The Astrophysical Journal, 625:L123–L126, 2005 June 1

DETECTING THE STOCHASTIC GRAVITATIONAL WAVE BACKGROUND USING PULSAR TIMING

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Received 2005 February 13; accepted 2005 April 19; published 2005 May 9

ABSTRACT

The direct detection of gravitational waves is a major goal of current astrophysics. We provide details of a new method for detecting a stochastic background of gravitational waves using pulsar timing data. Our results show that regular timing observations of 40 pulsars each with a timing accuracy of 100 ns will be able to make a direct detection of the predicted stochastic background from coalescing black holes within 5 years. With an improved prewhitening algorithm, or if the background is at the upper end of the predicted range, a significant detection should be possible with only 20 pulsars.

Subject headings: gravitational waves — pulsars: general

1. INTRODUCTION

Analysis of pulsar pulse time-of-arrival (TOA) data shows that pulsars, especially millisecond pulsars (MSPs), are very stable clocks. The measurement of timing residuals, that is, the differences between observed and predicted TOAs, enables the direct detection of gravitational waves (GWs; Estabrook & Wahlquist 1975; Sazhin 1978; Detweiler 1979). The fluctuating TOAs induced by a GW will be correlated between widely spaced pulsars. Hellings & Downs (1983) attempted to detect this correlation by cross-correlating the time derivative of the timing residuals for multiple pulsars. In our work, we have developed a similar cross-correlation technique and have, for the first time, a fully analyzed method for combining multiple pulsar observations in order to make an unambiguous detection of a GW background. We emphasize that, in contrast to Hellings & Downs (1983), our method is based entirely on the measured residuals.

Only the effects of a stochastic background of GWs are considered. Astrophysical sources of such a background include cosmological processes (e.g., Maggiore 2000) and coalescing massive black hole binary systems (Jaffe & Backer 2003; Wyithe & Loeb 2003; Enoki et al. 2004). We show that a direct detection of a stochastic GW background is possible using pulsar timing observations and that the significance of the detection depends on the number of pulsars observed, the rms timing noise achieved, the number of observations, and the power spectrum of the measured timing residuals. The results are applied to the case of the Parkes pulsar timing array (PPTA[4]).

In the next section, the analysis technique is described. In § 3 the significance of detecting a given stochastic background using this method is estimated. The effects of prefiltering the residual time series are also discussed. The results are summarized in § 4.

2. DETECTION TECHNIQUE

As a first step, the power spectra of the pulsar timing residuals are analyzed. If they all show a very red power-law spectrum, the residuals may be dominated by a GW background. However, such red spectra can also be due to period noise intrinsic to the pulsar, uncorrected interstellar delays, inaccuracies in the solar system ephemeris, or variations in terrestrial time standards (e.g., Foster & Backer 1990). A GW background produces a unique signature in the timing residuals that can only be confirmed by observing correlated signals between multiple pulsars widely distributed on the sky.

The presence of a stochastic GW background will cause the pulse TOAs to fluctuate randomly, but these fluctuations will be correlated between different pulsars. In order to detect the presence of a GW background, one needs to first calculate the correlation coefficient between the observed timing residuals of each pair of observed pulsars:

$$r(\theta) = \frac{1}{N}\sum_{i=0}^{N-1} R(t_i, \hat{\boldsymbol{k}}_1)R(t_i, \hat{\boldsymbol{k}}_2), \tag{1}$$

where $R(t_i, \hat{\boldsymbol{k}})$ is the time series of N pulsar residuals sampled regularly in time, $\hat{\boldsymbol{k}}_1$ and $\hat{\boldsymbol{k}}_2$ are the directions to the two pulsars, and $\cos(\theta) = \hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2$. It will be assumed that R has zero mean and that each pulsar pair has a unique angular separation; $r(\theta)$ is written only as a function of the angular separation since the GW background is expected to be isotropic. In the presence of an isotropic GW background, the ensemble-averaged value of $r(\theta)$ is given by[5]

$$\langle r(\theta)\rangle = \sigma_g^2 \zeta(\theta), \tag{2}$$

$$\zeta(\theta) = \frac{3}{2}x\log x - \frac{x}{4} + \frac{1}{2} + \frac{1}{2}\delta(x), \tag{3}$$

where $x = [1-\cos(\theta)]/2$, σ_g is the rms of the timing residuals induced by the stochastic GW background, and $\delta(x)$ equals 1 for $x = 0$ and 0 otherwise. The detection technique proposed here simply looks for the presence of the function $\zeta(\theta)$ in the measured correlation coefficients $r(\theta)$.

Since one cannot perform the ensemble average in practice, the measured statistic, $r(\theta)$, will be of the form $r(\theta) = \langle r(\theta)\rangle + \Delta r(\theta)$, where $\Delta r(\theta)$ is a "noise term." Since $r(\theta)$ is calculated by summing over a large ($\geq$20) number of data points, $\Delta r(\theta)$ will be a Gaussian random variable for practical purposes. The optimal way to detect the presence of a known functional form within random data is to calculate the corre-

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[4] See http://www.atnf.csiro.au/research/pulsar/psrtime.

[5] For an outline of the calculation of ζ, see Hellings & Downs (1983).

lation between the data and the known function. Hence, to detect the presence of the GW background, one needs to calculate

$$\rho = \frac{(1/N_p)\sum_{i=0}^{N_p-1}[r(\theta_i)-\bar{r}][\zeta(\theta_i)-\bar{\zeta}]}{\sigma_r\sigma_\zeta}, \tag{4}$$

where θ_i is the angle between the ith pair of pulsars and N_p is the number of distinct pairs of pulsars; $\bar{r}$ and $\bar{\zeta}$ indicate the mean values over all pairs of pulsars, and σ_r^2 and σ_ζ^2 are the variances of r and ζ, respectively. For M pulsars, $N_p = M(M-1)/2$.

From the definition of $r(\theta)$ and equation (4), one can show that the expected value of ρ is approximately

$$\rho \approx \frac{\sigma_g^2\sigma_\zeta}{\sqrt{\sigma_g^4\sigma_\zeta^2+\sigma_{\Delta r}^2}}, \tag{5}$$

$$\sigma_{\Delta r}^2 = \frac{1}{N_p}\sum_{i=0}^{Np-1}\langle[r(\theta_i)-\langle r(\theta_i)\rangle]^2\rangle. \tag{6}$$

For the case in which there is no correlation in the data, the statistics of ρ will be Gaussian with zero mean and variance given by $\sigma_\rho^2 = 1/N_p = 2/(M^2-M)$. Hence, the significance of a measured value of ρ may be defined as $S = \rho/\sigma_\rho$. The probability of measuring a correlation greater than or equal to ρ when no actual correlation is present is given by $[1-\mathrm{erf}\,(S/\sqrt{2})]/2$.

3. ESTIMATING THE DETECTION SIGNIFICANCE

In order to estimate the expected detection significance, S, one needs to estimate σ_g and $\sigma_{\Delta r}$. It is assumed that the timing residuals, $R(t,\hat{\boldsymbol{k}})$, are stationary Gaussian random variables that are sampled at regular intervals denoted by Δt. It is also assumed that terms proportional to t and t^2 (i.e., the period and period-derivative terms) have been subtracted from $R(t,\hat{\boldsymbol{k}})$.

The spacetime fluctuations induced by a stochastic GW background are described by a quantity known as the characteristic strain spectrum denoted by h_c (e.g., Maggiore 2000). Models of the GW background propose a power-law dependence between h_c and the GW frequency, f: $h_c(f) = Af^\alpha$ (Jaffe & Backer 2003; Wyithe & Loeb 2003; Maggiore 2000; Enoki et al. 2004). Using this form of the characteristic strain spectrum, the power spectrum of the induced residuals is given by $P_R(f) = \langle|\tilde{R}(f)|^2\rangle = (A^2/4\pi^2)f^{2\alpha-3}$, where $\tilde{R}(f)$ is the Fourier transform of $R(t)$. Given $P_R(f)$, the total rms fluctuation induced by the stochastic GW background is given by

$$\sigma_g^2 = \int_{f_l}^{f_h} P_R(f)df \tag{7}$$

$$= \frac{A^2}{2\pi^2(2-2\alpha)}(f_l^{2\alpha-2}-f_h^{2\alpha-2}), \tag{8}$$

where f_l is the lowest detectable frequency given by $1/T$ and f_h is the highest detectable frequency typically given by $\frac{1}{2}\Delta t$. T is the total time span of the data set. Since $\alpha<0$ for backgrounds of interest (Maggiore 2000), the term containing f_h is negligible.

Estimating $\sigma_{\Delta r}$ is slightly more complicated. To take into account the effects of subtracting linear and quadratic terms from the residuals, a semianalytic approach was adopted. As outlined below, an estimate for $\sigma_{\Delta r}$ is made analytically, but with one free parameter β. For a given value of β, S is calculated as a function of A for a given set of pulsars and timing parameters. $S(A)$ is compared to Monte Carlo simulations in order to determine the correct value of β. This showed that the value of β is insensitive to the values α, N, M, σ_g, and the rms residual noise level.

Using equation (1) together with the assumption that $R(t,\hat{\boldsymbol{k}})$ is a Gaussian random variable, one can show that

$$\sigma_{\Delta r}^2 \approx \overline{\frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}c_{ij}(\hat{\boldsymbol{k}}_1)c_{ij}(\hat{\boldsymbol{k}}_2)}, \tag{9}$$

where $c_{ij}(\hat{\boldsymbol{k}}) = \langle R(t+i\Delta t,\hat{\boldsymbol{k}})R(t+j\Delta t,\hat{\boldsymbol{k}})\rangle$. The bar above equation (9) represents an average over all pairs of pulsars. As the autocorrelation function and the power spectrum are Fourier transforms of one another, one can estimate $\sigma_{\Delta r}^2$ from the expected power spectrum of the residuals. The statistics of the residuals are assumed to be stationary so that $c_{ij}(\hat{\boldsymbol{k}})$ depends only on $i-j$. The expected discrete power spectrum of $R(t,\hat{\boldsymbol{k}})$, which includes both a GW component and a white-noise component, is given by

$$P_d(i,\hat{\boldsymbol{k}}) = \begin{cases} P_g(i)+2\sigma_n(\hat{\boldsymbol{k}})^2/N & \text{for } i>0, \\ 0 & \text{for } i=0, \end{cases} \tag{10}$$

where $P_g(i)$ is the discrete power spectrum of the GW-induced timing residuals, i is the discrete frequency bin number corresponding to frequency i/T, and $\sigma_n(\hat{\boldsymbol{k}})$ is the rms value of the residual fluctuations caused by all non-GW sources for the pulsar in the $\hat{\boldsymbol{k}}$-direction. It is assumed that all noise sources have a flat spectrum. This assumption is consistent with most observations of MSPs. $P_g(i)$ is given by

$$P_g(i) = \frac{A^2T^{2-2\alpha}}{(2\pi)^2(2-2\alpha)}m(i), \tag{11}$$

where

$$m = \begin{cases} 0 & \text{for } i=0, \\ \beta^{2\alpha-2}-(1.5)^{2\alpha-2} & \text{for } i=1, \\ (i-0.5)^{2\alpha-2}-(i+0.5)^{2\alpha-2} & \text{for } i>1. \end{cases}$$

Effectively, β is the lowest frequency used to calculate the correlation function c_{ij}. Monte Carlo simulations show that $\beta = 0.97$.

For the case in which all pulsars have the same noise level, the detection significance becomes

$$S = \sqrt{\frac{M(M-1)/2}{1+[\chi(1+\overline{\zeta^2})+2(\sigma_n/\sigma_g)^2+(\sigma_n/\sigma_g)^4]/N\sigma_\zeta^2}}. \tag{12}$$

Here $\chi = (1/\sigma_g^4 N)\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}c_{g_{ij}}^2$, where $c_{g_{ij}}$ is the correlation function for the GW-induced component of the timing residuals; χ is a measure of the "whiteness" of the residuals.

The solid curve in Figure 1*a* plots the detection significance versus power-law amplitude for $\alpha = -\frac{2}{3}$, the expected value for a background generated by an ensemble of supermassive black hole binaries (Jaffe & Backer 2003). This spectral index

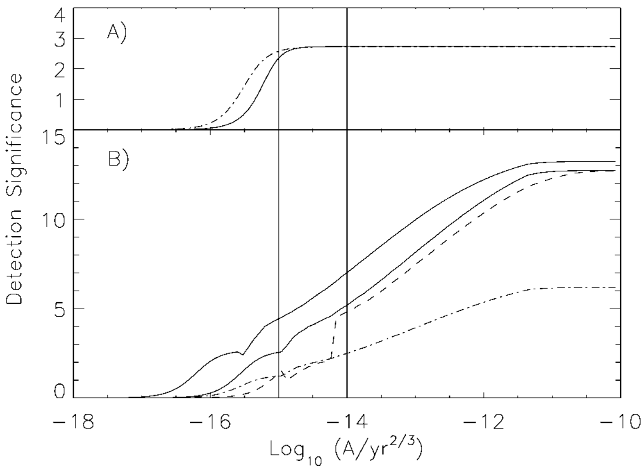

FIG. 1.—Detection significance, S, vs. the logarithm of the amplitude A of the characteristic strain amplitude $h_c(f)$. The strain spectral index $\alpha = -\frac{2}{3}$, corresponding to an astrophysical background of GWs generated by supermassive binary black holes. The vertical lines bound the values of A expected by models of the GW background (Jaffe & Backer 2003; Wyithe & Loeb 2003; Enoki et al. 2004). In panel a, the curves were calculated with 20 pulsars each with rms residual noise fluctuations of 100 ns. The solid line corresponds to the simple correlation technique. The dashed line includes the effect of low-pass filtering. Panel b shows the effects of the whitening technique. The solid line was calculated with the same parameters as in panel a. The remaining curves were generated using different noise levels and a number of pulsars. See text for further details.

together with the removal of the linear and quadratic terms from R effectively makes $\chi = 0.6N$. The parameters are set as follows: $N = 250$, $M = 20$, $\sigma_n = 100$ ns, and $T = 5$ yr. These values are the target values for the PPTA (Hobbs 2005). Note that the significance saturates for high values of A. This effect can easily be seen in equation (12) since all terms of the form σ_n/σ_g go to zero as σ_g gets very large. This saturation is due to the "self-noise" associated with the stochastic nature of the background, and its asymptotic value is independent of σ_n. The roll-off at low values of A occurs at $\sigma_g = \sigma_n$.

Since the power spectrum of the GW-induced timing residuals will be dominated by low frequencies, one can apply a low-pass filter to each of the residual time series before correlating. This is similar to fitting a low-order polynomial to the data and then correlating the resulting fits. To estimate the significance for this technique, one evaluates σ_g^2 and $\sigma_{\Delta r}^2$ using equations (8) and (10), but with a high-frequency cutoff $f_{\rm hc}$. For purposes of this discussion, $f_{\rm hc}$ was set to $4/T$. The dot-dashed line in Figure 1a shows the effect of using a low-pass filter on the residuals. All the other parameters are the same as those used to generate the solid line. Low-pass filtering effectively reduces σ_n while keeping σ_g relatively unchanged. It also has the effect of increasing χ/N when P_g is a red power-law spectrum. Hence, low-pass filtering will not increase the maximum attainable significance, but it will lower the value of σ_g where the roll-off starts to occur.

We next try to increase the maximum achievable significance. This method involves both low-pass filtering and a technique called "whitening." When correlating two time series that each have a steep power-law spectrum, an optimal signal-to-noise ratio is obtained if filters are applied to give each time series a flat spectrum before correlation. This will act to reduce χ in equation (12). In practice, starting from the lowest nonzero frequency bin, we give each Fourier component with significant power equal amplitude and set higher components to zero. In this way, we are correlating only that part of the signal that has a high signal-to-noise ratio and adjusting the power spectrum to optimize the measurement of the correlation function.

P_d and σ_g need to be calculated in order to estimate S using the whitening method. After whitening, $P_d(i, \hat{\boldsymbol{k}}) = 2\sigma_d(\hat{\boldsymbol{k}})^2/N$, where $\sigma_d(\hat{\boldsymbol{k}})$ is the rms of the residual data from the $\hat{\boldsymbol{k}}$th pulsar. The whitening also affects σ_g. In the general case in which the pulsars have different noise levels, σ_g will depend on the pulsar. The expression for ρ then becomes

$$\rho \approx \frac{(\overline{\sigma_g^2\zeta^2} - \overline{\sigma_g^2\zeta}\,\overline{\zeta})/\sigma_\zeta}{\sqrt{[\overline{\sigma_g^4\zeta^2} - (\overline{\sigma_g^2\zeta})^2] + \sigma_{\Delta r}^2}}, \tag{13}$$

with $\sigma_g(\theta)^2$ given by

$$\sigma_g(\theta)^2 = \frac{2}{N}\sigma_d(\hat{\boldsymbol{k}}_1)\sigma_d(\hat{\boldsymbol{k}}_2) \times \sqrt{\left[\sum_{i=0}^{N_{\rm max}} P_g(i)/P_d(i,\hat{\boldsymbol{k}}_1)\right]\left[\sum_{i=0}^{N_{\rm max}} P_g(i)/P_d(i,\hat{\boldsymbol{k}}_2)\right]}, \tag{14}$$

where $N_{\rm max}$ is the largest frequency bin used based on the criterion discussed above. The solid line in Figure 1b plots the significance using the whitening versus A. The same parameters were used for this case as in the previous cases.

The above discussion assumes that the noise levels were the same for all pulsars. Next, the case in which the pulsars have different noise levels will be considered. All curves in Figure 1b were generated using the whitening technique. Unless specified, 250 observations were taken on each pulsar over 5 years. The dashed line corresponds to 20 pulsars, 10 with $\sigma_n = 100$ ns and 10 with $\sigma_n = 500$ ns. The dot-dashed line has 10 pulsars each with $\sigma_n = 100$ ns and 500 observations. The triple–dot-dashed line has 20 pulsars with $\sigma_n = 100$ ns and 500 observations over 10 years.

When given a choice between observing a large sample of pulsars with different noise levels and observing only those pulsars with the lowest noise levels but for a longer time, the above curves demonstrate that one should actually observe the larger sample of pulsars. This is not a general statement, but rather it depends on the level of the GW background and the noise level. However, the levels chosen above are relevant to the PPTA (Jaffe & Backer 2003; Wyithe & Loeb 2003; Hobbs 2005). Note that for large M, the significance scales as M. Hence, doubling the number of pulsars will double the expected significance.

4. SUMMARY

The main goal of this work is to determine the effectiveness of an array of pulsars, such as the PPTA, for detecting a stochastic background of GWs. Using a simple correlation technique, the detection significance was calculated given the number of pulsars, the location of each pulsar, the TOA precision, the number of observations, the total time span of the data, and the amplitude and power-law index of the GW background. For the case in which all pulsars have the same white-noise spectrum, equation (12) may be used to calculate the detection significance. For the case of the PPTA, it was found that the maximum achievable significance will be about 3 for a background with spectral index $\alpha = -\frac{2}{3}$ and $A \sim 10^{-15}$, which is the expected level of the GW background from an ensemble of supermassive binary black holes in galaxies (Jaffe & Backer 2003; Wyithe & Loeb 2003; Enoki et al. 2004). Note that lowering the rms noise level will only decrease the minimum detectable value of A and not increase the maximum attainable significance.

Low-pass filtering the timing residuals, or, equivalently, fitting low-order polynomials (i.e., cubic terms) to the residuals and correlating the coefficients, does not increase the maximum attainable significance. The significance level is increased by prewhitening of the timing residuals. Using whitening, it is estimated that the PPTA could obtain a detection significance greater then 4 for $A \geq 3 \times 10^{-15}\ \mathrm{yr}^{-2/3}$ provided that efficient whitening filters can be designed and implemented. This is an area of further study and will be addressed in a later paper. With the same qualifiers, increasing the total time span of the PPTA to 10 years would yield a significance greater than 4 for $A \geq 10^{-15}\ \mathrm{yr}^{-2/3}$. Since the significance scales as the number of pulsars, doubling that number will double the expected significance. Hence, using the simple correlation technique described here without any prefiltering, a stochastic background with $A \geq 10^{-15}\ \mathrm{yr}^{-2/3}$ will be detectable at a significance of about 5.5 using 40 pulsars observed 250 times over 5 years and each having 100 ns timing precision.

Part of this research was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration and funded through the internal Research and Technology Development program. The authors wish to thank Russell Edwards for useful discussions.

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