Probability and Statistics
A quick wrap-up

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Sample Space

- Set of all possible events
  - Toss a coin: $S = \{\text{H}, \text{T}\}$
  - Roll two dice:

$$S = \left\{ \begin{matrix} (1,1), (1,2), \dots, (1,6) \\ (2,1), (2,2), \dots, (2,6) \\ \dots \\ (6,1), (6,2), \dots, (6,6) \end{matrix} \right\}$$

  - Flight time:

$$S = [0, \infty)$$

Events & Probabilities

event is a subset E of S

- Example: flight time b/w 1 and 2 hours:

$$E = [1, 2] \subset [0, \infty)$$

- Event of either E or F:

$$G = E \cup F$$

- Event of both E and F:

$$G = EF \equiv E \cap F$$

Probability

- Probability: $\{E : E \subseteq S\} \to \mathbb{R}$:
  - $0 \leqslant P(E) \leqslant 1$
  - $P(S) = 1$
  - For mutually exclusive events

$$P(\cup_n E_n) = \sum_n P(E_n) \tag{1}$$

Conditional Probability

- Bayes' theorem:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

- Example: family has 2 children. One of them is a boy. What is the probability that the other is a boy too?
- Example 1.13, p 13

Independent Events

- $P(EF) = P(E)P(F)$
- $P(E|F) = P(E)$
- Intuition: knowledge of F does not help to tell anything about E

Random Variable (RV)

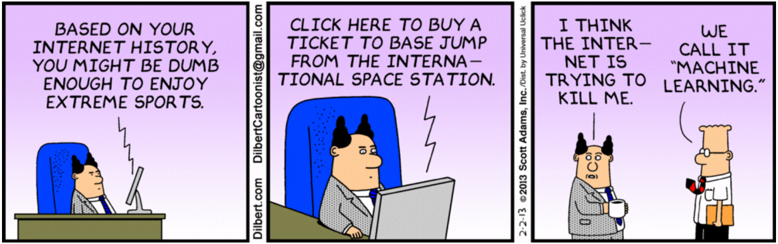
- Random Variable X is

$$X : S \to \mathbb{R}$$

(Note: it is neither random nor variable ☺)

- Examples:
  - number on die
  - sum of numbers on two dice
  - income
  - would this person be willing to buy a ticket?

Preferences are Modelled as RV



- Preferences are modelled as RV
- The probability is estimated by ML methods

Discrete RV

- Discrete: can take discrete (countable) values
- Probability mass function:

$$p(a) = P(X = a)$$

- Cumulative distribution function

$$F(a) = \sum_{i:x_i \leqslant a} p(x_i)$$

- Examples:
  - Bernoulli distribution:

$$\begin{aligned} p(0) &= P(X = 0) = 1 - p \\ p(1) &= P(X = 1) = p \end{aligned}$$

- Binomial Distribution

Continuous RV

- Continuous: can take values on continuum (uncountable) set $\mathbb{R}$
- Probability density function

$$f(x) : P(x \in B) = \int_B f(x) dx$$

- A single value occurs with probability zero (*almost never*):

$$P(X = a) = \int_a^a f(x) dx = 0$$

  Note: this is not the same as impossible event!
- Cumulative distribution function:

$$F(a) = P(X \leqslant a) = \int_{-\infty}^a f(x) dx$$

Continuous RV Examples

Uniform distribution: $X \sim \mathrm{Unif}(0,1)$:

$$f(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

Normal distribution: $X \sim N(\mu, \sigma^2)$:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$

Expectation

- Discrete case:

$$\mathbb{E}\,X \equiv \sum_x xp(x)$$

  - Example: expected value of coin toss (Bernoulli RV):

$$\mathrm{E}X = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}$$

- Continuous case:

$$\mathbb{E}\,X = \int_{-\infty}^{\infty} f(x)dx$$

  - Example: uniform distribution:

$$\mathbb{E}\,X = \int_0^1 x dx = \frac{1}{2}x^2\Big|_{x=0}^{x=1} = \frac{1}{2}$$

Notes:

- Expectation is a property of RV
- It is *not* sample mean!
- But they are related:

Theorem

Law of Large Numbers:

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathbb{E}\,X \quad \textit{as} \quad n \to \infty$$

Theorem

Expectation is linear operator

$$\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$$

Variance

- Variance

$$\mathrm{Var}\, X = \mathbb{E}[(X - \mathbb{E}\, X)^2] = \mathbb{E}[X^2] - (\mathrm{E}[X])^2$$

Joint Distribution Function

$$F(a, b) = P(X \leqslant a, Y \leqslant b)$$

- Marginal distributions:

$$F_X(a) = P(X \leqslant a) = P(X \leqslant a, Y < \infty) = F(a, \infty)$$

Discrete Caes

- joint probability mass function

$$p(x, y) = P(X = x, Y = y)$$

- Marginal mass function:

$$p_X(x) = \sum_y p(x, y)$$

Continuous Case

- joint probability density function

$$P(X \in A, Y \mathrm{in} B) = \int_A \int_B f(x,y) dX \, \mathrm{dif} Y$$

- Marginal density

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y) dy$$

Covariance

$$\mathrm{Cov} X, Y = \mathbb{E}[(X - \mathbb{E}\, X)(Y - \mathbb{E}\, Y)] = \mathbb{E}[XY] - E[X] \cdot \mathbb{E}[Y]$$

- Properties:
  - $\mathrm{Cov}(X, X) = \mathrm{Var}\, X$
  - $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$
  - $\mathrm{Cov}(\lambda X, Y) = \lambda \mathrm{Cov}(X, Y)$
  - $\mathrm{Cov}(X, Y + Z) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z)$

Conditional Probability

Conditional probability mass function

$$p_{X|Y} = P(X = x|Y = y) = \frac{P(X = x, Y = y}{P(Y = y)} = \frac{p(x, y)}{p_Y(y)}$$

Conditional expectation

$$\mathbb{E}[X|Y = y] = \sum_x P(X = X|Y = y) = \sum_x x p_{X|Y}(x|y)$$