Isabelle-Semitopology

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Contents

theory *Semi−Topology*
 imports *Main*
begin

This is a formalisation of a few notions of semitopology and related lemmas, following https://arxiv.org/abs/2303.09287

In general we try to use to decompose proof into readable steps which are each discharged automatically using Sledgehammer (which automatically inserts the lines starting with *by*)

no-notation
 — The default meaning of O (relation composition) is annoying so we remove it
 relcomp (**infixr** O *75*)

1 Definition of a semitopology

locale *semitopology* =
  fixes *open* :: $'a\ set \Rightarrow bool$
  assumes $\bigwedge\ Os\ .\ \forall\ O \in Os\ .\ open\ O \Longrightarrow open\ (\bigcup\ Os)$
    and *open UNIV*
begin

lemma *open* {}
  — The axioms imply that the empty set is open
  by (*metis Sup-empty empty-iff semitopology-axioms semitopology-def*)

2 Section 2

lemma *l-2-3-2*:
  fixes *S*
  shows $(\forall\ p \in S\ .\ \exists\ O\ .\ open\ O \wedge p \in O \wedge O \subseteq S) \longleftrightarrow open\ S$
proof −
  have *open S* **if** $\forall\ p \in S\ .\ \exists\ O\ .\ open\ O \wedge p \in O \wedge O \subseteq S$
  proof −
    from $\langle\forall\ p \in S\ .\ \exists\ O\ .\ open\ O \wedge p \in O \wedge O \subseteq S\rangle$
    obtain *f* **where** $1{:}\forall\ p \in S\ .\ open\ (f\ p) \wedge p \in f\ p \wedge f\ p \subseteq S$
      — first we skolemize the existential in the assumption
      by *metis*
    from *1* **have** $2{:}S = \bigcup\ \{f\ p \mid p\ .\ p \in S\}$ **by** *blast*
    from *1* **have** $3{:}open\ (f\ p)$ **if** $p \in S$ **for** *p* **using** $\langle p{\in}S\rangle$ **by** *auto*
    from *2 3* **show** *?thesis*
      by (*smt* (*verit*, *ccfv-SIG*) *mem-Collect-eq semitopology-axioms semitopology-def*)
  qed
  moreover
  have $\forall\ p \in S\ .\ \exists\ O\ .\ open\ O \wedge p \in O \wedge O \subseteq S$ **if** *open S*
    using *that* **by** *blast*
  ultimately show *?thesis* **by** *blast*
qed

3 Section 3

3.1 Subsection 3.1

definition *intersects* :: $'a\ set \Rightarrow 'a\ set \Rightarrow bool$ (**infix** *int 100*)
  where $intersects\ a\ b \equiv a \cap b \neq \{\}$

lemma *int-commutes*:
  fixes $S\ S'$
  shows $S\ int\ S' = S'\ int\ S$
  by (*simp add*: *inf-commute intersects-def*)

lemma *int-refl*:

fixes S
shows S *int* $S = (S \neq \{\})$
by (*simp add*: *local.intersects-def*)

lemma *int-union*:
fixes X Y Z
shows X *int* $(Y \cup Z) = (X$ *int* $Y \vee X$ *int* $Z)$
by (*simp add*: *Int-Un-distrib local.intersects-def*)

lemma *subset-int*:
fixes X X'
assumes $X \subseteq X'$ **and** $X \neq \{\}$
shows X *int* X'
by (*metis assms inf.orderE local.intersects-def*)

lemma *subset-int-other-1*:
fixes X X' Y
assumes $X \subseteq X'$ **and** X *int* Y
shows X' *int* Y
by (*metis assms int-commutes int-union subset-Un-eq*)

lemma *subset-int-other-2*:
fixes X X' Y
assumes $Y \subseteq Y'$ **and** X *int* Y
shows X *int* Y'
by (*metis assms int-union subset-Un-eq*)

definition *transitive*
where *transitive* $S \equiv \forall\ O\ O'$. *open* $O \wedge$ *open* $O' \wedge S$ *int* $O \wedge S$ *int* $O' \longrightarrow O$ *int* O'

definition *topen*
where *topen* $S \equiv S \neq \{\} \wedge$ *open* $S \wedge$ *transitive* S

3.2 Subsection 3.4

lemma *l-3-4-2-1*:
fixes S S'
assumes $S' \subseteq S$ **and** *transitive* S
shows *transitive* S'
by (*metis Int-assoc Int-empty-right assms le-iff-inf local.intersects-def local.transitive-def*)

lemma *l-3-4-2-2*:
fixes S S'
assumes $S' \subseteq S$ **and** *topen* S **and** $S' \neq \{\}$ **and** *open* S'
shows *topen* S'
using *assms l-3-4-2-1 topen-def* **by** *force*

lemma *l-3-4-3-1*:

fixes $S\ S'\ O\ O'$
assumes *transitive* S **and** *transitive* S' **and** *open* $S \vee$ *open* S'
and *open* O **and** *open* O' **and** O *int* S **and** S *int* S' **and** S' *int* O'
shows O *int* O'
by (*metis Int-commute assms local.intersects-def local.transitive-def*)

lemma *l-3-4-3-2*:
fixes $S\ S'$
assumes S *int* S' **and** *transitive* S **and** *transitive* S' **and** *open* $S \vee$ *open* S'
shows *transitive* $(S \cup S')$
by (*smt* (*verit*) *Un-iff assms disjoint-iff-not-equal local.intersects-def local.transitive-def*)

3.3 Subsection 3.5

definition *connected* **where**
connected $Ss \equiv \forall\ S \in Ss\ .\ \forall\ S' \in Ss\ .\ S$ *int* S'

lemma *l-3-5-2-1*:
fixes $S\ S'$
assumes S *int* S' **and** *topen* S **and** *topen* S'
shows *topen* $(S \cup S')$
by (*metis Un-empty Un-iff Un-upper1 Un-upper2 assms l-2-3-2 l-3-4-3-2 semitopology.topen-def semitopology-axioms*)

3.4 Subsection 3.6

definition *intertwined* **where**
intertwined $p\ p' ==$ *transitive* $\{p,\ p'\}$

lemma *def-3-6-1*:
fixes $p\ p'$
shows *intertwined* $p\ p' = (\forall\ O\ O'\ .$ *open* $O \wedge$ *open* $O' \wedge p \in O \wedge p' \in O' \longrightarrow O$ *int* $O')$
proof −
have *intertwined* $p\ p'$ **if** $\forall\ O\ O'\ .$ *open* $O \wedge$ *open* $O' \wedge p \in O \wedge p' \in O' \longrightarrow O$ *int* O'
by (*smt* (*verit, best*) *Int-commute inf-right-idem insert-absorb insert-disjoint*(*2*) *intersects-def intertwined-def that transitive-def*)
moreover
have $\forall\ O\ O'\ .$ *open* $O \wedge$ *open* $O' \wedge p \in O \wedge p' \in O' \longrightarrow O$ *int* O' **if** *intertwined* $p\ p'$
using *intersects-def intertwined-def that transitive-def* **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *l-3-6-4-a*:
fixes S
shows *transitive* $S = (\forall\ p \in S\ .\ \forall\ p' \in S\ .$ *intertwined* $p\ p')$
proof
show $\forall\ p \in S\ .\ \forall\ p' \in S\ .$ *intertwined* $p\ p'$ **if** *transitive* S

by (*simp add*: *intertwined-def l-3-4-2-1 that*)
show *transitive* S **if** $\forall\ p \in S\ .\ \forall\ p' \in S\ .$ *intertwined* $p\ p'$
by (*smt* (*verit*, *best*) *that disjoint-iff-not-equal def-3-6-1 semitopology.intersects-def semitopology.transitive-def semitopology-axioms*)
qed

lemma *l-3-6-4-b*:
fixes S
shows *transitive* $S = (\forall\ p \in S\ .\ \forall\ p' \in S\ .$ *transitive* $\{p, p'\})$
using *intertwined-def l-3-6-4-a* **by** *auto*

4 Section 4

definition *interior* **where**
interior $R \equiv \bigcup\ \{O\ .\ open\ O \wedge O \subseteq R\}$

4.1 Subsection 4.4

definition K **where**
— The community of a point
$K\ p \equiv$ *interior* $\{p'\ .\ intertwined\ p\ p'\}$

definition *regular* **where**
regular $p \equiv p \in K\ p\ \ \wedge\ \ topen(K\ p)$

lemma *interior-open*:
fixes R
shows *open* (*interior* R)
proof $-$
have $\exists\ O\ .\ open\ O \wedge O \subseteq interior\ R \wedge p \in O$ **if** $p \in interior\ R$ **for** p
using *interior-def that* **by** *auto*
thus *?thesis*
by (*meson l-2-3-2*)
qed

5 Section 5

5.1 Subsection 5.1

definition *closure* **where**
closure $R \equiv \{p\ .\ \forall\ O\ .\ open\ O \wedge p \in O \longrightarrow O\ int\ R\}$

lemma *closure-monotone*:
fixes $R\ R'$
assumes $R \subseteq R'$
shows *closure* $R \subseteq$ *closure* R'
using *assms closure-def subset-int-other-2* **by** *force*

lemma *closure-increasing*:

fixes R
shows $R \subseteq$ *closure* R
using *closure-def local.intersects-def* **by** *auto*

lemma *closure-idempotent*:
fixes R
shows *closure* $R =$ *closure* (*closure* R)
proof
show *closure* $R \subseteq$ *closure* (*closure* R)
by (*simp add*: *closure-increasing*)
next
show *closure* (*closure* R) $\subseteq$ *closure* R
proof $-$
{ fix p
assume $p \in$ *closure* (*closure* R) **and** $p \notin$ *closure* R
hence *False*
unfolding *closure-def intersects-def* **by** *auto* **}**
thus *?thesis*
by *blast*
qed
qed

definition *closed* **where**
closed $R \equiv R =$ *closure* R

definition *clopen* **where**
clopen $R \equiv$ *open* $R \wedge$ *closed* R

lemma *complement-closed-is-open*:
fixes R
assumes *closed* R
shows *open* $(-\ R)$
proof $-$
have $\exists\ O$. *open* $O \wedge O \subseteq -\ R \wedge p \in O$ **if** $p \in -R$ **for** p
proof $-$
from ‹$p \in -\ R$›
have $p \in -$ (*closure* R)
using *assms closed-def* **by** *auto*
hence $\exists\ O$. *open* $O \wedge O \cap R = \{\} \wedge p \in O$
using *closure-def local.intersects-def* **by** *auto*
thus *?thesis*
by (*simp add*: *inf-shunt*)
qed
thus *?thesis*
by (*meson l-2-3-2*)
qed

lemma *complement-open-is-closed*:
fixes O

assumes *open O*
shows *closed* $(-O)$
proof $-$
have $O \cap$ *closure* $(-O) = \{\}$
using *assms closure-def semitopology.intersects-def semitopology-axioms* **by** *fastforce*
thus *?thesis*
by (*meson closed-def closure-increasing compl-le-swap1 disjoint-eq-subset-Compl subset-antisym*)
qed

lemma *empty-closed*:
closed $\{\}$
by (*metis Compl-UNIV-eq complement-open-is-closed semitopology-axioms semitopology-def*)

lemma *univ-closed*: *closed UNIV*
using *closed-def closure-increasing* **by** *blast*

lemma *intersection-of-closed-is-closed*:
fixes *Rs*
assumes $\forall\ R \in Rs\ .\ closed\ R$
shows *closed* $(\bigcap\ Rs)$
proof $-$
have *1*:$\bigcap Rs = -\bigcup\{-R \mid R\ .\ R \in Rs\}$
by (*simp add*: *Setcompr-eq-image*)
thus *?thesis*
by (*smt* (*verit*, *del-insts*) *assms complement-closed-is-open complement-open-is-closed mem-Collect-eq semitopology-axioms semitopology-def*)
qed

lemma *closure-as-intersection*:
fixes *R*
shows *closure* $R = \bigcap\ \{C\ .\ closed\ C \wedge R \subseteq C\}$
by (*smt* (*z3*) *cInf-eq-minimum closure-idempotent closure-increasing closure-monotone mem-Collect-eq semitopology.closed-def semitopology-axioms*)

lemma *interior-of-closed-in-closed*:
fixes *C*
assumes *closed C*
shows *interior* $C \subseteq C$
using *semitopology.interior-def semitopology-axioms* **by** *auto*

lemma *open-in-interior-of-closure*:
fixes *O*
assumes *open O*
shows $O \subseteq$ *interior* (*closure O*)
using *assms closure-increasing interior-def* **by** *auto*

lemma *closure-of-interior-of-closed-in-closed*:
fixes C
assumes *closed* C
shows *closure* (*interior* C) $\subseteq$ C
by (*metis assms closed-def closure-monotone interior-of-closed-in-closed*)

5.2 Subsection 5.3

lemma *intertwined-iff-in-closure-of-neighbourhood*:
fixes p p'
shows *intertwined* p p' = ($\forall$ O . *open* O $\wedge$ $p \in O$ $\longrightarrow$ $p' \in$ *closure* O)
using *closure-def semitopology.int-commutes def-3-6-1 semitopology-axioms* **by** *fastforce*

lemma *intertwined-is-intersection-of-closures-of-neighbourhoods*:
fixes p
shows $\{p'$. *intertwined* p $p'\}$ = $\bigcap$ $\{$*closure* O | O . *open* O $\wedge$ $p \in O\}$
proof −
have $\bigcap$ $\{$*closure* O | O . *open* O $\wedge$ $p \in O\}$ = $\{p'$. $\forall$ O . *open* O $\wedge$ $p{\in}O$ $\longrightarrow$ p' $\in$ *closure* $O\}$
by *auto*
thus *?thesis*
by (*simp add*: *semitopology.intertwined-iff-in-closure-of-neighbourhood semitopology-axioms*)
qed

lemma *intertwined-is-intersection-of-closed*:
fixes p
shows $\{p'$. *intertwined* p $p'\}$ = $\bigcap$ $\{C$. *closed* C $\wedge$ $p \in$ *interior* $C\}$
proof −
have $\bigcap$ $\{$*closure* O | O . *open* O $\wedge$ $p \in O\}$ = $\bigcap$ $\{C$. *closed* C $\wedge$ $p \in$ *interior* $C\}$
proof −
have $\exists$ C . *closed* C $\wedge$ $p \in$ *interior* C $\wedge$ $C \subseteq$ *closure* O **if** *open* O **and** $p \in O$ **for** O
proof −
define C **where** $C \equiv$ *closure* O
have *closed* C
using *C-def closed-def closure-idempotent* **by** *auto*
moreover have $C \subseteq$ *closure* O
by (*simp add*: *C-def*)
moreover have $p \in$ *interior* C
using *C-def open-in-interior-of-closure that* **by** *blast*
ultimately show *?thesis*
by *blast*
qed
hence $\bigcap$ $\{C$. *closed* C $\wedge$ $p \in$ *interior* $C\}$ $\subseteq$ $\bigcap$ $\{$*closure* O | O . *open* O $\wedge$ $p \in O\}$
by *fastforce*

moreover
have $\exists\ O .\ open\ O \wedge p \in O \wedge closure\ O \subseteq C$ **if** *closed C* **and** $p \in interior\ C$ **for** *C*
proof −
define *O* **where** $O \equiv interior\ C$
have *open O*
by (*simp add*: *O-def interior-open*)
moreover have $p \in O$
using *O-def that*(*2*) **by** *auto*
moreover have $closure\ O \subseteq C$
by (*simp add*: *O-def closure-of-interior-of-closed-in-closed that*(*1*))
ultimately show *?thesis* **by** *blast*
qed
hence $\bigcap\ \{closure\ O \mid O .\ open\ O \wedge p \in O\} \subseteq \bigcap\ \{C .\ closed\ C \wedge p \in interior\ C\}$
by *fastforce*
ultimately
show *?thesis* **by** *auto*
qed
thus *?thesis*
by (*simp add*: *semitopology.intertwined-is-intersection-of-closures-of-neighbourhoods semitopology-axioms*)
qed

lemma *closed* $\{p' .\ intertwined\ p\ p'\}$
using *intersection-of-closed-is-closed intertwined-is-intersection-of-closed* **by** *auto*

lemma *closure-in-intertwined-set*:
$closure\ \{p\} \subseteq \{p' .\ intertwined\ p\ p'\}$
by (*smt* (*verit*) *closure-def insert-is-Un int-commutes int-union mem-Collect-eq mk-disjoint-insert*
semitopology.def-3-6-1 semitopology-axioms subsetI)

lemma *closure-is-intertwined-set*:
assumes $interior\ (closure\ \{p\}) \neq \{\}$
shows $closure\ \{p\} = \{p' .\ intertwined\ p\ p'\}$
oops

lemma *cascade*:
— This is a version of the Cascade Theorem
fixes *S O*
assumes *topen S* **and** *open O* **and** *O int S*
shows $S \subseteq closure\ O$ **using** *closure-increasing*
by (*smt* (*verit*, *best*) *assms int-commutes closure-def mem-Collect-eq semitopology.topen-def semitopology-axioms subsetD subsetI transitive-def*)

end

6 Finding examples with Nitpick

Here we restrict the union axiom to pairwise union to make it easier to find small models with Nitpick.

nitpick-params [*timeout* = *120*]

6.1 definitions

locale *semitopology-alt* =
fixes *open* :: $'a$ *set* $\Rightarrow$ *bool* — open seems to be reserved
assumes $\bigwedge$ O O' . $[\![$*open* O; *open* $O']\!]$ $\Longrightarrow$ *open* $(O \cup O')$ — using just pairwise union in an attempt to appease Nitpick
and *open* {} **and** *open* *UNIV*
begin

definition *intersects* :: $'a$ *set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ *bool* (**infix** *int* *100*)
where *intersects* a b $\equiv$ $a \cap b \neq \{\}$

definition *closure* **where**
closure $R \equiv \{p\ .\ \forall\ O\ .\ open\ O \land p \in O \longrightarrow O\ int\ R\}$

definition *transitive*
where *transitive* $S \equiv \forall\ O\ O'\ .\ open\ O \land open\ O' \land S\ int\ O \land S\ int\ O' \longrightarrow O\ int\ O'$

definition *topen* **where**
topen $S \equiv open\ S \land transitive\ S$

definition *strong-topen* **where**
strong-topen $S \equiv open\ S \land (\forall\ O\ O'\ .\ open\ O \land open\ O' \land S\ int\ O \land S\ int\ O' \longrightarrow O \cap O' \cap S \neq \{\})$

definition *intertwined* **where**
intertwined p p' == *transitive* $\{p,\ p'\}$

definition *intertwined-set-of* **where**
intertwined-set-of $p \equiv \{p'\ .\ intertwined\ p\ p'\}$

definition *closed* **where**
closed $R \equiv R = closure\ R$

definition *interior* **where**
interior $R \equiv \bigcup\ \{O\ .\ open\ O \land O \subseteq R\}$

definition *boundary* **where**
boundary $R \equiv R - interior\ R$

definition *edge* **where**

edge $p \equiv$ *boundary* (*intertwined-set-of* p)

definition K **where**
— The community of a point
$K\ p \equiv$ *interior* $\{p'\ .\ \textit{intertwined}\ p\ p'\}$

definition *regular* **where**
regular $p \equiv p \in K\ p \wedge$ *topen* $(K\ p)$

definition *weakly-regular* **where**
weakly-regular $p \equiv p \in (K\ p)$

definition *conflicted* **where**
conflicted $p \equiv \exists\ p'.\ \exists\ p''.\ ((\textit{intertwined}\ p'\ p) \wedge (\textit{intertwined}\ p\ p'') \wedge \neg (\textit{intertwined}\ p'\ p''))$

definition *unconflicted* **where**
unconflicted $p \equiv \forall\ p'.\ \forall\ p''.\ \textit{intertwined}\ p'\ p \wedge \textit{intertwined}\ p\ p'' \longrightarrow \textit{intertwined}\ p'\ p''$

definition *dense-in* **where**
dense-in $D\ P \equiv (D \subseteq P) \wedge (\forall\ O.\ \textit{open}\ O \wedge O \neq \{\} \wedge O \subseteq P \longrightarrow D\ \textit{int}\ O)$

definition *strongly-dense-in* **where**
strongly-dense-in $D\ P \equiv (D \subseteq P) \wedge (\forall\ O.\ \textit{open}\ O \wedge P\ \textit{int}\ O \longrightarrow D\ \textit{int}\ O)$

definition *resilient* **where**
resilient $S \equiv \forall\ C\ .\ \textit{closed}\ C \longrightarrow \textit{strong-topen}\ (S - C)$

definition *atom* **where**
atom $A \equiv A \neq \{\} \wedge \textit{open}\ A \wedge (\forall\ O\ .\ \neg\ (O \neq \{\} \wedge \textit{open}\ O \wedge O \subset A))$

definition *cover* **where**
cover $p\ O \equiv \textit{open}\ O \wedge p \in O \wedge (\forall\ O'\ .\ \neg(\textit{open}\ O' \wedge p{\in}O' \wedge O' \subset O))$

definition *kernel-atom* **where**
kernel-atom $p\ A \equiv \textit{atom}\ A \wedge A \subseteq K\ p$

definition *kernel* **where**
kernel $p \equiv \{A\ .\ \textit{kernel-atom}\ p\ A\}$

6.2 Finding models...

lemma *topen* $\{\}$
— Just checking...
by (*metis inf-bot-left local.intersects-def local.transitive-def semitopology-alt-axioms semitopology-alt-def topen-def*)

lemma

fixes *S1 S2*
assumes *resilient S1* **and** *resilient S2*
and $S1 \cap S2 \neq \{\}$
shows *resilient* $(S1 \cup S2)$
nitpick[*verbose*, *card 'a=5*, *timeout=30000*] — okay up to 8
oops

lemma
fixes *S1 O*
assumes *resilient S1* **and** *open O* **and** $O \cap S1 \neq \{\}$
shows $S1 \subseteq$ *closure O*
nitpick[*verbose*, *card 'a=5*, *timeout=3000*] — okay up to 8
oops

lemma — This is to find an example satisfying all the premises.
fixes *p I C Cl*
assumes $I =$ *intertwined-set-of p*
and $Cl =$ *closure* $\{p\}$
and *closed C* **and** $p \in$ *interior C* **and** $\forall\ C'.$ *closed* $C' \wedge p \in$ *interior* $C' \longrightarrow \neg\ C' \subset C$
and $Cl \subset I$ **and** $I \subset C$
shows *False*
nitpick[*card 'a=4*, *verbose*]
oops

lemma *closure-is-intertwined-set*:
fixes *p O*
assumes *open O* **and** $p \in O$ **and** $O \subseteq \{p'.$ *intertwined p p'*$\}$
shows *closure* $O = \{p'.$ *intertwined p p'*$\}$
oops

lemma — Space in which every point is unconflicted but not weakly regular'
assumes $(\forall\ p.\ \neg$ *weakly-regular p* $\wedge$ *unconflicted p*$)$
shows *False*
nitpick[*card 'a=4*]
oops

lemma — Space in which every point is weakly regular but conflicted (nitpick fails because impossible)
assumes $(\forall\ p.$ *weakly-regular p* $\wedge$ *conflicted p*$)$
shows *False*
nitpick[*card 'a=1*]
oops

lemma — Space in which every point is weakly regular but not regular (nitpick fails because impossible)
assumes $\forall\ p.$ *weakly-regular p* $\wedge \neg$ *regular p*

shows *False*
nitpick[*card 'a=1*]
oops

lemma — An unconflicted point on the boundary of a regular p
fixes *p q*
assumes *regular p* **and** $q \in$ *edge p* **and** $\neg$ *conflicted q*
shows *False*
nitpick[*card 'a=4*]
oops

lemma — A conflicted point on the boundary of a regular p
fixes *p q*
assumes *regular p* **and** $q \in$ *edge p* **and** *conflicted q* **and** *weakly-regular q*
shows *False*
nitpick[*card 'a=3*]
oops

lemma — A regular point on the boundary of a regular p (fails because impossible)
fixes *p q*
assumes *regular p* **and** $q \in$ *edge p* **and** *regular q*
shows *False*
nitpick[*card 'a=2*]
oops

lemma — A conflicted, not weakly regular point
fixes *p*
assumes *conflicted p* **and** $\neg$ *weakly-regular p*
shows *False*
nitpick[*card 'a=5*]
oops

lemma — A conflicted, not weakly regular point on a boundary
fixes *p q*
assumes *conflicted p* **and** $\neg$ *weakly-regular p* **and** $p \in$ *edge q*
shows *False*
nitpick[*card 'a=5*]
oops

lemma — A point on the boundary of a closed set that is not intertwined with any element of its interior
fixes *p C*
assumes *closed C* **and** *interior* $C \neq \{\}$ **and** $\forall\ q\ .\ \neg\ (q \in$ *interior* $C \wedge$ *intertwined* $p\ q)$ **and** $p \in$ *boundary C*
shows *False*
nitpick[*card 'a=3*]
oops

lemma — A point that is unconflicted but on the intersection of the boundary of two closed neighbourhoods whose intersections do not intersect
fixes $p\ C\ C'$
assumes *closed* C **and** *closed* C' **and** *interior* $C \neq \{\}$ **and** *interior* $C' \neq \{\}$ **and** $p \in$ *boundary* C **and** $p \in$ *boundary* C' **and** $\neg$ (*interior* C) *int* (*interior* C') **and** $\neg$ *conflicted* p
shows *False*
nitpick[*card* $'a=4$]
oops

lemma — A point that is regular but on the intersection of the boundary of two closed neighbourhoods whose intersections do not intersect, and p is not intertwined with any point in the interiors of either closed neighbourhoods
fixes $p\ C\ C'$
assumes *closed* C **and** *closed* C' **and** *interior* $C \neq \{\}$ **and** *interior* $C' \neq \{\}$ **and** $p \in$ *boundary* C **and** $p \in$ *boundary* C' **and** $\neg$ (*interior* C) *int* (*interior* C') **and** $\forall\ q\ .\ \neg$ ($q\in$ *interior* $C \wedge$ *intertwined* $p\ q$) **and** $\forall\ q\ .\ \neg$ ($q\in$ *interior* $C' \wedge$ *intertwined* $p\ q$) **and** *regular* p
shows *False*
nitpick[*card* $'a=4$]
oops

lemma — Just because D is dense in P does not mean it is strongly dense)
fixes $D\ P$
assumes *open* P **and** $D \neq \{\}$ **and** *dense-in* $D\ P$
shows $\neg$ *strongly-dense-in* $D\ P$
nitpick[*card* $'a=2$, *verbose*]
oops

Can p be in its community but the community of p not be a topen? As we see below, the answer is yes.

lemma
fixes $p\ Kp\ b$
assumes $Kp = K\ p$ **and** $p \in Kp$ **and** $\neg$ *topen* Kp **and** $b = (\forall\ p \in Kp\ .\ \forall\ p' \in Kp\ .$ *intertwined* $p\ p'$) **and** $\neg\ b$
shows *False*
nitpick[*card* $'a=3$]
oops

lemma — A (wrong) conjecture:
fixes p
assumes $\neg$ *regular* p **and** *open* O **and** $p \in O$ **and** $\forall\ p' \in O\ .$ *intertwined* $p\ p'$
shows *False*
nitpick[*card* $'a=4$, *verbose*]
oops

lemma
fixes p
assumes $K\ p \neq \{\}$ **and** $\neg$ *transitive* ($K\ p$) **and** $\neg\ p \in K\ p$

shows *False*
nitpick[*card 'a=5*, *verbose*]
oops

lemma — If p is regular and A is an atom in p's community, it does not follow that p has a cover that includes A:
fixes *p A Kp*
assumes *regular p* **and** *atom A* **and** $A \subseteq Kp$ **and** $Kp = K\ p$
shows $\exists\ O\ .\ cover\ p\ O \wedge A \subseteq O$
nitpick[*card 'a=3*]
oops

lemma — no counterexample with 7 points
fixes *p Kp*
assumes *regular p* **and** $Kp = K\ p$
shows $\exists\ O\ .\ cover\ p\ O \wedge (\forall\ O'\ .\ atom\ O' \wedge transitive\ O' \wedge O'\ int\ O \longrightarrow O' \subseteq Kp)$
oops

Seems like the above should hold: Take a cover O inside the community of p. Then an O' as above must be in p-intertwined, and thus it's in the community (which this the interior of p-intertwined.

lemma — Example of a cover O of p that intersects an atom that goes outside the community, even though all transitive atoms that intersect O are within the community
fixes *p A Kp O*
assumes *regular p* **and** $Kp = K\ p$
and $cover\ p\ O \wedge (\forall\ O'\ .\ atom\ O' \wedge transitive\ O' \wedge O'\ int\ O \longrightarrow O' \subseteq Kp)$
and *atom A* **and** *A int O* **and** $\neg\ A \subseteq Kp$
shows *False*
nitpick[*card 'a=4*]
oops

lemma
fixes *p A O O'*
assumes *regular p* **and** *cover p O* **and** *atom O'*
and *transitive O'* **and** *O' int O*
shows $O' \subseteq K\ p$
nitpick[*card 'a=3*]
oops

lemma — Second example requested by Jamie
fixes *C interior-C*
assumes *closed C* **and** $\forall\ C'\ .\ \neg(interior\ C' \neq \{\} \wedge closed\ C' \wedge C' \subset C)$
and $interior\ C \neq \{\}$ **and** $\neg$ *topen interior-C* **and** *interior-C = interior C*
shows *False*
nitpick[*card 'a=4*]
oops

end

end