Documentation of conservative 1D interpolation routine

Timothy D. Williams

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1 Interpolation element-to-element

N elements, nodes are $x_0, \ldots, x_N$

$$\tilde{f} = f_n \quad \text{for } x_n \leq x \leq x_{n+1}. \tag{1a}$$

Target grid: M elements, nodes are $y_0, \ldots, y_M$:

$$\tilde{F}(x) = F_m \quad \text{for } y_m \leq x \leq y_{m+1}. \tag{2}$$

Want average value of $\tilde{f}$ over $y_m < x < y_{m+1}$:

$$F_m(y_{m+1} - y_m) = \int_{y_m}^{y_{m+1}} \tilde{f}(x)\mathrm{d}x = \sum_{n=0}^{N-1} I^{(mn)} f_n, \tag{3}$$

where

$$\begin{aligned}
I^{(mn)} &= H_{mn} \int_{L_{mn}}^{U_{mn}} 1 \, \mathrm{d}x \\
&= H_{mn}(U_{mn} - L_{mn}), && \text{(4a)} \\
H_{mn} &= H(x_{n+1} - y_m) H(y_{m+1} - x_n), && \text{(4b)} \\
L_{mn} &= \max\{y_m, x_n\}, && \text{(4c)} \\
U_{mn} &= \min\{y_{m+1}, x_{n+1}\}. && \text{(4d)}
\end{aligned}$$

2 Interpolation node-to-element

Source grid: N elements, nodes are $x_0, \ldots, x_N$.

$$\begin{aligned}
\tilde{f}(x) &= f_n \xi_0(x; x_n, x_{n+1}) \\
&\qquad + f_{n+1}(x; x_n, x_{n+1}) \quad \text{for } x_n \leq x \leq x_{n+1}, && \text{(5a)} \\
\xi_1(x; x_n, x_{n+1}) &= \frac{x - x_n}{x_{n+1} - x_n}, && \text{(5b)} \\
\xi_0(x; x_n, x_{n+1}) &= 1 - \xi_1(x; x_n, x_{n+1}). && \text{(5c)}
\end{aligned}$$

Target grid: M elements, nodes are $y_0, \ldots, y_M$:

$$\tilde{F}(x) = F_m \quad \text{for } y_m \leq x \leq y_{m+1}, \tag{6}$$

Want average value of $\tilde{f}$ over $y_m < x < y_{m+1}$:

$$
\begin{aligned}
F_m(y_{m+1} - y_m) &= \int_{y_m}^{y_{m+1}} \tilde{f}(x)\mathrm{d}x \\
&= \sum_{n=0}^{N-1} \left(I_0^{(mn)} f_n + I_1^{(mn)} f_{n+1} \right) \\
&= I_0^{(m0)} f_0 + I_1^{(m,N-1)} f_N \\
&\quad + \sum_{n=1}^{N-1} f_n \left(I_0^{(mn)} + I_1^{(m,n-1)} \right),
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
I_1^{(mn)} &= H_{mn} \int_{L_{mn}}^{U_{mn}} \xi_j(x; x_n, x_{n+1})\mathrm{d}x \\
&= H_{mn} \left[\frac{(x - x_n)^2}{2(x_{n+1} - x_n)} \right]_{L_{mn}}^{U_{mn}},
\end{aligned} \tag{8a}
$$

$$
I_0^{(mn)} = H_{mn}\Big(U_{mn} - L_{mn}\Big) - I_1^{(mn)}, \tag{8b}
$$

3 Interpolation element-to-node

4 Interpolation node-to-node