2} = (-A)(B)$

$-\frac{1}{2} = A$

$\left(\frac{\cancel{dx}}{y-4}\right)\frac{dy}{\cancel{dx}} = \frac{\cancel{y^2-4}}{\cancel{y^2-4}}(dx)$ eq. 3 = $\frac{-1/2}{(x+1)} + \frac{1/2}{(x-1)}$

Date:

$\int \frac{dy}{y^2-4} = \int dx$

$\frac{1}{y^2-4}$ Factor: $y^2-4 = (y-2)(y+2)$

using partial function method

$(y^2-4)\left(\frac{1}{y^2-4}\right) = \left(\frac{A}{y-2} + \frac{B}{y+2}\right)$

$1 = A(y+2) + B(y-2)$

$1 = Ay + 2A + By - 2B$

$1 = 2A - 2B \to ①$

$0 = A + B \to ②$

$A = B \to 2.1$

subs eq. 2.1 to eq. 1

$= 2A - 2B$

$= 2(-B) - 2B$

$1 = -2B - 2B$

$\frac{1}{-4} = \frac{-4B}{-4}$

$-\frac{1}{4} = B \qquad A = \frac{1}{4}$

③ $\frac{1}{y-4} = \frac{1/4}{(y-2)} + \frac{-1/4}{(y+2)}$

$\int \frac{dy}{y^2-4} = x + C_1$

$\int \left(\frac{1}{y^2-4}\right) dy = x + C_1$

Recall eq. 3

$\int \left[\frac{1/4}{(y-2)} + \frac{-1/4}{(y+2)}\right] dy = x + C$

$\frac{1}{4}\ln|y-2| + C_2 - \frac{1}{4}\ln|y+2| + C$

$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + C_1 - C_2 - C_3$ let $C_1 - C_2 - C_3 = C_4$

④ $\left(\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2|\right) = (x + C_4)(4)$ let $4C_4 = C_5$

$\ln|y-2| - \ln|y+2| = 4x + 4C_4$

$\ln|y-2| - \ln|y+2| = 4x + C_5$

$e^{\ln\frac{|y-2|}{|y+2|}} = e^{4x+C_5} \to \frac{|y-2|}{y+2} = Ce^{4x}$

general sol'n

$\frac{|y-2|}{|y+2|} = e \to \frac{|y-2|}{|y+2|} = e^{4x} \cdot e^{C_5}$

EQUATIONS OF ORDER ONE

Date:

HOMOGENEOUS EQUATIONS

EXAMPLE 1

$$(x^2+y^2)dx+(x^2-xy)dy=0$$

we use $y=vx$

$dy=vdx+xdu$

STEP 1

Test of homogeinity:

$$((tx^2)+(ty)^2)dx+((tx)^2-(tx\cdot ty)^2)dy=0$$

$$t^2(x^2+y^2)dx+t^2(x^2-xy)dy=0$$

$$t^\alpha f(x,y)$$ this is a HDE

2 Substitute $y=vx$

$$(x^2+(vx)^2)dx+(x^2-x(vx))dy=0$$

note that $dy=vdx+xdv$

$$(x^2+(vx)^2dx+(x^2-x(vx))[vdx+xdv]=0$$

$$(\cancel{x^2}+v^2\cancel{x^2})dx+(\cancel{x^2}-\cancel{x^2}v)[vdx+xdv]=0$$ eliminate $x^2$

$$(1+v^2)dx+(1-v)[vdx+xdv]=0$$ separable DE

$$(1+v^2)dx+v(1-v)dx+x(1-v)dv=0$$

$$(1xv^2)dx+vdx-v^2dx+xdu-xudv=0$$

$$dx+\cancel{v^2dx}+vdx-\cancel{v^2dx}+xdv-xudv=0$$

$$dx+vdx+xdu-xvdy=0$$

$$(1+v)dx+x(1-v)dy=0$$

NOTE THAT: $\frac{v-1}{v+1}=1-\frac{2}{v+1}$

$$\frac{dx}{x}+\frac{(1-v)}{(v+1)}dv=0$$

$$\frac{dx}{x}-\frac{(v-1)}{v+1}dv=0$$

$$\frac{dx}{x}-\left[1-\frac{2}{v+1}\right]dv=0$$

$$\frac{dx}{x}-dv+\frac{2dv}{v+1}=0$$

3 Integrate & simplify

$$\int\frac{dx}{x}-\int dv+2\int\frac{dv}{v+1}=\int 0$$

$$\rightarrow \ln x+\ln C_1-v+\ln C_2+2\ln|v+1|+\ln C_3=\ln C_4$$

$$\ln x-v+2\ln|v+1|=\ln C_4-\ln C_1+\ln C_2-\ln C_3$$

let $\ln C_4-\ln C_1-\ln C_2-\ln C_3=\ln C_5$

$$\ln x-v+2\ln|v+1|=\ln C_5$$

$$\ln x+\ln|v+1|^2-\ln C_5=v$$

$$\ln\left|\frac{x(v+1)^2}{C_5}\right|=v$$

Recall: $v=\frac{y}{x}$

$$\ln\left|\frac{x(\frac{y}{x}+1)^2}{C_5}\right|=\frac{y}{x}$$

$$\ln\left|\frac{x\left(\frac{y+x}{x}\right)^2}{C_5}\right| = \frac{y}{x}$$

Date:

$$\ln\left|x\frac{(y+x)^2}{x^2 C_5}\right| = \frac{y}{x}$$

let $C_5 = C$

$$\ln_e\left|\frac{(y+x)^2}{C_5 x}\right| = e^{\frac{y}{x}}$$

$$\boxed{(x+y)^2 = Cxe^{\frac{y}{x}}}$$

$$(x+y)^2 = C_5 x e^{\frac{y}{x}}$$

EQUATIONS OF ORDER ONE

HOMOGENEOUS EQUATIONS

- substitution method

HDE → substitution method → SDE → END

(homogenity test under HDE; simplify and separable, integrate, beautify under → END; separable DE above SDE)

homogenity test    simplify    separable, integrate, beautify

$f(x,y) = x^3 + y^3$ → homogeneous

$\alpha = 3$    degree of 3

SUBSTITUTION

$x \to tx \qquad y \to ty$

function:

$t^{\alpha} f(x,y) = f(tx, ty)$

$f(tx+ty) = (tx)^3 + (ty)^3$

$f(tx, ty) = t^3x^3 + t^3y^3$

$f(tx, ty) = t^3(x^3+y^3)$

$f(tx, ty) = t^3 f(x,y)$

$\alpha = 3$

homogeneous

non-homogeneous

$f(x,y) = x^3+y^3+1$

$x \to tx \qquad y \to ty$

$f(tx, ty) = (tx)^3 + (ty)^3 + 1$

$f(tx, ty) = t^3(x^3+y^3+1$

$f(t\cancel{x}, ty)\ t^3 f(x,y) + 1$

$f(x,y) = x^2 + xy$

$f(tx, ty) = (tx^{\cancel{3}})^2 + (tx)(ty)$

$= t^2x^2 + t^2xy$

$f(tx, ty) = t^2(x^2+xy) = t^2 f(x,y)$

2    $2+1 = 3$

$\cancel{x}\ x^2 + x^2y$ (non-homo)

3    $2+1=3$

(homo) $x^3 + x^2y$

EXAMPLE 1 : $(x^2+y^2)dx + (x^2-xy)dy = 0$

$y = vx, \quad dy = vdx + xdv$

$(x^2 + v^2x^2)dx + (x^2 - vx^2)dy = 0$

$(x^2 + v^2x^2)dx + (x^2-vx^2)(vdx+xdv) = 0$

$\dfrac{\cancel{x^2}(1+v^2)dx + \cancel{x^2}(1-v)(vdx+xdv) = 0}{\cancel{x^2} \qquad \cancel{x^2} \qquad x^2}$

$(1+v^2)dx + (1-v)(vdx+xdv) = 0$

$(1+v^2)dx + v(1-v)dx + x(1-v)dv = 0$

$dx + \cancel{v^2dx} + vdx - \cancel{v^2dx} + xdv - xvdv = 0$

$dx + vdx + xdv - xvdv = 0$

$(1+v)dx + (x-xv)dv = 0$

$\dfrac{\cancel{(1+v)}dx}{x\cancel{(1+v)}} + \dfrac{\cancel{x}(1-v)dv}{\cancel{x}(1+v)} = 0$

$\dfrac{dx}{x} + \dfrac{(-v+1)}{(v+1)}dv = 0$

$\dfrac{dx}{x} - \dfrac{(v-1)}{(v+1)}dv = 0$

② $\dfrac{dx}{x} - \left(1 - \dfrac{2}{v+1}\right)dv = 0 \qquad \dfrac{dx}{x} - dv = \dfrac{2dv}{v+1} = 0$

$\int \dfrac{dx}{x} - \int dv + 2\int \dfrac{dv}{v+1} = \int 0$

$\ln|x| + \ln C_1 - v + \ln C_2 + 2\ln|v+1| + \ln C_3 = \ln C_4$

$\ln x - v + 2\ln|v+1| = \ln C_4 - \ln C_1 - \ln C_3 - \ln C_2$

let $\ln C_4 - \ln C_1 - \ln C_3 - \ln C_2$

$\ln x - v + 2\ln|v+1| = \ln c_5$

$\ln x + \ln|v+1|^2 - \ln C_5 = v$

$e^{\ln\left|\frac{x(v+1)^2}{C_5}\right|} = e^v$ $\qquad$ Recall: $v = \dfrac{y}{x}$ $\qquad y = vx$

$x\left(\dfrac{y}{x}+1\right)^2 = C_5 e^{y/x}$

$\dfrac{x(v+1)^2}{C_5} = e^v \qquad x\left(\dfrac{y+x}{x}\right)^2 = C_5 e^{y/x}$

$x\dfrac{(x+y)^2}{x^2} = C_5 e^{\frac{y}{x}}$

$\dfrac{(x+y)^2}{x} = C_5 e^{y/x}$

$(x+y)^2 = C_5 x e^{y/x}$

$C_5 = C$

$\boxed{(x+y)^2 = Cxe^{y/x}}$

EXERCISES

Date:

1) $\dfrac{d^2x}{dt^2} + k^2x = 0$. ORDINARY, non LINEAR, 2nd ORDER

2) $\dfrac{\partial^2 w}{\partial t^2} = a^2 \dfrac{\partial^2 w}{\partial x^2}$. PARTIAL, NONLINEAR, 2nd ORDER

3) $(x^2+y^2)dx + 2xydy = 0$. ORDINARY, LINEAR, 2nd ORDER

4) $y' + P(x)y = Q(x)$. ORDINARY, NONLINEAR, 1st ORDER

5) $y''' - 3y' + 2y = 0$. ORDINARY, LINEAR, 1st ORDER

6) $yy'' = x$. ORDINARY, non LINEAR, 1st ORDER

7) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$. PARTIAL, NONLINEAR, 2nd ORDER

8) $\dfrac{d^4y}{dx^4} = w(x)$. ORDINARY, LINEAR, 4th ORDER

9) $x\dfrac{d^2y}{dt^2} - y\dfrac{d^2x}{dt^2} = 0$. ORDINARY, non LINEAR, 2nd ORDER

10) $L\dfrac{di}{dt} + Ri = E$. ORDINARY, LINEAR, 1st ORDER

11) $(x+y)dx + (3x^2-1)dy = 0$. ORDINARY, NONLINEAR, 2nd ORDER

12) $x(y'')^3 + (y')^4 - y = 0$. ORDINARY, NONLINEAR, 1st ORDER

HOMOGENEOUS

Date:

- $(x+y)\,dx + x\,dy = 0$

$(x+vx)\,dx + x(v\,dx + x\,dv) = 0$

$\frac{\cancel{x}(1+v)\,dx}{\cancel{x}} + \frac{\cancel{x}(v\,dx + x\,dv)}{\cancel{x}} = 0$

$(1+v)\,dx + (v\,dx + x\,dv) = 0$

$dx + \cancel{v\,dx} + \cancel{v\,dx} + x\,dv = 0$

$\frac{dx}{x} + \frac{\cancel{x}\,dv}{\cancel{x}} = 0$

$\int \frac{dx}{x} + \int dv = 0$

$\ln|x| + \ln c_1 + v + \ln c_2 = \ln c_3$

$\ln|x+v| = \ln c_1 - \ln c_2 - \ln c_3$

$\ln|x+v| = \ln c_4$

$\ln x - \ln c_4 = v$

$\ln_e x - \ln c_4 = -v_e$

$x - \ln c_4 = e^{-y/x}$

$\frac{x}{c_4} = e^{-y/x}$ //

EXACT DE

Date:

Example 1

$(2xy - 3x^2)\,dx + (x^2 + 2y)\,dy = 0$

Test of "EXACTNESS"

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$M(x,y) = 2xy - 3x^2$

$N(x,y) = x^2 + 2y$

$$\frac{\partial M}{\partial y} = 2x - 0 = 2x$$

$$\frac{\partial N}{\partial x} = 2x + 0 = 2x$$

$dF = M\,dx + N\,dy$

$$\frac{dF}{\partial x} = M(x,y)$$

$$\frac{\partial F}{\partial x} = 2xy - 3x^2$$

INTEGRATE

$$\int \frac{\partial F}{\partial x}\,\partial x = \int 2xy\,\partial x - 3\int x^2\,dx$$

$F = x^2y + A(y) - 3x^2 + B(y)$

$F = x^2y - 3x^2 + A(y) + B(y)$

let $A(y) + B(y) = C(y)$

$F = x^2y - x^3 + C(y)$ we look for $(y)$

DIFFERENTIATE F WITH Y ONLY

$$\frac{\partial F}{\partial y} = \frac{\partial(x^2y)}{\partial y} - \frac{\partial(x^3)}{\partial y} + C'(y)$$

$$\frac{\partial F}{\partial y} = x^2 - 0 + C'(y) = x^2 + C'(y)$$

Recall: $\frac{\partial F}{\partial y} = N(x,y)$

$\cancel{x^2} + 2y = \cancel{x^2} + C'y$

$C'(y) = 2y\,dy$

$\int C'(y) = \int 2y\,dy$

$C(y) = y^2$

$F = x^2y - x^3 + y^2$

$$\boxed{x^2y - x^3 + y^2 = C}$$

Coralbreeze

EXAMPLE

Date:

① $(x+2y)dx + (2x+y)dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\quad M(x,y) = x + 2y \quad \frac{\partial M}{\partial y} = 0 + 2 = 2$

$N(x,y) = 2x+y \quad \frac{\partial N}{\partial x} = 2 + 0 = 2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow 2 = 2$

② Recall: $\frac{\partial F}{\partial x} = M(x,y) = x + 2y$

③ Integrate to get F

$\int \frac{\partial F}{\partial x} \partial x = \int x dx + \int 2y \, \partial x$

$F = \frac{x^2}{2} + A(y) + 2xy + B(y)$

$F = \frac{x^2}{2} + 2xy + A(y) + B(y)$

let $A(y) + B(y) = C(y)$

③ Differentiate F to get $C(y)$

$F = \frac{x^2}{2} + 2xy + C(y)$

$\frac{\partial F}{\partial y} = 0 + 2x + C'(y)$

③ Recall:

$\frac{\partial F}{\partial y} = N(x,y)$

$\frac{\partial F}{\partial y} = (2x+y)\,dy$

subs eq ① to eq ②

$\frac{\partial F}{\partial y} = \cancel{2x} + c'(y) = (\cancel{2x}+y)\,dy$

$c'(y) = y\,dy$

Integrate

$\int c'(y) = \int y\,dy$

$c(y) = \frac{y^2}{2}$

$F = \frac{x^2}{2} + 2xy + \frac{y^2}{2}$

④ let $F = c$

$c = \frac{x^2}{2} + 2xy + \frac{y^2}{2}$

⑤ Beautify

$(2)(c) = \left(\frac{x^2}{2} + 2xy + \frac{y^2}{2}\right)(2)$

$2c = x^2 + 4xy + y^2$

let $2c = C_2$

$\boxed{C_2 = x^2 + 4xy + y^2}$

separable ni MJ :))

Date:

1.)

$$\left(\frac{\cancel{dx}}{\cos y}\right)\frac{dy}{\cancel{dx}} = x^2 \cancel{\cos y}\left(\frac{dx}{\cancel{\cos y}}\right)$$

$$\int \frac{dy}{\cos y} = \int x^2\,dx$$

$$\int \frac{1}{\cos y}\,dy = \int x^2\,dx$$

$\sec y$

$\ln|\sec y + \tan y| + c$

$$\int \sec y\,dy = \int x^2\,dx \quad \ln|\sec y + \tan y| + c$$

$$\ln|\sec y + \tan y| + \ln C_1 = \frac{x^3}{3} + \ln C_2$$

$$\ln|\sec y + \tan y| = \frac{x^3}{3} + \underbrace{\ln C_2 - \ln C_1}_{\ln C_3}$$

$$\ln|\sec y + \tan y| = \frac{x^3}{3} + \ln C_3$$

$e \qquad e \qquad e$

$$\sec y + \tan y = e^{\frac{x^3}{3}} \cdot C_3$$

$$\sec y + \tan y = C_3\, e^{\frac{x^3}{3}} \quad //$$

2.) $y' = (x^2-4)(3y+2)$

$\ln C_4 - \ln C_5$

$\ln\left|\dfrac{e_4}{C_5}\right|$

$$\left(\frac{\cancel{dx}}{3y+2}\right)\frac{dy}{\cancel{dx}} = (x^2-4)\cancel{(3y+2)}\left(\frac{dx}{\cancel{3y+2}}\right)$$

$$\int \frac{dy}{(3y+2)} = \int (x^2-4)\,dx$$

let $u = 3y+2$

$$\frac{du}{3} = \frac{\cancel{3}\,dy}{\cancel{3}}$$

$$\int \frac{1}{3y+2}\,dy = \int (x^2-4)\,dx$$

$$\frac{du}{3} = dy$$

$$\int \frac{1}{u}\,\frac{dy}{3} = \int (x^2-4)\,dx$$

$$\frac{1}{3}\int \frac{1}{u}\,dy = \frac{x^3}{3} - 4x + c$$

$$\frac{1}{3}\ln u + \ln C_1 = \frac{x^3}{3} - 4x + \ln C_2$$

$$\frac{1}{3}\ln|3y+2| + \ln C_1 = \frac{x^3}{3} - 4x + \ln C_2$$

$$\frac{1}{3}\ln|3y+2| = \frac{x^3}{3} - 4x + \underbrace{\ln C_2 + \ln C_1}_{\ln C_3}$$

$$3\left(\frac{1}{\cancel{3}}\ln|3y+2|\right) = \left(\frac{x^3}{\cancel{3}} - 4x + \ln C_3\right)\cancel{3}$$

$$\ln|3y+2| = x^3 - 12x + \boxed{3\ln C_3} \to \ln C_4$$

$e \qquad e$

$$3y+2 = e^{x^3-12x} \cdot C_4$$

$$3y+2 = C_4\, e^{x^3-4x} \quad //$$

# Linear DE (1ST ORDER)

Date: $\int (p(x))dx$

$V = e^{\int (p(x))dx}$

$$\frac{dy}{dx} + P(x)y = Q(x)$$ — STANDARD FORM — LINEAR OF y

$$\frac{dx}{dy} + P(y)x = g(y)$$ — LINEAR OF x

INTEGRATING FACTOR

factor to solve the DE that you will multiply to the DE.

$$uv = \int (uv) = \int (uv' + vu') = uv$$

$$p(x) \cdot (q(x)) = \int (uv)' = uv$$

INTEGRATION FACTOR

$$\frac{dy}{dx} + \underbrace{(5x^2+1)}_{P(x)}y = 10x$$

STEP 1 LDE

1) Put the DE to the standard form to identify $P(x)$.

2) Solve for integrating factor (I.F.).

3) Multiply the I.F. to the DE.

$P(x) = 5x^2 + 1$

$$v = e^{\int p(x)dx}$$

$$\int p(x) = \int (5x^2+1)dx$$

$$\int p(x)dx = \frac{5}{3}x^3 + x$$

$$v = e^{\left(\frac{5}{3}x^3+x\right)}$$

$$v\left[\frac{dy}{dx} + (5x^2+1)y\right] = [10x]v$$

$$\underbrace{e^{\frac{5}{3}x^3+x}\frac{dy}{dx}}_{u \cdot v'} + \underbrace{e^{\frac{5}{3}x^3+x}(5x^2+1)}_{u'}\underbrace{y}_{v} = 10xe^{\frac{5}{3}x^3+x}$$

$$\int (uv' + u'v) = \int (uv)' = uv$$

$$\int \left(e^{\frac{5}{3}x^3+x} + e^{\frac{5}{3}x^3+x}(5x^2+1)y\right) = \int \left(10xe^{\frac{5}{3}x^3+x}\right)$$

$$ye^{\frac{5}{3}x^3+x} = \int \left(10xe^{\frac{5}{3}x^3+x}\right)dx$$

$$ye^{\frac{5}{3}x^3+x} = \int 10xe^{\frac{5}{3}x^3+x}dx$$

~~$\int u dv = uv - \int v du$~~

$$ye^{\frac{5}{3}x^3+x} = 10\int xe^{\frac{5}{3}x^3+x}dx$$

$$\boxed{y = \frac{10\int xe^{\frac{5}{3}x^3+x} + C}{e^{\frac{5}{3}x^3+x}}}$$

$$y = \frac{\int Q(x)\,V + C}{V}$$

$x = rx$

$y = ry$

## HOMOGENEOUS DE

Date:

$F(x,y)dx + G(x,y)dy = 0$

$F(x,y)$ & $G(x,y)$ are homogeneous of the same degree.

1) $x^2 + y^2 = 0$
$(rx)^2 + (ry)^2 = 0$
$r^2x^2 + r^2y^2 = 0$
$r(x^2 + y^2) = 0$
Homogeneous of 2nd degree

2) $x + y^2 = 0$
$(rx) + (ry)^2 = 0$
$rx + r^2y^2 = 0$
$r(x + ry^2) = 0$
Not homogeneous

3) $2x + y = 0$
$2rx + ry = 0$
$r^2(2x + y) = 0$
Homogeneous of 1st degree

IF 'F' IS SIMPLER
$x = vy$ : $dx = vdy + ydv$
IF 'G' IS SIMPLER
$y = ux$ : $dy = udx + xdu$

1. $xydx - (x+2y)^2 dy = 0$

$xydx - (x^2 + \underbrace{4xy}_{\text{double the product of the 1st \& 2nd term.}} + 4y^2)dy = 0$

$(rx)(ry)dx - ((rx)^2 + 4(rx)(ry) + 4(ry)^2)dy = 0$

$r^2xydx - (r^2x^2 + 4r^2xy + 4r^2y^2)dy = 0$

$r^2 xydx - (x^2 + 4xy + 4y^2)dy = 0$ > HOMOGENEOUS

$\underbrace{xydx}_{F} - \underbrace{(x^2 + 4xy + 4y^2)dy}_{G} = 0$

$(vy)(y)(vdy + ydv) - ((vy)^2 + 4(vy)(y) + 4y^2)dy = 0$

$vy^2(vdy + ydv) - (v^2y^2 + 4vy^2 + 4y^2)dy = 0$

$v^2y^2dy + vy^3dv - v^2y^2dy - 4vy^2dy - 4y^2dy = 0$

$vy^3dv - 4vy^2dy - 4y^2dy = 0$

$vy^3dv - 4y^2(vdy + 1dy) = 0$

$\dfrac{vy^3dv}{y^3(v+1)} - \dfrac{4y^2\cancel{(v+1)}dy}{y^3\cancel{(v+1)}} = 0$ (next is separate)

$\dfrac{v}{(v+1)}dv - \dfrac{4y^2}{y^3}dy = 0$ (SIMPLIFY)

$\dfrac{v}{v+1}dv - \dfrac{4}{y}dy = 0$

$$\begin{array}{r|l} & 1 - \frac{1}{v+1} \\ v+1 & v \\ & -(v+1) \\ \hline & -1 \end{array}$$

$\int\left(1 - \dfrac{1}{v+1}\right)dv - \int \dfrac{4}{y}dy = \int 0$

$v - \ln(v+1) - 4\ln y = C$

$\boxed{\dfrac{x}{y} - \ln\left(\dfrac{x}{y} + 1\right) - 4\ln y = C}$

$$\frac{x}{y} = \frac{vy}{y} \qquad v = \frac{x}{y}$$

$$v - \ln(v+1) - 4\ln y = c$$

Date:

$$\frac{x}{y} \ln\left(\frac{x}{y}+1\right) - 4\ln y = c$$

$$\frac{x}{y}\left(\frac{x}{y}+1\right) - 4y = Ce^{v}$$

$$\frac{x}{y}\left(\frac{x}{y}+1\right) = C4ye^{\frac{x}{y}}$$

IF 'F' IS SIMPLIER

$x = vy$ : $dx = vdy + ydv$

IF 'G' IS SIMPLIER

$y = ux$: $dy = udx + xdu$

2.) $2(2x^2+y^2)dx - xydy = 0$

$$(4x^2+2y^2)dx - xydy = 0$$

$$(4(rx)^2 + 2(ry)^2)dx - (rx)(ry)dy = 0$$

$$(4r^2x^2 + 2r^2y^2)dx - r^2xy\,dy = 0$$

$r^2(4x^2+2y^2)dx - xy\,dy = 0$ HOMOGENEOUS OF 2ND DEGREE

$$(4x^2 + 2y^2)dx - xydy = 0$$

F G

$$(4x^2 + 2(ux)^2)dx - x(ux)(udx + xdu)$$

$$(4x^2 + 2u^2x^2)dx - ux^2(udx + xdu)$$

$$4x^2dx + \underline{2u^2x^2dx} - \underline{u^2x^2dx} + ux^3du = 0$$

imagenary -1

$$4x^2dx + u^2x^2dx - ux^3du = 0$$

$$x^2(4+u^2)dx - ux^3du = 0$$

$$\frac{x^2\cancel{(u^2+4)}dx}{x^3\cancel{(u^2+4)}} - \frac{u\cancel{x^3}du}{\cancel{x^3}(u^2+4)} = 0$$

$$\frac{x^2}{x^3}dx - \frac{u}{(u^2+4)}du = 0$$

let $v = u^2+4$

$$\frac{dv}{2u} = \frac{2\cancel{u}du}{2\cancel{u}}$$

$$du = \frac{dv}{2u}$$

$$\int\frac{1}{x}dx - \int\frac{u}{u^2+4}du = \int 0$$

$$\int\frac{1}{x}dx - \int\frac{\cancel{u}}{v}\frac{dv}{2\cancel{u}} = \int 0$$

$$\int\frac{1}{x}dx - \int\frac{1}{2v}dv = \int 0$$

$$\ln x - \frac{1}{2}\int\frac{1}{v}dv = \int 0$$

$\ln x - \frac{1}{2}\int \frac{1}{v}\,dv = \int 0$ $\quad v = u^2+4$

$\ln x - \frac{1}{2}\ln(u^2+4) = C$ $\quad \frac{y}{x} = \frac{v\cancel{x}}{\cancel{x}} = u = \frac{y}{x}$

Date:

$\ln x - \frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2+4\right) = C$

$e^{\ln x - \frac{1}{2}\ln\left(\frac{y^2}{x^2}+4\right)} = e^{C}$ ✓

$x - \frac{1}{2}\left(\frac{y^2}{x^2}+4\right) = Ce^{v}$

$x - \frac{1}{2}\left(\frac{y^2}{x^2}+4\right) Ce^{\frac{y}{x}}$

$$\boxed{-\frac{1}{2}\left(\frac{y^2}{x^2}+4\right) Cxe^{\frac{y}{x}}}$$

3.) $\left(x\csc\left(\frac{y}{x}\right)-y\right)dx + x\,dy = 0$

F $\qquad$ G

$\left(x\csc\left(\frac{u\cancel{x}}{\cancel{x}}\right)-ux\right)dx + x(u\,dx + x\,du) = 0$

$(x\csc u - ux)dx + xu\,dx + x^2du = 0$

$x\csc u\,dx - \cancel{ux\,dx} + \cancel{ux\,dx} + x^2du = 0$

$\frac{x\cancel{\csc u}\,dx + \cancel{x^2}du = 0}{x^2(\cancel{\csc u}) \quad \cancel{x^2}(\csc u)}$

$\frac{x}{x^2}dx + \frac{1}{\csc u}du = 0$

$\int \frac{1}{x}dx + \int \frac{1}{\csc u}du = \int 0$ $\quad \frac{y}{x} = \frac{u\cancel{x}}{\cancel{x}} \quad u = \frac{y}{x}$

$\ln x + \ln \csc u = C_3$

$e^{\ln x} + e^{\ln \csc\left(\frac{y}{x}\right)} = e^{C_3}$ ✓

$x + \csc\left(\frac{y}{x}\right) = Ce^{v}$

$$\boxed{x + \csc\left(\frac{y}{x}\right) = Ce^{\frac{y}{x}}}$$

## EXACT DE

GENERAL FORM: $M(x,y)\,dx + N(x,y)\,dy = 0$

IF $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ IS EXACT, THEN THE SOL'N IS:

SOL'N 1:

$F(x,y) = \int M(x,y)\,dx + g(y)$

$\frac{\partial F}{\partial y} = \int (M(x,y)\,dx) + g'(y) = N(x,y)$

$g(y) = \int g'(y)\,dy$

SOL'N 2:

$F(x,y) = \int N(x,y)\,dy + f(x)$

$\frac{\partial F}{\partial x} = \int (N(x,y)\,dy) + f'(x) = M(x,y)$

$f(x) = \int f'(x)\,dx$

1.) $3x(xy-2)dx + (x^3+2y)\,dy = 0$

$(3x^2y - 6x)dx + (x^3+2y)dy = 0$

M N

Date:

TEST OF EXACTNESS

$\frac{\partial M}{\partial y} = 3x^2(1) - 0$

$= 3x^2$

$\frac{\partial N}{\partial x} = \cancel{x^3+2y}$

$= 3x^2 + 0$

$= 3x^2$

Sol'n 1:

$f(x,y) = \int (3x^2y - 6x)dx + g(y)$

$f = 3y\left(\frac{x^3}{3} - 6\left(\frac{x^2}{2}\right)dx\ g(y)\right)$

$f = x^3y - 3x^2\,dx\ g(y)$

$\cancel{\frac{\partial f}{\partial y} = x^3(1) - 0 + g(y) = x^3 + 2y}$

$\cancel{x^3} + g'(y) = \cancel{x^3} + 2y$

$\int g'(y) = \int 2y$

$g(y) = \frac{2y^2}{2}$

$g(y) = y^2$

$\boxed{x^3y - 3x^2 + y^2 = C}$

Sol'n 2: $\int (x^3+2y)dy + f(x)$

$f = x^3y + \frac{2y^2}{2} + f(x)$

$f = x^3y + y^2 + f(x)$

$\cancel{\frac{\partial f}{\partial x} = 3x^2y + 0 + f'(x) = 3x^2y - 6x}$

$\cancel{3x^2y}\ f'(x) = \cancel{3x^2y} - 6x$

$\int f'(x) = \int -6x$

$f(x) = -\frac{6x^2}{2}$

$f(x) = -3x^2$

$\boxed{x^3y + y^2 - 3x^2 = C}$