Solve the equations for $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ and $k_1, k_2, k_3 \in \mathbb{Z}$, here v is a fix parameter..

$$\begin{cases} v = 3\xi_1^2 - \frac{k_1^2}{\xi_1^2} \\ v = 3\xi_2^2 - \frac{k_2^2}{\xi_2^2} \\ v = 3\xi_3^2 - \frac{k_3^2}{\xi_3^2} \\ (\xi_1+\xi_2+\xi_3)^3 - \xi_1^3 - \xi_2^3 - \xi_3^3 + \frac{(k_1+k_2+k_3)^2}{\xi_1+\xi_2+\xi_3} - \frac{k_1^2}{\xi_1} - \frac{k_2^2}{\xi_2} - \frac{k_3^2}{\xi_3} = 0 \end{cases}$$

$$3(\xi_1+\xi_2+\xi_3)^2 - \frac{(k_1+k_2+k_3)^2}{(\xi_1+\xi_2+\xi_3)^2} = v + 12\frac{(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_3+\xi_1)}{\xi_1+\xi_2+\xi_3}$$

Therefore $3(\xi_1+\xi_2+\xi_3)^2 - \frac{(k_1+k_2+k_3)^2}{(\xi_1+\xi_2+\xi_3)^2} = v$ if and only if $\xi_1+\xi_2 = 0$ or $\xi_2+\xi_3 = 0$ or $\xi_3+\xi_1 = 0$.

We consider the factor

$$\begin{aligned} &\frac{(k_1+k_2+k_3)^2}{\xi_1+\xi_2+\xi_3} - \frac{k_1^2}{\xi_1} - \frac{k_2^2}{\xi_2} - \frac{k_3^2}{\xi_3} \\ =& -\frac{1}{\xi_1+\xi_2+\xi_3}\left[\xi_1\xi_2\left(\frac{k_1}{\xi_1} - \frac{k_2}{\xi_2}\right)^2 + \xi_2\xi_3\left(\frac{k_2}{\xi_2} - \frac{k_3}{\xi_3}\right)^2 + \xi_3\xi_1\left(\frac{k_3}{\xi_3} - \frac{k_1}{\xi_1}\right)^2\right]. \end{aligned}$$

For simplification, we denote $\frac{k_1}{\xi_1}$, $\frac{k_2}{\xi_2}$, $\frac{k_3}{\xi_3}$by α, β, γ. We have the equations

$$\begin{cases} 3\xi_1^2 - \alpha^2 = v \\ 3\xi_2^2 - \beta^2 = v \\ 3\xi_3^2 - \gamma^2 = v \\ \frac{1}{\xi_1+\xi_2+\xi_3}\Big[3(\xi_1+\xi_2+\xi_3)(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_3+\xi_1) - \xi_1\xi_2(\alpha-\beta)^2 \\ \qquad -\xi_2\xi_3(\beta-\gamma)^2 - \xi_3\xi_1(\gamma-\alpha)^2\Big] = 0 \end{cases}$$

If given $\xi_2, \xi_3, \beta, \gamma$, solves the second and third equations, here we try to find ξ_1 and α, satisfying the first and last equation.

$$\begin{aligned} &3(\xi_1+\xi_2+\xi_3)(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_3+\xi_1) - \xi_1\xi_2(\alpha-\beta)^2 \\ &- \xi_2\xi_3(\beta-\gamma)^2 - \xi_3\xi_1(\gamma-\alpha)^2 \\ =&3\big[\xi_1^3\xi_2 + \xi_3\xi_1^3 + \xi_1^2\left(\xi_2^2+\xi_3^2+2\xi_2\xi_3\right) + \xi_1\left(\xi_2^2\xi_3+\xi_2\xi_3^2\right) \\ &+ (\xi_2+\xi_3)^2\xi_1^2 + (\xi_2+\xi_3)^3\xi_1 + (\xi_2+\xi_3)\left(\xi_2^2\xi_3+\xi_2\xi_3^2\right)\big] \\ &- \xi_1\xi_2\left(\alpha^2+\beta^2-2\alpha\beta\right) - \xi_2\xi_3(\beta-\gamma)^2 - \xi_3\xi_1\left(\gamma^2-2\gamma\alpha+\alpha^2\right) \\ =&6(\xi_2+\xi_3)^2\xi_1^2 + 3(\xi_2+\xi_3)\xi_2\xi_3\xi_1 + (\xi_2+\xi_3)v\xi_1 + 9(\xi_2+\xi_3)\xi_2\xi_3\xi_1 \\ &+ 3(\xi_2+\xi_3)^2\xi_2\xi_3 + (\xi_2+\xi_3)v\xi_1 + 2\alpha\beta\xi_1\xi_2 + 2\gamma\alpha\xi_3\xi_1 - \xi_2\xi_3(\beta-\gamma)^2 \\ =&6(\xi_2+\xi_3)^2\xi_1^2 + \left[2(\beta\xi_2+\gamma\xi_3)\alpha + 2(v+6\xi_2\xi_3)(\xi_2+\xi_3)\right]\xi_1 \\ &+ \left[6\xi_2\xi_3 + 2\beta\gamma + 2v\right]\xi_2\xi_3 \end{aligned}$$

Therefore we have the euqation

$$3(\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1)^2 + v(\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1) + (k_1k_2+k_2k_3+k_3k_1) = 0.$$

Hence we convert the original equation into the form

$$\begin{cases} 3\xi_1^4 - v\xi_1^2 - k_1^2 = 0 & (1) \\ 3\xi_2^4 - v\xi_2^2 - k_2^2 = 0 & (2) \\ 3\xi_3^4 - v\xi_3^2 - k_3^2 = 0 & (3) \\ 3\left(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1\right)^2 + v\left(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1\right) + \left(k_1k_2 + k_2k_3 + k_3k_1\right) = 0 & (4) \end{cases}$$

For $v > 0$, we let $\xi_i = \sqrt{\frac{v}{3}}\sec\theta_i$ and $k_i = \frac{v}{\sqrt{3}}\sec\theta_i\tan\theta_i$, we assume that $\theta_i \in [0,2)\backslash\{\frac{\pi}{2},\frac{3\pi}{2}\}$. Plug the substitution into equation (4), and we will have the following equation:

$$\left(\sec\theta_1\sec\theta_2 + \sec\theta_2\sec\theta_3 + \sec\theta_3\sec\theta_1\right)^2 + \sec\theta_1\sec\theta_2 + \sec\theta_2\sec\theta_3 + \sec\theta_3\sec\theta_1$$
$$+ \sec\theta_1\sec\theta_2\tan\theta_1\tan\theta_2 + \sec\theta_2\sec\theta_3\tan\theta_2\tan\theta_3 + \sec\theta_3\sec\theta_1\tan\theta_3\tan\theta_1 = 0$$

Multiplying above equation by $\cos^2\theta_1\cos^2\theta_2\cos^2\theta_3$ we obtain the following equation

$$\begin{aligned}
&\left(\cos\theta_1 + \cos\theta_2 + \cos\theta_3\right)^2 + \left(\cos\theta_1 + \cos\theta_2 + \cos\theta_3\right)\cos\theta_1\cos\theta_2\cos\theta_3 \\
&+ \sin\theta_1\sin\theta_2\cos^2\theta_3 + \sin\theta_2\sin\theta_3\cos^2\theta_1 + \sin\theta_3\sin\theta_1\cos^2\theta_2 \\
=& \left(\cos\theta_1 + \cos\theta_2\right)^2 + 2\left(\cos\theta_1 + \cos\theta_2\right)\cos\theta_3 + \left(1 + \cos\left(\theta_1 - \theta_2\right)\right)\cos^2\theta_3 \\
&+ \left(\cos\left(\theta_2 - \theta_3\right) + \cos\left(\theta_3 - \theta_1\right)\right)\cos^2\theta_1 + \cos\left(\theta_3 - \theta_1\right)\left(\cos^2\theta_2 - \cos^2\theta_1\right) \\
=& 2\cos\left(\frac{\theta_1 - \theta_2}{2}\right)\left[\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\left(\cos\theta_1 + \cos\theta_2\right) + 2\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_3\right. \\
&+ \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\cos^2\theta_3 + \cos\left(\frac{\theta_2 + \theta_1}{2} - \theta_3\right)\cos^2\theta_1 \\
&\left.+ \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\theta_3 - \theta_1\right)\left(\cos\theta_2 - \cos\theta_1\right)\right]
\end{aligned}$$

Taking out the factor $2\cos\left(\frac{\theta_1 - \theta_2}{2}\right)$, and working on the remaining terms, we have

$$\begin{aligned}
&\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\left(\cos\theta_1 + \cos\theta_2\right) + 2\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_3 + \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\cos^2\theta_3 \\
&+ \cos\left(\frac{\theta_2 + \theta_1}{2} - \theta_3\right)\cos^2\theta_1 + \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\theta_3 - \theta_1\right)\left(\cos\theta_2 - \cos\theta_1\right) \\
=& \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\left(\cos\theta_1 + \cos\theta_3\right) + \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_2\left(1 + \cos\left(\theta_3 - \theta_1\right)\right) \\
&+ \left(\cos\left(\frac{\theta_2 + \theta_1}{2} - \theta_3\right) + \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\right)\cos^2\theta_1 + \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\left(\cos^2\theta_3 - \cos^2\theta_1\right) \\
&+ \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\left(\cos\theta_3 + \cos\theta_1\right) - \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_1\left(1 + \cos\left(\theta_3 - \theta_1\right)\right) \\
=& 2\cos\left(\frac{\theta_3 - \theta_1}{2}\right)\left[2\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_3 + \theta_1}{2}\right) + \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_2\cos\left(\frac{\theta_3 - \theta_1}{2}\right)\right. \\
&+ \cos\left(\frac{\theta_2 - \theta_3}{2}\right)\cos^2\theta_1 + \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\left(\cos\theta_3 - \cos\theta_1\right)\cos\left(\frac{\theta_3 + \theta_1}{2}\right) \\
&\left.- \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\theta_1\cos\left(\frac{\theta_3 - \theta_1}{2}\right)\right]
\end{aligned}$$

Taking out the facto $2\cos\left(\frac{\theta_3-\theta_1}{2}\right)$, and working on the remaing terms we have

$$\begin{aligned}
&\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos^2\theta_1+\cos\left(\frac{\theta_1-\theta_2}{2}\right)(\cos\theta_3-\cos\theta_1)\cos\left(\frac{\theta_3+\theta_1}{2}\right)\\
&+2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_3+\theta_1}{2}\right)+\cos\left(\frac{\theta_1+\theta_2}{2}\right)(\cos\theta_2-\cos\theta_1)\cos\left(\frac{\theta_3-\theta_1}{2}\right)\\
=&\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos^2\theta_1+\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)+\cos\left(\frac{\theta_2-\theta_3}{2}\right)\\
&-2\cos\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_3+\theta_1}{2}\right)\sin\left(\frac{\theta_3+\theta_1}{2}\right)\sin\left(\frac{\theta_3-\theta_1}{2}\right)\\
&+2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_3-\theta_1}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\\
=&\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos^2\theta_1+\cos\left(\frac{\theta_2-\theta_3}{2}\right)+\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\\
&-\cos\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_3-\theta_1}{2}\right)\sin(\theta_1+\theta_3)+\cos\left(\frac{\theta_3-\theta_1}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin(\theta_1+\theta_2)
\end{aligned}$$

The first two terms in the above equation have the desired factor $\cos\left(\frac{\theta_2-\theta_3}{2}\right)$ and we keep working the remaing factors

$$\begin{aligned}
&\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\cos\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_3-\theta_1}{2}\right)\sin(\theta_1+\theta_3)\\
&+\cos\left(\frac{\theta_3-\theta_1}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin(\theta_1+\theta_2)\\
=&\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)+\frac{1}{2}\sin(\theta_1+\theta_3)\left(\sin\left(\frac{\theta_2-\theta_3}{2}\right)-\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\right)\\
&-\frac{1}{2}\sin(\theta_1+\theta_2)\left(\sin\left(\frac{\theta_2-\theta_3}{2}\right)+\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\right)\\
=&\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)+\frac{1}{2}(\sin(\theta_1+\theta_3)-\sin(\theta_1+\theta_2))\sin\left(\frac{\theta_2-\theta_3}{2}\right)\\
&-\frac{1}{2}(\sin(\theta_1+\theta_3)+\sin(\theta_1+\theta_2))\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\\
=&\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\\
&-\sin\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\cos\left(\frac{\theta_2-\theta_3}{2}\right)\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\\
=&\cos\left(\frac{\theta_2-\theta_3}{2}\right)\left(\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\sin\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\right)
\end{aligned}$$

We take out the factor $\cos\left(\frac{\theta_2-\theta_3}{2}\right)$ and have the following equation:

$$\cos\left(\frac{\theta_2-\theta_3}{2}\right)\left(1+\cos^2\theta_1+\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\sin\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\right)$$

We work on the remaing terms and have the following equation which always greater or equals 0:

$$
\begin{aligned}
&1+\cos^2\theta_1+\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\sin\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\sin\left(\frac{\theta_2+\theta_3}{2}-\theta_1\right)\\
=&1+\cos^2\theta_1+\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)-\sin^2\left(\frac{\theta_2+\theta_3}{2}\right)\cos^2\theta_1+\sin^2\theta_1\cos^2\left(\frac{\theta_2+\theta_3}{2}\right)\\
=&1+\cos^2\theta_1\cos^2\left(\frac{\theta_2+\theta_3}{2}\right)+\sin^2\theta_1\cos^2\left(\frac{\theta_2+\theta_3}{2}\right)+\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}+\theta_1\right)\\
=&1+\cos^2\left(\frac{\theta_2+\theta_3}{2}\right)+\frac{1}{2}\cos\left(\theta_1+\theta_2\right)+\frac{1}{2}\cos\left(\theta_1+\theta_3\right)\\
=&\frac{1}{2}\left(3+\cos\left(\theta_1+\theta_2\right)+\cos\left(\theta_2+\theta_3\right)+\cos\left(\theta_3+\theta_1\right)\right)
\end{aligned}
$$

Hence we have equation (4) equals the following equation:

$$
2\cos\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_2-\theta_3}{2}\right)\cos\left(\frac{\theta_3-\theta_1}{2}\right)\left(3+\cos\left(\theta_1+\theta_2\right)+\cos\left(\theta_2+\theta_3\right)+\cos\left(\theta_3+\theta_1\right)\right)=0
$$

Since $|\theta_i-\theta_j|\le 2\pi$, hence if $\cos\left(\frac{\theta_1-\theta_2}{2}\right)=0$, which implies that $\theta_1-\theta_2=\pm\pi$, therefore $\cos\theta_1=-\cos\theta_2$, similarly if $\cos\left(\theta_1+\theta_2\right)=-1$ which also implies that $\cos\theta_1=-\cos\theta_2$. We obtain the conclusion we have $\xi_1+\xi_2=0$ or $\xi_2+\xi_3=0$ or $\xi_3+\xi_1=0$