Задача 1 (энтропия.)

① X равновероятно принимает $K \geq 2$ значений

$p_k = \frac{1}{K}$

1) индекс Джини: $H(R) = \sum_{k=1}^{K} \frac{1}{K}\left(1 - \frac{1}{K}\right) = \frac{K}{K}\left(1 - \frac{1}{K}\right) = \frac{K-1}{K}$

2) энтропия: $H(R) = -\sum_{k=1}^{K} \frac{1}{K} \log_2 \frac{1}{K} = -\frac{K}{K} \log_2 \frac{1}{K} = \log_2 K$

② X равномерно распределена на отрезке $[0;a]$

$$p_k(x) = \begin{cases} 0, & x \notin [0;a] \\ \frac{1}{a}, & x \in [0;a] \end{cases}$$

1) индекс Джини: $H(R) = \int_0^a \frac{1}{a}\left(1 - \frac{1}{a}\right) dx = \left(\frac{a-1}{a^2}\right) x \Big|_0^a =$

$= \frac{a-1}{a}$

2) энтропия: $H(R) = -\int_0^a \frac{1}{a} \log_2 \frac{1}{a} \cdot dx = -\frac{1}{a} \cdot \log_2 \frac{1}{a} \, x \Big|_0^a =$

$= -\log_2 \frac{1}{a} = \log_2 a.$

③ X нормально распределена $N(0, \sigma^2)$

$$p_k(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

1) индекс Джини:

$$H(R) = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \left(1 - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\right) dx =$$

$$= \int_{-\infty}^{+\infty} \left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} - \frac{1}{2\sigma^2\pi} e^{-\frac{x^2}{\sigma^2}}\right] dx =$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}} dx - \frac{1}{2\sigma^2\pi} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{\sigma^2}} dx = \left[\begin{matrix} x = \sigma z \\ dx = \sigma dz \end{matrix}\right] =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} dz - \frac{1}{2\sigma\pi} \int_{-\infty}^{+\infty} e^{-z^2} dz =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} - \frac{1}{2\sigma\pi} \cdot \sqrt{\pi} = 1 - \frac{1}{2\sigma\sqrt{\pi}}$$

г) энтропия

$$H(R) = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \cdot \log_2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx =$$

$$= -\log_2 \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx + \frac{\log_2 e}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{x^2}{2\sigma^2} dx =$$

$$= \begin{bmatrix} x = \sigma z \\ dx = \sigma dz \end{bmatrix} = \log_2(\sigma\sqrt{2\pi}) + \frac{\log_2 e}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} z^2 dz =$$

$$= \begin{bmatrix} u = z & du = dz \\ dv = z e^{-\frac{z^2}{2}} dz & v = -e^{-\frac{z^2}{2}} \end{bmatrix} = \log_2(\sigma\sqrt{2\pi}) +$$

$$+ \frac{\log_2 e}{2\sqrt{2\pi}} \cdot \left(-z e^{-\frac{z^2}{2}} \Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} dz \right) = \log_2(\sigma\sqrt{2\pi}) +$$

$$+ \frac{\log_2 e}{2\sqrt{2\pi}} (0 + \sqrt{2\pi}) = \log_2(\sigma\sqrt{2\pi}) + \log_2 \sqrt{e} =$$

$$= \log_2(\sigma\sqrt{2\pi e})$$