Day 2: Gravitational Wave Sources RESULTS

IPTA 2014 Student Workshop
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Question 1: h should vary as $1/r$, this way $L \sim r^2 h^2$ is non-zero at infinity.

Question 2: G/c^4 gives you a dimensionless h.

Question 3: G/c^4 is extremely small and GWs are extremely weak. You would need an astrophysical source with large mass and size that vary rapidly with time. A binary system is a good example of that.

Question 4: According to Kepler's third law, the angular orbital frequency Ω is defined by:

$$\Omega^2 = (2\pi/P)^2 = \frac{GM}{a^3} \tag{1}$$

which is true for any eccentricity e.

Question 5: We have:

$$h \sim \frac{1}{r}\frac{\partial^2(MR^2)}{\partial t^2} \sim \frac{1}{r}\Omega^2 \mu a^2 \sim \frac{1}{r} GM\mu/a \tag{2}$$

Adding in the factor G/c^4 we get the needed equation.

Question 6:

$$h \sim \left(\frac{G}{c^4}\right)\frac{1}{r}\mu GM\left(\frac{\Omega^2}{GM}\right)^{1/3} \sim \left(\frac{G}{c^4}\right)\frac{1}{r}\mu\Omega^{2/3}(GM)^{2/3} \tag{3}$$

Question 7:

$$\mu G(GM)^{2/3} = G^{5/3}\mathcal{M}^{5/3} \text{ and } \Omega^{2/3} = \pi^{2/3} f^{2/3} \tag{4}$$

so

$$h \sim \frac{(G\mathcal{M})^{5/3}}{rc^4}(\pi f)^{2/3} \tag{5}$$

Question 8: A rough estimate for the maximum frequency would be when the two black holes are "touching" each other, so that $a = 2R_s$. Then:

$$f_{\max} = \frac{1}{\pi}\left(\frac{GM}{8R_s^3}\right)^{1/2} \tag{6}$$

Question 9: Conservation of energy: Energy at surface of star (kinetic energy, zero gravitational potential) = Energy brought to infinity (zero kinetic energy, gravitational potential): $1/2\, mv^2 = GM_{BH}m/r$ so $v_{\text{escape}} = \sqrt{2GM_{BH}/r}$.

Question 10: The escape velocity at the surface of a black hole (the event horizon) is c. so $c^2 = 2GM_{BH}/R_s$ and $R_s = 2GM_{BH}/c^2$.

Question 11:

$$f_{\max} = \frac{1}{\pi}\left(\frac{c^6}{8G^2M^2}\right)^{1/2} = \frac{c^3}{2\sqrt{2}\pi GM} \tag{7}$$

Question 12: $\mathcal{M}^{5/3} = \mu M^{2/3} = \frac{M}{4}M^{2/3} = \frac{1}{4}M^{5/3}$

$$h = \frac{1}{2}\left(\frac{2}{5}\right)^{1/2}\frac{(GM)^{5/3}}{rc^4}\frac{c^2}{(GM)^{2/3}} = \frac{1}{2}\left(\frac{2}{5}\right)^{1/2}\frac{(GM)}{rc^2} \tag{8}$$

Question 13:
1) $h \sim 10^{-21}$, $f_{max} = 10^3$ Hz
2) $h \sim 10^{-18}$, $f_{max} = 10^{-2}$ Hz
3) $h \sim 10^{-14}$, $f_{max} = 10^{-5}$ Hz

Question 14: PTAs are in some way similar to LIGO, just a much longer interferometer arm (4 km at LIGO, distance Earth - pulsar for PTAs). The PTAs are therefore sensitive to longer wavelengths or shorter frequencies (case 3 for PTAs vs case 1 for LIGO).

Question 15: Flux $= dE/(dAdt)$ is in units of mass/time^3. We need a pre-factor of unit mass/time, so the pre-factor is c^3/G.

Question 16: Flux $= \frac{\pi}{4}\frac{c^3}{G}f^2h^2$ and so:

$$dE/dt = \pi^2 r^2 \frac{c^3}{G}\left(\frac{32}{5}\frac{(G\mathcal{M})^{10/3}}{r^2c^8}\right)\pi^{4/3}f^{10/3} = \frac{32}{5}\frac{\pi^{10/3}G^{7/3}}{c^5}(\mathcal{M}f)^{10/3} \tag{9}$$

$$dE/dt = \frac{32}{5}\frac{\mu^2G^{7/3}M^{4/3}}{c^5}\pi^{10/3}\frac{1}{\pi^{10/3}}(GM)^{5/3}\frac{1}{a^5} \tag{10}$$

so we have:

$$dE/dt = \frac{32}{5}\frac{G^4}{c^5}\frac{\mu^2M^3}{a^5} \tag{11}$$

Question 17: Equating Equations 23 and 24, we obtain for each frequency (we use $df/f = df_r/f_r$):

$$\frac{\pi}{4}\frac{c^2}{G}f^2h_c^2(f) = \int_0^\infty N(z)\frac{1}{1+z}\frac{dE_{gw}}{df_r}f_r dz \tag{12}$$

therefore:

$$h_c^2(f) = \frac{4G}{\pi c^2}\frac{1}{f^2}\int_0^\infty N(z)\frac{1}{1+z}\left(f_r\frac{dE_{gw}}{df_r}\right)|_{f_r=f(1+z)}dz \tag{13}$$

Question 18: We had previously found (Kepler's Third Law): $\Omega = \sqrt{\frac{GM}{a^3}}$, so that $f_r = \frac{\Omega}{\pi} = \frac{1}{\pi}\sqrt{\frac{GM}{a^3}}$ and:

$$a = \frac{(GM)^{1/3}}{(\pi f_r)^{2/3}} \tag{14}$$

and

$$\frac{da}{df_r} = -\frac{2}{3}\frac{(GM)^{1/3}}{\pi^{2/3}} f_r^{-5/3} \tag{15}$$

Question 19:

$$\frac{dE}{df_r} = \frac{dE}{dt}\left(\frac{1}{da/dt}\right)\frac{da}{df_r} = \left(\frac{32}{5}\frac{\pi^{10/3}G^{7/3}}{c^5}\left(\mathcal{M}f_r\right)^{10/3}\right)\left(-\frac{5}{64}\frac{c^5a^3}{G^3}\frac{1}{\mu M^2}\right)\left(-\frac{2}{3}\frac{(GM)^{1/3}}{\pi^{2/3}}f_r^{-5/3}\right) \tag{16}$$

with:

$$a^3 = \frac{GM}{(\pi f_r)^2} \text{ and } \frac{1}{\mu M^2} = M^{-4/3}\mathcal{M}^{-5/3} \tag{17}$$

so that:

$$\frac{dE}{df_r} = \frac{1}{3}(\pi G)^{2/3}\mathcal{M}^{5/3}f_r^{-1/3} \tag{18}$$

Question 20: Plugging the latest result into Equation 25:

$$h_c^2(f) = \frac{4G}{\pi c^2}\frac{1}{f^2}\int_0^\infty N(z)\frac{1}{1+z}\left(\frac{1}{3}(\pi G)^{2/3}\mathcal{M}^{5/3}f_r^{2/3}\right)dz \tag{19}$$

$$h_c^2(f) = \frac{4(G\mathcal{M})^{5/3}}{3\pi^{1/3}c^2}\frac{1}{f^2}\int_0^\infty N(z)\frac{1}{1+z}\left(f^{2/3}(1+z)^{2/3}\right)dz \tag{20}$$

so that:

$$h_c^2(f) = \frac{4(G\mathcal{M})^{5/3}}{3\pi^{1/3}c^2}f^{-4/3}\int_0^\infty N(z)\frac{1}{(1+z)^{1/3}}dz \tag{21}$$

Therefore:

$$h_c(f) = Af^{-2/3} \tag{22}$$

with:

$$A = \left[\frac{4(G\mathcal{M})^{5/3}}{3\pi^{1/3}c^2}\int_0^\infty N(z)\frac{1}{(1+z)^{1/3}}dz\right]^{1/2} \tag{23}$$