Day 2: Gravitational Wave Sources

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In today's activity, we study the emission of gravitational waves from individual black hole binary sources, and from a stochastic background of supermassive black hole binaries.

1. Gravitational wave as spacetime perturbation

In Einstein's theory of General Relativity, we characterize the warping of spacetime using the metric tensor $g_{\mu\nu}$. Consider two events A and B determined by four coordinates in a four-dimensional spacetime. If these events are very close to each other, the four-dimensional "distance" ds between these two events is given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu , \tag{1}$$

where greek indices μ and ν run over all four spacetime coordinates. Here we use the "Einstein summation notation": repeated indices are summed over all coordinates, so that:

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + \text{cross terms such as}\, g_{01} dx^0 dx^1 \text{etc.} \tag{2}$$

The "0" index refers to time, while 1, 2 and 3 represent the usual three-dimensional space. In Cartesian coordinates, these are (c t), x, y and z. Therefore in Cartesian coordinates, the previous equation can be rewritten as:

$$ds^2 = g_{00}\,(c\,dt)^2 + g_{11}\,dx^2 + g_{22}\,dy^2 + g_{33}\,dz^2 + \text{cross terms} \tag{3}$$

The metric tensor, a rank-2 tensor, can be represented as a 2 $\times$ 2 matrix:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \tag{4}$$

In flat spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

so that

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(c\,dt)^2 + dx^2 + dy^2 + dx^2 \tag{6}$$

Note that for a photon traveling at the speed of light, $ds = 0$. When we consider gravitational waves, we separate the spacetime into a component that is time-independent (the background spacetime) and a component that varies with time (the gravitational waves). If we are far enough from any mass, the background spacetime is flat, and we can write the metric as:

$$g_{\mu\nu} = \eta_{\mu\nu}(\text{flat spacetime}) + h_{\mu\nu}(\text{gravitational waves}), \text{where } h_{\mu\nu} \ll 1. \tag{7}$$

Since $h_{\mu\nu}$ is small, we only need to consider its linear contribution. The Einstein equations, which lead to the equations of motion of matter in curved spacetime, can then be simplified by the process of "linearization". The linearization of Einstein's equations leads to a wave equation:

$$\Box \overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \tag{8}$$

where $T_{\mu\nu}$ is the stress-energy tensor (representing matter). The solution is a wave traveling at the speed of light. We find that the waves are transverse and that, as in the case of electromagnetic waves, there are two polarizations. Unlike the up-down and left-right polarizations of an EM wave, the GW polarizations are represented by the effect they have on a circle of test masses oriented perpendicular to the direction of the wave. These are called the "+" and "×" polarizations. In this case, the dimensionless amplitude h of a metric perturbation represents the ratio of the "stretched part" of the circle to the size of the original circle. In the next section, we search for orders of magnitude of the GW amplitude, h.

2. Gravitational wave amplitude: order of magnitude

In this section, we try to estimate an approximate expression for the dimensionless amplitude h of a metric perturbation, a distance r from a source. The lowest-order radiation is quadrupolar and therefore depends on the mass quadrupole moment $I_{ij} = \int \rho r_i r_j d^3x$. The quadrupole moment has dimensions MR^2 (M is some mass and R a characteristic dimension). The amplitude should also be proportional to $1/r$ (so that energy is brought to infinity), so we have:

$$h \sim MR^2/r \tag{9}$$

We know that h is dimensionless, so something is missing. In GR we usually work with "natural units", where $G = c = 1$, meaning that mass, distance and time all have the same "unit". We also know that time derivatives need to be involved, since a static system cannot emit anything. Two time derivatives will cancel out the current units, so we have:

$$h \sim \frac{1}{r} \frac{\partial^2 (MR^2)}{\partial t^2} \tag{10}$$

To get back to physical units, we have to restore the factors of G and c.

Question 1: Find the correct factors of G and c in the above equation. Hint: if M is a mass, then GM/c^2 has units of distance, and GM/c^3 has units of time.

Moving on... Since G is small and c is large, this prefactor is absolutely tiny! This tells us that unless: M and R are large; the system is changing fast; and r is small, the metric perturbation is minuscule. What kinds of sources might emit the strongest gravitational waves? Binaries! Let us study the emission of GW waves in binaries in the next section.

2.1. GW sources: Keplerian orbits and circular motion in binaries

Let's consider binary systems with two masses m_1 and m_2. We can define the total mass $M = m_1 + m_2$ and the reduced mass $\mu = m_1 m_2 / M$. A system of two masses m_1 and m_2 in a bound orbit is equivalently described by a single mass μ orbiting in an external potential with semi-major axis a. According to Kepler's third law, the orbital frequency Ω is defined by:

$$\Omega^2 = (2\pi/P)^2 = \frac{GM}{a^3} \sim \frac{M}{a^3} \text{ in natural units} \tag{11}$$

Question 2: Approximating the mass quadrupole moment to be $I \sim MR^2 \sim \mu a^2$ and replacing $\partial^2/\partial t^2$ with Ω^2, show that:

$$h \sim \left(\frac{G}{c^4}\right) \frac{1}{r} \frac{\mu M}{a} \tag{12}$$

$$h \sim (GM)^{5/3} \frac{1}{c^4 r} (\pi f)^{2/3} \tag{13}$$

Therefore promising sources of gravitational waves are very massive systems with a varying mass quadrupole moment.

Question 2: Knowing that the gravitational wave frequency is twice the orbital frequency, derive the GW frequency: $f \sim \frac{c^3}{4\pi GM}$

Question 3: Find estimates for the amplitude h and frequency f of GWs for 1) a 10 solar mass binary at 100 Mpc 2) a 10^6 binary at 10 Gpc 3) a 10^9 binary at 1 Gpc

2.2. Precise derivation

Let's do this again in a more exact way. The GW amplitudes at a distance r from the source are, for each GW polarization:

$$h_+ = \frac{G^{5/3}}{c^4} \frac{1}{r} 2(1 + cos^2 i)(\pi f M)^{2/3} \mu cos(2\pi f t) \tag{14}$$

$$h_\times = \pm \frac{G^{5/3}}{c^4} \frac{1}{r} 4 cos i (\pi f M)^{2/3} \mu sin(2\pi f t) \tag{15}$$

Question: Averaging over the orbital period and the orientations of the binary orbital plane, find an exact expression for the averaged characteristic GW amplitude h(f, M, r).

3. Supermassive black hole binary population

In order to estimate the amplitude of a background of gravitational waves from supermassive black hole binaries, we also need to know how many such sources there are. This depends on 4 things:

- The galaxy merger rate
- The relation between SMBHs and their hosts
- The efficiency of SMBH coalescence following galaxy mergers
- when and how accretion is triggered during a merger event

4. Stochastic gravitational wave background

Estimating the amplitude of a background of gravitational waves from supermassive black hole binaries depends on:

- The amplitude of GW emission from a single binary, as seen in Section 1
- The SMBH binary population, as seen in Section 2

Consider a class of sources with differential number density $d^2n/dzdM$ emitting an energy spectrum $dE/dlnf$. The amplitude of the background of gravitational waves emitted by these sources is:

$$h_c^2(f) = \frac{4G}{\pi c^2 f^2} \int_0^\infty dz \int_0^\infty dM \frac{d^2n}{dzdM} \frac{1}{1+z} \frac{dE_{gw}}{dln f_r} \tag{16}$$

with:

$$\frac{d^2n}{dzdM} = \frac{d^3N}{dzdMdln f_r} \frac{dln f_r}{dt_r} \frac{dt_r}{dz} \frac{dz}{dV_c} \tag{17}$$

and:

$$\frac{dE}{dln f_r} = \frac{1}{1+z} \frac{dt_r}{dln f_r} \frac{\pi c^3}{4G} 4\pi r(z)^2 f_r^2 h^2 \tag{18}$$

Question 4: Using the last 3 equations, find that the amplitude of the GW background can be expressed as:

$$h_c^2(f) = \int_0^\infty dz \int_0^\infty dM \frac{d^3N}{dzdMdln f_r} h^2(f_r) \tag{19}$$

Question 5: For supermassive black hole binaries, we know $dN/dlnf \sim f^{-8/3}$. Derive the following:

$$h_c(f) = A\left(\frac{f}{yr^{-1}}\right)^{-2/3} \tag{20}$$

5. Black hole binary environment and GW emission

In a galaxy merger, the central SMBHs are initially inspiraling toward each other because of dynamical friction. The BHs are coupled to their environment, be it stellar-driven or gas-driven. This causes the BHs to lose energy and to get closer (more details on inspiral). Only when the BHs get within parsecs of each other does GW emission in the PTA range start to occur. However the eccentricity of the SMBH binary would greatly affect the expected GW emission signal. We usually assume circular orbits, while in fact the orbit could still have great eccentricity as the GW emission process starts.

That is, early on at large angular separations, GW do not dominate, and the angular separation at which GWs start to dominate depends on whether the BH binary is in a stellar or gaseous environment. Defining the GW timescale as:

$$t_{GW} = 7.84\times10^7\mathrm{yr}\, M_8 q_s^{-1} a_3^4 F(e)^{-1}, F(e) = (1-e^2)^{-7/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \tag{21}$$

where $q_s = 4q/(1+q^2)$, $M_8 = M/10^8 M_\odot$, and a_3 is the binary separation in units of $10^3 R_s$ with R_s the Schwarzschild radius ($R_s = 2GM/c^2$). In a gas-rich environment, the migration timescale is:

$$t_m = 2.09\times10^6\mathrm{yr}\,\alpha_{0.3}^{-1/2}\left(\frac{\dot{m}_{0.3}}{\epsilon_{0.1}}\right)^{5/8} M_8^{3/4} q_s^{3/8}\delta_a(e)^{7/8} a_3^{7/8} \tag{22}$$

In a star-dominated system, the MBHB hardening timescale is given by:

$$t_h = 2.89\times10^6\mathrm{yr}\,\sigma_{100} M_8^{-1}\rho_5^{-1} a_3^{-1} H_{15}^{-1} \tag{23}$$

Question: Find the characteristic angular separation a at which GW emission takes over in the binary evolution for a gas-rich environment.

Question: Find the characteristic angular separation a at which GW emission takes over in the binary evolution in a star-dominated environment.