# LONG TIME BEHAVIOR OF SMALL DATA SOLUTIONS TO THE CUBIC NONLINEARITY KP-I TYPE EQUATION

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Abstract. We consider the long-time behavior of solutions to the KP-I type equation on product space $\mathbb{R}\times\mathbb{T}$. Using the method of testing by wave packets, we prove small data global existence and modified scattering.

## 1. Introduction

In this article we consider the Kadomtsev-Petviashvili equation (KP-I) initial value problem

$$\begin{cases}\partial_t u+\partial_x^3 u-\partial_x^{-1}\partial_y^2 u+\partial_x\left(u^3/3\right)=0,\\ u(0,x,y)=u_0(x,y)\end{cases} \tag{1.1}$$

on the product space $(t,x,y)\in\mathbb{R}\times\mathbb{R}\times\mathbb{T}$.

The original KP-I equation

$$\partial_t u+\partial_x^3 u-\partial_x^{-1}\partial_y^2 u+\partial_x\left(u^2/3\right)=0,\quad u(0,x,y)=u_0(x,y) \tag{1.2}$$

is posed on $(t,x,y)\in\mathbb{R}\times\mathbb{R}^2$. The KP-I type equation and the KP-II type equation, in which the sign of the term $\partial_x^{-1}\partial_y^2 u$ in (1.1) is $+$ instead of $-$, were derived in [16] as models for the propagation of dispersive long waves with weak transverse effects. The Cauchy theory and asymptotic profile for (1.2) has been extensively studied [1, 4, 6, 10, 15, 17, 20, 21]. However, due to the important role played by the geometry, the properties of (1.2) are not very related to (1.1) here. Instead, the properties of (1.1) are intermediate between the short-pulse equation and the the mKdV equation, respectively (1.3), (1.4) on $(t,x)\in\mathbb{R}\times\mathbb{R}$. The short-pulse equation

$$\partial_t u-\partial_x^{-1}u+\partial_x\left(u^3/3\right)=0,\quad u(0,x)=u_0(x), \tag{1.3}$$

and the mKdV equation,

$$\partial_t u+\partial_x^3 u+\partial_x\left(u^3/3\right)=0,\quad u(0,x)=u_0(x). \tag{1.4}$$

The short-pulse equation exhibits wave-breaking phenomena and the existence of a blow-up solution [18, 19]. Hence we need to consider the small data case, Hayashi et al [13, 11]. [12] proved the nonexistence of the usual scattering states, and Niizato [22] showed the existence of a modified scattering state of (1.3) for small initial data $u_0\in H_x^s\cap\dot H_x^{-1}$ with $s>10$ and $x\partial_x u_0\in H_x^5$. Using the factorization technique, Hayashi and Naumkin [9] proved the existence of a modified scattering state for (1.3) for a larger class of initial data $u_0\in H_x^s\cap\dot H_x^{-1}$ and $x\partial_x u_0\in H_x^r$ with $s>\frac{5}{2}+r$ and $r>\frac{3}{2}$. In [23], Okamoto used wave packet method to prove the modified scattering for initial data $u_0\in H_x^s\cap\dot H_x^{-1}$ with $s>4$ and $x\partial_x u_0\in L_x^2$ in the following sense: If $u$ is a solution to (1.3) with suitable small initial data, there exists a unique modified final state $W\in L^\infty(\mathbb{R}^-)$ such that for large $t$,

$$u(t,x)=\frac{2}{\sqrt{t}}\mathbf{1}_{\mathbb{R}^-}(x)\,\mathrm{Re}\left\{W\left(\frac{x}{t}\right)\exp\left(-2i\sqrt{t\,|x|}+3i\sqrt{\frac{t}{|x|}}\left|W\left(\frac{x}{t}\right)\right|^2\log t\right)\right\}+O\left(\epsilon t^{-\frac{1}{2}+\kappa}\right). \tag{1.5}$$

The Cauchy problem for (1.4) has been studied extensively. Since the mKdV is completely integrable, the asymptotic behavior can be obtained by inverse scattering techniques such as in Deift and Zhou [2] and references therein. Hayashi and Naumkin [7, 8] proved global existence and modified asymptotics in a neighborhood of a self-similar solution without relying on the complete integrability, with errors bounded in $L_x^p$ for $4<p\leq\infty$. Harrop-Griffiths [5] derived modified scattering in $L_x^2\cap L_x^\infty$ with initial data $u_0\in H_x^{1,1}$ by using a wave packet method in the following sense: If $u$ is a solution to (1.4) with suitable small initial data, there exists a unique

modified final stat $W \in L^{\infty}(\mathbb{R}) \cap L^{2}(\mathbb{R})$ such that for large $t$,

$$\begin{aligned} u(t,x) &= t^{-\frac{1}{3}}\left(t^{-\frac{1}{3}}|x|\right)^{-\frac{1}{4}} \mathrm{Re}\left\{\exp\left(-i\frac{2}{3}\frac{|x|^{\frac{3}{2}}}{\sqrt{3t}}+i\frac{\pi}{4}+\frac{3i}{4\pi}\left|W\left(\sqrt{\frac{|x|}{t}}\right)\right|^{2}\log\left(\frac{|x|^{\frac{3}{2}}}{\sqrt{t}}\right)\right)W\left(\sqrt{\frac{|x|}{t}}\right)\right\} \\ &\quad + O\left(t^{-\frac{1}{3}}\left(t^{-\frac{1}{3}}|x|\right)^{-\frac{3}{8}}\right). \end{aligned} \tag{1.6}$$

We aim to establish global existence and asymptotics for solutions to (1.1) with sufficiently small, regular, and spatially localized initial data. To state our main result we begin with the conserved quantities

$$E_0(t) = \int u dxdy, \tag{1.7}$$

$$E_1(t) = \int u^2 dxdy, \tag{1.8}$$

$$E_2(t) = \int (\partial_x u)^2 + \left(\partial_x^{-1}\partial_y u\right)^2 + \frac{1}{6}u^4 dxdy. \tag{1.9}$$

We denote by $\mathcal{L}$ the linear operator of (1.1):

$$\mathcal{L} := \partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2. \tag{1.10}$$

We note that

$$\partial_x^{-1} f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} f(z)dz$$

holds provided that $f \in \dot{H}_x^{-1}(\mathbb{R})$, where $\partial_x^{-1} := \mathcal{F}^{-1}\frac{1}{i\xi}\mathcal{F}$.

Here we introduce the vector field which commutes with the linear operator $\mathcal{L}$

$$L_x := x - 3t\partial_x^2 - t\partial_x^{-2}\partial_y^2, \tag{1.11}$$

or the equivalent form when we restrict $y$ to a single frequency $k$

$$L_x(k) := x - 3t\partial_x^2 + t\partial_x^{-2}k^2. \tag{1.12}$$

The operator $L_x$ can not be directly applied when estimating the pointwise decay, hence we introduce the operator $\tilde{L}_x(k)$,

$$\tilde{L}_x(k) := \frac{1}{6}\left(-x + \sqrt{x^2 + 12t^2k^2}\right) + t\partial_x^2. \tag{1.13}$$

If we define the symbols

$$p(x,\xi,k) := x + 3t\xi^2 - t\xi^{-2}k^2, \quad \text{and} \quad q(x,\xi,k) := \frac{1}{6}\left(-x + \sqrt{x^2+12t^2k^2}\right) - t\xi^2,$$

we will have $L_x = p(x,D)$ and $\tilde{L}_x = q(x,D)$. Notice that the division satisfies

$$\left|\frac{q(x,\xi,k)}{p(x,\xi,k)}\right| = \left|\frac{t\xi^2\left[\frac{1}{6t}\left(-x+\sqrt{x^2+12t^2k^2}\right) - \xi^2\right]}{3t\left[\xi^2 - \frac{-x+\sqrt{x^2+12t^2k^2}}{6t}\right]\left[\xi^2 + \frac{x+\sqrt{x^2+12t^2k^2}}{6t}\right]}\right| = \left|\frac{\xi^2}{3\left[\xi^2 + \frac{x+\sqrt{x^2+12t^2k^2}}{6t}\right]}\right| \lesssim 1$$

for every $t, x, \xi \in \mathbb{R}$ and $k \in \mathbb{Z}$. Therefore by the pseudodifferential product formula [26], let $r(x,\xi,k) := \frac{q(x,\xi,k)}{p(x,\xi,k)}$ and $s \in \mathbb{R}$, we have the relation $q(x,\xi,k) = r(x,\xi,k)p(x,\xi,k)$

$$\left\|\tilde{L}_x(k) f(k)\right\|_{L_x^2} \lesssim \left\|r(x,D,k) L_x(k) f(k)\right\|_{L_x^2} + \|f(k)\|_{H_x^s} \lesssim \|L_x(k) f(k)\|_{L_x^2} + \|f(k)\|_{H_x^s}.$$

Summing over different $y$ frequencies, we obtain

$$\left\|\tilde{L}_x(k) f(k)\right\|_{L_x^2 l_k^2} \lesssim \|L_x f\|_{L_{x,y}^2} + \|f\|_{L_{x,y}^2}. \tag{1.14}$$

When $\xi$ is close to 0, $k \neq 0$, and $x > 0$, we have

$$\left|\frac{q(x,\xi,k)}{p(x,\xi,k)}\right| \lesssim \left|\frac{t\xi^2}{x + \sqrt{x^2+12t^2k^2}}\right| \lesssim \left(|k| + \langle x\rangle\right)^{-1}\xi^2.$$

Therefore we have the inequality

$$\left\|\tilde{L}_x(k)\,P_N f(k)\right\|_{L_x^2} \lesssim N^2|k|^{-1}\left(\|L_x P_N f(k)\|_{L_x^2} + \|kP_N f(k)\|_{L_x^2}\right) \tag{1.15}$$

when $N \ll 1$. We also define the operators $\tilde{L}_x^{\pm}(k)$ by factoring the symbol $q(x,\xi)$,

$$\tilde{L}_x^{\pm}(k) := \frac{1}{\sqrt{6}}\left(-x+\sqrt{x^2+12t^2k^2}\right)^{\frac{1}{2}} \mp i\sqrt{t}\partial_x. \tag{1.16}$$

If we examine the operators $\tilde{L}_x^{\pm}(k)$, the operators are hyperbolic when $\frac{1}{\sqrt{6}}\left(-\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}\right)^{\frac{1}{2}} \approx \mp\xi$. Let $v=\frac{x}{t}$. When $v\to\infty$, by Taylor expansion we have $\frac{1}{\sqrt{6}}\left(-v+\sqrt{v^2+12k^2}\right)^{\frac{1}{2}} \approx \frac{|k|}{v}$ . When $v\to-\infty$, we have $\frac{1}{\sqrt{6}}\left(-v+\sqrt{v^2+12k^2}\right)^{\frac{1}{2}} \approx \frac{1}{\sqrt{3}}\sqrt{-v}\left[1+\frac{3k^2}{v^2}\right] \approx \frac{|v|^{\frac{1}{2}}}{\sqrt{3}}$. Therefore when $t$ is large and $x \gg 1$, the equation has asymptotic profile similar to the short-pulse equation [23], and in the region $x \ll -1$, the equation has asymptotic profile similar to the mKdV equation [5].

We define the time-dependent space $X$ as

$$\|u\|_X^2 = \|u\|_{L^2}^2 + \left\|\partial_x^5 u\right\|_{L^2}^2 + \left\|\partial_y^4 u\right\|_{L^2}^2 + \|L_x\partial_x u\|_{L^2}^2 + \left\|\partial_x^{-1}\partial_y u\right\|_{L^2}^2. \tag{1.17}$$

**Theorem 1.** *Assume that the initial data $u_0$ at time 0 satisfies*

$$\|u_0\|_X \le \epsilon \ll 1. \tag{1.18}$$

*Then, there exists a unique global solution $u$ which satisfies the bound*

$$\|u(t)\|_X \le \epsilon\langle t\rangle^{C\epsilon}, \tag{1.19}$$

*as well as the pointwise bound*

$$\|u(t,x,y)\|_{L_x^\infty H_y^1}, \|u(t,x,y)\|_{L_x^\infty H_y^1} \lesssim \epsilon\langle t\rangle^{-\frac{1}{2}}. \tag{1.20}$$

*Further, we have the following asymptotics as $t\to+\infty$*

$$u(t,x,y) = \frac{2}{\sqrt{t}}\sum_{k\in\mathbb{Z}} Re\left\{\exp\left(\frac{2t}{3\sqrt{6}}\sqrt{-\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}}\left(2\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}\right)\right)G\left(t,\frac{x}{t},k\right)\right\}e^{iky}+O\left(t^{-\frac{2}{3}??}\right), \tag{1.21}$$

*where $G$ satisfying the equation*

$$i\partial_t G(t,v,k) = t^{-1}\sqrt{-v+\sqrt{v^2+12k^2}}\sum_{(k_1,k_2,k_3)\in\Upsilon(k)} G(t,v,k_1)\,\overline{G}(t,v,k_2)\,G(t,v,k_3), \tag{1.22}$$

*and*

$$\Upsilon(v,k) = \left\{\begin{array}{c}(k_1,k_2,k_3)\in\mathbb{Z}^3: \quad k_1-k_2+k_3=k,\\ \sqrt{-v+\sqrt{v^2+12k^2}}-\sqrt{-v+\sqrt{v^2+12k_1^2}}+\sqrt{-v+\sqrt{v^2+12k_2^2}}-\sqrt{-v+\sqrt{v^2+12k_3^2}}=0.\end{array}\right\}$$

## 2. Notations and local well-posedness

We define the Fourier transform of a Schwartz function $f\in\mathcal{S}_x(\mathbb{R})$ to be

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}}\int f(x)e^{-ix\xi}dx, \tag{2.1}$$

with inverse

$$\check{f}(x) = \mathcal{F}^{-1}f(x) = \frac{1}{\sqrt{2\pi}}\int f(\xi)e^{ix\xi}d\xi. \tag{2.2}$$

Let $\mathcal{X}(x)\in C_0^\infty(\mathbb{R})$ be a real-valued, even function satisfying $0\le\mathcal{X}\le1$, supported on $(-2,2)$ and identically 1 on $[-1,1]$. For $N\in 2^{\mathbb{Z}}$, we define

$$P_{\le N}u = \mathcal{F}^{-1}\mathcal{X}\left(N^{-1}\xi\right)\widehat{u}, \quad P_{>N}u = u-P_{\le N}u, \quad P_N u = P_{\le N}u - P_{\le\frac{N}{2}}u, \quad P_{N<\cdot\le M}u = P_{\le M}P_{>N}u. \tag{2.3}$$

We also define the projections to positive and negative frequencies

$$P_{\pm}u = \mathcal{F}^{-1}\mathbf{1}_{(0,\infty)}(\xi)\widehat{u}. \tag{2.4}$$

We recall the Bernstein inequality, for $1 \leq p \leq q \leq \infty$,

$$\|P_N u\|_{L_x^q} \lesssim N^{\frac{1}{p}-\frac{1}{q}} \|P_N u\|_{L_x^p}. \tag{2.5}$$

We define the frequency localization of $y$ frequency in a similar way, we use $u(t,x,y)$ to denote the original function and $u(t,x,k)$ to denote the Fourier trnasform with respect to $y$.

$$u(t,x,k) = \frac{1}{\sqrt{2\pi}} \int e^{-iky} u(t,x,y)dy.$$

We define the frequency projection of $y$ frequency by $Q$ in the following sense

$$Q_{\leq N} u(t,x,y) := \sum_{|k|\leq N} u(t,x,k)e^{iky}.$$

**Lemma 2.** *The equation (1.1) is locally-wellposed in the space $u \in H_x^5 L_y^2 \cap L_x^2 H_y^4$, if $u_0 \in X$.*

*Proof.* Here we use the frequency envelope technique introduced by Tao [25] to prove local well-posedness. We start from the iteration scheme such that

$$\partial_t u_1 + \partial_x^3 u_1 - \partial_x^{-1}\partial_y^2 u_1 = 0 \tag{2.6}$$

and for $n \geq 2$

$$\partial_t u_n + \partial_x^3 u_n - \partial_x^{-1}\partial_y^2 u_n = u_{n-1}^2 \partial_x u_n \tag{2.7}$$

with initial data $u_n(0) = u_0$. (2.6) and (2.7) are linear equations, hence the solutions $u_n$ exists for $n \in \mathbb{N}$. We apply the fractional Leibnitz rule

$$\left(\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2\right) \partial_x^5 u_n = 2u_{n-1}\left(\partial_x^5 u_{n-1}\right) \partial_x u_n + u_{n-1}^2 \partial_x \partial_x^5 u + \mathrm{err}_1(t,x,y).$$

$$\left(\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2\right) \partial_y^3 u_n = 2u_{n-1}\left(\partial_y^4 u_{n-1}\right) \partial_x u_n + u_{n-1}^2 \partial_x \partial_y^4 u + \mathrm{err}_2(t,x,y).$$

Multiplying the equation with $\partial_x^4 u_n$ and integrating over $x$ and $y$, and using the integration by parts

$$\int \left(\partial_x^5 u_n\right)\left(u_{n-1}^2 \partial_x \partial_x^5 u_n\right) dxdy = -\int u_{n-1}\left(\partial_x u_{n-1}\right)\left(\partial_x^5 u_n\right)^2 dxdy,$$

and

$$\int \left(\partial_y^4 u_n\right)\left(u_{n-1}^2 \partial_x \partial_y^3 u_n\right) dxdy = -\int u_{n-1}\left(\partial_x u_{n-1}\right)\left(\partial_y^4 u_n\right)^2 dxdy.$$

We obtain the bounds

$$\partial_t \left\|\partial_x^5 u_n\right\|_{L_{x,y}^2}^2 \lesssim \|u_{n-1}\|_{L_{x,y}^\infty} \left\|\partial_x^5 u_{n-1}\right\|_{L_{x,y}^2} \|\partial_x u_n\|_{L_{x,y}^\infty} \left\|\partial_x^5 u_n\right\|_{L_{x,y}^2} + \|u_{n-1}\|_{L_{x,y}^\infty} \|\partial_x u_{n-1}\|_{L_{x,y}^\infty} \left\|\partial_x^5 u_n\right\|_{L_{x,y}^2}^2,$$

$$\partial_t \left\|\partial_y^4 u_n\right\|_{L_{x,y}^2}^2 \lesssim \|u_{n-1}\|_{L_{x,y}^\infty} \left\|\partial_y^4 u_{n-1}\right\|_{L_{x,y}^2} \|\partial_x u_n\|_{L_{x,y}^\infty} \left\|\partial_y^4 u_n\right\|_{L_{x,y}^2} + \|u_{n-1}\|_{L_{x,y}^\infty} \|\partial_x u_{n-1}\|_{L_{x,y}^\infty} \left\|\partial_y^4 u_n\right\|_{L_{x,y}^2}^2.$$

Here we use the interpolation inequalities

$$\|f\|_{L_{x,y}^\infty} \lesssim \left\|\partial_x^5 f\right\|_{L_{x,y}^2}^{\frac{1}{10}} \left\|\partial_y^4 f\right\|_{L_{x,y}^2}^{\frac{1}{8}} \|f\|_{L_{x,y}^2}^{\frac{31}{40}}, \quad \|\partial_x f\|_{L_{x,y}^\infty} \lesssim \left\|\partial_x^5 f\right\|_{L_{x,y}^2}^{\frac{3}{10}} \left\|\partial_y^4 f\right\|_{L_{x,y}^2}^{\frac{1}{8}} \|f\|_{L_{x,y}^2}^{\frac{23}{40}},$$

hence we obtain

$$\begin{aligned}
&\partial_t \left(\left\|\partial_x^5 u_n\right\|_{L_{x,y}^2}^2 + \left\|\partial_y^4 u_n\right\|_{L_{x,y}^2}^2\right) \\
&\lesssim \|u_{n-1}\|_{L_{x,y}^2}^{\frac{31}{40}} \left(\left\|\partial_x^5 u_{n-1}\right\|_{L_{x,y}^2} + \left\|\partial_y^4 u_{n-1}\right\|_{L_{x,y}^2}\right)^{\frac{49}{40}} \|u_n\|_{L_{x,y}^2}^{\frac{23}{40}} \left(\left\|\partial_x^5 u_n\right\|_{L_{x,y}^2} + \left\|\partial_y^4 u_n\right\|_{L_{x,y}^2}\right)^{\frac{57}{40}} \\
&\quad + \|u_{n-1}\|_{L_{x,y}^2}^{\frac{44}{40}} \left(\left\|\partial_x^5 u_{n-1}\right\|_{L_{x,y}^2} + \left\|\partial_y^4 u_{n-1}\right\|_{L_{x,y}^2}\right)^{\frac{26}{40}} \left(\left\|\partial_x^5 u_n\right\|_{L_{x,y}^2}^2 + \left\|\partial_y^4 u_n\right\|_{L_{x,y}^2}^2\right).
\end{aligned}$$

By mathematical induction, if we have a large number $M$ such that

$$\left\|\partial_x^5 u_{n-1}\right\|_{L_{x,y}^2}^2 + \left\|\partial_y^4 u_{n-1}\right\|_{L_{x,y}^2}^2 \leq M^2, \tag{2.8}$$

then we have the following bound by applying Grönwall's inequality

$$\left\|\partial_x^5 u_n(t)\right\|_{L^2_{x,y}}^2 + \left\|\partial_y^4 u_n(t)\right\|_{L^2_{x,y}}^2 \leq \left(\left\|\partial_x^5 u_0\right\|_{L^2_{x,y}}^2 + \left\|\partial_y^4 u_0\right\|_{L^2_{x,y}}^2\right) \exp\left(CM^2 t\right). \tag{2.9}$$

The constant $C$ is a fixed number for every function $u_n$. Hence we may choose a time interval $I$ such that for $\forall t \in I$,

$$\left(\left\|\partial_x^5 u_0\right\|_{L^2_{x,y}}^2 + \left\|\partial_y^4 u_0\right\|_{L^2_{x,y}}^2\right) \exp\left(CM^2 t\right) \leq M^2.$$

For every $n \in \mathbb{N}$, we have

$$\left\|\partial_x^5 u_n(t)\right\|_{L^2_{x,y}}^2 + \left\|\partial_y^4 u_n(t)\right\|_{L^2_{x,y}}^2 \leq M^2 \tag{2.10}$$

for $t \in I$. We prove the convergence of $u_n$ by the following computations, by equation (2.7),

$$\begin{aligned}\left(\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2\right)(u_n - u_{n-1}) &= u_{n-1}^2 \partial_x u_n - u_{n-2}^2 \partial_x u_{n-1} \\ &= u_{n-1}^2 \partial_x (u_n - u_{n-1}) + \left(u_{n-1}^2 - u_{n-2}^2\right) \partial_x u_{n-1}.\end{aligned}$$

We obtain the bound

$$\begin{aligned}\partial_t \|u_n - u_{n-1}\|_{L^2_{x,y}}^2 &\lesssim \|u_{n-1}\|_{L^\infty_{x,y}} \|\partial_x u_{n-1}\|_{L^\infty_{x,y}} \|u_n - u_{n-1}\|_{L^2_{x,y}}^2 \\ &\quad + \left(\|u_{n-1}\|_{L^\infty_{x,y}} + \|u_{n-2}\|_{L^\infty_{x,y}}\right) \|\partial_x u_{n-1}\|_{L^\infty_{x,y}} \|u_n - u_{n-1}\|_{L^2_{x,y}} \|u_{n-1} - u_{n-2}\|_{L^2_{x,y}}.\end{aligned} \tag{2.11}$$

Applying the mathematical induction and Grönwall's inequality again, by (2.10), we have

$$\|(u_n - u_{n-1})(t)\|_{L^2_{x,y}} \leq CM^2 \exp\left(CM^2 t\right) \|u_{n-1} - u_{n-2}\|_{L^\infty_t(I; L^2_{x,y})}. \tag{2.12}$$

Again, if we pick the interval $I$ small enough, we will have

$$\|u_n - u_{n-1}\|_{L^\infty_t(I; L^2_{x,y})} \leq \frac{1}{2} \|u_{n-1} - u_{n-2}\|_{L^\infty_t(I; L^2_{x,y})}, \tag{2.13}$$

and by contraction principle, the functions $u_n$ converges to some function $u$ in $L^\infty_t\left(I; L^2_{x,y}\right)$.

The uniqueness can be obtained by the same inequality, if there exist two solution $u$, $v$ to the same initial data $u_0$,

$$\begin{aligned}\left(\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2\right)(u - v) &= u^2 \partial_x u - v^2 \partial_x v \\ &= u^2 \partial_x (u - v) + \left(u^2 - v^2\right) \partial_x v.\end{aligned}$$

Hence we have the inequality

$$\partial_t \|u - v\|_{L^2_{x,y}}^2 \lesssim \left(\|u\|_{L^\infty_{x,y}} + \|v\|_{L^\infty_{x,y}}\right)\left(\|\partial_x u\|_{L^\infty_{x,y}} + \|\partial_x v\|_{L^\infty_{x,y}}\right) \|u - v\|_{L^2_{x,y}}^2. \tag{2.14}$$

Applying Grönwall's inequality, we obtain the uniqueness

$$\|u - v\|_{L^\infty_t(I; L^2_{x,y})} \lesssim \|u(0) - v(0)\|_{L^2_{x,y}} = 0.$$

$\square$

## 3. Pointwise estimates

We decompose $u$ into positive and negative frequencies:

$$u = u^+ + u^-, \quad u^\pm := P_x^\pm u.$$

Because $u$ is real valued, $u^+ = \overline{u^-}$ and $u = \mathrm{Re} u^+$. Moreover,

$$\left\|u^+(t)\right\|_{L^2_{x,y}} = \left\|u^-(t)\right\|_{L^2_{x,y}} = \frac{1}{\sqrt{2}} \|u(t)\|_{L^2_{x,y}}.$$

For $t \geq 1$, we further decompose $u^+$ into its hyperbolic and elliptic parts for a single $y$-frequency $k$ and $k \neq 0$,

$$u^{\mathrm{hyp},+}(k) = \sum_N u_N^{\mathrm{hyp},+}(k), \quad u^{\mathrm{ell},+}(k) = u(k) - u^{\mathrm{hyp},+}(k).$$

The hyperbolic part $u_N^{\mathrm{hyp},+}(k)$ is given by

$$\text{when } N > \sqrt{\frac{|k|}{3}}, \quad u_N^{\mathrm{hyp},+}(k) := \mathcal{X}\left(-\nu 3tN^2 \leq \cdot \leq -\nu^{-1} 3tN^2\right) u_N^+(k), \tag{3.1}$$

$$\text{when } N < \sqrt{\frac{|k|}{3}}, \quad u_N^{\text{hyp},+}(k) := \mathcal{X}\left(\nu^{-1}\frac{tk^2}{N^2} \leq \cdot \leq \nu\frac{tk^2}{N^2}\right)u_N^+(k), \tag{3.2}$$

where $\nu$ is some positive parameter. We observe that $u^{\text{hyp}} = u^{\text{hyp},+} + u^{\text{hyp},-} = 2\text{Re}\left(u^{\text{hyp},+}\right)$.

**Lemma 3.** *If $\mathcal{X} \in C_0^\infty$ and $R > 0$, we define $\mathcal{X}_R(x) = \mathcal{X}\left(R^{-1}x\right)$, then for $1 \leq p \leq \infty$ we have the estimate*

$$\left\|\left(1 - P_{\frac{N}{4}\leq\cdot\leq 4N}\right)\mathcal{X}_R P_N u\right\|_{L_x^p} \lesssim_j \langle RN\rangle^{-j}\|P_N u\|_{L_x^p}. \tag{3.3}$$

See [5] for the proof. So $\mathcal{X}_R P_N$ is localized at frequncies $\sim N$ whenever $RN \gg 1$.

Here we define the frequency projection of $y$ by $Q$.

For $t \geq 1$, we have the following elliptic and hyperbolic estimates:

### 3.1. For $t \geq 1$, $N > \sqrt{\frac{|k|}{3}}$, and elliptic area.

**Lemma 4.** *For $t \geq 1$ we have the bounds for the elliptic area and mKdV-like case $|k| < 3N^2$ ( $|k| \neq 0$ ),*

$$\left\|Q_{\leq 3N^2}u_N^{\text{ell}}\right\|_{L_x^\infty H_y^1} \lesssim t^{-1}N^{-\frac{1}{2}}\|u(t)\|_X \tag{3.4}$$

$$\left\|Q_{\leq 3N^2}\partial_x u_N^{\text{ell}}\right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{2}{3}}N^{-\frac{1}{2}}\|u(t)\|_X, \tag{3.5}$$

*and*

$$\left\|Q_{\leq 3N^2}u_N^{\text{ell}}\right\|_{L_{x,y}^2} \lesssim t^{-1}N^{-3}\|u(t)\|_X. \tag{3.6}$$

*Proof.* When $N > \sqrt{\frac{|k|}{3}}$, we have

$$\left\|\left(|x| + 3tN^2\right)Nu_N^{\text{ell}}(k)\right\|_{L_x^2} \lesssim \left\|L_x(k)\partial_x u_N^{\text{ell}}(k)\right\|_{L_x^2} + \left\|u_N^{\text{ell}}(k)\right\|_{L_x^2}. \tag{3.7}$$

By the relation $|k| \lesssim N^2$

$$\begin{aligned}|k|\left\|u_N^{\text{ell}}(k)\right\|_{L_x^\infty} &\lesssim |k|N^{\frac{1}{2}}\left\|u_N^{\text{ell}}(k)\right\|_{L_x^2} \lesssim |k|t^{-1}N^{-\frac{5}{2}}\left\|tN^3u_N^{\text{ell}}(k)\right\|_{L_x^2}\\ &\lesssim t^{-1}N^{-\frac{1}{2}}\left(\left\|L_x(k)\partial_x u_N^{\text{ell}}(k)\right\|_{L_x^2} + \left\|u_N^{\text{ell}}(k)\right\|_{L_x^2}\right).\end{aligned}$$

Hence summing over $k$ in $l_k^2$, we obtain the bound for where the $y$ frequency less that $3N^2$ we obtain the bound. Similarly, we have

$$\begin{aligned}|k|\left\|\partial_x u_N^{\text{ell}}(k)\right\|_{L_x^\infty} &\lesssim |k|N^{\frac{3}{2}}\left\|u_N^{\text{ell}}(k)\right\|_{L_x^2} \lesssim t^{-\frac{2}{3}}N^{-\frac{1}{2}}\left\|tN^3u_N^{\text{ell}}(k)\right\|_{L_x^2}^{\frac{2}{3}}\left\||k|^3u_N^{\text{ell}}(k)\right\|_{L_x^2}^{\frac{1}{3}}\\ &\lesssim t^{-\frac{2}{3}}N^{-\frac{1}{2}}\left(\left\|L_x(k)\partial_x u_N^{\text{ell}}(k)\right\|_{L_x^2} + \left\|u_N^{\text{ell}}(k)\right\|_{L_x^2} + \left\||k|^3u_N^{\text{ell}}(k)\right\|_{L_x^2}\right).\end{aligned}$$

For where the $y$ frequency less that $3N^2$we summing over $k$.

Also, we have the $L_{x,y}^2$ bound

$$\left\|Q_{\leq 3N^2}u_N^{\text{ell}}\right\|_{L_{x,y}^2} \lesssim t^{-1}N^{-3}\left\|tN^3u_N^{\text{ell}}\right\|_{L_{x,y}^2} \lesssim t^{-1}N^{-3}\left(\|L_x\partial_x u\|_{L_{x,y}^2} + \left\|u_N^{\text{ell}}\right\|_{L_{x,y}^2}\right). \tag{3.8}$$

□

### 3.2. For $t \geq 1$, $N < \sqrt{\frac{|k|}{3}}$ and elliptic area.

**Lemma 5.** *For $t \geq 1$ we have the bounds for the elliptic and short-pulse like case $|k| > 3N^2$,*

$$\left\|Q_{\geq 3N^2}u_N^{\text{ell}}\right\|_{L_x^\infty H_y^1} \lesssim t^{-1}\min\left\{N^{\frac{3}{2}}, N^{-\frac{1}{2}}\right\}\|u(t)\|_X, \tag{3.9}$$

$$\left\|Q_{\geq 3N^2}\partial_x u_N^{\text{ell}}\right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{2}{3}}\min\left\{N^{\frac{5}{2}}, N^{-\frac{1}{2}}\right\}\|u(t)\|_X, \tag{3.10}$$

*and*

$$\left\|Q_{\geq 3N^2}u_N^{\text{ell}}\right\|_{L_{x,y}^2} \lesssim t^{-\frac{3}{4}}\min\left\{N^{-1}, N^{\frac{1}{2}}\right\}\|u(t)\|_X. \tag{3.11}$$

*Proof.* When $N < \sqrt{\frac{|k|}{3}}$, we have

$$\left\|\left(|x| + t\frac{k^2}{N^2}\right) N u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} \lesssim \left\|L_x(k)\partial_x u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} + \left\|u_N^{\mathrm{ell}}(k)\right\|_{L_x^2}.$$

For $N \le 1$,

$$\begin{aligned} |k| \left\|u_N^{\mathrm{ell}}(k)\right\|_{L_x^\infty} &\lesssim |k| N^{\frac{1}{2}} \left\|u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} \lesssim |k|^{-1} t^{-1} N^{\frac{3}{2}} \left\|t\frac{k^2}{N} u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} \\ &\lesssim t^{-1} N^{\frac{3}{2}} \left(\left\|L_x(k)\partial_x u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} + \left\|u_N^{\mathrm{ell}}(k)\right\|_{L_x^2}\right). \end{aligned}$$

For $1 < N \le \sqrt{\frac{|k|}{3}}$, we have $|k|^{-1} \lesssim N^{-2}$

$$\begin{aligned} |k| \left\|u_N^{\mathrm{ell}}(k)\right\|_{L_x^\infty} &\lesssim |k| N^{\frac{1}{2}} \left\|u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} \lesssim |k|^{-1} t^{-1} N^{\frac{3}{2}} \left\|t\frac{k^2}{N} u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} \\ &\lesssim t^{-1} N^{-\frac{1}{2}} \left(\left\|L_x(k)\partial_x u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} + \left\|u_N^{\mathrm{ell}}(k)\right\|_{L_x^2}\right). \end{aligned}$$

Therefore we have the bound for high $y$ frequency $|k| > 3N^2$after summing over different $k$ in $l_k^2$.

For $N \le 1$,

$$\begin{aligned} |k| \left\|\partial_x u_N^{\mathrm{ell}}(k)\right\|_{L_x^\infty} &\lesssim |k| N^{\frac{3}{2}} \left\|u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} \lesssim |k|^{-\frac{1}{2}} t^{-1} N^{\frac{5}{2}} \left\|t\frac{k^2}{N} u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} \\ &\lesssim t^{-1} N^{\frac{5}{2}} \left(\left\|L_x(k)\partial_x u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} + \left\|u_N^{\mathrm{ell}}(k)\right\|_{L_x^2}\right). \end{aligned}$$

For $1 < N \le \sqrt{\frac{|k|}{3}}$, we have $|k|^{-1} \lesssim N^{-2}$

$$\begin{aligned} |k| \left\|\partial_x u_N^{\mathrm{ell}}(k)\right\|_{L_x^\infty} &\lesssim |k| N^{\frac{3}{2}} \left\|u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} \lesssim |k|^{-\frac{4}{3}} t^{-\frac{2}{3}} N^{\frac{13}{6}} \left\|t\frac{k^2}{N} u_N^{\mathrm{ell}}(k)\right\|_{L_x^2}^{\frac{2}{3}} \left\||k|^3 u_N^{\mathrm{ell}}(k)\right\|_{L_x^2}^{\frac{1}{3}} \\ &\lesssim t^{-1} N^{-\frac{1}{2}} \left(\left\|L_x(k)\partial_x u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} + \left\|u_N^{\mathrm{ell}}(k)\right\|_{L_x^2} + \left\||k|^3 u_N^{\mathrm{ell}}(k)\right\|_{L_x^2}\right). \end{aligned}$$

Summing over different $k$ in $l_k^2$, we obtain the bound.

We have the $L_{x,y}^2$ bounds for $1 \le N < \sqrt{\frac{|k|}{3}}$

$$\left\|Q_{\ge 3N^2} u_N^{\mathrm{ell}}\right\|_{L_{x,y}^2} \lesssim t^{-1} N (3N^2)^{-2} \left\|tN^{-1}\partial_y^2 u_N^{\mathrm{ell}}\right\|_{L_{x,y}^2} \lesssim t^{-1} N^{-1} \left(\left\|L_x \partial_x u_N^{\mathrm{ell}}\right\|_{L_{x,y}^2} + \left\|u_N^{\mathrm{ell}}\right\|_{L_{x,y}^2}\right), \tag{3.12}$$

and for $N < 1$

$$\left\|Q_{\ge 3N^2} u_N^{\mathrm{ell}}\right\|_{L_{x,y}^2} \lesssim t^{-\frac{3}{4}} N^{\frac{1}{2}} \left\|tN^{-1}\partial_y^2 u_N^{\mathrm{ell}}\right\|_{L_{x,y}^2}^{\frac{3}{4}} \left\|Q_{\ge 3N^2} \partial_y^{-6} u_N^{\mathrm{ell}}\right\|_{L_{x,y}^2}^{\frac{1}{4}} \lesssim t^{-\frac{3}{4}} N^{\frac{1}{2}} \left(\left\|L_x \partial_x u_N^{\mathrm{ell}}\right\|_{L_{x,y}^2} + \left\|u_N^{\mathrm{ell}}\right\|_{L_{x,y}^2}\right). \tag{3.13}$$

□

**Lemma 6.** *We have the following bounds for the elliptic region*

$$\left\|u^{\mathrm{ell}}\right\|_{L_x^\infty H_y^1} \lesssim t^{-1} \left\|u(t)\right\|_X, \quad \left\|\partial_x u^{\mathrm{ell}}\right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{2}{3}} \left\|u(t)\right\|_X \tag{3.14}$$

*and*

$$\left\|u^{\mathrm{ell}}\right\|_{L_{x,y}^2} \lesssim t^{-\frac{3}{4}} \left\|u(t)\right\|_X. \tag{3.15}$$

*Proof.* By (3.4), (3.5), (3.6), (3.9),(3.10),(3.11), we obtain the inequalities:

$$\begin{aligned} &\left\|u_N^{\mathrm{ell}}\right\|_{L_x^\infty H_y^1} \lesssim t^{-1} \min\left\{N^{\frac{3}{2}}, N^{-\frac{1}{2}}\right\} \left\|u(t)\right\|_X, \quad \left\|\partial_x u_N^{\mathrm{ell}}\right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{2}{3}} \min\left\{N^{\frac{5}{2}}, N^{-\frac{1}{2}}\right\} \left\|u(t)\right\|_X, \\ &\left\|u_N^{\mathrm{ell}}\right\|_{L_{x,y}^2} \lesssim t^{-\frac{3}{4}} \min\left\{N^{\frac{1}{2}}, N^{-2}\right\} \left\|u(t)\right\|_X. \end{aligned} \tag{3.16}$$

Summing over different dyadic number $N$, we obtain the bounds.

□

### 3.3. For $t \geq 1$, $N > \sqrt{\frac{|k|}{3}}$, and hyperbolic area.

**Lemma 7.** *For $t \geq 1$ we have the bounds for the hyperbolic and mKdV-like case $|k| < 3N^2$, ( $|k| \neq 0$ )*

$$\left\| Q_{\leq 3N^2} u_N^{\mathrm{hyp},+} \right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{1}{2}} N^{-1} \left\| u(t) \right\|_X , \tag{3.17}$$

*and*

$$\left\| Q_{\leq 3N^2} \partial_x u_N^{\mathrm{hyp},+} \right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{1}{2}} N^{-\frac{1}{2}} \left\| u(t) \right\|_X . \tag{3.18}$$

*Proof.* For the hyperbolic part we use the formula

$$\begin{aligned} &\left\| \tilde{L}_x^- f \right\|_{L_x^2}^2 + 2\sqrt{t} \mathrm{Im} \int \frac{1}{\sqrt{6}} \left( -x + \sqrt{x^2 + 12t^2k^2} \right)^{\frac{1}{2}} f \overline{\partial_x f} dx \\ =& \left\| \frac{1}{\sqrt{6}} \left( -x + \sqrt{x^2 + 12t^2k^2} \right)^{\frac{1}{2}} f \right\|_{L_x^2}^2 + t \left\| \partial_x f \right\|_{L_x^2}^2 . \end{aligned} \tag{3.19}$$

Apply the equality to $f = \tilde{L}_x^+ (k) u_N^{\mathrm{hyp},+}(k)$. By a direct computation, we have

$$\tilde{L}_x^- \tilde{L}_x^+ = \tilde{L}_x - \frac{i}{2} \sqrt{\frac{t}{6}} \left( -x + \sqrt{x^2 + 12t^2k^2} \right)^{\frac{1}{2}} \left( x^2 + 12t^2k^2 \right)^{-\frac{1}{2}} , \tag{3.20}$$

$$\partial_x \tilde{L}_x^+ = \tilde{L}_x^+ \partial_x + \frac{1}{2} \sqrt{\frac{1}{6}} \left( -x + \sqrt{x^2 + 12t^2k^2} \right)^{\frac{1}{2}} \left( x^2 + 12t^2k^2 \right)^{-\frac{1}{2}} . \tag{3.21}$$

Here we assume that $k \neq 0$ ,we will have $\left| \left( -x + \sqrt{x^2 + 12t^2k^2} \right)^{\frac{1}{2}} \left( x^2 + 12t^2k^2 \right)^{-\frac{1}{2}} \right| \lesssim \left( x^2 + 12t^2k^2 \right)^{-\frac{1}{4}} \lesssim t^{-\frac{1}{2}}$.

We will use the computation

$$\begin{aligned} &\mathrm{Im} \int \left( -x + \sqrt{x^2 + 12t^2k^2} \right)^{\frac{1}{2}} \left( \tilde{L}_x^+ u_N^{\mathrm{hyp},+}(k) \right) \overline{\partial_x \tilde{L}_x^+ u_N^{\mathrm{hyp},+}(k)} dx \\ =& \mathrm{Im} \int \left( -x + \sqrt{x^2 + 12t^2k^2} \right)^{\frac{1}{4}} \left( \tilde{L}_x^+ u_N^{\mathrm{hyp},+}(k) \right) \overline{\partial_x \left[ \left( -x + \sqrt{x^2 + 12t^2k^2} \right)^{\frac{1}{4}} \tilde{L}_x^+ u_N^{\mathrm{hyp},+}(k) \right]} dx \\ =& - \mathrm{Re} \int \xi \left| \mathcal{F} \left[ \left( -x + \sqrt{x^2 + 12t^2k^2} \right)^{\frac{1}{4}} \tilde{L}_x^+ u_N^{\mathrm{hyp},+}(k) \right] (\xi) \right|^2 d\xi. \end{aligned}$$

Since $u_N^{\mathrm{hyp},+}$ is localized to positive frequencies, $\left( -x + \sqrt{x^2 + 12t^2k^2} \right)^{\frac{1}{4}} \tilde{L}_x^+ u_N^{\mathrm{hyp},+}$ is also localized to positive frequencies up to rapid decaying tails by commutator estimates of pseudo-differential operators. Therefore we have to estimate the bound , by (3.21), (3.19) and (3.20), we have

$$\begin{aligned} &\left\| \tilde{L}_x^+ \partial_x u_N^{\mathrm{hyp},+}(k) \right\|_{L_x^2} \\ \lesssim& \left\| \partial_x \tilde{L}_x^+ u_N^{\mathrm{hyp},+}(k) \right\|_{L_x^2} + \left\| \left( \frac{-x + \sqrt{x^2 + 12t^2k^2}}{x^2 + 12t^2k^2} \right)^{\frac{1}{2}} u_N^{\mathrm{hyp},+}(k) \right\|_{L_x^2} \\ \lesssim& t^{-\frac{1}{2}} \left( \left\| \tilde{L}_x^- \tilde{L}_x^+ u_N^{\mathrm{hyp},+}(k) \right\|_{L_x^2} + \left\| u_N^{\mathrm{hyp},+}(k) \right\|_{L_x^2} \right) \\ \lesssim& t^{-\frac{1}{2}} \left( \left\| \tilde{L}_x u_N^{\mathrm{hyp},+}(k) \right\|_{L_x^2} + \left\| u_N^{\mathrm{hyp},+}(k) \right\|_{L_x^2} \right) . \end{aligned} \tag{3.22}$$

For $N > \sqrt{\frac{|k|}{3}}$, by (3.22) we have the bound

$$\begin{aligned}
|k| \left\| u_N^{\text{hyp},+}(k) \right\|_{L_x^\infty} &\lesssim \left\| \partial_x e^{-i\phi} u_N^{\text{hyp},+}(k) \right\|_{L_x^2}^{\frac{1}{2}} \left\| |k|^2 u_N^{\text{hyp},+} \right\|_{L_x^2}^{\frac{1}{2}} \\
&\lesssim t^{-\frac{1}{4}} N^{-\frac{1}{2}} \left\| \tilde{L}_x^+ \partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2}^{\frac{1}{2}} \left\| |k|^2 u_N^{\text{hyp},+} \right\|_{L_x^2}^{\frac{1}{2}} \\
&\lesssim t^{-\frac{1}{2}} N^{-\frac{1}{2}} \left( \| u_N(k) \|_{L_x^2} + \left\| \tilde{L}_x u_N(k) \right\|_{L_x^2} \right)^{\frac{1}{2}} \left\| |k|^2 u_N \right\|_{L_x^2}^{\frac{1}{2}}.
\end{aligned}$$

Summing over $k$ in $l_k^2$, we obtain the bound by applying Cauchy inequality

$$\begin{aligned}
\left\| Q_{\leq 3N^2} u_N^{\text{hyp},+} \right\|_{L_x^\infty H_y^1} &\lesssim t^{-\frac{1}{2}} N^{-1} \left[ \sum_k \left( \| \partial_x u_N(k) \|_{L_x^2} + \left\| \tilde{L}_x \partial_x u_N(k) \right\|_{L_x^2} \right) \left\| |k|^2 u_N \right\|_{L_x^2} \right]^{\frac{1}{2}} \\
&\lesssim t^{-\frac{1}{2}} N^{-1} \left[ \sum_k \left( \| \partial_x u_N(k) \|_{L_x^2}^2 + \left\| \tilde{L}_x \partial_x u_N(k) \right\|_{L_x^2}^2 + \left\| |k|^2 u_N \right\|_{L_x^2}^2 \right) \right]^{\frac{1}{2}} \\
&\lesssim t^{-\frac{1}{2}} N^{-1} \left( \| \partial_x u_N \|_{L_{x,y}^2} + \| L_x \partial_x u_N \|_{L_{x,y}^2} + \| u_N \|_{L_x^2 H_y^2} \right).
\end{aligned} \tag{3.23}$$

Similarly, we have

$$|k| \left\| \partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^\infty} \lesssim t^{-\frac{1}{2}} N^{-\frac{1}{2}} \left( \| \partial_x u_N(k) \|_{L_x^2} + \left\| \tilde{L}_x \partial_x u_N(k) \right\|_{L_x^2} \right)^{\frac{1}{2}} \left\| |k|^2 u_N(k) \right\|_{L_x^2}^{\frac{1}{2}}.$$

Hence we have the bound by summing over $k$ in $l_k^2$

$$\left\| Q_{\leq 3N^2} \partial_x u_N^{\text{hyp},+} \right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{1}{2}} N^{-\frac{1}{2}} \left( \| \partial_x u_N \|_{L_{x,y}^2} + \| L_x \partial_x u_N \|_{L_{x,y}^2} + \| u_N \|_{L_x^2 H_y^2} \right). \tag{3.24}$$

Therefore we prove (3.17) and (3.18).

□

### 3.4. For $t \geq 1$, $N < \sqrt{\frac{|k|}{3}}$ and hyperbolic area.

**Lemma 8.** *For $t \geq 1$ we have the bounds for the hyperbolic area and mKdV like case $|k| > 3N^2$*

$$\left\| Q_{\geq 3N^2} u_N^{\text{hyp},+} \right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{1}{2}} \min\left\{ N^{-1}, N^{\frac{1}{2}} \right\} \| u(t) \|_X, \tag{3.25}$$

*and*

$$\left\| Q_{\geq 3N^2} \partial_x u_N^{\text{hyp},+} \right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{1}{2}} \min\left\{ N^{-\frac{1}{2}}, N \right\} \| u(t) \|_X. \tag{3.26}$$

*Proof.* For $1 \leq N < \sqrt{\frac{|k|}{3}}$, the estimatesa are the same as prevous case (3.17) and (3.18),

$$\left\| Q_{\geq 3N^2} u_N^{\text{hyp},+} \right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{1}{2}} N^{-1} \left( \| \partial_x u_N \|_{L_{x,y}^2} + \| L_x \partial_x u_N \|_{L_{x,y}^2} + \| u_N \|_{L_x^2 H_y^2} \right). \tag{3.27}$$

$$\left\| Q_{\geq 3N^2} \partial_x u_N^{\text{hyp},+} \right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{1}{2}} N^{-\frac{1}{2}} \left( \| \partial_x u_N \|_{L_{x,y}^2} + \| L_x \partial_x u_N \|_{L_{x,y}^2} + \| u_N \|_{L_x^2 H_y^2} \right). \tag{3.28}$$

For $N<1$, we use the same inequality with (1.15) and a different bound for the factor $\left\| \left( \frac{-x+\sqrt{x^2+12t^2k^2}}{x^2+12t^2k^2} \right)^{\frac{1}{2}} u_N^{\text{hyp},+}(k) \right\|_{L_x^2}$ in (3.22). In this case, $u_N^{\text{hyp},+}$ is supported in where $x \sim t\frac{k^2}{N^2} > 0$ , hence we have the following inequality

$$\begin{aligned}\left\| \left( \frac{-x+\sqrt{x^2+12t^2k^2}}{x^2+12t^2k^2} \right)^{\frac{1}{2}} u_N^{\text{hyp},+}(k) \right\|_{L_x^2} &\lesssim \left\| \left( \frac{12t^2k^2}{(x^2+12t^2k^2)\left(x+\sqrt{x^2+12t^2k^2}\right)} \right)^{\frac{1}{2}} u_N^{\text{hyp},+}(k) \right\|_{L_x^2} \\ &\lesssim \left\| \left( \frac{t^2k^2N^6}{(t^2k^4+t^2k^2N^4)\sqrt{tk^2+\sqrt{t^2k^4+N^4t^2k^2}}} \right)^{\frac{1}{2}} u_N^{\text{hyp},+}(k) \right\|_{L_x^2} \\ &\lesssim t^{-\frac{1}{2}}|k|^{-1}N^3 \left\| u_N^{\text{hyp},+}(k) \right\|_{L_x^2}.\end{aligned}$$

Therefore we obtain the bound

$$\begin{aligned}\left\| \tilde{L}_x^+ \partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2} &\lesssim t^{-\frac{1}{2}} \left( \left\| \tilde{L}_x^- \tilde{L}_x^+ u_N^{\text{hyp},+}(k) \right\|_{L_x^2} + \left\| \left( \frac{-x+\sqrt{x^2+12t^2k^2}}{x^2+12t^2k^2} \right)^{\frac{1}{2}} u_N^{\text{hyp},+}(k) \right\|_{L_x^2} \right) \\ &\lesssim t^{-\frac{1}{2}} \left( N^2|k|^{-1} \left\| L_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2} + |k|^{-1}N^3 \left\| u_N^{\text{hyp},+}(k) \right\|_{L_x^2} \right) \\ &\lesssim t^{-\frac{1}{2}}|k|^{-1}N \left( \left\| L_x\partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2} + \left\| \partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2} \right).\end{aligned} \tag{3.29}$$

By (3.29), we have

$$\begin{aligned}|k| \left\| u_N^{\text{hyp},+}(k) \right\|_{L_x^\infty} &\lesssim t^{-\frac{1}{4}}N^{-\frac{1}{2}} \left\| \tilde{L}_x^+ \partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2}^{\frac{1}{2}} \left\| |k|^2 u_N^{\text{hyp},+}(k) \right\|_{L_x^2}^{\frac{1}{2}} \\ &\lesssim t^{-\frac{1}{2}}N^{\frac{1}{2}} \left( \left\| L_x\partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2} + \left\| \partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2} \right)^{\frac{1}{2}} \left\| N^{-1}|k| u_N^{\text{hyp},+}(k) \right\|_{L_x^2}^{\frac{1}{2}} \\ &\lesssim t^{-\frac{1}{2}}N^{\frac{1}{2}} \left( \left\| L_x\partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2} + \left\| \partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2} \right)^{\frac{1}{2}} \left\| \partial_x^{-1}|k| u_N^{\text{hyp},+}(k) \right\|_{L_x^2}^{\frac{1}{2}}.\end{aligned}$$

Similarly, we have

$$|k| \left\| \partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^\infty} \lesssim t^{-\frac{1}{2}}N \left( \left\| L_x\partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2} + \left\| \partial_x u_N^{\text{hyp},+}(k) \right\|_{L_x^2} \right)^{\frac{1}{2}} \left\| \partial_x^{-1}|k| u_N^{\text{hyp},+}(k) \right\|_{L_x^2}^{\frac{1}{2}}. \tag{3.30}$$

Finally, summing over $k$ in $l_k^2$, we obtain the bound for the hyperbolic region:

$$\left\| Q_{\geq 3N^2} u_N^{\text{hyp},+} \right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{1}{2}}N^{\frac{1}{2}} \left( \|\partial_x u_N\|_{L_{x,y}^2} + \|L_x\partial_x u_N\|_{L_{x,y}^2} + \left\|\partial_x^{-1}\partial_y u_N\right\|_{L_{x,y}^2} \right). \tag{3.31}$$

$$\left\| Q_{\geq 3N^2} \partial_x u_N^{\text{hyp},+} \right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{1}{2}}N \left( \|\partial_x u_N\|_{L_{x,y}^2} + \|L_x\partial_x u_N\|_{L_{x,y}^2} + \left\|\partial_x^{-1}\partial_y u_N\right\|_{L_{x,y}^2} \right). \tag{3.32}$$

Combining (3.27), (3.28), (3.31) and (3.32) together, we obtain the desired bound.

□

**Lemma 9.** *For $t \geq 1$, in the hyperbolic region we have the estimates:*

$$\left\| u^{\text{hyp},+} \right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{1}{2}} \|u(t)\|_X, \tag{3.33}$$

*and*

$$\left\| \partial_x u^{\text{hyp},+} \right\|_{L_x^\infty H_y^1} \lesssim t^{-\frac{1}{2}} \|u(t)\|_X. \tag{3.34}$$

By (3.17), (3.18), (3.25), and (3.26), we sum over different dyadic $N$ and obtain the bounds.

## 4. Wave packets

We consider the Hamiltonian flow corresponding to (1.1), which is given by

$$(x,\xi) \mapsto \left(x - 3t\xi^2 + tk^2\xi^{-2}, \xi\right).$$

We expect solutions initially localized spatially near 0 and travel along the ray $\Gamma_v := \{x = vt\}$. Where the frequecy $\xi_v$

$$\xi_v = \pm\left(\frac{-v+\sqrt{v^2+12k^2}}{6}\right)^{1/2} = \pm\frac{1}{\sqrt{6}}\left(-\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}\right)^{1/2}.$$

This produces a phase function

$$\begin{aligned}\phi(t,x,k) &= \pm\frac{t}{\sqrt{6}}\int^{x/t}\left(-s+\sqrt{s^2+12k^2}\right)^{1/2}ds\\ &= \pm\frac{t}{\sqrt{6}}\frac{2}{3}\sqrt{-\frac{x}{t}+\sqrt{12k^2+\left(\frac{x}{t}\right)^2}}\left(2\frac{x}{t}+\sqrt{12k^2+\left(\frac{x}{t}\right)^2}\right).\end{aligned}$$

The size of the wave packet should associated with the quatity

$$a_{\xi\xi} = 6t\xi + 2tk^2\xi^{-3}.$$

Therefore the wave packet should have a scaling

$$\lambda_v = |k|^{-\frac{1}{2}}\left(v+\sqrt{v^2+12k^2}\right)^{1/4}\left(v^2+12k^2\right)^{1/4}$$

For $v \in \mathbb{R}$, the wave packet will have the form

$$\Psi_v(t,x,k) := \lambda_v^{-1}e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right),$$

where $\mathcal{X}$ is a smooth function with compact support and $\mathcal{X}(1) = 1$, and $\int_{\mathbb{R}}\mathcal{X}dx = 1$.

$$\begin{aligned}\partial_t\phi &= -\frac{1}{3}\sqrt{\frac{1}{6}}\sqrt{-\frac{x}{t}+\sqrt{12k^2+\left(\frac{x}{t}\right)^2}}\left(\frac{x}{t}+2\sqrt{12k^2+\left(\frac{x}{t}\right)^2}\right)\\ &= -\frac{1}{3}\sqrt{\frac{1}{6}}\sqrt{-\frac{x}{t}+\sqrt{12k^2+\left(\frac{x}{t}\right)^2}}\left[\frac{1}{2}\left(-\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}\right)+\frac{3}{2}\left(\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}\right)\right]\\ &= \phi_x^3 + k^2\phi_x^{-1}\\ \partial_x^3\phi &= -\partial_x^2\frac{1}{\sqrt{6}}\left(-\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}\right)^{1/2}\\ &= -\partial_x\frac{1}{\sqrt{6}}\frac{-1}{2t}\left(\left(\frac{x}{t}\right)^2+12k^2\right)^{-1/2}\sqrt{-\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}}\\ &= -\frac{1}{\sqrt{6}}\frac{1}{2t^2}\left(\left(\frac{x}{t}\right)^2+12k^2\right)^{-3/2}\sqrt{-\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}}\left(\frac{x}{t}+\frac{1}{2}\sqrt{\left(\frac{x}{t}\right)^2+12k^2}\right)\end{aligned}$$

By direct computations, we have the following equations:

$$\partial_t\lambda_v\Psi_v = -i\frac{2}{3}\sqrt{2}\sqrt{\frac{x}{t}+\sqrt{12k^2+\left(\frac{x}{t}\right)^2}}\,|k|\,e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right) - \frac{x+vt}{2\lambda_v t^{\frac{3}{2}}}e^{i\phi}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right).$$

$$\begin{aligned}\partial_x^3\lambda_v\Psi_v =& e^{i\phi}\left(i\phi_{xxx}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)-3\phi_{xx}\phi_x\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+3i\phi_{xx}\lambda_v^{-1}t^{-\frac{1}{2}}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right)\\
&+e^{i\phi}\left(-i\phi_x^3\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)-3\lambda_v^{-1}t^{-\frac{1}{2}}\phi_x^2\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right)\\
&+e^{i\phi}\left(3i\lambda_v^{-2}t^{-1}\phi_x\mathcal{X}''\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+\lambda_v^{-3}t^{-\frac{3}{2}}\mathcal{X}'''\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right).\end{aligned}$$

$$\begin{aligned}\partial_x^{-1}\lambda_v\Psi_v =& \frac{1}{i\phi_x}e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)-\partial_x^{-1}\left\{\frac{\phi_{xx}}{i\phi_x^2}e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+\frac{e^{i\phi}}{i\lambda_v\sqrt{t}\phi_x}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right\}\\
=&\frac{1}{i\phi_x}e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+\frac{e^{i\phi}}{\lambda_v\sqrt{t}\phi_x^2}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\\
&+\partial_x^{-1}\left\{\frac{i\phi_{xx}}{\phi_x^2}e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+2\frac{\phi_{xx}e^{i\phi}}{\lambda_v\sqrt{t}\phi_x^3}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)-\frac{e^{i\phi}}{\lambda_v^2t\phi_x^2}\mathcal{X}''\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right\}\\
=&\frac{-i}{\phi_x}e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+\frac{e^{i\phi}}{\lambda_v\sqrt{t}\phi_x^2}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+\frac{\phi_{xx}}{\phi_x^3}e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+\frac{ie^{i\phi}}{\lambda_v^2t\phi_x^3}\mathcal{X}''\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\\
&+\partial_x^{-1}\left\{-e^{i\phi}\partial_x\left\{\frac{\phi_{xx}}{\phi_x^3}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+\frac{i}{\lambda_v^2t\phi_x^3}\mathcal{X}''\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right\}+2\frac{\phi_{xx}e^{i\phi}}{\lambda_v\sqrt{t}\phi_x^3}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right\}\end{aligned}$$

Hence we have

$$\begin{aligned}&\left(\partial_t+\partial_x^3+\partial_x^{-1}k^2\right)\lambda_v\Psi_v\\
=&\left(\cancel{i\phi_t-i\phi_x^3-i\phi_x^{-1}k^2}\right)e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\\
&+\left(-3\phi_x^2\lambda_v^{-1}t^{-\frac{1}{2}}+k^2\phi_x^{-2}\lambda_vt^{-\frac{1}{2}}-\frac{x+vt}{2\lambda_vt^{\frac{3}{2}}}\right)e^{i\phi}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\\
&+\left(-3\phi_{xx}\phi_x+\frac{\phi_{xx}}{\phi_x^3}\right)e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+i\phi_{xxx}e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\\
&+i\lambda_v^{-2}t^{-1}\left(3\phi_x+\frac{1}{\phi_x^3}\right)e^{i\phi}\mathcal{X}''\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\\
&+3i\lambda_v^{-1}t^{-\frac{1}{2}}\phi_{xx}e^{i\phi}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+e^{i\phi}\lambda_v^{-3}t^{-\frac{3}{2}}\mathcal{X}'''\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\\
&+k^2\partial_x^{-1}\left\{-e^{i\phi}\partial_x\left\{\frac{\phi_{xx}}{\phi_x^3}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+\frac{i}{\lambda_v^2t\phi_x^3}\mathcal{X}''\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right\}+2\frac{\phi_{xx}e^{i\phi}}{\lambda_v\sqrt{t}\phi_x^3}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right\}.\end{aligned}$$

From above computations,$|\phi_{xxx}|\lesssim t^{-2}$, $|\phi_{xx}|\lesssim t^{-1}$, therefore we can see that except ther first and second line, other factors have at least $t^{-1}$ decay.

We have $i\phi_t-i\phi_x^3-i\phi_x^{-1}k^2=0$, and

$$\begin{aligned}&\left(-3\phi_x^2+k^2\phi_x^{-2}\right)-\frac{x+vt}{2t}\\
=&-\frac{1}{2}\left(-\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}\right)+k^2\frac{6}{12k^2}\left(\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}\right)-\frac{x+vt}{2t}\\
=&\frac{x}{t}-\frac{x+vt}{2t}=\frac{x-vt}{2t}.\end{aligned}$$

Hence we have the decay estimate for the approximate solution $\Psi_v$

$$\begin{aligned}&\left(\partial_t+\partial_x^3+\partial_x^{-1}k^2\right)\lambda_v\Psi_v\\
=&\frac{x-vt}{2\lambda_vt^{\frac{3}{2}}}e^{i\phi}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)-\frac{1}{2t}\left(\left(\frac{x}{t}\right)^2+12k^2\right)^{-1/2}\frac{x}{t}e^{i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\\
&+i\lambda_v^{-2}t^{-1}\phi_x^{-1}\frac{x}{t}e^{i\phi}\mathcal{X}''\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+\mathcal{O}_{L_v^\infty}\left(t^{-\frac{3}{2}}\right)\end{aligned}$$

$$\left(\partial_t+\partial_x^3+\partial_x^{-1}k^2\right)\lambda_v\Psi_v=\frac{x-vt}{2\lambda_v t^{\frac{3}{2}}}e^{i\phi}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)+\mathcal{O}_{L_v^\infty}\left(t^{-1}\right).$$

We show that $\Psi_v(t,x,k)$ is essentially frequency localized near $\xi_v$. For $v\in\Omega_\alpha(t):=\{-t^\alpha\le v\le -t^{-\alpha}\}\cup\{t^{-\alpha}\le v\le t^\alpha\}$ and $t\ge 1$, we define by $N_v\in 2^{\mathbb{Z}}$, the nearest dyadic number to $\xi_v$. Then $\frac{\xi_v}{4}<N_v<4\xi_v$ when $N_v>0$.

**Lemma 10.** *For $t\ge 1$ and $v\in\Omega_\alpha(t)$, we have*

$$\left\|\left(1-P_{\frac{N_v}{4}\le\cdot\le 4N_v}\right)\Psi_v(t,v,k)\right\|_{L_v^2}\lesssim_c\left(t^{\frac{1}{2}}\lambda_v\right)^{-c}\tag{4.1}$$

*for any $c\ge 0$.*

*Proof.* From Taylor's theorem, we can write

$$\begin{aligned}\phi(t,x,k)=&\,\phi(t,vt,k)+\phi_x(t,vt,k)(x-vt)+\frac{1}{2}\phi_{xx}(t,vt,k)(x-vt)^2\\&+\int_{vt}^x\frac{(x-r)^2}{2}\phi_{xxx}(t,r,k)\,dr\\=&\,\phi(t,vt,k)+\xi_v(x-vt)+\frac{1}{2}\phi_{xx}(t,vt,k)(x-vt)^2+R\left(\frac{x-tv}{\sqrt{t}\lambda_v},\frac{\sqrt{t}v}{\lambda_v},\frac{k}{v}\right).\end{aligned}$$

Here

$$\begin{aligned}R:=&-\frac{1}{\sqrt{6}}\frac{x^3\lambda_v^2}{4av^{\frac{1}{2}}}\\&\int_0^1(1-\theta)^2\left(\left(\frac{x}{a}\theta+1\right)^2+12k^2\right)^{-3/2}\sqrt{-\left(\frac{x}{a}\theta+1\right)+\sqrt{\left(\frac{x}{a}\theta+1\right)^2+12k^2}}\left(\frac{x}{a}\theta+1+\frac{1}{2}\sqrt{\left(\frac{x}{a}\theta+1\right)^2+12k^2}\right)d\theta.\end{aligned}$$

Changing the variable $r=\frac{x-vt}{t^{\frac{1}{2}}\lambda_v}$, we have

$$\begin{aligned}&\mathcal{F}\left[\Psi_v\right](t,\xi,k)\\=&\frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}\mathcal{X}\left(\frac{x-tv}{t^{\frac{1}{2}}\lambda_v}\right)e^{i\phi(t,x,k)}dx\\=&\frac{1}{\sqrt{2\pi}}t^{\frac{1}{2}}\lambda_v e^{i(\phi(t,tv,k)+tv\xi)}\int e^{-irt^{\frac{1}{2}}\lambda_v(\xi-\xi_v)}\mathcal{X}(r)\,e^{\frac{i}{2\sqrt{2}}r^2+R\left(r,\frac{\sqrt{t}v}{\lambda_v},\frac{k}{v}\right)}dr\\=&t^{\frac{1}{2}}\lambda_v\mathcal{X}_1\left(t^{\frac{1}{2}}\lambda_v(\xi-\xi_v)\right).\end{aligned}$$

Here we use the equality

$$\begin{aligned}&\phi_{xx}(t,vt,k)\,t\lambda_v^2\\=&\frac{1}{2\sqrt{6}}\left(-v+\sqrt{v^2+12k^2}\right)^{\frac{1}{2}}\left(v+\sqrt{v^2+12k^2}\right)^{\frac{1}{2}}|k|^{-1}\\=&\frac{1}{\sqrt{2}}.\end{aligned}$$

$\square$

We define

$$\gamma(t,v,k):=\int_{\mathbb{R}}u(t,x,k)\overline{\Psi_v(t,x,k)}dx.$$

Because $u^{\text{ell}}$ and $u^{\text{hyp},-}$ are frequency localized away form $\xi_v$, we can replace $u$ on the right hand side with $u^{\text{hyp},+}$.

$$\begin{aligned}&\left|\gamma(t,v,k)-\sum_{N\sim N_v}\int u_N^{\text{hyp},+}(t,x,k)\overline{\Psi_v(t,x,k)}dx\right|\\\lesssim&\left\|\left(1-P_{\frac{N_v}{4}\le\cdot\le 4N_v}\right)\Psi_v(t,v,k)\right\|_{L_v^1}\|u(t,x,k)\|_{L_x^\infty}\end{aligned}$$

Hence we have

$$\left\|\gamma(t,v,k)-\sum_{N\sim N_v}\int u_N^{\mathrm{hyp},+}(t,x,k)\overline{\Psi_v(t,x,k)}dx\right\|_{L_v^\infty h_k^1}\lesssim t^{-\frac12}\left(t^{\frac12}v^{\frac14}\right)^{-c}\|u(t)\|_X .$$

**Lemma 11.** *For $t\geq 1$,*

$$\|\gamma(t,v,k)\|_{L_v^\infty}\lesssim t^{\frac12}\lambda_k\|u(t,x,k)\|_{L_x^\infty}. \tag{4.2}$$

*For $t\geq1$, and $v\in\Omega_\alpha(t)$, we have the bounds*

$$\left\|u(t,vt,k)-2t^{-\frac12}Re\left\{e^{i\phi(t,vt,k)}\gamma(t,v,k)\right\}\right\|_{L_v^\infty h_k^1}\lesssim t^{-\frac23}\|u(t)\|_X. \tag{4.3}$$

$$\left\|u_x(t,vt,k)-\frac{2}{\sqrt6}t^{-\frac12}\left(-v+\sqrt{v^2+12k^2}\right)^{\frac12}Re\left\{e^{i\phi(t,vt,k)}\gamma(t,v,k)\right\}\right\|_{L_v^\infty h_k^1}\lesssim t^{-\frac34}\|u(t)\|_X. \tag{4.4}$$

*Proof.* From above computations, we set $w^{\mathrm{hyp},+}(t,x,k):=e^{-i\phi(t,x,k)}u^{\mathrm{hyp},+}(t,x,k)$. As $u=2\mathrm{Re}u^+$,

$$\begin{aligned}&u(t,vt,k)-2t^{-\frac12}\mathrm{Re}\left\{e^{i\phi(t,vt,k)}\gamma(t,v,k)\right\}\\=&2\mathrm{Re}\left[e^{i\phi(t,vt,k)}\sum_N\left(w_N^{\mathrm{hyp},+}(t,vt,k)-t^{-\frac12}\lambda_v^{-1}\int w_N^{\mathrm{hyp},+}(t,x,k)\mathcal{X}\left(\frac{x-vt}{\sqrt t\lambda_v}\right)dx\right)\right]\\&+u^{\mathrm{ell}}(t,vt,k)-2t^{-\frac12}\int u^{\mathrm{ell}}(t,x,k)\overline{\Psi_v}(t,x,k)dx.\end{aligned}$$

Hence by (3.14), we obtain

$$\begin{aligned}&\left\|u(t,vt,k)-2t^{-\frac12}\mathrm{Re}\left\{e^{i\phi(t,vt,k)}\gamma(t,v,k)\right\}\right\|_{L_v^\infty h_k^1}\\\lesssim&\sum_N\left\|w_N^{\mathrm{hyp},+}(t,vt,k)-t^{-\frac12}\lambda_v^{-1}\int w_N^{\mathrm{hyp},+}(t,x,k)\mathcal{X}\left(\frac{x-vt}{\sqrt t\lambda_v}\right)dx\right\|_{L_v^\infty h_k^1}\\&+t^{-\frac23}\|u\|_X+t^{-\frac23}v^{-\frac14}\left(t^{\frac12}v^{\frac14}\right)^{-c}\|u(t)\|_X.\end{aligned}$$

By Hölder's inequality we obtain

$$\left|w_N^{\mathrm{hyp},+}(t,vt,k)-w_N^{\mathrm{hyp},+}(t,(v+z)t,k)\right|\lesssim|z|^{\frac16}\left\|D_v^{\frac23}w_N^{\mathrm{hyp},+}(t,tv,k)\right\|_{L_v^2},$$

and the pointwise bound

$$\begin{aligned}&\left\|w_N^{\mathrm{hyp},+}(t,vt,k)-t^{-\frac12}\lambda_v^{-1}\int w_N^{\mathrm{hyp},+}(t,x,k)\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt t}\right)dx\right\|_{L_v^\infty h_k^1}\\=&t^{\frac12}\left\|\lambda_v^{-1}\int\left(w_N^{\mathrm{hyp},+}(t,vt,k)-w_N^{\mathrm{hyp},+}(t,(v+z)t,k)\right)\mathcal{X}\left(\sqrt t\lambda_v^{-1}z\right)dz\right\|_{L_v^\infty h_k^1}\\\lesssim&t^{\frac12}\left\|\lambda_v^{-1}D_v^{\frac23}w_N^{\mathrm{hyp},+}(t,vt,k)\int|z|^{\frac16}\left|\mathcal{X}\left(\sqrt t\lambda_v^{-1}z\right)\right|dz\right\|_{L_v^2h_k^1}\\\lesssim&t^{-\frac1{12}}\left\|\lambda_v^{\frac16}D_v^{\frac23}w_N^{\mathrm{hyp},+}(t,vt,k)\right\|_{L_v^2h_k^1}.\end{aligned}$$

Here we use the bound for $\lambda_v$:

$$\begin{aligned}|\lambda_v(k)|&\lesssim|k|^{\frac14}\langle v\rangle^{\frac14}\qquad\text{for } v<0,\\|\lambda_v(k)|&\lesssim|k|^{\frac14}\langle v\rangle^{\frac34}\qquad\text{for } v\geq0.\end{aligned}$$

For $v<0$, we apply the inequality and $w_N^{\mathrm{hyp},+}(t,x,k)$ supported in the region $\left[-\nu 3tN^2, -\nu^{-1}3tN^2\right]$,

$$\begin{aligned}
&t^{-\frac{1}{12}}\left\|\lambda_v^{\frac{1}{6}}D_v^{\frac{2}{3}}w_N^{\mathrm{hyp},+}(t,vt,k)\right\|_{L_v^2h_k^1}\lesssim t^{-\frac{1}{12}}\left\|\langle v\rangle^{\frac{1}{24}}D_v^{\frac{2}{3}}w_N^{\mathrm{hyp},+}(t,vt,k)\right\|_{L_v^2h_k^{\frac{25}{24}}}\\
\lesssim &t^{-\frac{1}{12}}\left\|\langle v\rangle^{\frac{1}{16}}\partial_v w_N^{\mathrm{hyp},+}(t,vt,k)\right\|_{L_v^2l_k^2}^{\frac{2}{3}}\left\|w_N^{\mathrm{hyp},+}(t,vt,k)\right\|_{L_v^2h_k^{\frac{25}{8}}}^{\frac{1}{3}}\\
\lesssim &t^{-\frac{1}{12}}\left\|\langle v\rangle^{-\frac{7}{16}}\left(-v+\sqrt{v^2+12k^2}\right)^{\frac{1}{2}}\partial_v w_N^{\mathrm{hyp},+}(t,vt,k)\right\|_{L_v^2l_k^2}^{\frac{2}{3}}\left\|w_N^{\mathrm{hyp},+}(t,vt,k)\right\|_{L_v^2h_k^{\frac{25}{8}}}^{\frac{1}{3}}\\
\lesssim &t^{-\frac{7}{12}}N^{-\frac{7}{12}}\left\|\left(-\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}\right)^{\frac{1}{2}}\tilde{L}_x^+u_N^{\mathrm{hyp},+}(t,x,k)\right\|_{L_x^2l_k^2}^{\frac{2}{3}}\left\|u_N^{\mathrm{hyp},+}(t,x,k)\right\|_{L_x^2h_k^{\frac{25}{8}}}^{\frac{1}{3}}\\
\lesssim &t^{-\frac{11}{12}}N^{-\frac{7}{12}}\left\|L_xu_N^{\mathrm{hyp},+}(t)\right\|_{L_{x,y}^2}^{\frac{2}{3}}\left\|u_N^{\mathrm{hyp},+}(t)\right\|_{L_x^2H_y^{\frac{25}{8}}}^{\frac{1}{3}}.
\end{aligned}$$

For $v\geq 0$, we apply the inequality and $w_N^{\mathrm{hyp},+}(t,x,k)$ supported in the region $\left[\nu^{-1}t\frac{k^2}{N^2}, \nu t\frac{k^2}{N^2}\right]$,

$$\begin{aligned}
&t^{-\frac{1}{12}}\left\|\lambda_v^{\frac{1}{6}}D_v^{\frac{2}{3}}w_N^{\mathrm{hyp},+}(t,vt,k)\right\|_{L_v^2h_k^1}\lesssim t^{-\frac{1}{12}}\left\|\langle v\rangle^{\frac{1}{8}}D_v^{\frac{2}{3}}w_N^{\mathrm{hyp},+}(t,vt,k)\right\|_{L_v^2h_k^{\frac{25}{24}}}\\
\lesssim &t^{-\frac{1}{12}}\left\|N^{-\frac{3}{8}}|k|\partial_v w_N^{\mathrm{hyp},+}(t,vt,k)\right\|_{L_v^2l_k^2}^{\frac{2}{3}}\left\|w_N^{\mathrm{hyp},+}(t,vt,k)\right\|_{L_v^2h_k^{\frac{9}{8}}}^{\frac{1}{3}}\\
\lesssim &t^{-\frac{7}{12}}N^{-\frac{11}{12}}\left\|N\tilde{L}_x^+u_N^{\mathrm{hyp},+}(t,x,k)\right\|_{L_v^2h_k^1}^{\frac{2}{3}}\left\|u_N^{\mathrm{hyp},+}(t,x,k)\right\|_{L_v^2h_k^{\frac{9}{8}}}^{\frac{1}{3}}\\
\lesssim &t^{-\frac{11}{12}}N^{-\frac{11}{12}}\left[\min\left\{1,N^2\right\}\left(\left\|L_xu_N^{\mathrm{hyp},+}(t)\right\|_{L_{v,y}^2}+\left\|\partial_x^{-1}u_N^{\mathrm{hyp},+}\right\|_{L_{x,y}^2}\right)\right]^{\frac{2}{3}}\left\|u_N^{\mathrm{hyp},+}(t)\right\|_{L_v^2H_y^{\frac{9}{8}}}^{\frac{1}{3}}\\
\lesssim &t^{-\frac{11}{12}}\min\left\{N^{-\frac{11}{12}},N^{\frac{5}{12}}\right\}\left(\left\|L_xu_N^{\mathrm{hyp},+}(t)\right\|_{L_{v,y}^2}+\left\|\partial_x^{-1}u_N^{\mathrm{hyp},+}\right\|_{L_{x,y}^2}\right)^{\frac{2}{3}}\left\|u_N^{\mathrm{hyp},+}(t)\right\|_{L_v^2H_y^{\frac{9}{8}}}^{\frac{1}{3}}.
\end{aligned}$$

We obtain the estimate

$$\begin{aligned}
&\left\|u(t,vt,k)-2t^{-\frac{1}{2}}\mathrm{Re}\left\{e^{i\phi(t,vt,k)}\gamma(t,v,k)\right\}\right\|_{L_v^\infty h_k^1}\\
\lesssim &t^{-\frac{11}{12}}\sum_N\min\left\{N^{-\frac{7}{12}},N^{\frac{5}{12}}\right\}\|u\|_X\lesssim t^{-\frac{11}{12}}\|u\|_X.
\end{aligned}$$

By applying integration by parts, we have

$$\begin{aligned}
&\int u^{\mathrm{hyp},+}e^{-i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\\
=&-\int\frac{\phi_{xx}}{i\phi_x^2}u^{\mathrm{hyp},+}e^{-i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx-\int\frac{1}{i\phi_x}u_x^{\mathrm{hyp},+}e^{-i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\\
&-\lambda_v^{-1}t^{-\frac{1}{2}}\int\frac{1}{i\phi_x}u^{\mathrm{hyp},+}e^{-i\phi}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\\
=&i\frac{1}{\sqrt{6}}\left(-v+\sqrt{v^2+12k^2}\right)^{-\frac{1}{2}}\int u_x^{\mathrm{hyp},+}e^{-i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx+t^{-\frac{1}{2}}|v|^{-?}\|u\|_X.
\end{aligned}$$

Therefore we have the following bounds: Here we use (3.33) and Hölder inequality and substitution, let $\epsilon_z = \lambda_v(k)zt^{-\frac{1}{2}}$

$$
\begin{aligned}
&\left\|\int \frac{\phi_{xx}}{i\lambda_v\phi_x^2}u^{\mathrm{hyp},+}e^{-i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\right\|_{h_k^1}\\
\lesssim&\left\|u^{\mathrm{hyp},+}\right\|_{L_v^\infty h_k^1}\left\|\lambda_v^{-1}(k)\int\left|\frac{\phi_{xx}}{\phi_x^2}\right|\left|\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right|dx\right\|_{h_k^1}\\
\lesssim& t^{-\frac{1}{2}}\left\|u(t)\right\|_X\left\|\lambda_v^{-1}(k)\int\frac{1}{2t}\frac{\sqrt{\frac{x}{t}+\sqrt{\left(\frac{x}{t}\right)^2+12k^2}}}{\sqrt{2}|k|\sqrt{\left(\frac{x}{t}\right)^2+12k^2}}\left|\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right|dx\right\|_{h_k^1}\\
\lesssim& t^{-1}\left\|u\right\|_X\left\||k|^{-1}\int\frac{\sqrt{v+\epsilon_z+\sqrt{(v+\epsilon_z)^2+12k^2}}}{\sqrt{(v+\epsilon_z)^2+12k^2}}\left|\mathcal{X}(z)\right|dz\right\|_{h_k^1}\\
\lesssim& t^{-1}|v|^{-\frac{1}{2}}\left\|u\right\|_X.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\left\|\lambda_v^{-2}(k)t^{-\frac{1}{2}}\int\frac{1}{i\phi_x}u^{\mathrm{hyp},+}e^{-i\phi}\mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\right\|_{h_k^1}\\
=&\left\|\lambda_v^{-1}(k)\int\frac{1}{i\phi_x}u^{\mathrm{hyp},+}e^{-i\phi}\partial_x\left[\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right]dx\right\|_{h_k^1}\\
\lesssim&\left\|\lambda_v^{-1}(k)\int\left[\frac{\phi_{xx}}{\phi_x^2}\left(e^{-i\phi}u^{\mathrm{hyp},+}\right)+\frac{1}{\phi_x}\partial_x\left(e^{-i\phi}u^{\mathrm{hyp},+}\right)\right]\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\right\|_{h_k^1}\\
\lesssim& t^{-1}|v|^{-\frac{1}{2}}\left\|u\right\|_X+\left\|\lambda_v^{-1}(k)t^{-\frac{1}{2}}\int\left[\frac{e^{-i\phi}}{\phi_x}\tilde{L}_x^+(k)u^{\mathrm{hyp},+}\right]\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\right\|_{h_k^1}.
\end{aligned}
$$

The first term we can use the previous estimate, and for the second factor, we have the following estimate

$$
\begin{aligned}
&\left\|\lambda_v^{-1}(k)t^{-\frac{1}{2}}\int\left[\frac{e^{-i\phi}}{\phi_x}\tilde{L}_x^+(k)u^{\mathrm{hyp},+}\right]\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\right\|_{h_k^1}\\
\lesssim&\left\|\lambda_v^{-1}(k)t^{-\frac{1}{2}}\left\|\tilde{L}_x^+(k)u^{\mathrm{hyp},+}\right\|_{L_x^2}\left\|\frac{1}{\phi_x}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right\|_{L_x^2}\right\|_{h_k^1}\\
\lesssim&\left\|\lambda_v^{-\frac{1}{2}}(k)t^{-\frac{1}{4}}\left\|\tilde{L}_x^+(k)u^{\mathrm{hyp},+}\right\|_{L_x^2}\left\|\frac{\sqrt{v+\epsilon_z+\sqrt{(v+\epsilon_z)^2+12k^2}}}{|k|}\mathcal{X}(z)\right\|_{L_z^2}\right\|_{h_k^1}\\
\lesssim& t^{-\frac{3}{4}}|v|^{-\frac{7}{8}}\left\|L_x\partial_x u^{\mathrm{hyp},+}\right\|_{L_x^2}.
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\lambda_v^{-1}(k)\int\left[\frac{1}{\phi_x(t,x,k)}-\frac{1}{\phi_x(t,tv,k)}\right]u_x^{\mathrm{hyp},+}e^{-i\phi}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\right\|_{L_v^\infty h_k^1}\\
\lesssim&\left\|\lambda_v^{-1}(k)t\int\left[\int_0^1\frac{\phi_{xx}}{\phi_x^2}(t,vt+(1-\theta)tz,k)tzd\theta\right]\left(e^{-i\phi}u_x^{\mathrm{hyp},+}\right)(t,t(z+v),k)\mathcal{X}\left(\lambda_v^{-1}\sqrt{t}z\right)dz\right\|_{L_v^\infty h_k^1}\\
\lesssim& t^{-1}|v|^{-\frac{1}{2}}\left\|u\right\|_X.
\end{aligned}
$$

We have the computations of asymptotic profile for $u_x$, let $\tilde{w}^{\mathrm{hyp},+} := u_x^{\mathrm{hyp},+}$ and $\tilde{w}_N^{\mathrm{hyp},+} := \partial_x u_N^{\mathrm{hyp},+}$, we will have the relation $\partial_x \tilde{w}^{\mathrm{hyp},+}(t,x,k) = e^{-i\phi(t,x,k)} \tilde{L}_x^+(k) \partial_x u^{\mathrm{hyp},+}$.

$$\begin{aligned}
&u_x(t,vt,k) - \frac{2}{\sqrt{6}} t^{-\frac{1}{2}} \left(-v + \sqrt{v^2+12k^2}\right)^{\frac{1}{2}} \operatorname{Re}\left(i e^{-i\phi(t,vt,k)} \gamma(t,v,k)\right) \\
=&\operatorname{Re} \sum_N \left\{ e^{i\phi(t,vt,k)} \left[ \tilde{w}_N^{\mathrm{hyp},+}(t,vt,k) - t^{-\frac{1}{2}} \lambda_v^{-1} \int \tilde{w}_N^{\mathrm{hyp},+}(t,x,k) \mathcal{X}\left(\frac{x-vt}{\lambda_v \sqrt{t}}\right) dx \right] \right\} \\
&+ t^{-\frac{3}{4}} |v|^{-\frac{7}{8}} \left\| L_x \partial_x u^{\mathrm{hyp},+} \right\|_{L_x^2}.
\end{aligned}$$

Applying a similar computation as the asymptotic profile for $u$, we can obtain

$$\left\| \tilde{w}_N^{\mathrm{hyp},+}(t,vt,k) - t^{-\frac{1}{2}} \lambda_v^{-1} \int \tilde{w}_N^{\mathrm{hyp},+}(t,x,k) \mathcal{X}\left(\frac{x-vt}{\lambda_v \sqrt{t}}\right) dx \right\|_{L_v^\infty h_k^1} \lesssim t^{-\frac{11}{12}} \sum_N \min\left\{ N^{-\frac{7}{12}}, N^{\frac{5}{12}} \right\} \|u\|_X .$$

□

**Lemma 12.** *If $u$ solves (1.1), for $t \in [0,T]$, and $v \in \Omega_\alpha(t)$, we have*

$$\partial_t \gamma(t,v,k) = t^{-1} \sum_{\Upsilon(v,k)} \xi_v(k) \gamma(t,v,k_1) \overline{\gamma}(t,v,k_2) \gamma(t,v,k_3) + O(??). \tag{4.5}$$

*Proof.* $\gamma$ satisfies the following equation

$$\begin{aligned}
\dot{\gamma}(t,v,k) &= \int u_t \overline{\Psi_v} + u \overline{\Psi_{vt}} dx \\
&= \int u^2 u_x(k) \overline{\Psi_v}(k) dx + \int u \overline{(\partial_t + \partial_x^3 + \partial_x^{-1} k^2) \Psi_v} dx.
\end{aligned}$$

$$\begin{aligned}
&\left\| t^{-1} \int u(t,x,k) \frac{x-vt}{2\lambda_v \sqrt{t}} e^{-i\phi} \mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right) dx \right\|_{L_v^\infty h_k^1} \\
\lesssim& t^{-1} \|u(t)\|_{L_v^\infty h_k^1} \left\| \frac{x-vt}{2\lambda_v\sqrt{t}} e^{-i\phi} \mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right) \right\|_{L_x^1 h_k^1} \lesssim t^{-\frac{3}{2}} \|u(t)\|_X .
\end{aligned}$$

$$\begin{aligned}
&\left\| \int u(t,x,k) e^{-i\phi} \left( \left(\frac{x}{t}\right)^2 + 12k^2 \right)^{-\frac{1}{2}} \frac{x}{t} \mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right) dx \right\|_{L_v^\infty h_k^1} \\
=& \left\| \int u(t,x,k) e^{-i\phi} \left[ \partial_x \left( \sqrt{x^2+12t^2k^2} \mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right) \right) - \frac{\sqrt{x^2+12t^2k^2}}{\lambda_v\sqrt{t}} \mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right) \right] dx \right\|_{L_v^\infty h_k^1} \\
\lesssim& \left\| \int \sum_{N\sim\xi_v} u_N(t,x,k) e^{-i\phi} \partial_x \left( \sqrt{x^2+12t^2k^2} \mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right) \right) dx \right\|_{L_v^\infty h_k^1} \\
&+ \left\| \int \sum_{N\nsim\xi_v} u_N(t,x,k) e^{-i\phi} \partial_x \left( \sqrt{x^2+12t^2k^2} \mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right) \right) dx \right\|_{L_v^\infty h_k^1} \\
&+ \left\| \int u(t,x,k) e^{-i\phi} \frac{\sqrt{x^2+12t^2k^2}}{\lambda_v\sqrt{t}} \mathcal{X}'\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right) dx \right\|_{L_v^\infty h_k^1}
\end{aligned}$$

We have

$$\begin{aligned}
&\int \partial_x \left( \frac{1}{3} u^3(t,x,k) \right) \overline{\Psi_v}(t,x,k) dx \\
=& \frac{i}{3\sqrt{6}} \left(-v + \sqrt{v^2+12k^2}\right)^{\frac{1}{2}} \int u^3(t,x,k) \overline{\Psi}(t,x,k) dx + O\left(t^{-\frac{3}{2}} \|u(t)\|_E^3\right)
\end{aligned}$$

If the frequency support of

$$\left(u^{\text{hyp}}\right)^3 - 3\left|u^{\text{hyp},+}\right|^2 u^{\text{hyp},+} = \left(u^{\text{hyp},+}\right)^3 + 3\left|u^{\text{hyp},+}\right|^2 \overline{u^{\text{hyp},+}} + \overline{u^{\text{hyp},+}}^3$$

is contained in $[\frac{N_v}{4}, 4N_v]$, then at least one of $u^{\text{hyp},+}$ on the right hand side is $\left(1 - P_{\frac{N_v}{4}\leq\cdot\leq 4N_v}\right) u_N^{\text{hyp},+}$. Therefore we may reduce the estimate to the factor

$$\frac{i}{\sqrt{6}}\left(-v+\sqrt{v^2+12k^2}\right)^{\frac{1}{2}} \sum_{k_1-k_2+k_3=k} \int u^{\text{hyp},+}(t,x,k_1)\,\overline{u^{\text{hyp},+}}(t,x,k_2)\,u^{\text{hyp},+}(t,x,k_3)\,\overline{\Psi}(t,x,k)dx.$$

Define the resonance level

$$\Upsilon(v,k) = \left\{(k_1,k_2,k_3)\in\mathbb{Z}^3 : \phi_x(t,vt,k) - \phi_x(t,vt,k_1) + \phi_x(t,vt,k_2) - \phi_x(t,vt,k_3) = 0, k_1-k_2+k_3 = k\right\},$$

we will have the equation

$$\begin{aligned} i\partial_t\gamma(t,v,k) = t^{-1}\sum_{\Upsilon(v,k)} \xi_v(k)\,\gamma(t,v,k_1)\,\overline{\gamma}(t,v,k_2)\,\gamma(t,v,k_3) \\ + \text{lower order term} \end{aligned}$$

Here we define the resonance phase function

$$\Phi = \phi(t,x,k_1) - \phi(t,x,k_2) + \phi(t,x,k_3) - \phi(t,x,k)$$

$$\begin{aligned} &\sum_{(k_1,k_2,k_3)\in\mathcal{M}(k)} \lambda_v^{-1}\int u^{\text{hyp},+}(t,x,k_1)\,\overline{u^{\text{hyp},+}}(t,x,k_2)\,u^{\text{hyp},+}(t,x,k_3)\,e^{-i\phi(t,x,k)}\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx \\ =&\sum_{(k_1,k_2,k_3)\in\mathcal{M}(k)} \lambda_v^{-1}\int e^{i\Phi}\left(e^{-i\phi}u^{\text{hyp},+}\right)(t,x,k_1)\,\overline{\left(e^{-i\phi}u^{\text{hyp},+}\right)}(t,x,k_2)\left(e^{-i\phi}u^{\text{hyp},+}\right)(t,x,k_3)\,\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx \\ =&\sum_{(k_1,k_2,k_3)\in\mathcal{M}(k)} \lambda_v^{-1}\int e^{i\Phi}w^{\text{hyp},+}(t,x,k_1)\,\overline{w^{\text{hyp},+}}(t,x,k_2)\,w^{\text{hyp},+}(t,x,k_3)\,\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx \end{aligned}$$

Therefore we can change the first factor into a product of functions $\gamma(t,v,k)$

$$\begin{aligned} &\left\|\sum_{(k_1,k_2,k_3)\in\mathcal{M}(k)} \lambda_v^{-1}\int e^{i\Phi}\left[w^{\text{hyp},+}(t,x,k_1) - t^{-\frac{1}{2}}\gamma(t,v,k_1)\right]\overline{w^{\text{hyp},+}}(t,x,k_2)\,w^{\text{hyp},+}(t,x,k_3)\,\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\right\|_{l_k^2} \\ \lesssim&\left\|\int\lambda_v^{-1}\left|w^{\text{hyp},+}(t,x,k) - t^{-\frac{1}{2}}\gamma(t,v,k)\right|\left\|w^{\text{hyp},+}(t,x,k)\right\|_{l_k^1}^2\left|\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)\right|dx\right\|_{l_k^2} \\ \lesssim& t^{-\frac{17}{12}}|v|^{-\frac{11}{12}}\|u\|_X^3\ (**Check) \end{aligned}$$

Similarly, we can obtain the estimate for

$$\left\|\sum_{(k_1,k_2,k_3)\in\mathcal{M}(k)} \lambda_v^{-1}t^{-\frac{1}{2}}\gamma(t,v,k_1)\int e^{i\Phi}\left[\overline{w^{\text{hyp},+}(t,x,k_2) - t^{-\frac{1}{2}}\gamma(t,v,k_2)}\right]w^{\text{hyp},+}(t,x,k_3)\,\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\right\|_{l_k^2}$$

and

$$\left\|\sum_{(k_1,k_2,k_3)\in\mathcal{M}(k)} \lambda_v^{-1}t^{-1}\gamma(t,v,k_1)\,\overline{\gamma}(t,v,k_2)\int e^{i\Phi}\left[w^{\text{hyp},+}(t,x,k_3) - t^{-\frac{1}{2}}\gamma(t,v,k_3)\right]\mathcal{X}\left(\frac{x-vt}{\lambda_v\sqrt{t}}\right)dx\right\|_{l_k^2}.$$

Since we prove in the previous section that $\gamma(t,v,k)$ is frequence localization around $\xi_v(k)$, we can further decompostion the product into two parts:

$$\begin{aligned} &\sum_{(k_1,k_2,k_3)\in\mathcal{M}(k)} t^{-1}\gamma(t,v,k_1)\,\overline{\gamma}(t,v,k_2)\,\gamma(t,v,k_3) \\ =&\sum_{(k_1,k_2,k_3)\in\Upsilon(k)} t^{-1}\gamma(t,v,k_1)\,\overline{\gamma}(t,v,k_2)\,\gamma(t,v,k_3) + \sum_{(k_1,k_2,k_3)\in\mathcal{M}(k)\setminus\Upsilon(k)} t^{-1}\gamma(t,v,k_1)\,\overline{\gamma}(t,v,k_2)\,\gamma(t,v,k_3). \end{aligned}$$

By the fact that $\gamma(t,v,k_1)\overline{\gamma}(t,v,k_2)\gamma(t,v,k_3)$ is frequency localized around $\xi_v(k_1)-\xi_v(k_2)+\xi_v(k_3)$, we know that the second term should be decaying faster.

$$\left\|(1-P_{N\sim\xi_v})\gamma(t,v,k_1)\overline{\gamma}(t,v,k_2)\gamma(t,v,k_3)\right\|_{L_v^2 l_k^2}$$

□

When $v \ll -1$, we have

$$\phi(t,vt,k) \approx t\frac{2}{3\sqrt{3}}|v|^{\frac{3}{2}},$$

hence

$$\Upsilon(v,k)=\left\{(k_1,k_2,k_3)\in\mathbb{Z}^3 : k_1-k_2+k_3=k\right\},$$

which is trivial.

When $v \gg 1$, we have

$$\phi(t,vt,k)\approx 2t|k|\sqrt{v}.$$

$$\Upsilon(v,k)=\left\{(k_1,k_2,k_3)\in\mathbb{Z}^3 :, k_1-k_2+k_3=k, \quad |k_1|-|k_2|+|k_3|=|k|\right\}.$$

**Lemma 13.** *For $(t,v,k)\in[1,\infty)\times\mathbb{R}\times\mathbb{Z}$, if $G(t,v,k)$ satisfies the equation*

$$i\partial_t G(t,v,k)=t^{-1}\xi_v(k)\sum_{(k_1,k_2,k_3)\in\Upsilon(k)} G(t,v,k_1)\overline{G}(t,v,k_2)G(t,v,k_3), \tag{4.6}$$

*we have the following conservation laws:*

$$\|v^\alpha G(t,v,k)\|_{l_k^2}\equiv\|v^\alpha G(1,v,k)\|_{l_k^2}, \tag{4.7}$$

$$\left\|v^\alpha\left|3\xi_v^2(k)-k^2\xi_v^{-2}(k)\right|^{\frac{1}{2}}G(t,v,k)\right\|_{l_k^2}\equiv\left\|v^\alpha\left(3\xi_v^2(k)-k^2\xi_v^{-2}(k)\right)^{\frac{1}{2}}G(1,v,k)\right\|_{l_k^2}, \tag{4.8}$$

*here $\alpha\in\mathbb{R}$.*

*Proof.* Here define the symmetry 4-tuples

$$\Upsilon^4(v)=\left\{(k_1,k_2,k_3,k_4)\in\mathbb{Z}^4 :, k_1-k_2+k_3-k_4=0, \quad \xi_v(k_1)-\xi_v(k_2)+\xi_v(k_3)-\xi_v(k_4)=0\right\}.$$

By a direct computation and the symmetry between $k_1$, $k_2$, $k_3$, $k_4$, we obtain

$$\begin{aligned}
&\partial_t\|v^\alpha G(t,v,k)\|_{l_k^2}^2\\
=&2|v|^{2\alpha}t^{-1}\mathrm{Im}\sum_{(k_1,k_2,k_3,k_4)\in\Upsilon^4(v)}\xi_v(k_4)G(t,v,k_1)\overline{G}(t,v,k_2)G(t,v,k_3)\overline{G}(t,v,k_4)\\
=&\frac{1}{2}|v|^{2\alpha}t^{-1}\mathrm{Im}\sum_{(k_1,k_2,k_3,k_4)\in\Upsilon^4(v)}(\xi_v(k_1)-\xi_v(k_2)+\xi_v(k_3)-\xi_v(k_4))G(t,v,k_1)\overline{G}(t,v,k_2)G(t,v,k_3)\overline{G}(t,v,k_4)\\
=&0
\end{aligned}$$

Using the fact that $3\xi_v^4(k)+v\xi_v^2(k)-k^2=0$, we have

$$\partial_t\left\|v^\alpha\left|3\xi_v^2(k)-k^2\xi_v^{-2}(k)\right|^{\frac{1}{2}}G(t,v,k)\right\|_{l_k^2}^2=\partial_t\left\|v^\alpha|v|^{\frac{1}{2}}G(t,v,k)\right\|_{l_k^2}^2=0.$$

□

Hence we consider the $h_k^1$ norm of $G(t,v,k)$, using the equation $3\xi_v^4(k)-k^2=-v\xi_v^2(k)$, if $v$ is away from 0, we will have

$$\|kG(t,v,k)\|_{l_k^2}\lesssim\left\|\left(3\xi_v^4(k)-k^2\right)^{\frac{1}{2}}G(t,v,k)\right\|_{l_k^2}=\left\||v|^{\frac{1}{2}}\xi_v(k)G(t,v,k)\right\|_{l_k^2}\lesssim\left\|v^{\frac{1}{2}}\left|3\xi_v^2(k)-k^2\xi_v^{-2}(k)\right|^{\frac{1}{2}}G(t,v,k)\right\|_{l_k^2}\lesssim 1,$$

for $t\in[1,T)$.

## 5. Energy Estimates

In this section, we will complete the energy estimate of $L_x\partial_x u$, therefore we can obtain the bound for $\|u(t)\|_X$. First we make the assumptions on $u$:

**Hypothesis 1.** *The solution $u$ for (1.1) exists in a time interval $[0,T]$, and satisfies the bounds*

$$\|u(t)\|_{L_x^\infty H_y^1}, \|u_x(t)\|_{L_x^\infty H_y^1} \le D\epsilon |t|^{-\frac{1}{2}} \tag{5.1}$$

*for $t\in[0,T]$. Here $D$ is a sufficiently large positive number which does not depend on $u$.*

Then we have the following bound for the $L_x\partial_x u$:

**Proposition 14.** *Suppose $u$ is a solution to (1.1) satisfying Hypothesis 1 on $[0,T]$, then we will have that*

$$\|u(t)\|_X \lesssim \epsilon (1+|t|)^{D^3\epsilon^2} \tag{5.2}$$

*for $t\in[0,T]$.*

We start with the equation of $L_x\partial_x u$, by the property that $L_x\partial_x$ commutes with the linear operator, we have that

$$\begin{aligned}\left(\partial_t+\partial_x^3-\partial_x^{-1}\partial_y^2\right)(L_x\partial_x u) &= -\left(x\partial_x^2-3t\partial_x^4-t\partial_y^2\right)\left(u^3/3\right)\\ &= -x\left(2uu_x^2+u^2u_{xx}\right)+3t\left(12u_x^2u_{xx}+6uu_{xx}^2+8uu_xu_{xxx}+u^2u_{xxxx}\right)\\ &\quad +t\left(2uu_y+u^2u_{yy}\right).\end{aligned}$$

Hence we have the equation for $L_x\partial_x u$:

$$\begin{aligned}\left(\partial_t+\partial_x^3-\partial_x^{-1}\partial_y^2\right)(L_x\partial_x u) &= -u^2\partial_x(L_xu_x)-2u_xu(L_xu_x)+u^2u_x\\ &\quad -18t\partial_x(u_{xx}u_xu)-6t\partial_x\left(u_x^3\right)-2tu_y^2u+2t\left(\partial_x^{-1}u_{yy}\right)u_xu\end{aligned}$$

Since the factors

$$-18t\partial_x(u_{xx}u_xu)-6t\partial_x\left(u_x^3\right)-2tu_y^2u+2t\left(\partial_x^{-1}u_{yy}\right)u_xu$$

do not have decay prproperty, we have to apply normal form correction technique to obtain faster decaying. The normal form correction is developed in the work [24, 3, 14], the main idea is to apply a trilinear correction to the function in order to replace a nonresonant cubic nonlinearity by a milder quadrillion nonlinearity.

Here we consider the trilinear normal form correction:

$$L^3(f,g,h) := \int e^{ix(\xi_1+\xi_2+\xi_3)}\frac{1}{i\Omega}\widehat{f}(\xi_1,k_1)\,\widehat{g}(\xi_2,k_2)\,\widehat{h}(\xi_3,k_2)\,d\xi_1 d\xi_2 d\xi_3$$

where the resonance function $\Omega$ is given by

$$\Omega := \omega(\xi_1+\xi_2+\xi_3,k_1+k_2+k_3)-\omega(\xi_1,k_1)-\omega(\xi_2,k_2)-\omega(\xi_3,k_3)$$

and

$$\omega(\xi,k) := \xi^3+\frac{k^2}{\xi}.$$

If we consider where $(\xi_1,\xi_2,\xi_3,k_1,k_2,k_3)$ such that $|\Omega(\xi_1,\xi_2,\xi_3,k_1,k_2,k_3)|\ge 1$ for all $\xi_1\in N_1$, $\xi_2\in N_2$, $\xi_3\in N_3$, there is the multilinear estimate by Coifman-Meyer multiplier theorem

$$\left\|L^3\left(f_{N_1}(k_1),g_{N_2}(k_2),h_{N_3}(k_3)\right)\right\|_{L_x^p} \lesssim \|f_{N_1}(k_1)\|_{L_x^{p_1}}\|g_{N_2}(k_2)\|_{L_x^{p_2}}\|h_{N_3}(k_3)\|_{L_x^{p_3}},$$

where $1\le p,p_1,p_2,p_3\le\infty$, and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}$.

5.1. **Nonresonance frequencies.** The correction form

$$
\begin{aligned}
L_3^t(u,u,u) = 18t\left[2L_3(u_x,u_x,u_{xx}) + L_3(u,u_{xx},u_{xx}) + L_3(u,u_x,u_{xxx})\right] \\
+ 2t\left[L_3(u,u_yu_y) - L_3\left(u,u_x,\partial_x^{-1}u_{yy}\right)\right].
\end{aligned} \tag{5.3}
$$

Therefor we will have the following quadrillion linear form

$$
\begin{aligned}
&L_5^t(u,u,u,u,u)\\
:=&\left(\partial_t+\partial_x^3-\partial_x^{-1}\partial_y^2\right)L_3^t(u,u,u)\\
=&6t\left[4L_3\left(\partial_x^2u^3,u_x,u_{xx}\right)+2L_3\left(u_x,u_x,\partial_x^3u^3\right)+L_3\left(\partial_xu^3,u_x,u_{xx}\right)+L_3\left(u,\partial_x^2u^3,u_{xxx}\right)\right]\\
&+6t\left[L_3\left(u,u_x,\partial_x^4u^3\right)+L_3\left(\partial_x^2u^3,u_{xx},u_{xx}\right)+2L_3\left(u_x,\partial_x^3u^3,u_{xx}\right)\right]\\
&+\frac{2}{3}t\left[L_3\left(\partial_xu^3,u_y,u_y\right)+2L_3\left(u,\partial_x\partial_yu^3,u_y\right)\right]\\
&-\frac{2}{3}t\left[L_3\left(\partial_xu^3,u_x,\partial_x^{-1}u_{yy}\right)+L_3\left(u,\partial_x^2u^3,\partial_x^{-1}u_{yy}\right)+L_3\left(u,u_x,\partial_y^2u^3\right)\right].
\end{aligned} \tag{5.4}
$$

Together we obtained the differential equation

$$
\left(\partial_t+\partial_x^3-\partial_x^{-1}\partial_y^2\right)\left[L_x\partial_xu+L_3^t(u,u,u)\right]=-u^2\partial_x\left(L_x\partial_xu\right)-2u\left(\partial_xu\right)\left(L_x\partial_xu\right)+u^2u_x+L_5^t(u,u,u,u,u). \tag{5.5}
$$

Instead of working on the energy norm $X$ directly, we will work on the modified energy norm $\tilde{X}$:

$$
\|u\|_{\tilde{X}}^2=\|u\|_{L^2}^2+\left\|\partial_x^5u\right\|_{L^2}^2+\left\|\partial_y^3u\right\|_{L^2}^2+\left\|L_x\partial_xu+L_3^t(u,u,u)\right\|_{L^2_{x,y}}^2+\left\|\partial_x^{-1}\partial_yu\right\|_{L^2}^2. \tag{5.6}
$$

**Lemma 15.** *Suppose $u$ is a solution to (1.1) satisfying Hypothesis 1 on $[0,T]$, then we will have that*

$$
\left\|L_3^t(u,u,u)\right\|_{L^2_{x,y}}\lesssim D^2\epsilon^2\|u(t)\|_{\tilde{X}}, \tag{5.7}
$$

$$
\left\|L_5^t(u,u,u,u,u)\right\|_{L^2_{x,y}}\lesssim t^{-1}D^4\epsilon^4\|u(t)\|_{\tilde{X}.}. \tag{5.8}
$$

*Proof.* To start our estimate with $L_3^t(u,u,u)$ and $L_5^t(u,u,u,u,u)$, first we use the linear property of $L_t^3$ and $L_t^5$ separate each $u$ into diadic freqency regions, and without loss of generality, we assume that

$$
1\le N_1\le N_2\le N_3\le N_4\le N_5,
$$

Then we have the property that

$$
\begin{aligned}
&\left\|L_3(u_x,u_x,u_{xx})\right\|_{L^2_{x,y}}\\
\lesssim&\left\|\sum_{N_1<N_2<N_3}\left|L_3\left(P_{N_1}u_x,P_{N_2}u_x,P_{N_3}u_{xx}\right)\right|\right\|_{L^2_{x,y}}+\left\|\sum_{N_1<N_2<N_3}\left|L_3\left(P_{N_1}u_x,P_{N_2}u_{xx},P_{N_3}u_x\right)\right|\right\|_{L^2_{x,y}}\\
\lesssim&\left\|\sum_{N_1<N_2}\frac{N_1}{N_3}\left|L_3\left(P_{N_1}u,P_{N_2}u_x,P_{N_3}u_{xxx}\right)\right|\right\|_{L^2_{x,y}}+\left\|\sum_{N_1<N_2<N_3}\frac{N_2^2}{N_3^2}\left|L_3\left(P_{N_1}u_xP_{N_2},u,P_{N_3}u_{xxx}\right)\right|\right\|_{L^2_{x,y}}\\
\lesssim&\sum_{N_1<N_2}\frac{N_1}{N_3}\frac{1}{N_2}\left\|\left(P_{N_1}u\right)\left(P_{N_2}u_x\right)P_{N_3}u_{xxxx}\right\|_{L^2_{x,y}}+\sum_{N_1<N_2<N_3}\frac{N_2^2}{N_3^2}\frac{1}{N_1}\left\|\left(P_{N_1}u_x\right)\left(P_{N_2}u\right)\left(P_{N_3}u_{xxxx}\right)\right\|_{L^2_{x,y}}\\
\lesssim&\left\|uu_xu_{xxxx}\right\|_{L^2_{x,y}}\lesssim\left\|uu_x\right\|_{L^\infty_xH^1_y}\|u(t)\|_{\tilde{X}}
\end{aligned}
$$

Similarly we also have by summation over different diadic number $N_1$, $N_2$ and $N_3$

$$
\left\|L_3(u,u_{xx},u_{xx})\right\|_{L^2_{x,y}}\lesssim\left\|uu_xu_{xxxx}\right\|_{L^2_{x,y}}\lesssim\left\|uu_x\right\|_{L^\infty_xH^1_y}\|u(t)\|_{\tilde{X}}.
$$

For the quadrillion linear form, we can show that

$$\begin{aligned}
&\left\|L_3\left(\partial_x^2 u^3, u_x, u_{xx}\right)\right\|_{L^2_{x,y}}\\
\lesssim &\left\|L_5\left(u, u, u_x, u_{xx}, u_{xx}\right)\right\|_{L^2_{x,y}} + \left\|L_5\left(u, u_x, u_x, u_x, u_{xx}\right)\right\|_{L^2_{x,y}}\\
\lesssim &\left\|\sum_{N_1<N_2<N_3<N_4<N_5} \frac{N_4}{N_5}\left|L_5\left(P_{N_1}u, P_{N_2}u, P_{N_3}u_x, P_{N_4}u_x, P_{N_5}u_{xxx}\right)\right|\right\|_{L^2_{x,y}}\\
&+\left\|\sum_{N_1<N_2<N_3<N_4<N_5} \frac{N_2}{N_5}\left|L_5\left(P_{N_1}u, P_{N_2}u, P_{N_3}u_x, P_{N_4}u_x, P_{N_5}u_{xxx}\right)\right|\right\|_{L^2_{x,y}}\\
\lesssim &\left\|\sum_{N_1<N_2<N_3<N_4<N_5} \frac{N_4}{N_5}N_1^{-\frac{1}{3}}N_2^{-\frac{1}{3}}N_3^{-\frac{1}{3}}\left|L_5\left(P_{N_1}u, P_{N_2}u, P_{N_3}u_x, P_{N_4}u_x, P_{N_5}u_{xxxx}\right)\right|\right\|_{L^2_{x,y}}\\
&+\left\|\sum_{N_1<N_2<N_3<N_4<N_5} \frac{N_2}{N_5}N_1^{-\frac{1}{3}}N_3^{-\frac{1}{3}}N_4^{-\frac{1}{3}}\left|L_5\left(P_{N_1}u, P_{N_2}u, P_{N_3}u_x, P_{N_4}u_x, P_{N_5}u_{xxxx}\right)\right|\right\|_{L^2_{x,y}}\\
\lesssim &\left\|u^2u_x^2u_{xxxx}\right\|_{L^2_{x,y}} \lesssim \left\|uu_x\right\|^2_{L^\infty_{x,y}}\left\|u(t)\right\|_{\tilde{X}}.
\end{aligned}$$

Applying the similar arguments, we obtain the bounds:

$$\begin{aligned}
&\left\|L_3\left(u_x, u_x, \partial_x^3 u^3\right)\right\|_{L^2_{x,y}}, \left\|L_3\left(\partial_x u^3, u_x, u_{xx}\right)\right\|_{L^2_{x,y}}, \left\|L_3\left(u, \partial_x^2 u^3, u_{xxx}\right)\right\|, \left\|L_3\left(\partial_x^2 u^3, u_{xx}, u_{xx}\right)\right\|, \left\|L_3\left(u_x, \partial_x^3 u^3, u_{xx}\right)\right\|\\
\lesssim &\left\|u^2u_x^2u_{xxxx}\right\|_{L^2_{x,y}} \lesssim \left\|uu_x\right\|^2_{L^\infty_{x,y}}\left\|u(t)\right\|_{\tilde{X}}.
\end{aligned}$$

and

$$\left\|L_3\left(u, u_x, \partial_x^4 u^3\right)\right\|_{L^2_{x,y}} \lesssim \left\|u^2u_x^2u_{xxxx}\right\|_{L^2_{x,y}} + \left\|u^3u_xu_{xxxxx}\right\|_{L^2_{x,y}} \lesssim \left(\left\|uu_x\right\|^2_{L^\infty_{x,y}} + \left\|u^3u_x\right\|_{L^\infty_{x,y}}\right)\left\|u(t)\right\|_{\tilde{X}}.$$

For the factors $tuu_y^2$ and $tuu_x\partial_x^{-1}u_{yy}$, we will apply a similar argument to $tuu_y^2$ and switch to estimate $tu^2u_{yy}$. For the normal form correction we will show the bounds grows slow in time: Here we may move one derivative with respect to $y$ to the larger freqency in $y$, and summation over $k$:

$$\begin{aligned}
&\left\|L_3\left(u, u_yu_y\right)\right\|_{L^2_{x,y}}\\
=&\left\|\sum_{(k_1,k_2,k_3)\in\mathcal{M}(k)} L_3\left(u\left(k_1\right), k_2u\left(k_2\right), k_3u\left(k_3\right)\right)\right\|_{L^2_x l^2_k}\\
\lesssim&\left\|\sum_{(k_1,k_2,k_3)\in\mathcal{M}(k)}\sum_{|k_2|\leq|k_3|}\left|L_3\left(u\left(k_1\right), u\left(k_2\right), k_3^2u\left(k_3\right)\right)\right|\right\|_{L^2_x l^2_k}\\
\lesssim&\left\|u\right\|^2_{L^\infty_x H^1_y}\left\|u(t)\right\|_{\tilde{X}}.
\end{aligned}$$

The quadrillion linear estimates as following:

$$\begin{aligned}
&\left\|L_3\left(\partial_x u^3, u_y, u_y\right)\right\|_{L^2_{x,y}}\\
=&\left\|L_5\left(u, u, u_x, u_y, u_y\right)\right\|_{L^2_{x,y}}\\
=&\left\|\sum_{(k_1,k_2,k_3,k_4,k_5)\in\mathcal{M}(k)} L_5\left(u\left(k_1\right), u\left(k_2\right), u_x\left(k_3\right), k_4u\left(k_4\right), k_5u\left(k_5\right)\right)\right\|_{L^2_x l^2_k}\\
\lesssim&\left\|\sum_{(k_1,k_2,k_3,k_4,k_5)\in\mathcal{M}(k)}\sum_{|k_4|\leq|k_5|}\left|L_5\left(u\left(k_1\right), u\left(k_2\right), u_x\left(k_3\right), u\left(k_4\right), k_5^2u\left(k_5\right)\right)\right|\right\|_{L^2_x l^2_k}\\
\lesssim&\left\|u\right\|^3_{L^\infty_x H^1_y}\left\|u_x\right\|_{L^\infty_x H^1_y}\left\|u(t)\right\|_{\tilde{X}}.
\end{aligned}$$

Similarly, we can obtain the bound

$$\begin{aligned}
&\left\|L_3\left(u, \partial_x \partial_y u^3, u_y\right)\right\|_{L^2_{x,y}} \\
\lesssim &\left\|L_5\left(u, u, u_x, u_y, u_y\right)\right\|_{L^2_{x,y}}+\left\|L_5\left(u, u, u, u_{xy}, u_y\right)\right\|_{L^2_{x,y}} \\
\lesssim &\|u\|^3_{L^\infty_x H^1_y}\left\|u_x\right\|_{L^\infty_x H^1_y}\|u(t)\|_{\tilde{X}} \\
&+\left\|\sum_{\left(k_1, k_2, k_3, k_4, k_5\right) \in \mathcal{M}(k)} \sum_{\left|k_4\right| \leq\left|k_5\right|}\left|L_5\left(u\left(k_1\right), u\left(k_2\right), u\left(k_3\right), u\left(k_4\right), k_5^2 u_x\left(k_5\right)\right)\right|\right\|_{L^2_x l^2_k} \\
\lesssim &\|u\|^3_{L^\infty_x H^1_y}\left\|u_x\right\|_{L^\infty_x H^1_y}\|u(t)\|_{\tilde{X}}+\|u\|^4_{L^\infty_x H^1_y}\|u(t)\|_{\tilde{X}} .
\end{aligned}$$

The bound of $tuu_x\partial_x^{-1}u_{yy}$ needs more discussion when in the worst case, if in a frequency localization of $x$, $\partial_x^{-1}u_{yy}$ has $x$ frequency $N$ and $N \ll 1$. In this case we need to use the bound from the function $\Omega$, since $\xi_3^{-1}k_3^2\Omega^{-1}\left(\xi_1,\xi_2,\xi_3,k_1,k_2,k_3\right)$ is bounded, hence we have a new trilinear form

$$\begin{aligned}
&\left\|L_3\left(u, u_x, \partial_x^{-1} u_{yy}\right)\right\|_{L^2_{x,y}} \\
=&\left\|\int_{\xi_1-\xi_2+\xi_3=\xi} \sum_{\left(k_1, k_2, k_3\right) \in \mathcal{M}(k)} \widehat{u}\left(\xi_1, k_1\right) \xi_2 \widehat{u}\left(\xi_2, k_2\right) \xi_3^{-1} k_3^2 \widehat{u}\left(\xi_3, k_3\right) \Omega^{-1}\left(\xi_1, \xi_2, \xi_3, k_1, k_2, k_3\right) d \xi_1 d \xi_2 d \xi_3\right\|_{L^2_\xi l^2_k} \\
=&\left\|\tilde{L}_3\left(u, u_x u\right)\right\|_{L^2_{x,y}} \lesssim\left\|u u_x\right\|_{L^\infty_{x,y}}\|u\|_{L^2_{x,y}} \lesssim\left\|u u_x\right\|_{L^\infty_{x,y}}\|u(t)\|_{\tilde{X}} .
\end{aligned}$$

We can reduce the rest fifth order linear term

$$\begin{aligned}
& L_3\left(\partial_x u^3, u_x, \partial_x^{-1} u_{yy}\right)+L_3\left(u, \partial_x^2 u^3, \partial_x^{-1} u_{yy}\right) \\
=& \tilde{L}_3\left(\partial_x u^3, u_x, u\right)+\tilde{L}_3\left(u, \partial_x^2 u^3, u\right) .
\end{aligned}$$

We have

$$\begin{aligned}
&\left\|L_3\left(\partial_x u^3, u_x, \partial_x^{-1} u_{yy}\right)\right\|_{L^2_{x,y}}+\left\|L_3\left(u, \partial_x^2 u^3, \partial_x^{-1} u_{yy}\right)\right\| \\
\lesssim &\left\|u u_x\right\|^2_{L^\infty_{x,y}}\|u\|_{L^2_{x,y}}+\|u\|^4_{L^\infty_{x,y}}\left\|u_{xx}\right\|_{L^2_{x,y}} \lesssim\left(\left\|u u_x\right\|^2_{L^\infty_{x,y}}+\|u\|^4_{L^\infty_{x,y}}\right)\|u(t)\|_{\tilde{X}} .
\end{aligned}$$

Recall the hypothesis (5.1) and summing over all diadic number, we obtain (5.7) and (5.8).

□

5.2. **Resonance frequencies.** For the resonance frequencies, we are unable to use the normal form corrections. But we can take advantage of the elliptic region due to a faster decay in the elliptic area. Hence we first work with the case all functions are supported in the hyperbolic region. We consider the frequencies where resonance occurs,

$$\mathit{\Omega}=\left[\left(\xi_1+\xi_2+\xi_3\right)^3-\xi_1^3-\xi_2^3-\xi_3^3\right]+\left[\frac{\left(k_1+k_2+k_3\right)^2}{\xi_1+\xi_2+\xi_3}-\frac{k_1^2}{\xi_1}-\frac{k_2^2}{\xi_2}-\frac{k_3^2}{\xi_3}\right] \approx 0. \tag{5.9}$$

Hence we have the relation

$$\left[\left(\xi_1+\xi_2+\xi_3\right)^3-\xi_1^3-\xi_2^3-\xi_3^3\right] \approx-\left[\frac{\left(k_1+k_2+k_3\right)^2}{\xi_1+\xi_2+\xi_3}-\frac{k_1^2}{\xi_1}-\frac{k_2^2}{\xi_2}-\frac{k_3^2}{\xi_3}\right]. \tag{5.10}$$

Define the resonance set

$$\operatorname{Res}(\xi, k):=\left\{\left(\xi_1, \xi_2, \xi_3, k_1, k_2, k_3\right): \xi_1-\xi_2+\xi_3=\xi, k_1-k_2+k_3=k,\left|\Omega\left(\xi_1, \xi_2, \xi_3, k_1, k_2, k_3\right)\right| \leq 1\right\}. \tag{5.11}$$

**Lemma 16.** *Suppose $u$ is a solution to (1.1) satisfying Hypothesis 1 on $[0,T]$, then we will have that*

$$\left\langle L_x \partial_x u(k), \sum_{Res(\xi, k)} L_x \partial_x^2\left(u_{N_1}\left(k_1\right) \overline{u_{N_2}\left(k_2\right)} u_{N_3}\left(k_3\right)\right)\right\rangle_{L^2_x l^2_k} \lesssim t^{-1} D^2 \epsilon^2\left(\left\|L_x \partial_x u\right\|_{L^2_{x,y}}+\|u\|_{L^2_{x,y}}\right). \tag{5.12}$$

*Proof.* By using the hyperbolic equation is supported in the region $x \sim -3t\xi_i^2$ when $3\xi_i^4 > k_i^2$ and supported in the region $x \sim t\frac{k_i^2}{\xi_i^2}$ when $3\xi_i^4 < k_i^2$, we obtain that: If $u_{N_1}^{\text{hyp}}(k_1)\, u_{N_2}^{\text{hyp}}(k_2)\, u_{N_3}^{\text{hyp}}(k_3)$ is nonzero, then we either have

$$(\xi_1+\xi_2+\xi_3)^3 - \xi_1^3 - \xi_2^3 - \xi_3^3 \approx 0$$

and the functions supported in the area $x > 0$, or

$$\frac{(k_1+k_2+k_3)^2}{\xi_1+\xi_2+\xi_3} - \frac{k_1^2}{\xi_1} - \frac{k_2^2}{\xi_2} - \frac{k_3^2}{\xi_3} \approx 0$$

and the functions supported in the area $x < 0$. If the frequencies satisfying (5.9), we will have

$$\begin{aligned}
&L_x\partial_x^2\left(u_{N_1}(k_1)\,\overline{u_{N_2}(k_2)}u_{N_3}(k_3)\right)\\
=&\partial_x\left[L_x\partial_x\left(u_{N_1}(k_1)\,\overline{u_{N_2}(k_2)}u_{N_3}(k_3)\right)\right] - \partial_x\left(u_{N_1}(k_1)\,\overline{u_{N_2}(k_2)}u_{N_3}(k_3)\right)\\
\approx&\partial_x\left[\left(L_x\partial_x u_{N_1}(k_1)\right)\overline{u_{N_2}(k_2)}u_{N_3}(k_3) - u_{N_1}(k_1)\,\overline{\left(L_x\partial_x u_{N_2}(k_2)\right)}u_{N_3}(k_3) + u_{N_1}(k_1)\,\overline{u_{N_2}(k_2)}L_x\partial_x u_{N_3}(k_3)\right]\\
&+ L_3^{\text{res}}\left(u_{N_1}(k_1), \overline{u_{N_2}(k_2)}, u_{N_3}(k_3)\right) - \partial_x\left(u_{N_1}(k_1)\,\overline{u_{N_2}(k_2)}u_{N_3}(k_3)\right).
\end{aligned}$$

Here $L_3^{\text{res}}$ is given by

$$\begin{aligned}
&L_3^{\text{res}}\left(f_1(k_1), \overline{f_2(k_2)}, f_3(k_3)\right)\\
:=&2t\int e^{ix(\xi_1-\xi_2+\xi_3)}(\xi_1-\xi_2+\xi_3)\left[(\xi_1-\xi_2+\xi_3)^3 - \xi_1^3 + \xi_2^3 - \xi_3^3\right]\widehat{f}_1(k_1)\,\overline{\widehat{f}_2}(k_2)\,\widehat{f}_3(k_3)\,d\xi_1 d\xi_2 d\xi_3\\
:=&6t\int e^{ix(\xi_1-\xi_2+\xi_3)}(\xi_1-\xi_2+\xi_3)(\xi_1-\xi_2)(-\xi_2+\xi_3)(\xi_1+\xi_3)\,\widehat{f}_1(k_1)\,\overline{\widehat{f}_2}(k_2)\,\widehat{f}_3(k_3)\,d\xi_1 d\xi_2 d\xi_3.
\end{aligned}$$

Combine above condition with the hyperbolic region equations, we obtain

$$L_3^{\text{res}}\left(u_{N_1}^{\text{hyp}}(k_1), \overline{u_{N_2}^{\text{hyp}}(k_2)}, u_{N_3}^{\text{hyp}}(k_3)\right) \approx 0.$$

Therefore we have

$$\begin{aligned}
&\left\langle L_x\partial_x u(k), \sum_{\text{Res}(\xi,k)} L_x\partial_x^2\left(u_{N_1}^{\text{hyp}}(k_1)\,\overline{u_{N_2}^{\text{hyp}}(k_2)}u_{N_3}^{\text{hyp}}(k_3)\right)\right\rangle_{L_x^2 l_k^2}\\
\lesssim&\|u(t)\|_{L_x^\infty H_y^1}\|\partial_x u(t)\|_{L_x^\infty H_y^1}\|L_x\partial_x u\|_{L_x^2} + \|u(t)\|_{L_x^\infty H_y^1}\|\partial_x u(t)\|_{L_x^\infty H_y^1}\|u(t)\|_{L_{x,y}^2}.
\end{aligned}$$

The rest case are the functions with at least one is supported in the elliptic area. Since the function supported in ellptic area has better decay property, it is suffice to estimate the case that one of the functions is supported in the elliptic area and others are supported in the hyperbolic area. We need to prove that the symmetry tri-linear form $L_3^{\text{res}}\left(u_{N_1}^{\text{ell}}(k_1), \overline{u_{N_2}^{\text{hyp}}(k_2)}, u_{N_3}^{\text{hyp}}(k_3)\right)$ has a proper decay. It is suffice to consider the worst case and other cases can follow the same manner: We assume that $N_1 \le N_2 \le N_3$, and cosidering the factor $tu_{N_1}^{\text{ell}}(k_1)\,\overline{\left(\partial_x u_{N_2}^{\text{hyp}}(k_2)\right)}\partial_x^3 u_{N_3}^{\text{hyp}}(k_3)$. If the fator is nonzero, we must have either (i) $N_2^2 \approx N_3^2 \approx N^2$, $3N_2^4 > k_2^2$, $3N_3^4 > k_3^2$ and the factor supported in the region $x \sim -3tN^2$, or (ii) $\frac{k_2^2}{N_2^2} \approx \frac{k_3^2}{N_3^2} \approx M^2$, $3N_2^4 < k_2^2$, $3N_3^4 < k_3^2$ and the factor supported in the region $x \sim tM^2$.

Case (i):

$$\begin{aligned}
tu_{N_1}^{\text{ell}}(k_1)\,\overline{\left(\partial_x u_{N_2}^{\text{hyp}}(k_2)\right)}\partial_x^3 u_{N_3}^{\text{hyp}}(k_3) &\approx -tN^2 u_{N_1}^{\text{ell}}(k_1)\,\overline{\left(\partial_x u_N^{\text{hyp},\pm}(k_2)\right)}\partial_x u_N^{\text{hyp},\pm}(k_3)\\
&\approx -xu_{N_1}^{\text{ell}}(k_1)\,\overline{\left(\partial_x u_N^{\text{hyp},\pm}(k_2)\right)}\partial_x u_N^{\text{hyp},\pm}(k_3)
\end{aligned}$$

If $N_1 \geq 1$, we have

$$
\begin{aligned}
&\left\| \sum_{\mathrm{Res}(\xi,k)} \sum_{1\leq N_1\leq N} t u^{\mathrm{ell}}_{N_1}(k_1) \overline{\left(\partial_x u^{\mathrm{hyp},\pm}_N(k_2)\right)} \partial_x^3 u^{\mathrm{hyp},\pm}_N(k_3) \right\|_{L^2_x l^2_k} \\
\lesssim &\left\| \sum_{\mathrm{Res}(\xi,k)} \sum_{1\leq N_1\leq N} x u^{\mathrm{ell}}_{N_1}(k_1) \overline{\left(\partial_x u^{\mathrm{hyp},\pm}_N(k_2)\right)} \partial_x u^{\mathrm{hyp},\pm}_N(k_3) \right\|_{L^2_x l^2_k} \\
\lesssim &\left( \sum_{1\leq N_1\leq N} \left\| \partial_x u^{\mathrm{hyp},\pm}_N \right\|^4_{L^\infty_x H^1_y} N_1^{-2} \left\| \mathcal{X}\left(x\sim -3tN^2\right) x N_1 u^{\mathrm{ell}}_{N_1} \right\|^2_{L^2_{x,y}} \right)^{\frac{1}{2}} \\
\lesssim &\sum_{1\leq N_1} N_1^{-1} \|\partial_x u\|^2_{L^\infty_x H^1_y} \left\| x\partial_x u^{\mathrm{ell}}_{N_1} \right\|_{L^2_{x,y}} \lesssim \|\partial_x u\|^2_{L^\infty_x H^1_y} \|L_x\partial_x u\|_{L^2_{x,y}}.
\end{aligned}
$$

If $N_1 < 1$, we apply the inequality (1.15) and the condition $x \sim -3tN^2$,

$$
\begin{aligned}
&\left\| \sum_{\mathrm{Res}(\xi,k)} \sum_{N_1\leq N, N_1<1} t u^{\mathrm{ell}}_{N_1}(k_1) \overline{\left(\partial_x u^{\mathrm{hyp},\pm}_N(k_2)\right)} \partial_x^3 u^{\mathrm{hyp},\pm}_N(k_3) \right\|_{L^2_x l^2_k} \\
\lesssim &\left( \sum_{N_1\leq N, N_1<1} \left\| \partial_x u^{\mathrm{hyp},\pm}_N \right\|^4_{L^\infty_x H^1_y} N_1^{-2} \left\| \mathcal{X}\left(x\sim -3tN^2\right) \left( \frac{-x+\sqrt{x^2+12t^2k^2}}{6} \right) N_1 u^{\mathrm{ell}}_{N_1}(k) \right\|^2_{L^2_x l^2_k} \right) \\
\lesssim &\sum_{1\leq N_1} N_1^{-1} \|\partial_x u\|^2_{L^\infty_x H^1_y} \left\| \tilde{L}_x(k)\,\partial_x u^{\mathrm{ell}}_{N_1}(k) \right\|_{L^2_x l^2_k} \\
\lesssim &\sum_{1\leq N_1} N_1 \|\partial_x u\|^2_{L^\infty_x H^1_y} \left( \left\| L_x\partial_x u^{\mathrm{ell}}_{N_1} \right\|_{L^2_{x,y}} + \left\| u^{\mathrm{ell}}_{N_1} \right\|_{L^2_{x,y}} \right) \\
\lesssim &\|\partial_x u\|^2_{L^\infty_x H^1_y} \left( \|L_x\partial_x u\|_{L^2_{x,y}} + \|u\|_{L^2_{x,y}} \right).
\end{aligned}
$$

Case (ii): In this case, we have $N_3^2 \lesssim \frac{k_3^2}{N_3^2} \sim \frac{x}{t}$, for $N_1 > \frac{1}{2}$,

$$
\begin{aligned}
&\left\| \sum_{\mathrm{Res}(\xi,k)} \sum_{\frac{1}{2}\leq N_1\leq N_2\leq N_3} t u^{\mathrm{ell}}_{N_1}(k_1) \overline{\left(\partial_x u^{\mathrm{hyp}}_{N_2}(k_2)\right)} \partial_x^3 u^{\mathrm{hyp}}_{N_3}(k_3) \right\|_{L^2_x l^2_k} \\
\lesssim &\left( \sum_{\mathrm{Res}(\xi,k)} \sum_{\frac{1}{2}\leq N_1\leq N_2\leq N_3} \left\| x u^{\mathrm{ell}}_{N_1}(k_1) \overline{\left(\partial_x u^{\mathrm{hyp}}_{N_2}(k_2)\right)} \partial_x u^{\mathrm{hyp}}_{N_3}(k_3) \right\|^2_{L^2_x} \right)^{\frac{1}{2}} \\
\lesssim &\left( \sum_{\frac{1}{2}\leq N_1} \sum_{M\geq 1} \left\| \partial_x u^{\mathrm{hyp}} \right\|^4_{L^\infty_x H^1_y} N_1^{-2} \left\| \mathcal{X}\left(x\sim tM^2\right) x N_1 u^{\mathrm{ell}}_{N_1} \right\|^2_{L^2_{x,y}} \right)^{\frac{1}{2}} \\
\lesssim &\|\partial_x u\|^2_{L^\infty_x H^1_y} \|L_x\partial_x u\|_{L^2_{x,y}}.
\end{aligned}
$$

For where the frequency $N_1 < \frac{1}{2}$, we must have $3N_1^4 < k_1^2$. If $N_3 < 1$, we have

$$
\begin{aligned}
&\left\|\sum_{\operatorname{Res}(\xi,k)}\sum_{N_1\le N_2\le N_3,N_3<1} t u_{N_1}^{\mathrm{ell}}(k_1)\overline{\left(\partial_x u_{N_2}^{\mathrm{hyp}}(k_2)\right)}\partial_x^3 u_{N_3}^{\mathrm{hyp}}(k_3)\right\|_{L_x^2 l_k^2}\\
\lesssim&\left\|\sum_{\operatorname{Res}(\xi,k)}\sum_{N_1\le N_2\le N_3,N_3<1} \frac{N_1}{k_1^2}N_2N_3^2\left|t\frac{k_1^2}{N_1}u_{N_1}^{\mathrm{ell}}(k_1)\right|\left|u_{N_2}^{\mathrm{hyp}}(k_2)\right|\left|\partial_x u_{N_3}^{\mathrm{hyp}}(k_3)\right|\right\|_{L_x^2 l_k^2}\\
\lesssim&\sum_{N_1\le N_2\le N_3,N_3<1} N_1N_2N_3^2\left\|L_x\partial_x u_{N_1}^{\mathrm{ell}}\right\|_{L_{x,y}^2}\left\|u_{N_2}^{\mathrm{hyp}}\right\|_{L_x^\infty H_y^1}\left\|\partial_x u_{N_3}^{\mathrm{hyp}}\right\|_{L_x^\infty H_y^1}\\
\lesssim&\|u\|_{L_x^\infty H_y^1}\|\partial_x u\|_{L_x^\infty H_y^1}\|L_x\partial_x u\|_{L_{x,y}^2}.
\end{aligned}
$$

For where the frequency $N_1<\frac{1}{2}$, and $N_3\ge 1$, we use the inequality $3N_3^2<\frac{k_3^2}{N_3^2}\sim\frac{x}{t}$, and , $x\lesssim t$

$$
\begin{aligned}
&\left\|\sum_{\operatorname{Res}(\xi,k)}\sum_{N_1\le N_2\le N_3,N_1<1,N_3\ge 1} t u_{N_1}^{\mathrm{ell}}(k_1)\overline{\left(\partial_x u_{N_2}^{\mathrm{hyp}}(k_2)\right)}\partial_x^3 u_{N_3}^{\mathrm{hyp}}(k_3)\right\|_{L_x^2 l_k^2}\\
\lesssim&\left\|\sum_{\operatorname{Res}(\xi,k)}\sum_{N_1\le N_2\le N_3,N_1<1,N_3\ge 1}\left|x u_{N_1}^{\mathrm{ell}}(k_1)\right|\left|\partial_x u_{N_2}^{\mathrm{hyp}}(k_2)\right|\left|\partial_x u_{N_3}^{\mathrm{hyp}}(k_3)\right|\right\|_{L_x^2 l_k^2}\\
\lesssim&\sum_{N_1\le N_2\le N_3,N_3\ge1,N_1<1}\left\|\frac{xN_1}{xN_1^2-tk^2}\left(\left(xN_1-t\frac{k^2}{N_1}\right)u_{N_1}^{\mathrm{ell}}(k)\right)\right\|_{L_x^\infty l_k^2}\left\|\partial_x u_{N_2}^{\mathrm{hyp}}\right\|_{L_x^\infty H_y^1}\left\|\partial_x u_{N_3}^{\mathrm{hyp}}\right\|_{L_x^\infty H_y^1}\\
\lesssim&\sum_{N_1\le N_2\le N_3,N_3\ge1,N_1<1}\left\|N_1\left(xN_1-t\frac{k^2}{N_1}\right)u_{N_1}^{\mathrm{ell}}(k)\right\|_{L_x^\infty l_k^2}\left\|\partial_x u_{N_2}^{\mathrm{hyp}}\right\|_{L_x^\infty H_y^1}\left\|\partial_x u_{N_3}^{\mathrm{hyp}}\right\|_{L_x^\infty H_y^1}\\
\lesssim&\sum_{N_1\le N_2\le N_3,N_3\ge1,N_1<1}N_1\left\|L_x\partial_x u_{N_1}^{\mathrm{ell}}\right\|_{L_x^\infty l_k^2}\left\|\partial_x u_{N_2}^{\mathrm{hyp}}\right\|_{L_x^\infty H_y^1}\left\|\partial_x u_{N_3}^{\mathrm{hyp}}\right\|_{L_x^\infty H_y^1}.
\end{aligned}
$$

For where the frequency $N_1<\frac{1}{2}$, and $N_3\ge1$, and , $x\gtrsim t$, if we solve the equation $\Omega(\xi_1,\xi_2,\xi_3,k_1,k_2,k_3)\approx 0$ with the constrains $|\xi_1|\le\frac{1}{2}$, $|\xi_3|\ge 1$, $3\xi_i^4\le k_i^2$ for $i=1,2,3$, and $\frac{k_2^2}{\xi_2^2}\approx\frac{k_3^2}{\xi_3^2}\approx a\gtrsim 1$, $\frac{k_1^2}{\xi_1^2}\approx b\neq a$, this will lead to the equation $\frac{(k_1+k_2+k_3)^2}{\xi_1+\xi_2+\xi_3}-\frac{k_1^2}{\xi_1}-\frac{k_2^2}{\xi_2}-\frac{k_3^2}{\xi_3}\approx 0$. Which means $L_3^{\mathrm{res}}\left(u_{N_1}^{\mathrm{hyp}}(k_1),\overline{u_{N_2}^{\mathrm{hyp}}(k_2)},u_{N_3}^{\mathrm{hyp}}(k_3)\right)\approx 0$ in this case. □

5.3. **end of energy estimate\.**

**Lemma 17.** *Suppose $u$ is a solution to (1.1) satisfying Hypothesis 1 on $[0,T]$, then we will have that*

$$\|u(t)\|_{\tilde{X}}\lesssim\epsilon(1+|t|)^{D^3\epsilon^2} \tag{5.13}$$

*for $t\in[0,T]$.*

*Proof.* By (5.7), we have the following equality

$$\|u(t)\|_X\le\|u(t)\|_{\tilde{X}}+\left\|L_3^t(u,u,u)\right\|_{L_{x,y}^2}\lesssim\left(1+D^2\epsilon^2\right)\|u(t)\|_{\tilde{X}}\lesssim\|u(t)\|_{\tilde{X}}.$$

By (5.7), (5.8) we have the following bound for $L^2_{x,y}$ norm:

$$\begin{aligned}
&\partial_t \left\| L_x\partial_x u + L_3^t(u,u,u)\right\|^2_{L^2_{x,y}} \\
\lesssim &\left\|u\partial_x u\right\|_{L^\infty_{x,y}} \left\|L_x\partial_x u\right\|^2_{L^2_{x,y}} + \|u\|^2_{L^\infty_{x,y}} \left\|\partial_x L_3^t(u,u,u)\right\|_{L^2_{x,y}} \left\|L_x\partial_x u\right\|_{L^2_{x,y}} \\
&+ \left\|u\partial_x u\right\|_{L^\infty_{x,y}} \left\|L_x\partial_x u + L_3^t(u,u,u)\right\|_{L^2_{x,y}} \|u\|_{L^2_{x,y}} \\
&+ \left\|L_x\partial_x u + L_3^t(u,u,u)\right\|_{L^2_{x,y}} \left\|L_5^t(u,u,u,u,u)\right\|_{L^2_{x,y}} \\
&+ \left\langle L_x\partial_x u(k) + L_3^t(u,u,u), \sum_{\mathrm{Res}(\xi,k)} L_x\partial_x^2\left(u^{\mathrm{hyp}}_{N_1}(k_1)\overline{u^{\mathrm{hyp}}_{N_2}(k_2)}u^{\mathrm{hyp}}_{N_3}(k_3)\right)\right\rangle_{L^2_x l^2_k} \\
\lesssim & t^{-1}\left(D^2\epsilon^2 + D^4\epsilon^4\right)\|u(t)\|^2_{\tilde{X}}.
\end{aligned} \tag{5.14}$$

By the conservation laws (1.8) and (1.9), obviously there is the bound $\|u(t)\|_{L^2_{x,y}} + \left\|\partial_x^{-1}\partial_y u(t)\right\| \leq \epsilon$ for all $t \in [0,T]$. By the fractional Leibniz rule again, there are the equations for $\partial_x^5 u$ and $\partial_y^3 u$:

$$\left(\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2\right)\partial_x^5 u = 2u\left(\partial_x u\right)\partial_x^5 u + u^2\partial_x\partial_x^5 u + \mathrm{err}_3(t,x,y).$$

$$\left(\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2\right)\partial_y^3 u = 2u\left(\partial_x u\right)\partial_y^3 u + u^2\partial_x\partial_y^3 u + \mathrm{err}_4(t,x,y).$$

Applying (5.1), it is easy to show that

$$\begin{aligned}
\partial_t\left(\left\|\partial_x^5 u\right\|^2_{L^2_{x,y}} + \left\|\partial_y^3 u\right\|^2_{L^2_{x,y}}\right) &\lesssim \|uu_x\|_{L^\infty_{x,y}}\left(\left\|\partial_x^5 u\right\|^2_{L^2_{x,y}} + \left\|\partial_y^3 u\right\|^2_{L^2_{x,y}}\right) \\
&\lesssim D^2\epsilon^2 t^{-1}\|u(t)\|^2_{\tilde{X}}.
\end{aligned} \tag{5.15}$$

For large enough positive number $D$, we obtain the following bound by (5.14), (5.15), and Grönwall's inequality.

$$\|u(t)\|_{\tilde{X}} \lesssim \epsilon(1+t)^{D\left(D^2\epsilon^2 + D^4\epsilon^4\right)}.$$

If we further assume that $0 < \epsilon \ll D^{-\frac{3}{2}}$, we obtain the desired bound.

□

Therefore we finish the proof of Proposition 14 by the fact that $\|u(t)\|_X \lesssim \|u(t)\|_{\tilde{X}}$.

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