Solve the equations for $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ and $k_1, k_2, k_3 \in \mathbb{Z}$, here $v$ is a fix parameter..

$$\begin{cases} v = 3\xi_1^2 - \frac{k_1^2}{\xi_1^2} \\ v = 3\xi_2^2 - \frac{k_2^2}{\xi_2^2} \\ v = 3\xi_3^2 - \frac{k_3^2}{\xi_3^2} \\ (\xi_1+\xi_2+\xi_3)^3 - \xi_1^3 - \xi_2^3 - \xi_3^3 + \frac{(k_1+k_2+k_3)^2}{\xi_1+\xi_2+\xi_3} - \frac{k_1^2}{\xi_1} - \frac{k_2^2}{\xi_2} - \frac{k_3^2}{\xi_3} = 0 \end{cases}$$

$$3\left(\xi_1+\xi_2+\xi_3\right)^2 - \frac{\left(k_1+k_2+k_3\right)^2}{\left(\xi_1+\xi_2+\xi_3\right)^2} = v + 12\frac{\left(\xi_1+\xi_2\right)\left(\xi_2+\xi_3\right)\left(\xi_3+\xi_1\right)}{\xi_1+\xi_2+\xi_3}$$

Therefore $3\left(\xi_1+\xi_2+\xi_3\right)^2 - \frac{\left(k_1+k_2+k_3\right)^2}{\left(\xi_1+\xi_2+\xi_3\right)^2} = v$ if and only if $\xi_1+\xi_2=0$ or $\xi_2+\xi_3=0$ or $\xi_3+\xi_1=0$.

We consider the factor

$$\begin{aligned} &\frac{\left(k_1+k_2+k_3\right)^2}{\xi_1+\xi_2+\xi_3} - \frac{k_1^2}{\xi_1} - \frac{k_2^2}{\xi_2} - \frac{k_3^2}{\xi_3} \\ =& -\frac{1}{\xi_1+\xi_2+\xi_3}\left[\xi_1\xi_2\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)^2 + \xi_2\xi_3\left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)^2 + \xi_3\xi_1\left(\frac{k_3}{\xi_3}-\frac{k_1}{\xi_1}\right)^2\right]. \end{aligned}$$

For simplification, we denote $\frac{k_1}{\xi_1}$, $\frac{k_2}{\xi_2}$, $\frac{k_3}{\xi_3}$by $\alpha$, $\beta$, $\gamma$. We have the equations

$$\begin{cases} 3\xi_1^2 - \alpha^2 = v \\ 3\xi_2^2 - \beta^2 = v \\ 3\xi_3^2 - \gamma^2 = v \\ \frac{1}{\xi_1+\xi_2+\xi_3}\Big[3\left(\xi_1+\xi_2+\xi_3\right)\left(\xi_1+\xi_2\right)\left(\xi_2+\xi_3\right)\left(\xi_3+\xi_1\right) - \xi_1\xi_2\left(\alpha-\beta\right)^2 \\ \qquad -\xi_2\xi_3\left(\beta-\gamma\right)^2 - \xi_3\xi_1\left(\gamma-\alpha\right)^2\Big] = 0 \end{cases}$$

If given $\xi_2, \xi_3, \beta, \gamma$, solves the second and third equations, here we try to find $\xi_1$ and $\alpha$, satisfying the first and last equation.

$$\begin{aligned} &3\left(\xi_1+\xi_2+\xi_3\right)\left(\xi_1+\xi_2\right)\left(\xi_2+\xi_3\right)\left(\xi_3+\xi_1\right) - \xi_1\xi_2\left(\alpha-\beta\right)^2 \\ &-\xi_2\xi_3\left(\beta-\gamma\right)^2 - \xi_3\xi_1\left(\gamma-\alpha\right)^2 \\ =&3\Big[\xi_1^3\xi_2 + \xi_3\xi_1^3 + \xi_1^2\left(\xi_2^2+\xi_3^2+2\xi_2\xi_3\right) + \xi_1\left(\xi_2^2\xi_3+\xi_2\xi_3^2\right) \\ &+\left(\xi_2+\xi_3\right)^2\xi_1^2 + \left(\xi_2+\xi_3\right)^3\xi_1 + \left(\xi_2+\xi_3\right)\left(\xi_2^2\xi_3+\xi_2\xi_3^2\right)\Big] \\ &-\xi_1\xi_2\left(\alpha^2+\beta^2-2\alpha\beta\right) - \xi_2\xi_3\left(\beta-\gamma\right)^2 - \xi_3\xi_1\left(\gamma^2-2\gamma\alpha+\alpha^2\right) \\ =&6\left(\xi_2+\xi_3\right)^2\xi_1^2 + 3\left(\xi_2+\xi_3\right)\xi_2\xi_3\xi_1 + \left(\xi_2+\xi_3\right)v\xi_1 + 9\left(\xi_2+\xi_3\right)\xi_2\xi_3\xi_1 \\ &+3\left(\xi_2+\xi_3\right)^2\xi_2\xi_3 + \left(\xi_2+\xi_3\right)v\xi_1 + 2\alpha\beta\xi_1\xi_2 + 2\gamma\alpha\xi_3\xi_1 - \xi_2\xi_3\left(\beta-\gamma\right)^2 \\ =&6\left(\xi_2+\xi_3\right)^2\xi_1^2 + \left[2\left(\beta\xi_2+\gamma\xi_3\right)\alpha + 2\left(v+6\xi_2\xi_3\right)\left(\xi_2+\xi_3\right)\right]\xi_1 \\ &+\left[6\xi_2\xi_3+2\beta\gamma+2v\right]\xi_2\xi_3 \end{aligned}$$

Therefore we have the euqation

$$3\left(\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1\right)^2 + v\left(\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1\right) + \left(k_1k_2+k_2k_3+k_3k_1\right) = 0.$$

Hence we convert the original equation into the form

$$\begin{cases} 3\xi_1^4 - v\xi_1^2 - k_1^2 = 0 & (1) \\ 3\xi_2^4 - v\xi_2^2 - k_2^2 = 0 & (2) \\ 3\xi_3^4 - v\xi_3^2 - k_3^2 = 0 & (3) \\ 3\left(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1\right)^2 + v\left(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1\right) + \left(k_1k_2 + k_2k_3 + k_3k_1\right) = 0 & (4) \end{cases}$$

For $v > 0$, we let $\xi_i = \sqrt{\frac{v}{3}}\sec\theta_i$ and $k_i = \frac{v}{\sqrt{3}}\sec\theta_i\tan\theta_i$ , we assume that $\theta_i \in [0,2)\backslash\{\frac{\pi}{2},\frac{3\pi}{2}\}$. Plug the substitution into equation (4), and we will have the following equation:

$$\begin{aligned} &\left(\sec\theta_1\sec\theta_2 + \sec\theta_2\sec\theta_3 + \sec\theta_3\sec\theta_1\right)^2 + \sec\theta_1\sec\theta_2 + \sec\theta_2\sec\theta_3 + \sec\theta_3\sec\theta_1 \\ &+ \sec\theta_1\sec\theta_2\tan\theta_1\tan\theta_2 + \sec\theta_2\sec\theta_3\tan\theta_2\tan\theta_3 + \sec\theta_3\sec\theta_1\tan\theta_3\tan\theta_1 = 0 \end{aligned}$$

Multiplying above equation by $\cos^2\theta_1\cos^2\theta_2\cos^2\theta_3$ we obtain the following equation

$$\begin{aligned} &\left(\cos\theta_1 + \cos\theta_2 + \cos\theta_3\right)^2 + \left(\cos\theta_1 + \cos\theta_2 + \cos\theta_3\right)\cos\theta_1\cos\theta_2\cos\theta_3 \\ &+ \sin\theta_1\sin\theta_2\cos^2\theta_3 + \sin\theta_2\sin\theta_3\cos^2\theta_1 + \sin\theta_3\sin\theta_1\cos^2\theta_2 \\ =& \left(\cos\theta_1 + \cos\theta_2\right)^2 + 2\left(\cos\theta_1 + \cos\theta_2\right)\cos\theta_3 + \left(1 + \cos\left(\theta_1 - \theta_2\right)\right)\cos^2\theta_3 \\ &+ \left(\cos\left(\theta_2 - \theta_3\right) + \cos\left(\theta_3 - \theta_1\right)\right)\cos^2\theta_1 + \cos\left(\theta_3 - \theta_1\right)\left(\cos^2\theta_2 - \cos^2\theta_1\right) \\ =& 2\cos\left(\frac{\theta_1 - \theta_2}{2}\right)\left[\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\left(\cos\theta_1 + \cos\theta_2\right) + 2\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_3\right. \\ &+ \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\cos^2\theta_3 + \cos\left(\frac{\theta_2 + \theta_1}{2} - \theta_3\right)\cos^2\theta_1 \\ &\left. + \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\theta_3 - \theta_1\right)\left(\cos\theta_2 - \cos\theta_1\right)\right] \end{aligned}$$

Taking out the factor $2\cos\left(\frac{\theta_1 - \theta_2}{2}\right)$, and working on the remaining terms, we have

$$\begin{aligned} &\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\left(\cos\theta_1 + \cos\theta_2\right) + 2\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_3 + \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\cos^2\theta_3 \\ &+ \cos\left(\frac{\theta_2 + \theta_1}{2} - \theta_3\right)\cos^2\theta_1 + \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\theta_3 - \theta_1\right)\left(\cos\theta_2 - \cos\theta_1\right) \\ =& \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\left(\cos\theta_1 + \cos\theta_3\right) + \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_2\left(1 + \cos\left(\theta_3 - \theta_1\right)\right) \\ &+ \left(\cos\left(\frac{\theta_2 + \theta_1}{2} - \theta_3\right) + \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\right)\cos^2\theta_1 + \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\left(\cos^2\theta_3 - \cos^2\theta_1\right) \\ &+ \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\left(\cos\theta_3 + \cos\theta_1\right) - \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_1\left(1 + \cos\left(\theta_3 - \theta_1\right)\right) \\ =& 2\cos\left(\frac{\theta_3 - \theta_1}{2}\right)\left[2\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_3 + \theta_1}{2}\right) + \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_2\cos\left(\frac{\theta_3 - \theta_1}{2}\right)\right. \\ &+ \cos\left(\frac{\theta_2 - \theta_3}{2}\right)\cos^2\theta_1 + \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\left(\cos\theta_3 - \cos\theta_1\right)\cos\left(\frac{\theta_3 + \theta_1}{2}\right) \\ &\left. - \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_1\cos\left(\frac{\theta_3 - \theta_1}{2}\right)\right] \end{aligned}$$

Taking out the facto $2\cos\left(\frac{\theta_3-\theta_1}{2}\right)$, and working on the remaing terms we have

$$\begin{aligned}
&\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos^2\theta_1+\cos\left(\frac{\theta_1-\theta_2}{2}\right)(\cos\theta_3-\cos\theta_1)\cos\left(\frac{\theta_3+\theta_1}{2}\right)\\
&+2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_3+\theta_1}{2}\right)+\cos\left(\frac{\theta_1+\theta_2}{2}\right)(\cos\theta_2-\cos\theta_1)\cos\left(\frac{\theta_3-\theta_1}{2}\right)\\
=&\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos^2\theta_1+\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)+\cos\left(\frac{\theta_2-\theta_3}{2}\right)\\
&-2\cos\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_3+\theta_1}{2}\right)\sin\left(\frac{\theta_3+\theta_1}{2}\right)\sin\left(\frac{\theta_3-\theta_1}{2}\right)\\
&+2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_3-\theta_1}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\\
=&\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos^2\theta_1+\cos\left(\frac{\theta_2-\theta_3}{2}\right)+\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\\
&-\cos\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_3-\theta_1}{2}\right)\sin(\theta_1+\theta_3)+\cos\left(\frac{\theta_3-\theta_1}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin(\theta_1+\theta_2)
\end{aligned}$$

The first two terms in the above equation have the desired factor $\cos\left(\frac{\theta_2-\theta_3}{2}\right)$ and we keep working the remaing factors

$$\begin{aligned}
&\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\cos\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_3-\theta_1}{2}\right)\sin(\theta_1+\theta_3)\\
&+\cos\left(\frac{\theta_3-\theta_1}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin(\theta_1+\theta_2)\\
=&\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)+\frac{1}{2}\sin(\theta_1+\theta_3)\left(\sin\left(\frac{\theta_2-\theta_3}{2}\right)-\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\right)\\
&-\frac{1}{2}\sin(\theta_1+\theta_2)\left(\sin\left(\frac{\theta_2-\theta_3}{2}\right)+\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\right)\\
=&\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)+\frac{1}{2}(\sin(\theta_1+\theta_3)-\sin(\theta_1+\theta_2))\sin\left(\frac{\theta_2-\theta_3}{2}\right)\\
&-\frac{1}{2}(\sin(\theta_1+\theta_3)+\sin(\theta_1+\theta_2))\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\\
=&\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\\
&-\sin\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\cos\left(\frac{\theta_2-\theta_3}{2}\right)\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\\
=&\cos\left(\frac{\theta_2-\theta_3}{2}\right)\left(\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\sin\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\right)
\end{aligned}$$

We take out the factor $\cos\left(\frac{\theta_2-\theta_3}{2}\right)$ and have the following equation:

$$\cos\left(\frac{\theta_2-\theta_3}{2}\right)\left(1+\cos^2\theta_1+\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\sin\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\right)$$

We work on the remaing terms and have the following equation which always greater or equals 0:

$$
\begin{aligned}
&1+\cos^2\theta_1+\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\sin\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\\
=&1+\cos^2\theta_1+\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\sin^2\left(\frac{\theta_2+\theta_3}{2}\right)\cos^2\theta_1+\sin^2\theta_1\cos^2\left(\frac{\theta_2+\theta_3}{2}\right)\\
=&1+\cos^2\theta_1\cos^2\left(\frac{\theta_2+\theta_3}{2}\right)+\sin^2\theta_1\cos^2\left(\frac{\theta_2+\theta_3}{2}\right)+\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\\
=&1+\cos^2\left(\frac{\theta_2+\theta_3}{2}\right)+\frac{1}{2}\cos\left(\theta_1+\theta_2\right)+\frac{1}{2}\cos\left(\theta_1+\theta_3\right)\\
=&\frac{1}{2}\left(3+\cos\left(\theta_1+\theta_2\right)+\cos\left(\theta_2+\theta_3\right)+\cos\left(\theta_3+\theta_1\right)\right)
\end{aligned}
$$

Hence we have equation (4) equals the following equation:

$$
2\cos\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_3-\theta_1}{2}\right)sin\left(3+\cos\left(\theta_1+\theta_2\right)+\cos\left(\theta_2+\theta_3\right)+\cos\left(\theta_3+\theta_1\right)\right)=0
$$

Since $|\theta_i-\theta_j|\le 2\pi$, hence if $\cos\left(\frac{\theta_1-\theta_2}{2}\right)=0$, which implies that $\theta_1-\theta_2=\pm\pi$, therefore $\cos\theta_1=-\cos\theta_2$, similarly if $\cos\left(\theta_1+\theta_2\right)=-1$ which also implies that $\cos\theta_1=-\cos\theta_2$. We obtain the conclusion we have $\xi_1+\xi_2=0$ or $\xi_2+\xi_3=0$ or $\xi_3+\xi_1=0$

For where $v<0$, we will let $\xi_i=\sqrt{\frac{-v}{3}}\tan\theta_i$, and $k_i=\frac{-v}{\sqrt{3}}\tan\theta_i\sec\theta_i$, the equation (4) will have the form

$$
\begin{aligned}
&\left(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\tan\theta_3\tan\theta_1\right)^2-\left(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\tan\theta_3\tan\theta_1\right)\\
&+\tan\theta_1\tan\theta_2\sec\theta_1\sec\theta_2+\tan\theta_2\tan\theta_3\sec\theta_2\sec\theta_3+\tan\theta_3\tan\theta_1\sec\theta_3\sec\theta_1=0
\end{aligned}
$$

Multiplying by $\cos^2\theta_1\cos^2\theta_2\cos^2\theta_3$, we obtained

$$
\begin{aligned}
&\sin^2\left(\theta_1+\theta_2\right)\sin^2\theta_3+2\sin\left(\theta_1+\theta_2\right)\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_3+\sin^2\theta_1\sin^2\theta_2\cos^2\theta_3\\
&-\sin\left(\theta_1+\theta_2\right)\sin\theta_3\cos\theta_1\cos\theta_2\cos\theta_3-\sin\theta_1\sin\theta_2\cos\theta_1\cos\theta_2\cos^2\theta_3\\
&+\sin\theta_1\sin\theta_2\cos^2\theta_3+\sin\theta_2\sin\theta_3\cos^2\theta_1+\sin\theta_3\sin\theta_1\cos^2\theta_2
\end{aligned}
$$

We use the formula

$$
\begin{aligned}
&\sin^2\theta_1\sin^2\theta_2\cos^2\theta_3-\sin\theta_1\sin\theta_2\cos\theta_1\cos\theta_2\cos^2\theta_3+\sin\theta_1\sin\theta_2\cos^2\theta_3\\
=&\sin\theta_1\sin\theta_2\cos^2\theta_3\left(1-\cos\left(\theta_1+\theta_2\right)\right)=2\sin\theta_1\sin\theta_2\sin^2\left(\frac{\theta_1+\theta_2}{2}\right)\cos^2\theta_3
\end{aligned}
$$

$$
\begin{aligned}
&\sin\theta_2\sin\theta_3\cos^2\theta_1+\sin\theta_3\sin\theta_1\cos^2\theta_2\\
=&\sin\theta_3\cos\theta_2\left(\sin\theta_2\cos\theta_1+\sin\theta_1\cos\theta_2\right)+\sin\theta_2\sin\theta_3\cos\theta_1\left(\cos\theta_1-\cos\theta_2\right)\\
=&\sin\theta_3\cos\theta_2\sin\left(\theta_1+\theta_2\right)+2\sin\theta_2\sin\theta_3\cos\theta_1\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_2-\theta_1}{2}\right)
\end{aligned}
$$

The original equation has the following form

$$2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\left[\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin(\theta_1+\theta_2)\sin^2\theta_3+2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_3\right.$$
$$-\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\theta_3\cos\theta_1\cos\theta_2\cos\theta_3+\sin\theta_1\sin\theta_2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos^2\theta_3$$
$$\left.+\sin\theta_3\cos\theta_2\cos\left(\frac{\theta_1+\theta_2}{2}\right)-\sin\theta_2\sin\theta_3\cos\theta_1\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right]$$

After taking out the factor $2\sin\left(\frac{\theta_1+\theta_2}{2}\right)$, we work on the remaining term

$$\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\theta_2\sin\theta_3\left[(\sin\theta_1\sin\theta_3-\cos\theta_1\cos\theta_3)+1\right]$$
$$+\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\theta_2\sin\theta_3\left(\sin\theta_1\cos\theta_3+\sin\theta_3\cos\theta_1\right)$$
$$+\sin\theta_1\sin\theta_2\cos\theta_3\left[\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\theta_3+\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\theta_3\right]$$
$$-\sin\theta_2\sin\theta_3\cos\theta_1\sin\left(\frac{\theta_1-\theta_2}{2}\right)$$
$$=\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\theta_2\sin\theta_3\left(1-\cos(\theta_1+\theta_3)\right)+\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\theta_2\sin\theta_3\sin(\theta_1+\theta_3)$$
$$+\sin\theta_1\sin\theta_2\cos\theta_3\sin\left(\frac{\theta_1+\theta_2}{2}+\theta_3\right)-\sin\theta_2\sin\theta_3\cos\theta_1\sin\left(\frac{\theta_1-\theta_2}{2}\right)$$

We look through the last two terms in the above equation

$$\sin\theta_1\sin\theta_2\cos\theta_3\sin\left(\frac{\theta_1+\theta_2}{2}+\theta_3\right)-\sin\theta_2\sin\theta_3\cos\theta_1\sin\left(\frac{\theta_1-\theta_2}{2}\right)$$
$$=\sin\theta_1\sin\theta_2\cos\theta_3\left(\sin\left(\frac{\theta_1+\theta_2}{2}+\theta_3\right)+\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)$$
$$-\sin\theta_2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\left(\sin\theta_1\cos\theta_3+\sin\theta_3\cos\theta_1\right)$$
$$=2\sin\theta_1\sin\theta_2\cos\theta_3\sin\left(\frac{\theta_1+\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}\right)-\sin\theta_2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin(\theta_1+\theta_3)$$

The remaing terms have the following common factor

$$2\sin\left(\frac{\theta_1+\theta_3}{2}\right)\left[\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\theta_2\sin\theta_3\sin\left(\frac{\theta_1+\theta_3}{2}\right)+\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\theta_2\sin\theta_3\cos\left(\frac{\theta_1+\theta_3}{2}\right)\right.$$
$$\left.+\sin\theta_1\sin\theta_2\cos\theta_3\cos\left(\frac{\theta_2+\theta_3}{2}\right)-\sin\theta_2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_3}{2}\right)\right]$$

After taking out the common factor $2\sin\left(\frac{\theta_1+\theta_3}{2}\right)$, we work on the remaining terms, the first two terms in above equation can be written as

$$\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\theta_3\sin\left(\frac{\theta_1+\theta_2}{2}+\frac{\theta_2+\theta_3}{2}\right)$$
$$=\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\theta_3\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}\right)+\cos^2\left(\frac{\theta_1+\theta_2}{2}\right)\sin\theta_3\sin\left(\frac{\theta_2+\theta_3}{2}\right)$$

We work on the terms without $\sin\left(\frac{\theta_2+\theta_3}{2}\right)$,

$$\frac{1}{2}\sin\theta_3\sin\left(\theta_1+\theta_2\right)\cos\left(\frac{\theta_2+\theta_3}{2}\right)-\sin\theta_2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_3}{2}\right)$$
$$+\frac{1}{2}\sin\theta_1\left(\sin\left(\theta_2+\theta_3\right)+\sin\left(\theta_2-\theta_3\right)\right)\cos\left(\frac{\theta_2+\theta_3}{2}\right)$$

$$\frac{1}{2}\sin\theta_3\left(\cancel{\sin\theta_1\cos\theta_2}+\sin\theta_2\cos\theta_1\right)\cos\left(\frac{\theta_2+\theta_3}{2}\right)$$
$$+\frac{1}{2}\sin\theta_1\left(\sin\theta_2\cos\theta_3-\cancel{\cos\theta_2\sin\theta_3}\right)\cos\left(\frac{\theta_2+\theta_3}{2}\right)$$
$$-\sin\theta_2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_3}{2}\right)$$
$$=\sin\theta_2\sin\left(\theta_1+\theta_3\right)\cos\left(\frac{\theta_2+\theta_3}{2}\right)-\sin\theta_2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_3}{2}\right)$$
$$=\sin\theta_2\cos\left(\frac{\theta_1+\theta_3}{2}\right)\left(\sin\left(\frac{\theta_1+\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}\right)-\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)$$
$$=\sin\theta_2\cos\left(\frac{\theta_1+\theta_3}{2}\right)\frac{1}{2}\left(\sin\left(\frac{\theta_1+\theta_2}{2}+\theta_3\right)-\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)$$
$$=\sin\theta_2\cos^2\left(\frac{\theta_1+\theta_3}{2}\right)\sin\left(\frac{\theta_2+\theta_3}{2}\right)$$

Hence equation (4) has the form

$$4\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1+\theta_3}{2}\right)\sin\left(\frac{\theta_2+\theta_3}{2}\right)\left[\sin\theta_1\cos^2\left(\frac{\theta_2+\theta_3}{2}\right)+\sin\theta_2\cos^2\left(\frac{\theta_1+\theta_3}{2}\right)\right.$$
$$\left.+\sin\theta_3\cos^2\left(\frac{\theta_1+\theta_2}{2}\right)\right]=0$$

Therefore we need to verify whether the equation has solution or not

$$\sin\theta_1\cos^2\left(\frac{\theta_2+\theta_3}{2}\right)+\sin\theta_2\cos^2\left(\frac{\theta_1+\theta_3}{2}\right)+\sin\theta_3\cos^2\left(\frac{\theta_1+\theta_2}{2}\right)=0$$