Solve the equations for $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ and $k_1, k_2, k_3 \in \mathbb{Z}$, here v is a fix parameter..

$$\begin{cases} v = 3\xi_1^2 - \frac{k_1^2}{\xi_1^2} \\ v = 3\xi_2^2 - \frac{k_2^2}{\xi_2^2} \\ v = 3\xi_3^2 - \frac{k_3^2}{\xi_3^2} \\ (\xi_1+\xi_2+\xi_3)^3 - \xi_1^3 - \xi_2^3 - \xi_3^3 + \frac{(k_1+k_2+k_3)^2}{\xi_1+\xi_2+\xi_3} - \frac{k_1^2}{\xi_1} - \frac{k_2^2}{\xi_2} - \frac{k_3^2}{\xi_3} = 0 \end{cases}$$

$$3\left(\xi_1+\xi_2+\xi_3\right)^2 - \frac{\left(k_1+k_2+k_3\right)^2}{\left(\xi_1+\xi_2+\xi_3\right)^2} = v + 12\frac{\left(\xi_1+\xi_2\right)\left(\xi_2+\xi_3\right)\left(\xi_3+\xi_1\right)}{\xi_1+\xi_2+\xi_3}$$

Therefore $3\left(\xi_1+\xi_2+\xi_3\right)^2 - \frac{(k_1+k_2+k_3)^2}{(\xi_1+\xi_2+\xi_3)^2} = v$ if and only if $\xi_1+\xi_2 = 0$ or $\xi_2+\xi_3 = 0$ or $\xi_3+\xi_1 = 0$.

We consider the factor

$$\begin{aligned} &\frac{\left(k_1+k_2+k_3\right)^2}{\xi_1+\xi_2+\xi_3} - \frac{k_1^2}{\xi_1} - \frac{k_2^2}{\xi_2} - \frac{k_3^2}{\xi_3} \\ =& -\frac{1}{\xi_1+\xi_2+\xi_3}\left[\xi_1\xi_2\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)^2 + \xi_2\xi_3\left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)^2 + \xi_3\xi_1\left(\frac{k_3}{\xi_3}-\frac{k_1}{\xi_1}\right)^2\right] \end{aligned}$$

From the first three equations we have

$$3\left(\xi_1+\xi_2\right)\left(\xi_1-\xi_2\right) = \left(\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}\right)\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right),$$

and similar equations. We will assume that $\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2} \neq 0$.

$$\begin{aligned} &\xi_1\xi_2\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)^2 + \xi_2\xi_3\left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)^2 + \xi_3\xi_1\left(\frac{k_3}{\xi_3}-\frac{k_1}{\xi_1}\right)^2 \\ =&\xi_1\xi_2\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)\frac{3\left(\xi_1+\xi_2\right)\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}} + \xi_3\left(\xi_2+\xi_1\right)\left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)^2 \\ &+\xi_3\xi_1\left[\left(\frac{k_3}{\xi_3}-\frac{k_1}{\xi_1}\right)^2 - \left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)^2\right] \\ =&\xi_1\xi_2\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)\frac{3\left(\xi_1+\xi_2\right)\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}} + \xi_3\left(\xi_2+\xi_1\right)\left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)^2 \\ &+\xi_3\xi_1\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)\left(\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}-2\frac{k_3}{\xi_3}\right) \\ =&\left(\xi_1+\xi_2\right)\left[\xi_1\xi_2\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}} + \xi_3\left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)^2\right. \\ &\left.+\xi_3\xi_1\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\left(\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}-2\frac{k_3}{\xi_3}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&\xi_1\xi_2\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}+\xi_3\left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)^2+\xi_3\xi_1\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\left(\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}-2\frac{k_3}{\xi_3}\right)\\
=&\xi_1\left(\xi_2+\xi_3\right)\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}+\xi_1\xi_3\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\left[\frac{k_1}{\xi_1}-\frac{k_3}{\xi_3}-\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)\right]\\
&+\xi_3\left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)^2+\xi_3\xi_1\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)\\
=&\left(\xi_2+\xi_3\right)\left[\xi_1\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}+2\xi_1\xi_3\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\right.\\
&\left.+\xi_3\left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\xi_1\left(\frac{k_1}{\xi_1}-\frac{k_2}{\xi_2}\right)\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}+2\xi_1\xi_3\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}+\xi_3\left(\frac{k_2}{\xi_2}-\frac{k_3}{\xi_3}\right)\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\\
=&\xi_1\left[\frac{k_1}{\xi_1}-\frac{k_3}{\xi_3}+\frac{k_3}{\xi_2}-\frac{k_2}{\xi_2}\right]\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}+\xi_3\left[\frac{k_2}{\xi_2}-\frac{k_1}{\xi_1}+\frac{k_1}{\xi_1}-\frac{k_3}{\xi_3}\right]\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\\
&+2\xi_1\xi_3\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\\
=&-\xi_1\left(\frac{k_3}{\xi_3}-\frac{k_1}{\xi_1}\right)\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}-\xi_3\left(\frac{k_3}{\xi_3}-\frac{k_1}{\xi_1}\right)\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\\
&-\xi_1\left(\xi_2+\xi_3\right)\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}-\xi_3\left(\xi_2+\xi_1\right)\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\\
&+2\xi_1\xi_3\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\\
=&-\left(\xi_1+\xi_3\right)\left[\xi_2\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}+\xi_3\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\frac{3\left(\xi_3-\xi_1\right)}{\frac{k_3}{\xi_3}+\frac{k_1}{\xi_1}}+\xi_1\frac{3\left(\xi_3-\xi_1\right)}{\frac{k_3}{\xi_3}+\frac{k_1}{\xi_1}}\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\right]
\end{aligned}
$$

Eventually we have the equations

$$
\begin{aligned}
&\frac{\left(k_1+k_2+k_3\right)^2}{\xi_1+\xi_2+\xi_3}-\frac{k_1^2}{\xi_1}-\frac{k_2^2}{\xi_2}-\frac{k_3^2}{\xi_3}\\
=&\frac{1}{\xi_1+\xi_2+\xi_3}\left(\xi_1+\xi_2\right)\left(\xi_2+\xi_3\right)\left(\xi_3+\xi_1\right)\\
&\left[\xi_2\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}+\xi_3\frac{3\left(\xi_2-\xi_3\right)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\frac{3\left(\xi_3-\xi_1\right)}{\frac{k_3}{\xi_3}+\frac{k_1}{\xi_1}}+\xi_1\frac{3\left(\xi_3-\xi_1\right)}{\frac{k_3}{\xi_3}+\frac{k_1}{\xi_1}}\frac{3\left(\xi_1-\xi_2\right)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\right]
\end{aligned}
$$

The last equation has the form

$$
\begin{aligned}
&(\xi_1+\xi_2+\xi_3)^3-\xi_1^3-\xi_2^3-\xi_3^3+\frac{(k_1+k_2+k_3)^2}{\xi_1+\xi_2+\xi_3}-\frac{k_1^2}{\xi_1}-\frac{k_2^2}{\xi_2}-\frac{k_3^2}{\xi_3}\\
=&(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_3+\xi_1)\left[3+\frac{1}{\xi_1+\xi_2+\xi_3}\left(\xi_2\frac{3(\xi_1-\xi_2)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\frac{3(\xi_2-\xi_3)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\right.\right.\\
&\left.\left.+\xi_3\frac{3(\xi_2-\xi_3)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\frac{3(\xi_3-\xi_1)}{\frac{k_3}{\xi_3}+\frac{k_1}{\xi_1}}+\xi_1\frac{3(\xi_3-\xi_1)}{\frac{k_3}{\xi_3}+\frac{k_1}{\xi_1}}\frac{3(\xi_1-\xi_2)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\right)\right]=0
\end{aligned}
$$

If $v>0$, let $\xi_i=\frac{\sqrt{v}}{\sqrt{3}}\sec\theta_i,\ k_i=\frac{v}{\sqrt{3}}\sec\theta_i\tan\theta_i$, we have

$$
\begin{aligned}
&3+\frac{1}{\xi_1+\xi_2+\xi_3}\left(\xi_2\frac{3(\xi_1-\xi_2)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\frac{3(\xi_2-\xi_3)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}+\xi_3\frac{3(\xi_2-\xi_3)}{\frac{k_2}{\xi_2}+\frac{k_3}{\xi_3}}\frac{3(\xi_3-\xi_1)}{\frac{k_3}{\xi_3}+\frac{k_1}{\xi_1}}+\xi_1\frac{3(\xi_3-\xi_1)}{\frac{k_3}{\xi_3}+\frac{k_1}{\xi_1}}\frac{3(\xi_1-\xi_2)}{\frac{k_1}{\xi_1}+\frac{k_2}{\xi_2}}\right)\\
=&3+\frac{3}{\xi_1+\xi_2+\xi_3}\left(\xi_2\frac{2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\cdot\frac{2\sin\left(\frac{\theta_2-\theta_3}{2}\right)\sin\left(\frac{\theta_2+\theta_3}{2}\right)}{2\sin\left(\frac{\theta_2+\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}\right)}\right.\\
&+\xi_3\frac{2\sin\left(\frac{\theta_2-\theta_3}{2}\right)\sin\left(\frac{\theta_2+\theta_3}{2}\right)}{2\sin\left(\frac{\theta_2+\theta_3}{2}\right)\cos\left(\frac{\theta_2+\theta_3}{2}\right)}\cdot\frac{2\sin\left(\frac{\theta_3-\theta_1}{2}\right)\sin\left(\frac{\theta_3+\theta_1}{2}\right)}{2\sin\left(\frac{\theta_3+\theta_1}{2}\right)\cos\left(\frac{\theta_3+\theta_1}{2}\right)}\\
&\left.+\xi_1\frac{2\sin\left(\frac{\theta_3-\theta_1}{2}\right)\sin\left(\frac{\theta_3+\theta_1}{2}\right)}{2\sin\left(\frac{\theta_3+\theta_1}{2}\right)\cos\left(\frac{\theta_3+\theta_1}{2}\right)}\cdot\frac{2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\right)\\
=&3+\frac{3}{\sec\theta_1+\sec\theta_2+\sec\theta_3}\left(\sec\theta_2\frac{\sin\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\cdot\frac{\sin\left(\frac{\theta_2-\theta_3}{2}\right)}{\cos\left(\frac{\theta_2+\theta_3}{2}\right)}\right.\\
&\left.+\sec\theta_3\frac{\sin\left(\frac{\theta_2-\theta_3}{2}\right)}{\cos\left(\frac{\theta_2+\theta_3}{2}\right)}\cdot\frac{\sin\left(\frac{\theta_3-\theta_1}{2}\right)}{\cos\left(\frac{\theta_3+\theta_1}{2}\right)}+\sec\theta_1\frac{\sin\left(\frac{\theta_3-\theta_1}{2}\right)}{\cos\left(\frac{\theta_3+\theta_1}{2}\right)}\cdot\frac{\sin\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\right)
\end{aligned}
$$