2D solid mechanics problems by RKPM

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  - L2-norm

Problem statement

2D solid mechanics problem (form 1)

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0; \text{ on domain } \Omega$$

$$\mathbf{u} = \bar{\mathbf{u}}; \text{ on essential boundary } \Gamma_u$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}}; \text{ on natural boundary } \Gamma_t$$

If we use this formula, the symbols are:

$$\nabla \cdot (\mathbf{c} \nabla^S \mathbf{u}) + \mathbf{b} = 0$$

$$\nabla^s \mathbf{u} = \frac{1}{2}(\nabla \otimes \mathbf{u} + \mathbf{u} \otimes \nabla)$$

$$= \frac{1}{2}\left(\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} + \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \right)$$

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{xx} \end{pmatrix}$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{xx} \end{pmatrix}$$

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \end{pmatrix}$$

Problem statement

2D solid mechanics problem (form 2)

$$\mathbf{L}\cdot\boldsymbol{\sigma}+\mathbf{b}=0;\ \text{on domain }\Omega$$

$$\mathbf{u}=\bar{\mathbf{u}};\ \text{on essential boundary }\Gamma_u$$

$$\boldsymbol{\sigma}\cdot\mathbf{n}=\bar{\mathbf{t}};\ \text{on natural boundary }\Gamma_t$$

$$\boldsymbol{\varepsilon}=\mathbf{L}\mathbf{u};\ \ \boldsymbol{\sigma}=\mathbf{c}\boldsymbol{\varepsilon};$$

$$\mathbf{L}\cdot\boldsymbol{\sigma}=\mathbf{L}^T\mathbf{c}\boldsymbol{\varepsilon}=\mathbf{L}^T\mathbf{c}\mathbf{L}\mathbf{u}$$

$$\mathbf{c}=\frac{E}{1-\nu^2}\begin{pmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \dfrac{1-\nu}{2}\end{pmatrix}\ \text{(Plane stress);}$$

$$\mathbf{c}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{pmatrix}1 & \dfrac{\nu}{1-\nu} & 0\\ \dfrac{\nu}{1-\nu} & 1 & 0\\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)}\end{pmatrix}\ \text{(Plane strain).}$$

$$\boldsymbol{\sigma}=\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix},\ \ \boldsymbol{\varepsilon}=\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{pmatrix};$$

$$\mathbf{L}=\begin{pmatrix}\dfrac{\partial}{\partial x} & 0\\ 0 & \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x}\end{pmatrix},\ \ \mathbf{u}=\begin{pmatrix}u\\ v\end{pmatrix}.$$

Infinitesimal strain tensor, epsilon (half)
Engineering strain tensor, gamma (no half)

Steps: strong form to weak form (transformation method)

$$\int_\Omega \mathbf{w}\cdot(\nabla\cdot\boldsymbol{\sigma}+\mathbf{b})d\Omega=\mathbf{0}\ \text{(S)}\ \Rightarrow$$

$$\int_\Omega \mathbf{w}^T\nabla\cdot\boldsymbol{\sigma}d\Omega+\int_\Omega \mathbf{w}^T\mathbf{b}d\Omega=\mathbf{0}\ \Rightarrow$$

$$\mathbf{W}^T=(w_1,w_2),\mathbf{b}^T=(b_1,b_2),\boldsymbol{\sigma}=\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}$$

$$\Rightarrow$$

$$\int_\Omega [w_1(\sigma_{11,1}+\sigma_{12,2}+b_1)+w_2(\sigma_{21,1}+\sigma_{22,2}+b_2)]d\Omega=0\ \Rightarrow$$

$$\int_\Omega [w_1(\sigma_{11,1}+\sigma_{12,2})+w_2(\sigma_{21,1}+\sigma_{22,2})]d\Omega+\int_\Omega (w_1b_1+w_2b_2)d\Omega=0\ \Rightarrow$$

$$\int_\Omega [w_1\nabla\cdot\begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix}+w_2\nabla\cdot\begin{pmatrix}\sigma_{21}\\ \sigma_{22}\end{pmatrix}]d\Omega+\int_\Omega (w_1b_1+w_2b_2)d\Omega=0$$

$$\int_\Omega w_1\nabla\cdot\begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix}d\Omega=\int_\Omega \nabla\cdot(w_1\begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix})d\Omega-\int_\Omega \nabla w_1\cdot\begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix}d\Omega$$

$$=\int_\Gamma (w_1\begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix})\cdot\mathbf{n}d\Gamma-\int_\Omega \nabla w_1\cdot\begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix}d\Omega$$

$$\sigma_{ij,j}+b_i=0\ \Rightarrow$$

$$\int_\Omega w_i(\sigma_{ij,j}+b_i)d\Omega=0\ \Rightarrow$$

$$\int_\Omega w_i\sigma_{ij,j}d\Omega+\int_\Omega w_ib_id\Omega=0\ \Rightarrow$$

$$\int_\Omega \nabla\cdot(w_i\sigma_{ij})d\Omega-\int_\Omega w_{i,j}\sigma_{ij}d\Omega+\int_\Omega w_ib_id\Omega=0$$

Steps: strong form to weak form (transformation method)

$$\int_\Gamma (w_1 \begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix}) \cdot \mathbf{n} d\Gamma - \int_\Omega \nabla w_1 \cdot \begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix} d\Omega + \int_\Gamma (w_2 \begin{pmatrix}\sigma_{21}\\ \sigma_{22}\end{pmatrix}) \cdot \mathbf{n} d\Gamma - \int_\Omega \nabla w_2 \cdot \begin{pmatrix}\sigma_{21}\\ \sigma_{22}\end{pmatrix} d\Omega + \int_\Omega (w_1 b_1 + w_2 b_2) d\Omega = 0$$

$$\Leftrightarrow$$

$$\int_\Omega \nabla w_1 \cdot \begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix} d\Omega + \int_\Omega \nabla w_2 \cdot \begin{pmatrix}\sigma_{21}\\ \sigma_{22}\end{pmatrix} d\Omega = \int_\Gamma (w_1 \begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix}) \cdot \mathbf{n} d\Gamma + \int_\Gamma (w_2 \begin{pmatrix}\sigma_{21}\\ \sigma_{22}\end{pmatrix}) \cdot \mathbf{n} d\Gamma + \int_\Omega (w_1 b_1 + w_2 b_2) d\Omega$$

$$\Leftrightarrow \Gamma = \Gamma_u + \Gamma_t, \ \mathbf{W=0} \ @ \ \Gamma_u, \ \boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \ @ \ \Gamma_t$$

$$\int_\Omega \begin{pmatrix} w_{1,1}\\ w_{1,2}\end{pmatrix} \cdot \begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix} d\Omega + \int_\Omega \begin{pmatrix} w_{2,1}\\ w_{2,2}\end{pmatrix} \cdot \begin{pmatrix}\sigma_{21}\\ \sigma_{22}\end{pmatrix} d\Omega = \int_{\Gamma_u+\Gamma_t} (w_1 \begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix}) \cdot \mathbf{n} d\Gamma + \int_{\Gamma_u+\Gamma_t} (w_2 \begin{pmatrix}\sigma_{21}\\ \sigma_{22}\end{pmatrix}) \cdot \mathbf{n} d\Gamma + \int_\Omega (w_1 b_1 + w_2 b_2) d\Omega$$

$$\int_\Omega \nabla \mathbf{W} : \boldsymbol{\sigma} d\Omega = \int_{\Gamma_t} \mathbf{W} \cdot \bar{\mathbf{t}} d\Gamma + \int_\Omega \mathbf{W} \cdot \mathbf{b} d\Omega \ \Leftrightarrow \ \int_\Omega \nabla^S \mathbf{W} : \boldsymbol{\sigma} d\Omega = \int_{\Gamma_t} \mathbf{W} \cdot \bar{\mathbf{t}} d\Gamma + \int_\Omega \mathbf{W} \cdot \mathbf{b} d\Omega \ \text{(W)}$$

$$s_{ij} t_{ij} = s_{(ij)} t_{ij} \text{ (S is non-symmetric, t is symmetric)} \Rightarrow w_{i,j}\sigma_{ij} = w_{(i,j)}\sigma_{ij}$$

$$\boldsymbol{\sigma} = \mathbf{C} : \nabla^S \mathbf{u}$$

$$\int_\Gamma (w_i \sigma_{ij}) n_j d\Gamma - \int_\Omega w_{i,j}\sigma_{ij} d\Omega + \int_\Omega w_i b_i d\Omega = 0 \ \Rightarrow$$

$$\int_\Omega w_{i,j}\sigma_{ij} d\Omega = \int_\Gamma (w_i \sigma_{ij}) n_j d\Gamma + \int_\Omega w_i b_i d\Omega$$

$$\Leftrightarrow$$

$$\int_\Omega \nabla \mathbf{W} : \boldsymbol{\sigma} d\Omega = \int_{\Gamma_t} \mathbf{W} \cdot \bar{\mathbf{t}} d\Gamma + \int_\Omega \mathbf{W} \cdot \mathbf{b} d\Omega \ \text{(W)}$$

Transformation method

$$\int_{\Omega} \nabla^S \mathbf{W} : \mathbf{C} : \nabla^S \mathbf{U} d\Omega = \int_{\Gamma_t} \mathbf{W} \cdot \bar{\mathbf{t}} d\Gamma + \int_{\Omega} \mathbf{W} \cdot \mathbf{b} d\Omega \ \text{(W)}$$

$$\Rightarrow$$

$$a(\mathbf{w}^h, \mathbf{u}^h) = (\mathbf{w}^h, \bar{\mathbf{t}})_{\Gamma_t} + (\mathbf{w}^h, \mathbf{b}) \ \text{(G)}$$

$$\mathbf{u}^h = \mathbf{N}^T \mathbf{U}, \mathbf{w}^h = \mathbf{N}^T \mathbf{W}$$

$$\mathbf{KU} = \mathbf{f}$$

where,

$$\mathbf{K}_{ij} = \int_{\Omega} \mathbf{B}_i^T \mathbf{c} \mathbf{B}_j d\Omega;$$

$$\mathbf{f}_i = \int_{\Omega} \mathbf{N}_i \mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{N}_i \bar{\mathbf{t}} d\Gamma$$

Transformation method

$$\begin{pmatrix} \mathbf{u}(\mathbf{x}_1) \\ \mathbf{u}(\mathbf{x}_2) \\ \vdots \\ \mathbf{u}(\mathbf{x}_n) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Psi}_1(\mathbf{x}_1) & \boldsymbol{\Psi}_2(\mathbf{x}_1) & \cdots & \boldsymbol{\Psi}_n(\mathbf{x}_1) \\ \boldsymbol{\Psi}_1(\mathbf{x}_2) & \boldsymbol{\Psi}_2(\mathbf{x}_2) & \cdots & \boldsymbol{\Psi}_n(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\Psi}_1(\mathbf{x}_n) & \boldsymbol{\Psi}_2(\mathbf{x}_n) & \cdots & \boldsymbol{\Psi}_n(\mathbf{x}_n) \end{pmatrix} \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_n \end{pmatrix}$$

$$\boldsymbol{\Psi}_1 = \begin{pmatrix} \Psi_I & 0 \\ 0 & \Psi_I \end{pmatrix}$$

$$\Rightarrow$$

$$\begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_n \end{pmatrix} = \boxed{\begin{pmatrix} \boldsymbol{\Psi}_1(\mathbf{x}_1) & \boldsymbol{\Psi}_2(\mathbf{x}_1) & \cdots & \boldsymbol{\Psi}_n(\mathbf{x}_1) \\ \boldsymbol{\Psi}_1(\mathbf{x}_2) & \boldsymbol{\Psi}_2(\mathbf{x}_2) & \cdots & \boldsymbol{\Psi}_n(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\Psi}_1(\mathbf{x}_n) & \boldsymbol{\Psi}_2(\mathbf{x}_n) & \cdots & \boldsymbol{\Psi}_n(\mathbf{x}_n) \end{pmatrix}}^{-1} \begin{pmatrix} \mathbf{u}(\mathbf{x}_1) \\ \mathbf{u}(\mathbf{x}_2) \\ \vdots \\ \mathbf{u}(\mathbf{x}_n) \end{pmatrix} = \boldsymbol{\Lambda}^{-1}\mathbf{U}(\mathbf{x}_I)$$

$$\Rightarrow$$

$$\mathbf{u}(\mathbf{x}) = \sum \boldsymbol{\Psi}_I(\mathbf{x})\mathbf{d}_I = \mathbf{N}^T(\mathbf{x})\boldsymbol{\Lambda}^{-1}\mathbf{U}(\mathbf{x}_I)$$

Transformation method – continued

Now the matrix form becomes:

$$\begin{pmatrix} \mathbf{k}_{11} & \cdots & \mathbf{k}_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{k}_{n1} & \cdots & \mathbf{k}_{nn} \end{pmatrix} \boldsymbol{\Lambda}^{-1} \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{pmatrix}, \quad \mathbf{u}_i = \mathbf{g}_i \qquad \mathbf{K}' = \mathbf{K}\boldsymbol{\Lambda}^{-1}$$

$$\left(\begin{array}{c|c} \mathbf{K}'_{aa} & \mathbf{K}'_{ab} \\ \hline \mathbf{K}'_{ba} & \mathbf{K}'_{bb} \end{array}\right) \left(\begin{array}{c} \mathbf{U}_a \\ \hline \mathbf{U}_b \end{array}\right) = \left(\begin{array}{c} \mathbf{b}_a \\ \hline \mathbf{b}_b \end{array}\right)$$

where u_a is the inner unknown variables

and u_b is the known nodal value at EB.

$$\mathbf{K}'_{aa}\mathbf{U}_a = (\mathbf{b}_a - \mathbf{K}'_{ab}\mathbf{U}_b)$$

Ultimate matrix form

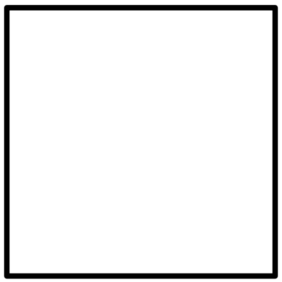
Steps

- Using 11-11, 21-21, 41-41 evenly spaced nodes
- Kernel function: rectangular kernel $\Phi_a(x, y) = \Phi_a(x) \bullet \Phi_a(y)$
- Using $P = [1, x, y]^T$

Numerical test

2D patch test:

1



$$\Omega = [0,1]\cdot[0,1];$$

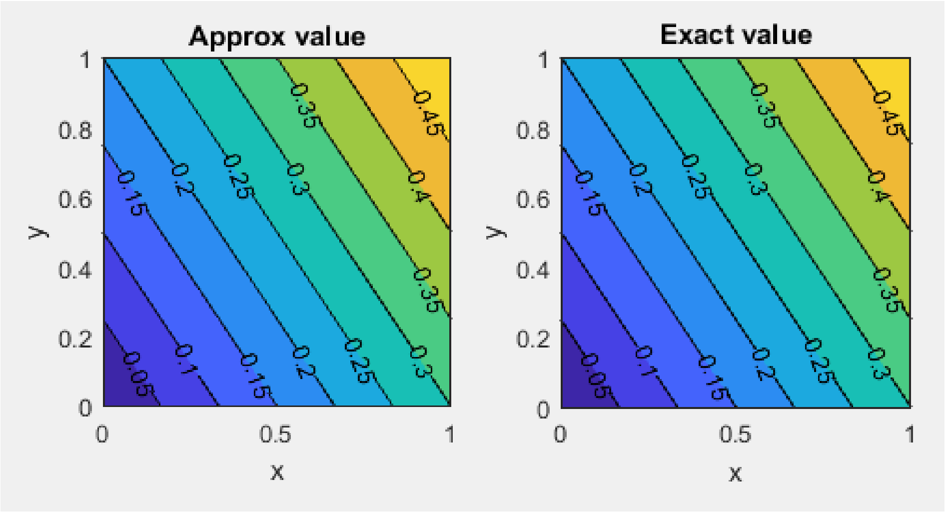
$$\bar{\mathbf{u}} = \begin{pmatrix} 0.2x + 0.3y \\ 0.1x + 0.4y \end{pmatrix} \text{ all essential boundary.}$$

Exact solution:

$$u = 0.2x + 0.3y;$$

$$v = 0.1x + 0.4y.$$

Results



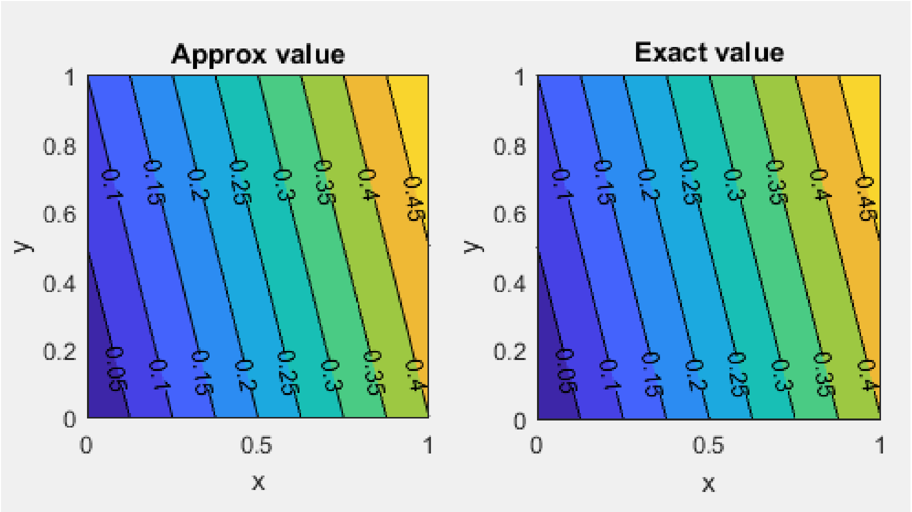
Displacement in X direction (4*4 nodes)

4*4 nodes:

L2 error = 1.933E-06

11*11 nodes:

L2 error = 1.092E-07



Displacement in Y direction (4*4 nodes)

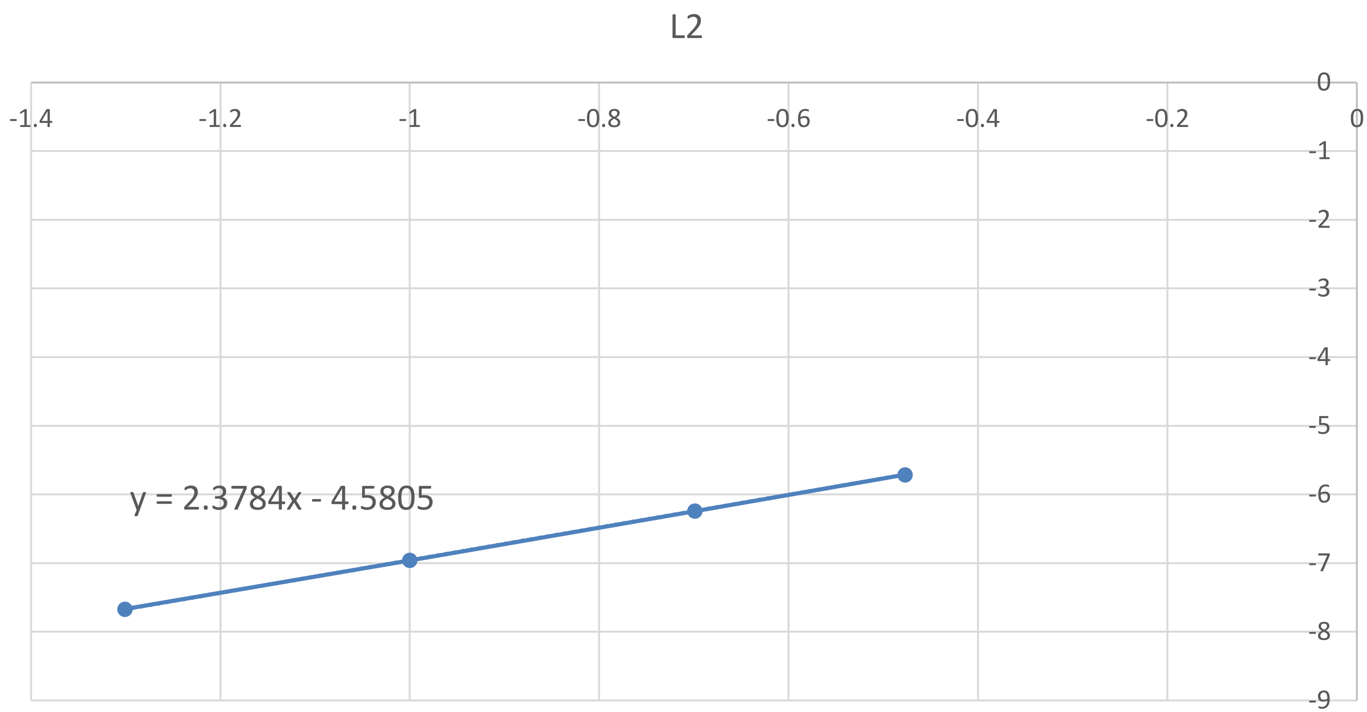
4*4 nodes:

L2 error = 1.126E-06

11*11 nodes:

L2 error = 5.393E-08

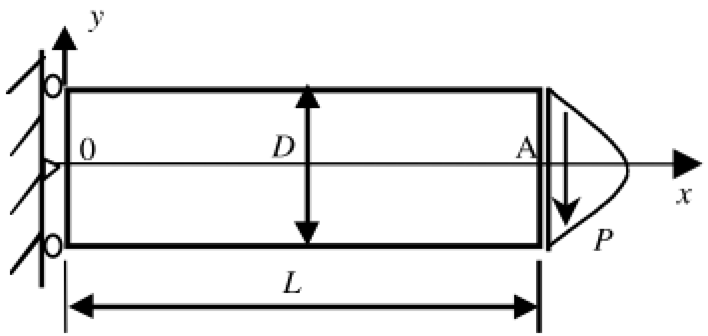
Results—L2 norm



Numerical test

Cantilever beam:

2



Exact solution:

$$u=\frac{Py}{6EI}[(6L-3x)x+(2+\nu)(y^2-\frac{D^2}{4})];$$

$$v=-\frac{P}{6EI}[3\nu y^2(L-x)+(4+5\nu)\frac{D^2x}{4}+(3L-x)x^2];$$

$$\sigma_{xx}=\frac{P(L-x)y}{I};$$

$$\sigma_{yy}=0;$$

$$\sigma_{xy}=-\frac{P}{2I}(\frac{D^2}{4}-y^2).$$

Essential BC:

$$\bar{\mathbf{u}}=\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\frac{Py}{6EI}(2+\nu)(y^2-\frac{D^2}{4})\\-\frac{P\nu L}{2EI}y^2\end{pmatrix}$$

Natural BC:

$$\bar{\mathbf{t}}=\begin{pmatrix}t_x\\t_y\end{pmatrix}=\mathbf{n}\boldsymbol{\sigma}=\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{pmatrix}=\begin{pmatrix}0\\-\frac{P}{2I}(\frac{D^2}{4}-y^2)\end{pmatrix}$$

$$I=\frac{D^3}{12};$$

$$P=1000\text{N};$$

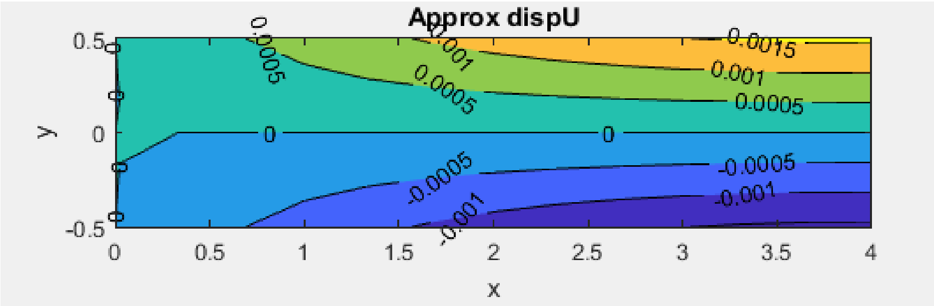
$$E=3\text{E}7\text{N/m}^2;$$

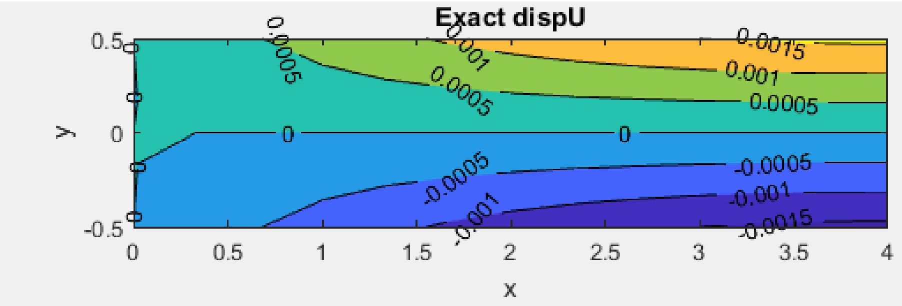
$$\nu=0.3;$$

$$D=1m;$$

$$L=4m.$$

Results





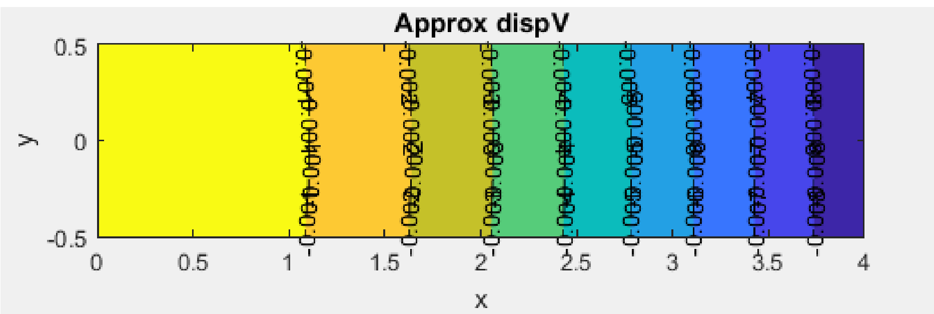
Displacement in X direction (13*4 nodes)

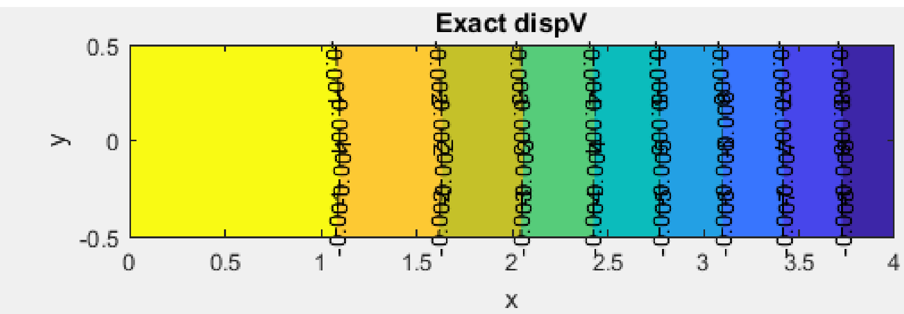
13*4 nodes:

L2 error = 5.583E-06

21*6 nodes:

L2 error = 2.368E-06





Displacement in Y direction (13*4 nodes)

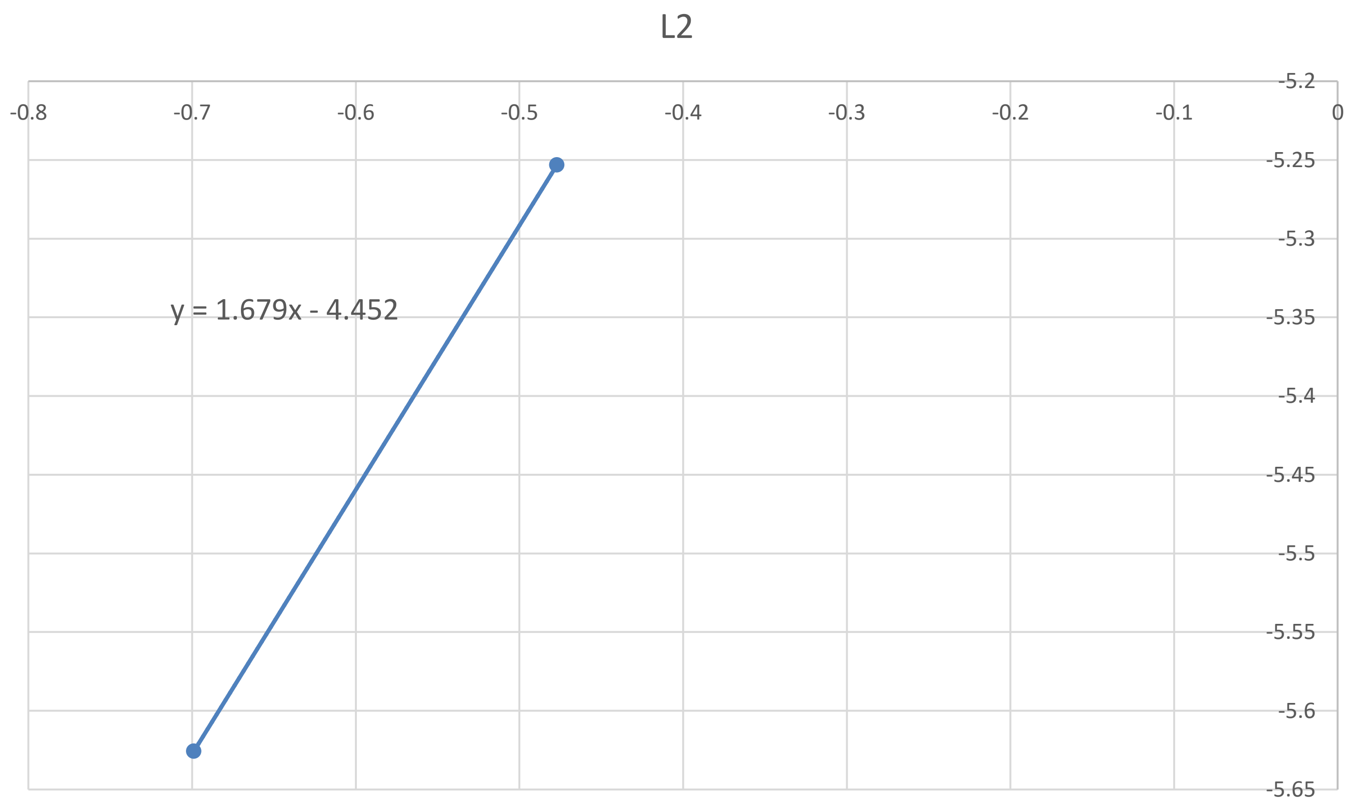
13*4 nodes:

L2 error = 4.417E-05

21*6 nodes:

L2 error = 1.622E-05

Results—L2 norm



Results – to get strain/stress

$$\mathbf{u}(\mathbf{x}) = \sum \boldsymbol{\Psi}_I(\mathbf{x})\mathbf{d}_I = [\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \ldots, \boldsymbol{\Psi}_n]\boldsymbol{\Lambda}_{n*n}^{-1}[\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_3]$$

$$= \sum_1^{np} \mathbf{N}_I \mathbf{u}_I$$

$\mathbf{u}_I, \mathbf{d}_I$: 2*1 vector;

$\boldsymbol{\Psi}_I, \mathbf{N}_I$: 2*2 matrix;

$\mathbf{L}, \mathbf{B_I}$: 3*2 matrix.

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{L}\mathbf{u}(\mathbf{x}) = \mathbf{L}\sum_1^{np} \mathbf{N}_I \mathbf{u}_I = \sum_1^{np} \mathbf{L}\mathbf{N}_I \mathbf{u}_I = \sum_1^{np} \mathbf{B}_I \mathbf{u}_I$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{x})$$

Indices derivative

$$\sigma_{ij,j} + b_i = 0 \;\Rightarrow$$

$$\int_\Omega w_i(\sigma_{ij,j} + b_i) d\Omega = 0 \;\Rightarrow$$

$$\int_\Omega w_i \sigma_{ij,j} d\Omega + \int_\Omega w_i b_i d\Omega = 0 \;\Rightarrow$$

$$\int_\Omega \nabla \cdot (w_i \sigma_{ij}) d\Omega - \int_\Omega w_{i,j} \cdot \sigma_{ij} d\Omega + \int_\Omega w_i b_i d\Omega = 0 \;\Rightarrow$$

$$\int_\Gamma (w_i \sigma_{ij}) n_j d\Gamma - \int_\Omega w_{i,j} \cdot \sigma_{ij} d\Omega + \int_\Omega w_i b_i d\Omega = 0 \;\Rightarrow$$

$$\int_\Omega w_{i,j} \cdot \sigma_{ij} d\Omega = \int_\Gamma (w_i \sigma_{ij}) n_j d\Gamma + \int_\Omega w_i b_i d\Omega$$

$$\Leftrightarrow$$

$$\int_\Omega \nabla \mathbf{W} : \boldsymbol{\sigma} d\Omega = \int_{\Gamma_t} \mathbf{W} \cdot \bar{\mathbf{t}} d\Gamma + \int_\Omega \mathbf{W} \cdot \mathbf{b} d\Omega \text{ (W)}$$