THE GRÖBNER BASIS OF A CATALAN PATH IDEAL

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ABSTRACT. For the ideal $I = \langle y_1 + \cdots + y_n, y_1^2, \ldots, y_n^2 \rangle$ in $R = \Bbbk[y_1, \ldots, y_n]$ with $\mathrm{char}(\Bbbk) = 0$, we show that the reduced Gröbner basis with lex-order consists of polynomials g_α that are represented in terms of paths moving northeast in the Cartesian plane that stay above the diagonal and cross the diagonal at the last step. This implies that a linear basis for the quotient ring R/I is given by a set of Catalan paths. We show that the dimension is the number of standard Young tableaux of size n and height at most two. The graded Frobenious characteristic of R/I as a symmetric group module is given by $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} s_{n-k,k} q^k$.

1. Introduction

Let $R = \Bbbk[y_1, \ldots, y_n]$ be the polynomial ring over a field $\Bbbk$ with $\mathrm{char}(\Bbbk) = 0$ and consider the ideal: $I = \langle\, y_1 + y_2 + \cdots + y_n, y_1^2, y_2^2, \ldots, y_n^2 \,\rangle \subseteq R$. We show that the graded quotient $N = R/I$ has Hilbert series given by

$$H_N(q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} f^{(n-k,k)} q^k \,,$$

where $f^{n-k,k}$ is the number of standard Young tableaux of shape $(n-k,k)$, equivalently, the dimension of the symmetric group irreducible indexed by that shape. These number are related to paths that remain above the diagonal such as Catalan paths (see Definition 1.3). We then show that N is indeed a symmetric group module and its irreducible representation decomposition is as expected (see Section 3). This problem was suggested to us by John Machacek [Mac19]. This is one among a family of ideals related to those studied extensively in [HRS18] and was motivated as a commutative version of the exterior portion of super-space quotient defined in [Zab19]. Our main goal here is to find explicitly the Gröbner basis for our case.

Theorem 1.1. *The reduced Gröbner basis with respect to $>_{lex}$ for the ideal I consists of $\{y_2^2, \ldots, y_n^2\}$ and of the polynomials*

$$g_\alpha = y^\alpha + \sum_{\beta \in P_\alpha} y^\beta \tag{1}$$

Where $\alpha \in \{0,1\}^n$ is a Modified Catalan Path (see Definition 1.7) and P_α is a set of paths weakly above α (see Definition 1.9).

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The paper is divided as follow. In §1.1 we introduce some definitions, notations and a loop version of our main theorem; namely Theorem 1.13. We span §2 to construct the Gröbner and in §3, we will compute some independent elements in the orthogonal complement of I. A dimension argument will allow us to conclude they form a basis of the quotient and that Theorem 1.13 holds. As a byproduct, we show that each homogeneous component is of degree $k \leq \lfloor \frac{n}{2} \rfloor$. It is in fact an irreducible symmetric group module indexed by the shape $(n-k, k)$.

Acknowledgement. We would like to thank John Machacek for his prior work on the project and for his suggestion to work on this project.

1.1. **Definitions, notations and a loop version of our main theorem.** In the following we need some notation and definitions relevant to the construction of our Gröbner basis.

Definition 1.2. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ we denote a monomial $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n} \in \Bbbk[y_1, \ldots, y_n]$. For any square free monomial y^α we call the $\{0,1\}$–sequence α the **bit-string** of the monomial. We often remove comas and parenthesis to represent α.

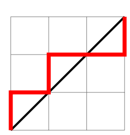
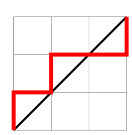
Definition 1.3. A **Catalan path** as a series of unit steps forming a staircase walk from $(0,0)$ to (l,l), which is weakly above the line $y = x$ in the Cartesian plane. Each step is either $(0,1)$ an **east** step, or $(1,0)$ a **north** step. Such a path contain $2l$ steps.

Remark 1.4. An arbitrary Northeast path correspond to a bit string $\alpha \in \{0,1\}^n$, where $\alpha_i = 0$ indicates the i^{th} step is North, and $\alpha_j = 1$ indicates the j^{th} step is East. We say that α a path of length n. We have the following correspondence:

$$\text{Square-Free Monomial} \quad \Longleftrightarrow \quad \text{Bitstring} \quad \Longleftrightarrow \quad \text{Path}$$

We will often reference a monomial, path or bistring interchangeably as the same object.

Example 1.5. Consider the monomials $y_2y_4y_5, y_2y_4y_6 \in \Bbbk[y_1, \ldots, y_6]$; their bitstring representations are `010110` and `010101` respectively. Their corresponding paths are respectively

 and  .

For definitions and theorems pertaining to Gröbner basis theory, we follow [CLO13]. Throughout this paper we will use the following lexicographic order.

Definition 1.6. Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We say that $\alpha >_{lex} \beta$ if the leftmost nonzero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n_{\geq 0}$ is positive. We will write $y^\alpha >_{lex} y^\beta$ if $\alpha >_{lex} \beta$, and it is a monomial order [CLO13, §2 —Proposition 4].

Theorem 1.1 is our main theorem. To prove it, we decompose the Buchberger's algorithm into loops. The division of S-polynomials produced in each loop is either the next generation of MCP polynomials or has a zero remainder.

Definition 1.7. A sequence $\alpha \in \{0,1\}^n$ is a **Modified Catalan Path (MCP)** if there exists integers $l, m \geq 0$ and a Catalan path $w \in \{0,1\}^{2l}$ such that $\alpha = w1(0)^m$. Note that α is equivalent to a Catalan path with an additional East step after the final step. We let $d = 2l + 1$ the position of the last 1 in α.

Definition 1.8. For a sequence $\alpha \in \{0, 1\}^n$ we say that $\ell(\alpha) = n$ is the **length** of α. The number of entries 0 and the number of entries 1 in α are denoted $\ell_0(\alpha)$ and $\ell_1(\alpha)$, respectively. Remark that $\ell_1(\alpha)$ is the degree of the corresponding monomial y^α. Moreover, if $\alpha = w1(0)^m$ is MCP, then $d = \ell(w) + 1 = 2\ell_1(\alpha) - 1$ is the position of the last entry 1 in α.

Definition 1.9. For a fixed MCP α, let $d = 2\ell_1(\alpha) - 1$. We define the set:

$$P_\alpha = \left\{\beta \in \{0,1\}^n \middle| \beta \neq \alpha,\ \ell_1(\beta) = \ell_1(\alpha) \text{ and } (\alpha_i = 0 \implies \beta_i = 0)_{1\leq i\leq d}\right\}$$

Thus, any $\beta \in P_\alpha$ will correspond to a northeast path that is weakly above α and ends in the same position as α.

Remark 1.10. Note that $\alpha >_{lex} \beta$ for all $\beta \in P_\alpha$. Hence, the polynomial g_α in Equation (1) has leading term given by y^α.

Definition 1.11. For k such that $\alpha_k = 1$, let $P_\alpha^{(k)} = \left\{\beta \in P_\alpha \mid \beta_k = 0\right\}$.

Definition 1.12. Given an integer $\ell \geq -1$ we let

$$F_\ell = \left\{y_2^2, \ldots, y_n^2\right\} \cup \left\{g_\alpha \middle| \alpha \text{ is an MCP and } \ell_1(\alpha) - 1 \leq \ell\right\},$$

where g_α is as indicated in Equation (1). Here $F_{-1} = \left\{y_2^2, \ldots, y_n^2\right\}$.

Theorem 1.13. *Fix the integer $n > 0$ and let $\ell = \lfloor \frac{n-1}{2} \rfloor$. The set F_ℓ is the reduced Gröbner basis for the ideal $I = \langle\, y_1 + \cdots + y_n, y_1^2, \ldots, y_n^2 \,\rangle$*

2. Constructing F_ℓ as S-polynomials

To prove Theorem 1.13, we proceed by iteration, inputing F_ℓ into a special loop of Buchberger's algorithm to obtain $F_{\ell+1}$. Each loop use only well selected S-polynomials to produce $F_{\ell+1}$, and Lemma 2.2 is the key to this process. Once we reach $F_{\lfloor \frac{n-1}{2} \rfloor}$, we turn to §3 to conclude that all other S-polynomials must have zero remainder after division by $F_{\lfloor \frac{n-1}{2} \rfloor}$, concluding our proof. This will be done using a dimension argument. Remark that $I \neq \langle\, F_\ell \,\rangle$ for $\ell \leq 0$, hence we must first show the following lemma.

Lemma 2.1. *Fix an integer $n > 0$ we have that $I = \langle\, F_1 \,\rangle$.*

Proof. We have that $F_1 = \{g_1, g_{011}, y_2^2, \ldots, y_n^2\}$ where

$$g_1 = y_1 + \sum_{\beta \in P_1} y^\beta = y_1 + y_2 + \cdots + y_n \in I$$

$$g_{011} = y_2 y_3 + \sum_{\beta \in P_{011}} y^\beta = \sum_{2\leq i<j\leq n} y_i y_j$$

To show that $\langle\, F_1 \,\rangle \subseteq I$ we only need to show that $g_{011} \in I$. Indeed, we have

$$g_{011} = \frac{1}{2}(g_1^2 - (y_1^2 + \cdots + y_n^2)) - y_1 g_1 + y_1^2 \in I\,.$$

Conversely, we have $y_1^2 = (y_2^2 + \cdots + y_n^2) - g_1^2 + 2(y_1 g_1 + g_{011}) \in \langle\, F_1 \,\rangle$. □

Now that this is established, we want to show that the division algorithm with respect to F_ℓ on certain S-polynomials of F_ℓ produce $F_{\ell+1}$. The next lemma will be very useful for this. For $g_\alpha \in F_\ell$, we compute *only* the S-polynomials of g_α and $y_j^2 \in F_\ell$ for all j such that $\alpha_j = 1$ and divide each results by $F_{\ell_1(\alpha)-1}$. Let us denote the result by

$$\mathcal{S}_{\alpha,j} = \overline{S(g_\alpha, y_j^2)}^{F_{\ell_1(\alpha)-1}}$$

Lemma 2.2. *Let $\ell \le \lfloor \frac{n-1}{2} \rfloor - 1$ and let $g_\alpha \in F_\ell$. Let $1 < k_1 < \cdots < k_{\ell_1(\alpha)} = d$ be the positions of all $\alpha_{k_i} = 1$. By definition of F_ℓ, $\ell_1(\alpha) - 1 \le \ell$ and $d = 2\ell_1(\alpha) - 1$. There exists an invertible linear map ϕ such that:*

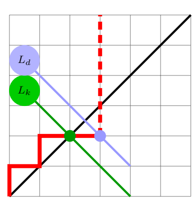
$$\phi(\mathcal{S}_{\alpha,k_i}) = \sum_{j=1}^{\ell_1(\alpha)} c_{ij} \mathcal{S}_{\alpha,k_j} = g_{\alpha^{(k_i)}} \,,$$

where $\alpha^{(k_i)} = \alpha_1 \cdot\cdot \alpha_{k_i-1} \mathtt{0} \alpha_{k_i+1} \cdot\cdot \alpha_d \mathtt{11}(\mathtt{0})^{n-d-2}$. The $\Bbbk$-matrix $M_\alpha = [c_{ij}]$ is invertible.

The proof of this lemma is technical and will be done by cases in the sections §2.1–§2.6. Once it is established, the invertibility of the matrix M_α shows that the $g_{\alpha^{(k_i)}}$ can be obtained from the $\mathcal{S}_{\alpha,k_i}$ by a sequence of well chosen elementary Gaussian operations that mimic S-polynomials and divisions. The $g_{\alpha^{(k_i)}}$ have distinct leading terms and are fully reduced, they are in the results of this loop of the Buchberger's algorithm and nothing else is produced from the $\mathcal{S}_{\alpha,k_i}$. To visualize our case analysis, it will be useful to develop good graphical representation of the paths.

Visualization. Given $g_\alpha \in F_\ell$, we have $\alpha = w1(0)^{n-2l-1}$ is an MCP for $\ell(w) = 2l \le \ell$. Let $d = 2l + 1 = 2\ell_1(\alpha) - 1$ be the position of the last 1 in α. For $\beta \in P_\alpha^{(k)}$ recall that $\beta_k = 0$ and k is chosen such that $\alpha_k = 1$. Let L_k be the anti-diagonal line intersecting the end of k^{th} position of α, and similarly for L_d which represents the last 1 in α. Then any such β we must have a north step, ending at some position on the line L_k. This is visualized as in Example 2.3.

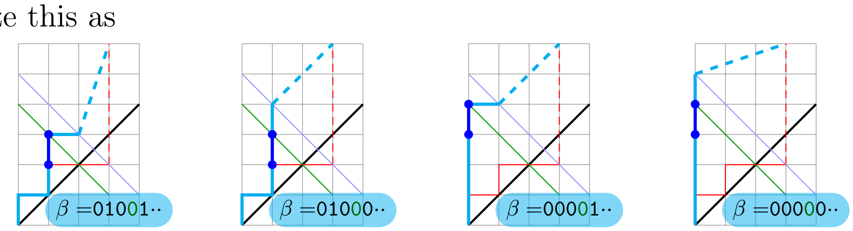
Example 2.3. Suppose, $k = 4$, and $\alpha = \mathtt{010110}\cdots\mathtt{0}$, then $d = 5$. The path α and the two lines L_k and L_d are visualized as



The set $P_\alpha^{(4)}$ decomposes into four different possibilities depending on the first d entries of $\beta = \beta_1\beta_2\beta_3\beta_4\beta_5\cdots$:

$$P_\alpha^{(4)} = \{\mathtt{01001}|\cdot\cdot\mathtt{1}\cdots\,;\ \mathtt{01000}|\cdot\cdot\mathtt{1}\cdot\cdot\mathtt{1}\cdots\,;\ \mathtt{00001}|\cdot\cdot\mathtt{1}\cdot\cdot\mathtt{1}\cdots\,;\ \mathtt{00000}|\cdot\cdot\mathtt{1}\cdot\cdot\mathtt{1}\cdot\cdot\mathtt{1}\cdots\}$$

We visualize this as



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From Example 2.3, it is evident that $P_\alpha^{(4)}$ decomposes according to the first d entries of $\beta \in P_\alpha^{(4)}$. The remaining entry of such β are only restricted by the number of ones. Also note that the number of zeros in $\beta_1 \cdots \beta_d$ determines the position where β crosses the line L_d. Indeed, let

$$\ell_0(\beta_1 \cdots \beta_d) = q \quad \text{and} \quad \ell_1(\beta_1 \cdots \beta_d) = (d) - q\,,$$

then the path β intersects the line L_d at the point $(q, d-q)$.

2.1. **Decomposing P_α.** In this section we decompose the set P_α according to first $d = 2\ell_1(\alpha) - 1$ entries of the path $\beta \in P_\alpha$

Definition 2.4. Let α be an MCP with $d = 2\ell_1(\alpha) - 1$. For $1 \le r \le \ell_1(\alpha)$ we define

$$U_{\alpha,r} = \left\{ u \in \{0,1\}^d \middle| (\alpha_i = 0 \implies u_i = 0)_{1 \le i \le d} \text{ and } \ell_1(u) = \ell_1(\alpha) - r \right\} \tag{2}$$

Given $u \in U_{\alpha,r}$ we define

$$P_{u,\alpha} = \left\{ \beta \in P_\alpha \middle| \beta_1 \cdots \beta_d = u \right\} \tag{3}$$

Example 2.5. Let $\alpha = \mathtt{010110} \cdots \mathtt{0}$, then $\ell_1(\alpha) = 3$. for $r = 1$, we have

$$U_{\mathtt{010110}\cdots\mathtt{0},1} = \{\mathtt{01010},\ \mathtt{01001},\ \mathtt{00011}\}\,.$$

The subsets $P_{u,\alpha}$ are disjoint, so we can partition $P_\alpha = \bigcup_{r=1}^{\ell_1(\alpha)} \bigcup_{u \in U_{\alpha,r}} P_{u,\alpha}$. Therefore

$$\sum_{\beta \in P_\alpha} y^\beta = \sum_{r=1}^{\ell_1(\alpha)} \sum_{u \in U_{\alpha,r}} y^u \sum_{d+1 \le i_1 < \cdots < i_r \le n} y_{i_1} \cdots y_{i_r}\,. \tag{4}$$

We remark that for $\beta \in P_{u,\alpha}$ the only restriction on $\beta_{d+1} \cdots \beta_n$ is that $\ell_1(\beta_{d+1} \cdots \beta_n) = \ell_1(\beta) - \ell_1(u) = r$. Hence in the last summation in Equation (4) we are summing over all square-free monomials of degree r in the variables $y_{d+1}, \dots, y_n$.

Definition 2.6. For $\alpha_k = 1$, let $u(\alpha, k) = \alpha_1 \cdots \alpha_{k-1} 0 \alpha_{k+1} \cdots \alpha_d \in U_{\alpha,1}$.

2.2. **computing $\overline{S(g_\alpha, y_k^2)}^{F_{-1}}$.** We first investigate the S-polynomial we want to compute dividing only by $F_{-1} = \{y_2^2, \dots, y_n^2\}$.

Lemma 2.7. *Let $g_\alpha \in F_\ell$ and given any $1 < k \le 2\ell_1(\alpha) - 1$ be such that $\alpha_k = 1$. (α_1 is always 0). We have*

$$\overline{S(g_\alpha, y_k^2)}^{F_{-1}} = \sum_{\beta \in P_\alpha^{(k)}} y^\beta y_k. \tag{5}$$

Proof. Since $\alpha_k = 1$, we have $\mathrm{LCM}(\mathrm{LM}(g_\alpha), y_k^2) = y^\alpha y_k$ and the result is as follow

$$\overline{S(g_\alpha, y_k^2)}^{F_{-1}} = \overline{y_k g_\alpha - y^\alpha y_k}^{F_{-1}} = \overline{\sum_{\beta \in P_\alpha} y^\beta y_k}^{F_{-1}} = \sum_{\beta \in P_\alpha^{(k)}} y^\beta y_k\,.$$

The last equality follow from the fact that any monomial $\overline{y^\beta y_k}^{F_{-1}} = 0$ if and only if $\beta \in P_\alpha \setminus P_\alpha^{(k)}$. $\square$

For $g_\alpha \in F_\ell$ and k such that $\alpha_k = 1$, Lemma 2.7 and $P_\alpha^{(k)} = \bigcup_{r=1}^{\ell_1(\alpha)} \bigcup_{u\in U_{\alpha,r},\, u_k=0} P_{u,\alpha}$ gives

$$\overline{S(g_\alpha, y_k^2)}^{F_{-1}} = \sum_{\beta\in P_\alpha^{(k)}} y^\beta y_k = \sum_{r=1}^{\ell_1(\alpha)} \sum_{\substack{u\in U_{\alpha,r}\\ u_k=0}} y^u y_k \sum_{d+1\le i_1<\cdots<i_r\le n} y_{i_1}\cdots y_{i_r} \tag{6}$$

where as before $d = 2\ell_1(\alpha)-1$. We need now to divide this by $F_{\ell_1(\alpha)-1}$. We will do this in two separate cases: the terms of this equation for $r=1$ and those for $r>1$.

2.3. **Division of term $r = 1$ in Equation (6) by $F_{\ell_1(\alpha)-1}$.** When $r=1$ in Equation (6), there is only a single $u \in U_{\alpha,1}$ such that $u_k = 0$, namely $u(\alpha, k)$ as in Definition 2.6. The term $r=1$ in Equation (6) is

$$y^{u(\alpha,k)} y_k \sum_{d+1\le j\le n} y_j = \sum_{j=d+1}^{n} y^\alpha y_j$$

The terms in the sum above are monomial factors of $\mathrm{LT}(g_\alpha) = y^\alpha$. Using the Division Algorithm we can divide the expression further using g_α:

$$\begin{aligned}\overline{\sum_{j=d+1}^{n} y^\alpha y_j}^{\{g_\alpha\}} &= \sum_{j=d+1}^{n} y^\alpha y_j - g_\alpha \sum_{j=d+1}^{n} y_j = -\sum_{\gamma\in P_\alpha} y^\gamma \sum_{j=d+1}^{n} y_j \\ &= -\sum_{s=1}^{\ell_1(\alpha)} \sum_{v\in U_{\alpha,s}} y^v \sum_{d+1\le i_1<\cdots<i_s\le n} y_{i_1}\cdots y_{i_s} \sum_{j=d+1}^{n} y_j\end{aligned}$$

We use Equation (4) in the last equality. We can further divide this last expression using F_{-1} and we get

$$\overline{\sum_{j=d+1}^{n} y^\alpha y_j}^{\{g_\alpha, F_{-1}\}} = -\sum_{s=1}^{\ell_1(\alpha)} (s+1) \sum_{v\in U_{\alpha,s}} y^v \sum_{d+1\le i_1<\cdots<i_{s+1}\le n} y_{i_1}\cdots y_{i_s} y_{i_{s+1}} \tag{7}$$

This results follows from the fact that

$$\sum_{d+1\le i_1<\cdots<i_s\le n} y_{i_1}\cdots y_{i_s} \sum_{j=d+1}^{n} y_j = (s+1) \sum_{d+1\le i_1<\cdots<i_{s+1}\le n} y_{i_1}\cdots y_{i_s} y_{i_{s+1}} + \text{terms containing } y_j^2$$

Once again, we want to separate the terms $r=1$ in Equation (7) and those for $r \ge 2$

Case 2.8 ($s=1$ *in Equation* (7))**.** In the case of $s=1$ in Equation (7) and $v \in U_{\alpha,1}$, we can represent any path $y^v y_{i_1} y_{i_2}$ visually as in Figure 1. For each $v \in U_{\alpha,1}$ we have $\ell_1(v) = \ell_1(\alpha) - 1$ and let k be the unique position where $v_k = 0$ and $\alpha_k = 1$. The paths of the monomial $y^v y_{i_1} y_{i_2}$ in the sum have two ones after the d^{th} position. If we select $i_1 = d+1$ and $i_2 = d+2$ this will form a *new* MCP path $\alpha^{(k)} = \alpha_1 \cdot\cdot \alpha_{k-1} \mathtt{0} \alpha_{k+1} \cdot\cdot \alpha_d \mathtt{11}(\mathtt{0})^{n-d-2}$, as displayed above. All the other paths in this case remain above the diagonal. There is no more division possible using $F_{\ell_1(\alpha)-1}$. The MCP path is actually new for F_ℓ if $\ell_1(\alpha) - 1 = \ell$ as $\ell_1(\alpha^{(k)}) - 1 > \ell$.

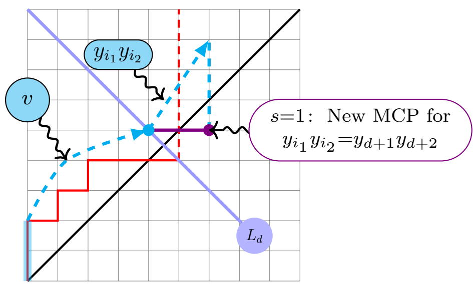

FIGURE 1. On the left is Case 2.8 and on the right is Case 2.9

Case 2.9 ($s \geq 2$ *in Equation* (7))**.** Similarly, in the case of $s \geq 2$ and $v \in U_{\alpha,s}$, we visualize any such path $y^v y_{i_1} \cdots y_{i_{s+1}}$ as in Figure 1. Any such paths clearly remain above the diagonal and no further divisions in $F_{\ell_1(\alpha)-1}$ is possible.

2.4. **Division of terms $r > 1$ in Equation (6) by $F_{\ell_1(\alpha)-1}$.** We now focus on the terms $r > 1$ in Equation (6). Fix $r > 1$ and fix $u \in U_{\alpha,r}$ such that $u_k = 0$. We are considering monomial of the form $y^u y_k y_{i_1} \cdots y_{i_r}$ where $d+1 \leq i_1 < \cdots < i_r \leq n$. Note that $\alpha_k = 1$ and $u_k = 0$, and there is at least one more entry such that $\alpha_j = 1$ and $u_j = 0$ for $j \neq k$. Hence, there are *at least* two entries in which u differs from α. We will split again in two cases depending if $r = 2$ or $r > 2$. We visualize this in Figure 2. When we multiply $y^\beta = y^u y_{i_1} \cdots y_{i_r}$ by y_k, the portion of the path corresponding to

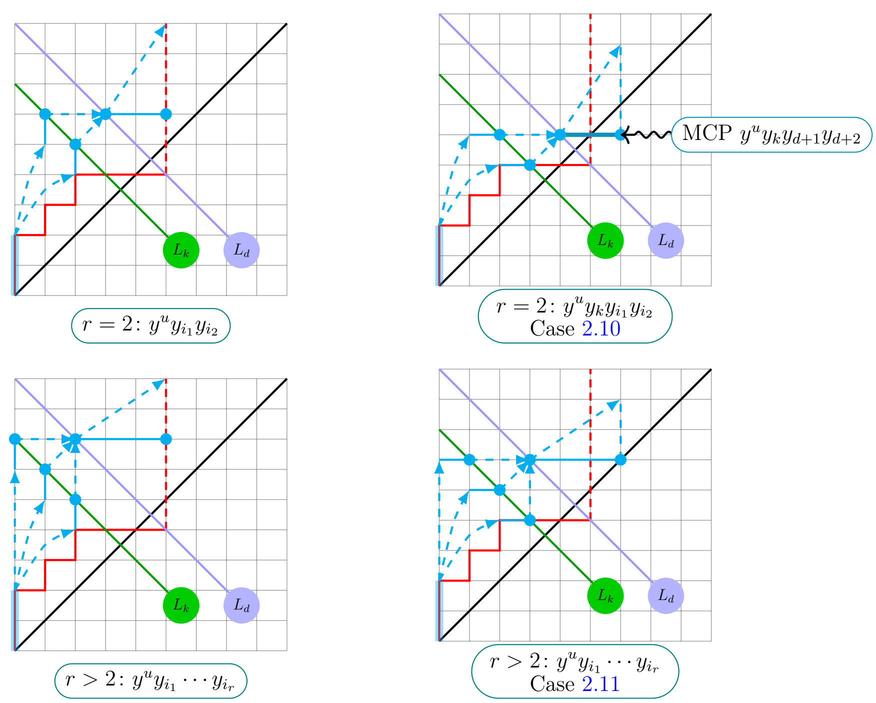

FIGURE 2. On the left are the path of monomials y^β for $\beta \in P_\alpha^{(k)}$. On the right are the path of the terms in Equation (6) for $r > 1$

$\beta_{k+1}\beta_{k+2}\cdots\beta_n$ will be translated down by one unit and translated right by one unit. All the paths remain above the diagonal except when $r=2$ and $y^\beta = y^u y_{d+1}y_{d+2}$. All paths will match a subset of the paths we encounter in §2.3 as we see bellow. In particular, no further division can be perform with polynomials in $F_{\ell_1(\alpha)-1}$.

Case 2.10 ($r=2$ *in Equation* (6))**.** As Visualized in Figure 2, when $r=2$ in Equation (6) the path we get are exactly the path in Case 2.8 for which $v_k=1$. here we have some of the special new MCP.

Case 2.11 ($r\geq 3$ *in Equation* (6))**.** Also Visualized in Figure 2, when $r>2$ in Equation (6) the path we get are exactly the path in Case 2.9 for which $v_k=1$.

2.5. **Combining all divisions.** The analysis of §2.3 and §2.4 show that to divide $S(g_\alpha, y_k^2)$ with respect to $F_{\ell_1(\alpha)-1}$ we need to substitute Equation (7) in Equation (6) for $r=1$. We also substitute $s=r-1$ to match up the terms. We obtain

$$\overline{S(g_\alpha,y_k^2)}^{F_{\ell_1(\alpha)-1}} = -\sum_{r=2}^{\ell_1(\alpha)+1}\sum_{v\in U_{\alpha,r-1}} r y^v \sum_{d+1\leq i_1<\cdots<i_r\leq n} y_{i_1}\cdots y_{i_r} \tag{8}$$
$$+\sum_{r=2}^{\ell_1(\alpha)}\sum_{\substack{u\in U_{\alpha,r}\\ u_k=0}} y^u y_k \sum_{d+1\leq i_1<\cdots<i_r\leq n} y_{i_1}\cdots y_{i_r}$$
$$=\sum_{r=2}^{\ell_1(\alpha)+1}\sum_{\substack{v\in U_{\alpha,r-1}\\ v_k=0}} (-r) y^v \sum_{d+1\leq i_1<\cdots<i_r\leq n} y_{i_1}\cdots y_{i_r} \tag{9}$$
$$+\sum_{r=2}^{\ell_1(\alpha)}\sum_{\substack{v\in U_{\alpha,r-1}\\ v_k=1}} (-r+1) y^v \sum_{d+1\leq i_1<\cdots<i_r\leq n} y_{i_1}\cdots y_{i_r}$$

In the second equality, we combined terms using the fact that for $u\in U_{\alpha,r}$ and $u_k=0$ the term $y^u y_k = y^v$ for $v\in U_{\alpha,r-1}$ and $v_k=1$. When $r=2$, the term $v\in U_{\alpha,r-1}$ and $v_k=1$ is unique and is exactly $v=u(\alpha,k)$ as in Definition 2.6.

2.6. **The Matrix the linear transformation.** The process until now has focused on a single S-polynomial reduction $\overline{S(g_\alpha,y_k^2)}^{F_{\ell_1(\alpha)-1}} = \mathcal{S}_{\alpha,k}$ for a fixed k and fixed α. In this section, we examine all possible $\mathcal{S}_{\alpha,k_j}$, letting k_j vary while keeping α fixed. Also, since α has $\ell_1(\alpha)$ entries equal to 1, we have that $U_{\alpha,1} = \{u(\alpha,k_i) \big| 1\leq i\leq \ell_1(\alpha)\}$ has cardinality $\ell_1(\alpha)$. we let

$$A_\alpha = \quad \Big[\text{Coefficient of } y^{u(\alpha,k_i)}y_{d+1}y_{d+2} \text{ in } \mathcal{S}_{\alpha,k_j}\Big]_{1\leq i,j\leq l+1}$$

When $r=2$ in Equation (9) we readily see that we have -2 in the diagonal of A_α and -1 off the diagonal. This matrix is invertible and we explicitly get obtain

$$A_\alpha = \begin{bmatrix} -2 & -1 & \cdots & -1 \\ -1 & -2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -2 \end{bmatrix} \qquad M_\alpha = A_\alpha^{-1} = \frac{1}{\ell_1(\alpha)+1}\begin{bmatrix} -\ell_1(\alpha) & 1 & \cdots & 1 \\ 1 & -\ell_1(\alpha) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -\ell_1(\alpha) \end{bmatrix}$$

Since A_α is invertible, we have that $\{\mathcal{S}_{\alpha,k_i}\}_{1\le i\le \ell_1(\alpha)}$ is linearly independent. Let V be the $\Bbbk$-span of $\{\mathcal{S}_{\alpha,k_i}\}_{1\le i\le \ell_1(\alpha)}$ and we define $\phi\colon V\to V$ as

$$\phi(\mathcal{S}_{\alpha,k_i}) = \sum_{j=1}^{\ell_1(\alpha)} c_{ij}\mathcal{S}_{\alpha,k_j}\,, \tag{10}$$

where $M_\alpha = [c_{ij}]$. This is the linear transformation we need in Lemma 2.2. To conclude the proof, we need to show that $\phi(\mathcal{S}_{\alpha,k_i}) = g_{\alpha^{(k_i)}}$. To compute this, we substitute Equation (9) in (10) for each k_j. When $r=2$ and fixed $d+2\le i_1<i_2\le n$, then coefficient of $y^{u(\alpha,k_i)}y_{i_1}y_{i_2}$ in the expansion on $\mathcal{S}_{\alpha,k_i}$ is exactly the matrix A_α. Therefore when we make the linear combination using M_α (the inverse of A_α), we get

$$\text{Coeff of } y^{u(\alpha,k_{i'})}y_{i_1}y_{i_2} \text{ in } \phi(\mathcal{S}_{\alpha,k_i}) = \begin{cases} 1 & \text{if } i=i'\,,\\ 0 & \text{otherwise.}\end{cases} \tag{11}$$

For the other terms, fix r,v and $d+1\le i_1<\cdots<i_r\le n$ such that $3\le r\le \ell_1(\alpha)+1$ and $v\in U_{\alpha,r-1}$. We want to determine the coefficient of $y^v y_{i_1}\cdots y_{i_r}$ in $\phi(\mathcal{S}_{\alpha,k_i})$. This will depend on the value of v_{k_i}.

Case 2.12 ($v_{k_i}=0$)**.** Since $v_{k_i}=0$ and $v\in U_{\alpha,r-1}$, there are $(r-1)-1=r-2$ other entries of v such that $\alpha_k=1$ and $v_k=0$. On the other hand, the number of positions where $\alpha_k=v_k=1$ is $\ell_1(\alpha)-r+1$. The coefficient of $y^v y_{i_1}\cdots y_{i_r}$ in $\phi(\mathcal{S}_{\alpha,k_i})$ is

$$\begin{aligned}-rc_{ii} + \sum_{\substack{v_{k_j}=0\\ j\ne i}} -rc_{ij} + \sum_{\substack{v_{k_j}=1\\ j\ne i}} (-r+1)c_{ij} &= \frac{r\ell_1(\alpha)-r(r-2)+(-r+1)(\ell_1(\alpha)-r+1)}{\ell_1(\alpha)+1}\\ &= 0\,.\end{aligned}$$

The coefficient $-r$ and $-r+1$ are from Equation (9) and depend on the value of v_k. The entry $[c_{ij}]=M_\alpha$.

Case 2.13 ($v_{k_i}=1$)**.** Now, $\alpha_{k_i}=v_{k_i}=1$. There are $r-1$ entries of v such that $\alpha_k=1$ and $v_k=0$ and $\ell_1(\alpha)-r$ entry $\alpha_k=v_k=1$ other than $k=k_i$. The coefficient of $y^v y_{i_1}\cdots y_{i_r}$ in $\phi(\mathcal{S}_{\alpha,k_i})$ is now

$$\begin{aligned}&(-r+1)c_{ii} + \sum_{\substack{v_{k_j}=0\\ j\ne i}} -rc_{ij} + \sum_{\substack{v_{k_j}=1\\ j\ne i}} (-r+1)c_{ij}\\ &\qquad = \frac{(r-1)\ell_1(\alpha)-r(r-1)+(-r+1)(\ell_1(\alpha)-r)}{\ell_1(\alpha)+1} = 1\,.\end{aligned}$$

It follows that

$$\phi(\mathcal{S}_{\alpha,k_i}) = y^{u(\alpha,k_i)}\sum_{d+1\le i_1<i_2\le n} y_{i_1}y_{i_2} + \sum_{r=3}^{\ell_1(\alpha)+1}\sum_{\substack{v\in U_{\alpha,r-1}\\ v_{k_i}=0}} y^v \sum_{d+1\le i_1<\cdots<i_r\le n} y_{i_1}\cdots y_{i_r} \tag{12}$$

This sum is non zero since $\ell_1(\alpha)-1\le \ell\le \lfloor\frac{n-1}{2}\rfloor-1$, therefore $d+2\le n$. We want to compare this with $g_{\alpha^{(k_i)}}$ where $\alpha^{(k_i)} = u(\alpha,k_i)110^{n-d-2}$ for $d=2\ell_1(\alpha)-1$. Using

Equation (4)

$$g_{\alpha^{(k_i)}} = y^{\alpha^{(k_i)}} + \sum_{s=1}^{\ell_1(\alpha^{(k_i)})} \sum_{u \in U_{\alpha^{(k_i)},s}} y^u \sum_{d+3 \le i_1 < \cdots < i_s \le n} y_{i_1} \cdots y_{i_s} \tag{13}$$

Note that $\ell_1(\alpha^{(k_i)}) = \ell_1(\alpha) + 1$ and for any $u \in U_{\alpha^{(k_i)},s}$ we must have $u_{k_i} = 0$ since $\alpha^{(k_i)}_{k_i} = 0$. For any other position $1 \le k \le d$, we have $\alpha_k = \alpha^{(k_i)}_k$. Hence $\alpha_k = 0 \implies u_k = 0 \implies v_k = 0$ for any $v \in U_{\alpha,r-1}$. For $k = d+1$ or $k = d+2$ there are no constrain on u as $\alpha^{(k_i)}_{d+1} = \alpha^{(k_i)}_{d+2} = 1$. Moreover the term $y^{\alpha^{(k_i)}} = y^{u(\alpha,k_i)} y_{d+1} y_{d+2}$ appears in the first sum of Equation (12). This shows that every term in Equation (13) appears in Equation (12).

For the converse, note that any term $y^{u(\alpha,k_i)} y_{i_1} y_{i_2}$ for $d+1 \le i_1 < i_2 \le n$ correspond to $y^{u(\alpha,k_i)} y_{d+1} y_{d+2} = y^{\alpha^{(k_1)}}$; or $y^{u(\alpha,k_i)} y_{d+1} y_i$ and $y^{u(\alpha,k_i)} y_{d+2} y_i$ for $u(\alpha,k_i)10, u(\alpha,k_i)01 \in U_{\alpha^{(k_i)},1}$ and $d+3 \le i \le n$; or $y^{u(\alpha,k_i)} y_{i_1} y_{i_2}$ for $u(\alpha,k_i)00 \in U_{\alpha^{(k_i)},2}$ and $d+3 \le i_1 < i_2 \le n$. Similarly, for $v \in U_{\alpha,r-1}$ with $v_{k_i} = 0$, the terms $y^v y_{i_1} \cdots y_{i_r}$ for $d+1 \le i_1 < \cdots < i_r \le n$ correspond to $y^v y_{d+1} y_{d+2} y_{j_3} \cdots y_{j_{r-1}}$, for $v11 \in U_{\alpha^{(k_i)},r}$ and $d+3 \le j_3 < \cdots < j_r \le n$; or $y^v y_{d+1} y_{j_2} \cdots y_{j_{r-1}}$ and $y^v y_{d+2} y_{j_2} \cdots y_{j_{r-1}}$ for $v01, v10 \in U_{\alpha^{(k_i)},r+1}$ and $d+3 \le j_2 < \cdots < j_r \le n$; or $y^v y_{j_1} \cdots y_{j_{r-1}}$ for $v00 \in U_{\alpha^{(k_i)},r+2}$ and $d+3 \le j_1 < \cdots < j_r \le n$. This show the reverse inclusion and conclude our proof of Lemma 2.2. $\square$

2.7. **Termination of Buchberger's Algorithm.** In the previous subsections, we computed the set of remainders of S-polynomials $\overline{S(g_\alpha, k_1)}^{F_{\ell_1(\alpha)-1}}, \ldots, \overline{S(g_\alpha, k_{\ell_1(\alpha)})}^{F_{\ell_1(\alpha)-1}}$ and used Lemma 2.2 to conclude that $\{g_{\alpha^{(k_i)}}\}_{1 \le i \le \ell_1(\alpha)}$ is obtained as an intermediate step in the Buchberger's Algorithm. If $\ell_1(\alpha) - 1 < \ell$ these polynomial are already in F_ℓ and division with respect to F_ℓ will give zero (no new polynomials). For $\ell_1(\alpha) - 1 = \ell$ we will get new polynomial and construct in this way $F_{\ell+1}$ as long as $\ell_1(\alpha) - 1 \le \lfloor \frac{n-1}{2} \rfloor - 1$ We need show that for $\ell = \lfloor \frac{n-1}{2} \rfloor$ the set F_ℓ is the reduced Gröbner basis. To this end the next lemma shows that the set

$$B = \left\{ y^\gamma \middle| \textit{square free and } \gamma \textit{ stays above the diagonal} \right\}, \tag{14}$$

$\Bbbk$-span the quotient R/I using division by F_ℓ only. In the next section we will sow that $\dim(R/I)$ is at least the cardinality of B. From this we will conclude that B is a basis, and therefore F_ℓ is the full reduced Gröbner basis of I. This will complete the proof of Theorem 1.13.

Lemma 2.14. *Let $\ell = \lfloor \frac{n-1}{2} \rfloor$. For any monomial y^δ, the remainder $\overline{y^\delta}^{F_\ell}$ is a $\Bbbk$-linear combination of B. In particular, B span R/I.*

Proof. If y^δ is not square free, then it is divisible by y_k^2 for some k and we get $\overline{y^\delta}^{F_\ell} = 0$. If y^δ crosses the diagonal, Let k be the smallest integer such that $\alpha = \delta_1 \delta_2 \cdots \delta_k 0^{n-k}$ crosses under the diagonal. It is clear that α is MCP and $\ell_1(\alpha) - 1 \le \ell$, hence $g_\alpha \in F_\ell$. Since we have $y^\delta = y^\alpha y^\epsilon$, we obtain

$$\overline{y^\delta}^{F_\ell} = \overline{y^\alpha y^\epsilon}^{F_\ell} = \overline{-\sum_{\beta \in P_\alpha} y^\beta y^\epsilon}^{F_\ell}. \tag{15}$$

All the terms $y^\beta y^\epsilon <_{lex} y^\alpha y^\epsilon$. we repeat the process on any monomials in the sum that cross the diagonal. We know this process stop (division algorithm) and we are left only with a linear combination of monomials in B. $\square$

3. Orthogonal complement, dimension, and irreducible decomposition

In this Section, we quickly introduce inverse system to compute the orthogonal complement $H = I^\perp$ of I. We have that $H \cong R/I$. We then construct a set B' of independent element inside H that has the same cardinality as B in Equation (14). This will show that both B and B' are a bases of the quotient, concluding the proof of Theorem 1.13. We then remark that since I is homogeneous and remain invariant under the permutation of variables, we can define an action of the symmetric group on the (graded) quotient R/I. Using irreducible representation theory of the symmetric group (see [Sag01] for example) we show that B' exhibit a unique irreducible for each homogeneous component of R/I. This will complete our full understanding of this quotient.

3.1. **Inverse System and Ortogonal complement of I.** At the root of commutative algebra, one find the theory of inverse system developed by Macaulay [Mac94]. This is classical theory and we only review (without proof) the needed ingrediens. Naively, we want to study the quotient R/I via the orthogonal complement of I under the following scalar product. For $P, Q \in R$ we define

$$\langle P, Q \rangle = \Big(P\big(\tfrac{\partial}{\partial y_1}, \tfrac{\partial}{\partial y_2}, \dots, \tfrac{\partial}{\partial y_n}\big)Q\Big)(0,0,\dots,0). \tag{16}$$

That is we substitute the partial derivatives in P and apply this operator to Q, after we evaluate the resulting polynomial at $(0,0,\dots,0)$. This is a scalar product on R. We then compute the orthogonal complement $H = I^\perp$ in R with respect to the scalar product in Equation (16). A wonderful lemma (a consequence of Taylor expansion) give us

Lemma 3.1.

$$H = \Big\{Q \in R \Big| P\big(\tfrac{\partial}{\partial y_1}, \dots, \tfrac{\partial}{\partial y_n}\big)Q = 0,\ P \textit{ generators of } I\Big\}$$

The difference is that now H is the solution set of a system of differential equations, we do not need the evaluation at zero. Since H is the orthogonal complement of I, we automatically get that

$$H \cong R/I\,.$$

3.2. **Standard Young Tableaux.** To define elements in H, we need to introduce the notion of Standard Young Tableaux. This will also be useful for the representation theory of the symmetric group aspect of our investigation. We shall also see that the set B in in Equation (14) is related to some standard Young tableaux.

Definition 3.2. Given a sequence of integer $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and $n = \lambda_1 + \cdots + \lambda_r$, we construct the set

$$D_\lambda = \Big\{(i,j) \in \mathbb{Z} \times \mathbb{Z} \Big| 1 \leq j \leq r;\ 1 \leq i \leq \lambda_j\Big\}\,.$$

We say that λ is an **partition** of n and D_λ is the **diagram** of λ. A bijection

$$T\colon \{1,2,\dots,n\} \to D_\lambda\,,$$

is called a **Standard Young Tableaux** (abbreviated SYT) if

$$T(i,j) < T(i+1,j) \qquad \text{and} \qquad T(i,j) < T(i+1,j),$$

whenever $T(i,j)$, $T(i+1,j)$ and/or $T(i,j) < T(i+1,j)$ are defined. We usually put the value $T(i,j)$ in the position (i,j) forming a *tableau* in the plane. The entries increase in rows and columns. We say that λ is the **shape** of T and n is the **size** of T. We write $T \in SYT_\lambda$ in this case.

Example 3.3. Let $\lambda = (5,3)$, the picture below is a visualization of a single $T \in SYT_\lambda$.

$$T = \begin{array}{|c|c|c|c|c|}\cline{1-3} 3 & 6 & 8 \\ \hline 1 & 2 & 4 & 5 & 7 \\ \hline\end{array}$$

Proposition 3.4. *For $k \le \lfloor \frac{n}{2} \rfloor$, let $\lambda = (n-k,k)$. The number $f^\lambda = \left|SYT_\lambda\right|$ is equal to the number of square free monomial y^γ that stays above the diagonal and $\ell_1(\gamma) = k$*

Proof. This is a very classical result and some version of it can be found in [Sta15]. The bijection $T \mapsto y^\gamma$ for $T \in SYT_\lambda$ is given by $\gamma_i = 0$ if i lies in the first row of T, otherwise $\gamma_i = 1$. One then check that staying above the diagonal is equivalent to the inequalities defining standard Young tableaux. □

In the proof above, let us denote by y^T the monomial we obtain from T. We thus have that B in Equation (14) is also given by the following disjoint union

$$B = \biguplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left\{ y^T \middle| T \in SYT_{(n-k,k)} \right\}. \tag{17}$$

3.3. **Independent element in H.** We introduce some special Specht polynomials [Spe35]. This will give us linearly independent polynomials in H. Fix $k \le \lfloor \frac{n}{2} \rfloor$ and let $\lambda = (n-k,k)$. Given $T \in SYT_\lambda$ we let

$$G_T(y_1, \dots y_n) = (y_{T(1,1)} - y_{T(2,1)})(y_{T(1,2)} - y_{T(2,2)}) \cdots (y_{T(1,k)} - y_{T(2,k)}). \tag{18}$$

Here, we only define G_T for shape λ with two parts, but it can be done for any shape using Vandermonde polynomials for each column of T. These are the polynomial originally defined by Specht [Spe35] to construct irreducible representations of S_n.

Example 3.5. Let T be as in Example 3.3, we have $G_T = (y_1 - y_3)(y_2 - y_6)(y_4 - y_8)$

Let

$$B' = \biguplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left\{ G_T \middle| T \in SYT_{(n-k,k)} \right\}. \tag{19}$$

Theorem 3.6. *We have that $B' \subseteq H = I^\perp$ is linearly independent.*

Proof. For the inclusion, using Lemma 3.1, we need to show that for $G_T \in B'$ we have

$$P\left(\tfrac{\partial}{\partial y_1}, \dots, \tfrac{\partial}{\partial y_n}\right) G_T = 0$$

for all generator P of I. If $P = y_k^2$ the we clearly get zero since G_T is linear in any single variable (square free). For $P = y_1 + y_2 + \cdots + y_n$, we use the product rule and reduce to the case

$$\left(\tfrac{\partial}{\partial y_1} + \tfrac{\partial}{\partial y_2} + \cdots + \tfrac{\partial}{\partial y_n}\right)(y_i - y_j) = 0$$

for all i, j.

To see that the polynomial are independent, expand each polynomial G_T and looks for its trailing term in lex-order, we obtain $LT(G_T) = y_{T(2,1)}y_{T(2,2)}\cdots y_{T(2,k)} = y^T$ which is clear by always taking the largest variable in each term $(y_i - y_j)$. Since all trailing term are distinct, we have linear independence by triangularity. $\square$

3.4. **Proof of theorem.** We have established that

$$|B| = |B'| \leq \dim(H) = \dim(R/I) \leq |B|.$$

The first equality is by construction, the inequality is Theorem 3.6, the next equality is the isomorphism of $H \cong R/I$ and the last inequality is Lemma 2.14. We must have equality everywhere, which show that both B and B' are basis of R/I. To see that Theorem 1.13 is true, assume we can compute new S-polynomials. This would produce new leading term and reduce the size of B as a spanning set for R/I which is absurbe. Hence, Theorem 1.13 holds true.

3.5. **Hilbert Seriies of R/I.** Since the ideal I is homogeneous, there is a well defined notion of degree on $N = R/I$, Hence $R/I = \bigoplus_{k\geq 0} N^{(k)}$ where $N^{(k)}$ is the homogeneous component of degree k in N. We know that the maximal degree in N is $k = \lfloor \frac{n}{2} \rfloor$. Considering the disjoint decomposition of the basis B in Equation 17 we immediately obtain

Theorem 3.7. *The Hilbert series of $N = R/I$ is given by*

$$H_N(q) = \sum_{k\geq 0} \dim\left(N^{(k)}\right)q^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} f^{(n-k,k)}q^k\,.$$

3.6. **Irreducible graded decomposition of R/I.** The symmetric group $S_n = \big\{\sigma \big| \sigma\colon \{1,\dots n\} \to \{1,\dots,n\} \text{ bijection}\big\}$ acts on polynomials in R with $\sigma P(y_1,\dots,y_n) = P(y_{\sigma(1)},\dots,y_{\sigma(n)})$. The generator of the ideal I are map to other generator of I under this action. Hence the Ideal I is invariant under the action of S_n. We can thus have a well define $\Bbbk$-linear action of S_n on the quotient $N = R/I$. In fact, since the action preserve degree, we have an action of S_n on each graded piece $N^{(k)}$ and we are interested in describing the decomposition of that action in term of irreducible representation. For more detail about representation of the symmetric group, see for instance [Sag01]. For our need, the irreducible representation of the symmetric can be nicely constructed directly in the space of polynomials using Specht polynomials [Spe35] which we have already encounter in §3.3.

It is known [Sag01, Spe35] that for a fix shape λ the set

$$\big\{G_T \big| T \in SYT_\lambda\big\} \tag{20}$$

is a basis for an irreducible representation of S_n indexed by λ. Considering the disjoint decomposition of the basis B' in Equation 19 we immediately obtain

Theorem 3.8. *For $k \leq \lfloor \frac{n}{2} \rfloor$, we have that Equation (20) is a basis of $N^{(k)}$. In particular, $N^{(k)}$ is an irreducible representation of S_n indexed by $\lambda = (n-k, k)$. We can thus write*

$$Frob_q(R/I) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} s_{(n-k,k)} q^k$$

where s_λ is the schur function used to encode irreducible representation of S_n and $Frob_q$ is the graded Frobeneous map that maps representation to symmetric functions (see [Sag01, Mac95]).

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