Задача №2 (Критерий информативности)

1. $$L(y,c)=\sum_{k=1}^{K}(c_k-[y=k])^2$$

$$c=\arg\min_{c} H(R)$$

$$H(R)=\min \frac{1}{|R|}\sum_{(x,y)\in R}\sum_{k=1}^{K}(c_k-[y=k])^2$$

$$\begin{cases}\frac{1}{|R|}\sum_{(x,y)\in R}\sum_{k=1}^{K}(c_k-[y=k])^2\to\min_{c_k}\\ \sum_{k=1}^{K}c_k=1\end{cases}$$

$$L(c,\lambda)=\frac{1}{|R|}\sum_{(x,y)\in R}\sum_{k=1}^{K}(c_k-[y=k])^2+\lambda\left(\sum_{k=1}^{K}c_k-1\right)$$

$$\frac{\partial L}{\partial c_k}=\frac{2}{|R|}\sum_{(x,y)\in R}c_k-\frac{1}{|R|}\sum_{(x,y)\in R}2[y=k]+\lambda=0.$$

$$\frac{2}{|R|}\sum_{(x,y)\in R}c_k=\frac{1}{|R|}\sum_{(x,y)\in R}2[y=k]-\lambda$$

$$c_k=\underbrace{\frac{1}{|R|}\sum_{(x,y)}[y=k]}_{p_k}-\frac{\lambda}{2}\Rightarrow c_k=p_k-\frac{\lambda}{2}$$

$$\underbrace{\sum_{k=1}^{K}c_k}_{1}=\sum_{k=1}^{K}\left(p_k-\frac{\lambda}{2}\right)\Rightarrow\lambda=0.\Rightarrow c_k=p_k$$

$$H(R)=\frac{1}{|R|}\sum_{(x,y)}\sum_{k=1}^{K}\left(p_k^2-2p_k[y=k]+[y=k]^2\right)=$$

$$=\sum_{k=1}^{K}(p_k-p_k^2)=\sum_{k=1}^{K}p_k(1-p_k)$$

Ответ: $\boxed{H(R)=\sum_{k=1}^{K}p_k(1-p_k)}$

② $L(y,c) = -\sum_{k=1}^{K} [y=k] \log c_k$

$$H(R) = \min_{c} -\frac{1}{|R|} \sum_{(x,y)\in R} \sum_{k=1}^{K} [y=k] \log c_k.$$

$$\begin{cases} -\frac{1}{|R|} \sum_{(x,y)\in R} \sum_{k=1}^{K} [y=k] \log c_k \longrightarrow \min_{c_k} \\ \sum_{k=1}^{K} c_k = 1 \end{cases}$$

$$L(c,\lambda) = -\frac{1}{|R|} \sum_{(x,y)\in R} \sum_{k=1}^{K} [y=k] \log c_k + \lambda \left(\sum_{k=1}^{K} c_k - 1\right)$$

$$\frac{\partial L}{\partial c_k} = -\frac{1}{|R|} \sum_{(x,y)\in R} \frac{[y=k]}{c_k} + \lambda = 0.$$

$$c_k = \frac{1}{\lambda} \cdot \underbrace{\frac{1}{|R|} \sum_{(x,y)\in R} [y=k]}_{p_k}$$

$$\Rightarrow c_k = \frac{1}{\lambda} \cdot p_k$$

$$\underbrace{\sum_{k=1}^{K} c_k}_{1} = \sum_{k=1}^{K} \frac{1}{\lambda} p_k = \frac{1}{\lambda} \underbrace{\sum_{k=1}^{K} p_k}_{1} \Rightarrow \lambda = 1.$$

$$\Rightarrow c_k = p_k.$$

$$H(R) = -\frac{1}{|R|} \sum_{(x,y)\in R} \sum_{k=1}^{K} [y=k] \log p_k =$$

$$= -\sum_{k=1}^{K} p_k \log p_k.$$

Ответ: $\boxed{H(R) = -\sum_{k=1}^{K} p_k \log p_k}$