The excedance quotient of the Bruhat order, Quasisymmetric Varieties and Temperley-Lieb algebras

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1 Introduction

2 Preliminaries

2.1 Noncrossing partitions

A *noncrossing partition* of size n is a diagram λ consisting of:

1. the positive integers $1, \ldots, n$, placed from left to right along a horizontal axis; and

2. a set of left-to-right arcs $(i, j) = i \frown j$, $i < j$ drawn above the axis with no intersections or coterminal points: λ contains no pair $i \frown k$, $j \frown l$ with $i \leqslant j < k \leqslant l$.

For example,

$$\lambda = \quad 1 \quad 2 \frown 3 \frown 4 \quad 5 \frown 6 \quad 7 \tag{2.1}$$

is a noncrossing partition of size 7 containing the arcs $2 \frown 7$, $3 \frown 5$, and $5 \frown 6$.

Considering a noncrossing partition λ as an (undirected) graph, the connected components of λ give a partition of the set $[n] = \{1, \ldots, n\}$, which is the origin of the term. For example, the noncrossing partition shown in Equation (2.1) corresponds to the set partition $\{\{1\}, \{3, 5, 6\}, \{2, 7\}, \{4\}\}$. Let

$$\mathrm{NCP}_n = \{\text{noncrossing partitions of size } n\}.$$

The number of noncrossing partitions of size n is the nth Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$ (see [9]).

Given an arc $i \frown j \in \lambda$, say that i is the *left endpoint* and j is the *right endpoint*, and let

$$\lambda^+ = \{i \in [n] \mid i \text{ is a left endpoint in } \lambda\}$$

and

$$\lambda^- = \{i \in [n] \mid i \text{ is a right endpoint in } \lambda\}$$

For example, with the noncrossing partition λ in (2.1), $\lambda^+ = \{2,3,5\}$ and $\lambda^- = \{5,6,7\}$. The arcs in λ give a bijection between the sets λ^+ and λ^-, so that

$$|\lambda^+| = |\lambda^-|.$$

The following lemma is classic in the literature about non-crossing partitions (for example, see [9]). We include a proof for the sake of exposition and the way we will use it in the following.

Lemma 2.2. *Each noncrossing partition λ of size n is uniquely determined by the sets λ^+ and λ^-. Moreover, given two subsets L and R of $[n]$ with equal size, the inequalities*

$$|[k-1] \cap L| \geqslant |[k] \cap R| \qquad \text{for } k \geqslant 1$$

hold if and only if we have $L = \lambda^+$ and $R = \lambda^-$ for some noncrossing partition λ of size n.

Proof. Given L and R, draw the elements of $[n]$ along the horizontal axis, increasing from left to right, and draw a half-arc starting at each vertex in L and and half-arc ending at each vertex in R. For example, with $n = 8$, $L = \{2,4,7\}$, and $R = \{5,6,8\}$, the resulting diagram is
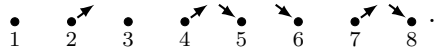
These half-arcs determine a unique noncrossing partition, which can be obtained by recursively connecting pairs of half-edges which have no incomplete edges between them. This is the same process as matching open and closed parenthesis, the difference here is that there are displayed in prescribe positions. The condition on L and R guaranties that there is always a starting half-arc in L available to match any ending half-arc in R. The non-crossing condition follows from the choosing process: a crossing would contradict a choice we made earlier. Continuing the preceding example, this successively gives the diagrams
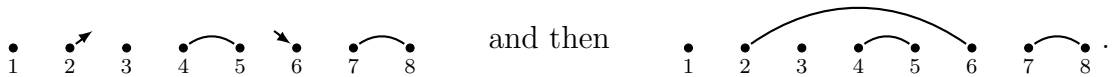
For the converse, if $L = \lambda^+$ and $R = \lambda^-$ for a nonnesting partition λ, the given inequalities must hold: the left endpoint of each arc must be in a position strictly less than the right endpoint. $\square$

2.2 Permutations and the Bruhat order

Let S_n denote the set of permutations of $[n]$ and for $w \in S_n$ we denote by $\ell(w)$ the length of w, that is the number of inversions $i < j$ where $w_i > w_j$. The Bruhat ordering of S_n is generated by the covering relation

$$v \lessdot w \quad \text{if } wv^{-1} \text{ is a reflection and } \ell(v) < \ell(w).$$

This definition above is somewhat unwieldy, so the following *tableau criterion* will be used. For $w \in S_n$, let $\mathcal{T}(w)$ be the diagram whose $n - k$th row $\mathcal{T}_k(w)$ consists of the elements

$w_1, w_2, \ldots, w_k$ appearing in increasing order. For example,

$$
\mathcal{T}(52314) = \begin{matrix}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 5 & \\
2 & 3 & 5 & & \\
2 & 5 & & & \\
5 & & & &
\end{matrix}
\quad \text{and} \quad
\mathcal{T}(41235) = \begin{matrix}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & \\
1 & 2 & 4 & & \\
1 & 4 & & & \\
4 & & & &
\end{matrix} \ .
$$

Proposition 2.3 ([4, Theorem 2.6.3]). *For $v, w \in S_n$, we have $v \leqslant w$ if and only if each entry of $\mathcal{T}(v)$ is less than or equal to to corresponding entry of $\mathcal{T}(w)$.*

For example, with $w = 52314$ and $v = 41235$ and $\mathcal{T}(v)$, $\mathcal{T}(w)$ shown above, the Tableau criterion shows that $v \leqslant w$.

3   The set QSV_n

This section will define the set $\mathrm{QSV}_n \subseteq S_n$ and establish its elementary properties. Section 3.1 recalls the description of the restriction of the Bruhat order to QSV_n given in [6].

Let λ be a noncrossing partition of size n and recall the sets λ^+ and λ^- from Section 2.1. Define a permutation $Q_\lambda \in S_n$ by

$$
Q_\lambda(j) = \begin{cases}
i & \text{if } j \in \lambda^- \text{ and } i \overset{\frown}{} j \in \lambda \\
k & \text{if } j \notin \lambda^- \text{ and } k \text{ is the largest element connected to } i \text{ in } \lambda
\end{cases}
$$

Thus, Q_λ sends $j \in [n]$ to its leftward neighbor in λ, if such a neighbor exists, and otherwise sends j to the rightmost element of its connected component.

Let

$$
\mathrm{QSV}_n = \{Q_\lambda \mid \lambda \in \mathrm{NCP}_n\}.
$$

For example, the elements of QSV_3 are:

$$
Q_{\underset{1\ 2\ 3}{\overgroup{\ \cdot\ \cdot\ \cdot\ }}} = 321, \qquad
Q_{\underset{1\ 2\ 3}{\overgroup{\cdot}\overgroup{\cdot}}} = \{312\}, \qquad
Q_{\underset{1\ 2\ 3}{\overgroup{\cdot\ \cdot}\ \cdot}} = 213,
$$

$$
Q_{\underset{1\ 2\ 3}{\cdot\ \overgroup{\cdot\ \cdot}}} = 132, \qquad \text{and} \qquad
Q_{\underset{1\ 2\ 3}{\cdot\ \cdot\ \cdot}} = 123.
$$

Lemma 3.1. *Let λ be a noncrossing partition of size n with connected components $C_1, C_2, \ldots, C_s$, and enumerate each C_r, $1 \leqslant r \leqslant s$ in increasing order as $\{c_{r,1} < c_{r,2} < \cdots < c_{r,|C_r|}\}$. We then have the disjoint cycle decomposition*

$$
Q_\lambda = \prod_{r=1}^{s} (c_{r,1} c_{r,|C_r|} \cdots c_{r,2}).
$$

Proof. The statement follows directly from the definition given above: $[n] - \lambda^- = \{c_1 \mid 1 \leqslant r \leqslant s\}$, so $Q_\lambda(c_{r,1}) = c_{r,|C_r|}$ for each r, and for $1 < i \leqslant |C_r|$, we have $c_{r,i} = c_{r,i-1}$. $\qquad\square$

For example, when λ has a single connected component, Q_λ is a single cycle: with

$$\lambda = {}_1 \frown {}_2 \frown {}_3 \frown {}_4 \frown {}_5 \frown {}_6 \frown {}_7 \qquad \text{we have} \qquad Q_\lambda = (1765432) = 7123456.$$

Considering the noncrossing partition shown in Equation (2.1) gives a more complicated example: with

$$\lambda = {}_1 \quad {}_2 \diagup \overparen{{}_3 \ \frown {}_4 \ {}_5 \frown {}_6} \searrow {}_7 \qquad \text{we have} \qquad Q_\lambda = (1)(27)(365)(4) = 1764352.$$

Remark 3.2. The set QSV_n is a particular case of a more general phenomenon involving non-crossing partitions and permutations. Given any n-cycle $c \in S_n$, [3] gives a bijection between NCP_n and the interval between the identity and c in the absolute order on S_n, and our construction realizes this bijection for $c = (1n \cdots 2)$.

3.1 The Bruhat order on QSV_n

The Bruhat order restricts to a partial order on the set QSV_n. This turns out to be a very natural order, as is described in the paper [6]. This section will recall the description from this source for use in later sections.

For $n \geqslant 0$, define a partial order $\leq$ on the set NCP_n of noncrossing partitions as the extension of the covering relation: λ is covered by μ if and only if λ is obtained from μ in one of the following ways:

1. removing an arc of the form ${}_i \frown {}_{i+1}$ from μ, or

2. replacing any arc ${}_i \frown {}_k$ in μ with two arcs ${}_i \frown {}_j$ and ${}_j \frown {}_k$ for some $i < j < k$ which do not intersect or share a left or right endpoint with any other arc in μ.

It is difficult to describe the non-covering relations of $\leq$ on NCP_n—and of the Bruhat order on QSV_n—in a direct and intuitive manner. Instead, these relations are best understood through an intermediary object. A *ballot sequence* of length $2n$ is a sequence $b \in \{\pm 1\}^{2n}$ for which each partial sum of b is nonnegative and the final sum is 0. A well-known bijection between noncrossing partitions is used in [6, Section 5.1]: for $\lambda \in \mathrm{NCP}_n$ define a ballot sequence $b^\lambda = (b_1^\lambda, b_2^\lambda, \ldots, b_{2n}^\lambda)$ by

$$b_{2k-1}^\lambda = \begin{cases} 1 & \text{if } k \notin \lambda^- \\ -1 & \text{if } k \in \lambda^- \end{cases} \qquad \text{and} \qquad b_{2k}^\lambda = \begin{cases} 1 & \text{if } k \in \lambda^+ \\ -1 & \text{if } k \notin \lambda^+, \end{cases}$$

for each $1 \leqslant k \leqslant n$.

Proposition 3.3 ([6, Theorem 1.1 and Corollary 7.5]). *Let λ and μ be noncrossing partitions of size n. The following are equivalent:*

1. *$\lambda \leq \mu$,*

2. *$Q_\lambda \leqslant Q_\mu$ in the Bruhat order, and*

3. *for all $1 \leqslant k \leqslant 2n$, $\sum_{i=1}^k b_k^\lambda \leqslant \sum_{i=1}^k b_k^\mu$.*
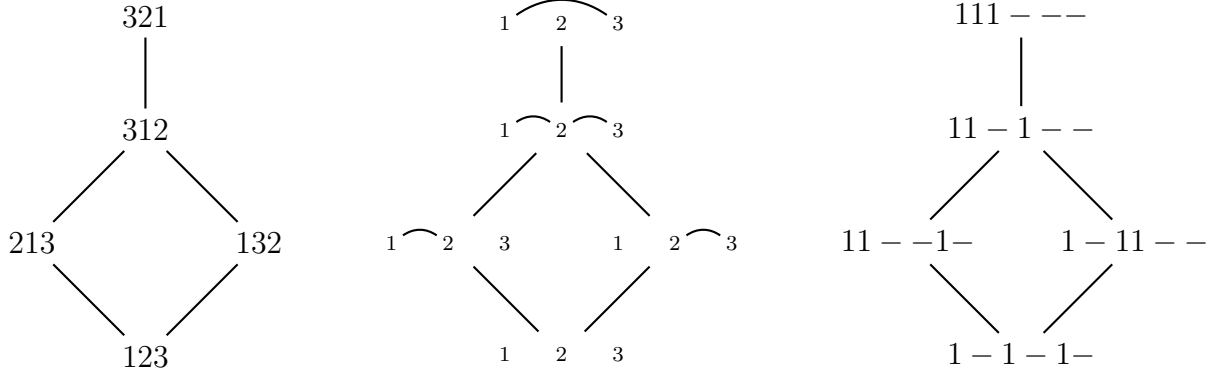
Figure 1: The Hasse diagrams of: QSV_3 with the Bruhat order; NCP_3 with $\leq$; and ballot sequences (for which each -1 is represented as $-$) with the term-wise order on partial sums.

As an example, for $n = 3$ the (isomorphic) orders on QSV_n, NCP_n, and ballot sequences are shown in Figure 1.

Remark 3.4. There are two superficial differences between our presentation and that of [6].

1. The results of [6] describe the Bruhat ordering of the set $\{w^{-1} \mid w \in \mathrm{QSV}_n\}$, rather than QSV_n. In the terminology of Remark 3.2, these are the noncrossing partitions associated to the cycle $(12 \ldots n)$ rather than $(1n \ldots 2)$. Inversion give an automorphism of the Bruhat order, so the results are equivalent.

2. The results of [6] use Dyck paths rather than ballot sequences for item 3. in Proposition 3.3. One can translate between the two by interchanging each 1 in a ballot sequence with an up step in a Dyck path, and likewise each -1 with a down step.

The final result of the section follows easily from the results of [6], but is not stated explicitly. For the sake of completeness, a proof is included.

Corollary 3.5. *Let λ and μ be noncrossing partitions of size n. Then $\lambda \leq \mu$ if and only if*

$$|\lambda^+ \cap [k-1]| - |\lambda^- \cap [k]| \leqslant |\mu^+ \cap [k-1]| - |\mu^- \cap [k]|$$

and

$$|\lambda^+ \cap [k]| - |\lambda^- \cap [k]| \leqslant |\mu^+ \cap [k]| - |\mu^- \cap [k]|$$

for all $1 \leqslant k \leqslant n$.

Proof. By Proposition 3.3, it is sufficient to show that for $1 \leqslant k \leqslant n$,

$$\sum_{i=1}^{2k-1} b_i^\lambda = 1 + 2|\lambda^+ \cap [k-1]| - 2|\lambda^- \cap [k]| \qquad \text{and} \qquad \sum_{i=1}^{2k} b_i^\lambda = 2|\lambda^+ \cap [k]| - 2|\lambda^- \cap [k]|.$$

This can be established inductively: for $k = 1$ both equations clearly hold, and for $k > 1$, we consider the differences between the $k - 1$st expression and the kth:

$$(1 + 2|\lambda^+ \cap [k-1]| - 2|\lambda^- \cap [k]|) - (2|\lambda^+ \cap [k-1]| + 2|\lambda^- \cap [k-1]|) = 1 - 2|\lambda^- \cap \{k\}|,$$

which is b_{2k-1}^λ, and

$$(2|\lambda^+ \cap [k]| - 2|\lambda^- \cap [k]|) - (1 + 2|\lambda^+ \cap [k-1]| + 2|\lambda^- \cap [k]|) = 2|\lambda^+ \cap \{k\}| - 1,$$

which is b_{2k}^λ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4   The excedance quotient of the Bruhat order

This section will describe a quotient of the Bruhat order on S_n under an equivalence relation defined by the weak excedances of a permutation. This quotient order has a number of desirable properties, which are also explored in the section. One particularly noteworthy result is that each equivalence class contains a unique element of the set QSV_n defined in Section 3, so the construction of this order can be seen as dual to the results of [6] described in Section 3.1.

Given a permutation $w \in S_n$, a *weak excedance* of w is a pair (i, w_i) for which $i \leqslant w_i$. Disaggregating, define the *excedance values* $E_{val}(w)$ and *excedance positions* $E_{pos}(w)$ to be

$$E_{val}(w) = \{w_i \mid (i, w_i) \text{ is a weak excedance of } w\}, \text{ and}$$

$$E_{pos}(w) = \{i \mid (i, w_i) \text{ is a weak excedance of } w\}.$$

Excedances and the sets $E_{val}(w)$ and $E_{pos}(w)$ are easiest seen using two-line notation for permutations. For example, marking non-excedancs in red,

$$w = \overset{1\,2\,3\,4\,5\,6\,7\,8}{35142658}, \qquad E_{pos}(w) = \{1, 2, 4, 6, 8\}, \qquad \text{and} \qquad E_{val}(w) = \{3, 4, 5, 6, 8\}.$$

We define the *excedance relation* $\sim$ on S_n by:

$$v \sim w \qquad \text{if and only if} \qquad E_{val}(v) = E_{val}(w) \text{ and } E_{pos}(v) = E_{pos}(w), \qquad (4.1)$$

and say that each equivalence class of $S_n/\sim$ is an *excedance class*.

This section will investigate the properties of excedance classes and their interaction with the Bruhat order; a summary of the main results follows. Each noncrossing partition λ of size n determines an excedance class: recall the sets λ^+ and λ^- defined in Section 2.1, and let

$$\mathcal{C}_\lambda = \{w \in S_n \mid E_{val}(w) = [n] - \lambda^+ \text{ and } E_{pos}(w) = [n] - \lambda^-\}.$$

In Section 4.1, it is shown that every excedance class of S_n is of the form $\mathcal{C}_\lambda$ for some noncrossing partition. As an example, the are five excedance classes of S_3 are given below:

$$\mathcal{C}_{\underset{1\;\;2\;\;3}{\frown}} = \{\overset{1\,2\,3}{321}, \overset{1\,2\,3}{231}\}, \qquad \mathcal{C}_{\underset{1\;\;2\;\;3}{\cdot\frown\frown}} = \{\overset{1\,2\,3}{312}\}, \qquad \mathcal{C}_{\underset{1\;\;2\;\;3}{\frown\cdot}} = \{\overset{1\,2\,3}{213}\},$$

$$\mathcal{C}_{\underset{1\;\;2\;\;3}{\cdot\;\;\frown}} = \{\overset{1\,2\,3}{132}\}, \qquad \text{and} \qquad \mathcal{C}_{\underset{1\;\;2\;\;3}{\cdot\;\;\cdot\;\;\cdot}} = \{\overset{1\,2\,3}{123}\}.$$

The Bruhat order descends to a relation on excedance classes, and we show that this relation is a partial order; in particular Section 4.3 proves the following result. Recall the order $\preceq$ from Section 3.1.

Theorem 4.2. *Writing $\leqslant$ for the partial order on excedance classes $S_n/\sim$ induced by the Bruhat order, $\mathcal{C}_\lambda \leqslant \mathcal{C}_\mu$ if and only if $\lambda \preceq \mu$.*

Two important intermediate results stated in Section 4.2 show that each excedance class $\mathcal{C}_\lambda$ contains a unique minimal and maximal elements, which are respectively a 321-avoiding permutation and the element $Q_\lambda \in \mathrm{QSV}_n$. Combined with Theorem 4.2, this gives the following corollary.

Corollary 4.3. *Each excedance class $\mathcal{C}_\lambda$ is an interval in the Bruhat order, with upper bound Q_λ and lower bound given by a 321-avoiding permutation.*

4.1 Excedance classes and noncrossing partitions

This section will establish some basic results which relate excedance classes to noncrossing partitions.

Proposition 4.4. *For $n \geqslant 0$, the map*

$$\begin{array}{ccc} \mathrm{NCP}_n & \longrightarrow & S_n/\sim \\ \lambda & \longmapsto & \mathcal{C}_\lambda \end{array}$$

is a bijection.

Proof. We will first show that every excedance class of S_n has the form $\mathcal{C}_\lambda$ for some $\lambda \in \mathrm{NCP}_n$. This is equivalent to showing that for any $w \in S_n$, the criterion of Lemma 2.2 holds:

$$|\{i \in [k-1] \mid i \notin E_{val}(w)\}| \geqslant |\{j \in [k] \mid j \notin E_{pos}(w)\}| \qquad \text{for all } 1 \leqslant k \leqslant n.$$

We will establish the above inequality directly. Fix $1 \leqslant k \leqslant n$ and suppose that $i \in [k]$ is not an excedance position, so that that $i > w_i$. By assumption, $w_i < k$ and w_i is not an excedance value. Thus,

$$\{w_i \mid i \in [k-1] \text{ and } i \notin E_{val}(w)\} \subseteq \{j \in [k] \mid j \notin E_{pos}(w)\},$$

giving the claim. Now, we must show that each $\mathcal{C}_\lambda$ is nonempty. This is established by Lemma 4.5 below, which completes the proof. $\qquad\square$

Lemma 4.5. *For any noncrossing partition λ, $Q_\lambda \in \mathcal{C}_\lambda$. In particular, if $Q_\lambda \in S_n$,*

$$E_{val}(Q_\lambda) = [n] - \lambda^+ \qquad \text{and} \qquad E_{pos}(Q_\lambda) = [n] - \lambda^-.$$

Proof. By definition, $Q_\lambda(j) < j$ if and only if the arc $Q_\lambda(j) \frown j$ appears in λ, in which case $j \in \lambda^-$ and $Q_\lambda(j) \in \lambda^+$; this establishes the claim. $\qquad\square$

The final result relates certain properties of the elements of $\mathcal{C}_\lambda$ to the partial sums of the ballot sequence b^λ defined in Section 3.1, by way of Corollary 3.5. This will be key to a number of arguments in subsequent sections.

Lemma 4.6. *Let λ be a noncrossing partition of size $n \geqslant 0$ and take $w \in \mathcal{C}_\lambda$. For all $1 \leqslant k \leqslant n$,*

$$|\{w_i \mid n \geqslant i > k \text{ and } w_i < k\}| = |\lambda^+ \cap [k-1]| - |\lambda^- \cap [k]|$$

and

$$|\{w_i \mid 1 \leqslant i \leqslant k \text{ and } w_i > k\}| = |\lambda^+ \cap [k]| - |\lambda^- \cap [k]|.$$

Proof. To show the first equation, note that by definition

$$|\lambda^+ \cap [k-1]| - |\lambda^- \cap [k]| = |\{j \in [k-1] \mid j \notin E_{val}(w)\}| - |\{i \in [k] \mid i \notin E_{pos}(w)\}|.$$

For $i \in [k]$ with $i \notin E_{pos}(w)$, it must be the case that $w_i < i$ and so $w_i \in [k-1]$ with $w_i \notin E_{val}(w)$. Therefore, the equation above counts the $w_i \in [k-1]$ for which $w_i \notin E_{val}(w)$ and $i \notin [k]$; this is exactly $\{w_i \mid n \geqslant i > k \text{ and } w_i < k\}$.

The second equation follows from a similar but somewhat more complicated argument. We begin by manipulating the right side into a more suitable form:

$$\begin{aligned} |\lambda^+ \cap [k]| - |\lambda^- \cap [k]| &= (k - |\lambda^- \cap [k]|) - (k - |\lambda^+ \cap [k]|) \\ &= |E_{pos}(w) \cap [k]| - |E_{val}(w) \cap [k]|. \end{aligned}$$

Now, for $j \in E_{val}(w) \cap [k]$, we have $j = w_i$ for some $i \leqslant j$, so that $i \in E_{pos}(w) \cap [k]$. Since $i \in E_{pos}(w) \cap [k]$ implies that $w_i \in E_{val}(w)$, the equation above counts the positions $i \in E_{pos}(w) \cap [k]$ for which $w_i \notin [k]$; this set is equinumerous to $\{w_i \mid 1 \leqslant i \leqslant k \text{ and } w_i > k\}$. $\quad\square$

4.2 Minimal and maximal elements

This section will show that each excedance class contains a unique Bruhat minimum and maximum. This is a key intermediate step to showing that excedance classes are Bruhat intervals with a well-defined quotient order. We will begin with the maximal elements, which are the elements of QSV_n, while the minimal (321-avoiding) elements are discussed in Subsection 4.2.1

Proposition 4.7. *For all noncrossing partitions λ, Q_λ is the Bruhat maximum element of $\mathcal{C}_\lambda$.*

Proof. Lemma 4.5 shows that $Q_\lambda \in \mathcal{C}_\lambda$, so we need only show that Q_λ is an upper bound for $\mathcal{C}_\lambda$; to this end, fix $w \in \mathcal{C}_\lambda$. Using the tableau criterion (Proposition 2.3), it is sufficient to show that $\mathcal{T}_k(w)$ is entry-wise less than or equal to $\mathcal{T}_k(Q_\lambda)$ for each $1 \leqslant k \leqslant n$.

The argument consists of two distinct parts, first conducting an element-by-element comparison of the entries of $\mathcal{T}_k(w)$ and $\mathcal{T}_k(Q_\lambda)$ which are strictly greater than k, and then doing the same for the elements which are at most k. The validity of this hinges on the fact that these collections of entires have the same cardinality for w and Q_λ: by Lemma 4.6,

$$\begin{aligned} |\{w_i \mid 1 \leqslant i \leqslant k \text{ and } w_i > k\}| &= |\{Q_\lambda(i) \mid 1 \leqslant i \leqslant k \text{ and } w_i > k\}| \\ &= |\lambda^+ \cap [k]| - |\lambda^- \cap [k]|, \end{aligned}$$

and consequently $|\{w_i \mid 1 \leqslant i \leqslant k \text{ and } w_i \leqslant k\}| = |\{Q_\lambda(i) \mid 1 \leqslant i \leqslant k \text{ and } w_i \leqslant k\}|.$

We begin with the first part. Enumerate the entries of $\mathcal{T}_k(w)$ and $\mathcal{T}_k(Q_\lambda)$ which are greater than k in increasing order as $x_1 < x_2 < \cdots < x_r$ and $q_1 < q_2 < \ldots < q_r$. We aim to show that $x_i \leqslant q_i$ for each $1 \leqslant i \leqslant r$. Fixing one such i, Lemma 4.6 gives that

$$|\{w_t \mid 1 \leqslant t \leqslant q_i \text{ and } w_t > q_i\}| = |\lambda^+ \cap [q_i]| - |\lambda^- \cap [q_i]|.$$

It is sufficient to show that the above quantity is equal to $r - i$, as this implies that each of $x_1, \ldots x_i$ must be less than q_i.

Let C_i denote the connected component of λ containing y_i. From the definition of Q_λ, q_i must be the maximal element of C_i, and $Q_\lambda^{-1}(q_i)$ the minimal element. Combinatorially, the difference $|\lambda^+ \cap [q_i]| - |\lambda^- \cap [q_i]|$ counts the number of arcs in λ with left endpoint in $[q_i]$ and right endpoint in $[n] - [q_i]$, and as $q_i \in E_{val}(w)$, this is the number of arcs in λ which are above q_i. Every arc in λ which lies above q_i must fully contain C_i between its left and right endpoints, and so in particular, the left endpoint of such an arc is contained in $[k]$, and the right endpoint is greater than q_i. Thus, each such arc belongs to the connected component of one of the elements $q_{i+1}, \ldots, q_r$; there are precisely $r - i$ such connected components.

For the second part of the argument, we aim to show that the entries of $\mathcal{T}_k(w)$ which are at most k are entry-wise less than or equal to the analogous entries of $\mathcal{T}_k(Q_\lambda)$, and we establish this in an indirect manner described below. Writing $s = |\lambda^+ \cap [k-1]| - |\lambda^- \cap [k]|$, Lemma 4.6 states that there are exactly s elements of $[k]$ which do not appear in $\mathcal{T}_k(w)$, and likewise for $\mathcal{T}_k(Q_\lambda)$. Respectively enumerate these elements in increasing order as $x_1 < x_2 < \cdots < x_s$ and $q_1 < q_2 < \cdots < q_s$. We will show that $q_i \leqslant x_i$ for each $1 \leqslant i \leqslant s$, as this implies the opposite comparison for the remaining elements of $[k] - \{q_1, q_2, \ldots, q_s\}$ and $[k] - \{x_1, x_2, \ldots, x_s\}$ as desired.

Fixing $1 \leqslant i \leqslant s$, Lemma 4.6 gives that

$$|\{w_t \mid n \geqslant t > k \text{ and } w_t < q_i\}| = |\lambda^+ \cap [q_i - 1]| - |\lambda^- \cap [q_i]|.$$

It is therefore sufficient to show that the above quantity $i - 1$, so that q_i is bounded above by each of $x_i, x_{i+1}, \ldots, x_s$.

Combinatorially, the difference $|\lambda^+ \cap [q_i - 1]| - |\lambda^- \cap [q_i]|$ counts the number of arcs in λ with a left endpoint $[q_i - 1]$ and a right endpoint in $[n] - [q_i]$, or equivalently, the arcs above q_i in λ. Writing C_i for the connected component containing q_i, each such arc must contain C_i between its left and right endpoints, so the left endpoint is less that q_i and the right endpoint lies somewhere in $[n] - [k]$. Thus, each such arc must belong to the connected component of one of $q_1, q_2, \ldots, q_{i-1}$, and there are $i - 1$ such connected components. $\qquad\square$

4.2.1 The minimal element of an excedance class

We now turn to the minimal element of each excedance class described in Theorem 4.2. For a noncrossing partition λ of size n, write

$$\lambda^+ = \{a_1 < a_2 < \cdots < a_s\}, \qquad \lambda^- = \{b_1 < b_2 < \cdots < b_s\},$$

$$[n] - \lambda^+ = \{x_1 < x_2 < \cdots < x_{n-s}\}, \qquad \text{and} \qquad [n] - \lambda^- = \{y_1 < y_2 < \cdots < y_{n-s}\},$$

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so that the elements of each set are enumerated in increasing order. Let $T_\lambda \in S_n$ be the permutation with

$$T_\lambda(i) = \begin{cases} a_s & \text{if } i = b_s \\ x_s & \text{if } i = y_s. \end{cases}$$

Thus, the one-line notation for T_λ can be obtained by placing the elements of λ^+ in increasing left-to-right order in the positions λ^-, and placing the remaining elements of $[n]$ in the remaining positions in the same manner. For example, with $n = 8$ and

$$\lambda = \overset{\displaystyle\frown}{1 \underset{\displaystyle\frown}{2 \quad 3} \quad 4 \quad 5} \quad 6 \quad 7 \quad 8$$

we have $\lambda^+ = \{1, 2, 5\}$ and $\lambda^- = \{3, 5, 7\}$, $[8] - \lambda^+ = \{3, 4, 6, 7, 8\}$, and $[8] - \lambda^- = \{1, 2, 4, 6, 8\}$, and consequently

$$T_\lambda = \overset{1\,2\,3\,4\,5\,6\,7\,8}{34\textcolor{red}{16}2\textcolor{red}{658}},$$

where non-excedances are marked in red, as at the beginning of Section 4.

Proposition 4.8. *For all noncrossing partitions λ, $T_\lambda \in \mathcal{C}_\lambda$, and this is the Bruhat-minimum element of $\mathcal{C}_\lambda$.*

Proof. To see that $T_\lambda \in \mathcal{C}_\lambda$, recall the elements a_i, b_i, x_i, and y_i defined above for λ. For $1 \leqslant r \leqslant s$,

$$a_r = \min\{k \mid r \geqslant |[k] \cap \lambda^+|\} \qquad \text{and} \qquad b_r = \min\{k \mid r \geqslant |[k] \cap \lambda^-|\}.$$

By Lemma 2.2, it is always the case that $|[k] \cap \lambda^-| \leqslant |[k-1] \cap \lambda^+|$, and so $b_r < a_r$. Thus the (b_r, a_r) is not a weak excedence of T_λ. A similar argument shows that every pair (y_r, x_r) is a weak excedence, so that

$$\lambda^+ = [n] - E_{val}(T_\lambda) \qquad \text{and} \qquad \lambda^- = [n] - E_{pos}(T_\lambda).$$

To see that $T_\lambda \leqslant w$ for all $w \in \mathcal{C}_\lambda$, recall the Tableau Criterion, Proposition 2.3. For $1 \leqslant k \leqslant n$, the row $\mathcal{T}_k(T_\lambda)$ will consist of the $|[k] \cap \lambda^-|$ smallest elements of $E_{val}(w)$ along with the $k - |[k] \cap \lambda^-|$ smallest elements of $[n] - E_{val}(w)$. For w, the row $\mathcal{T}_k(w)$ will also consist of $|[k] \cap \lambda^-|$ elements of $E_{val}(w)$ and $k - |[k] \cap \lambda^-|$ elements of $[n] - E_{val}(w)$, but these elements need not be the minimal ones. Thus, by assumption of minimality, $\mathcal{T}_k(T_\lambda)$ is entry-wise less than or equal to $\mathcal{T}_k(w)$. $\square$

For the next result, recall that a permutation $w \in S_n$ is 321-*avoiding* if there do not exist indices $i < j < k$ for which $w_i > w_j > w_k$. The number of 321-avoiding permutations is known to be the nth Catalan number, so the following result establishes that each excedance class contains a unique 321-avoiding permutation.

Proposition 4.9. *For all noncrossing partitions λ, the permutation T_λ is 321-avoiding.*

Proof. Let $i, j, k \in [n]$ and assume without loss of generality that $i < j < k$. Since any element of $[n]$ must be contained in either λ^+ or its complement, we must have two elements of $\{i, j, k\}$ which belong to one of λ^- or $[n] - \lambda^-$. Since T_λ restricts to an order-preserving bijection from λ^+ to λ^-, and from $[n] - \lambda^+$ to $[n] - \lambda^-$, this implies that no 321-pattern can exist in T_λ, giving the first claim. $\square$

4.3 Comparing excedance classes

This section will prove Theorem 4.2, following an intermediate result.

To begin, recall the order $\preceq$ on noncrossing partitions defined in Section 3.1

Proposition 4.10. *Let μ be a noncrossing partition of size n. Then*

$$\{w \in S_n \mid w \leqslant Q_\mu\} = \bigsqcup_{\lambda \preceq \mu} \mathcal{C}_\lambda.$$

Proof. To begin, assume that $\lambda \preceq \mu$, so that by Proposition 3.3, $Q_\lambda \leqslant Q_\mu$. By Proposition 4.7, we have $w \leqslant Q_\lambda$ for any $w \in C_\lambda$, so we have

$$\{w \in S_n \mid w \leqslant Q_\mu\} \supseteq \bigsqcup_{\lambda \preceq \mu} \mathcal{C}_\lambda.$$

To see the opposite containment, suppose that w is a permutation with $w \leqslant Q_\mu$ and let λ be the unique noncrossing partition for which $w \in \mathcal{C}_\lambda$. By way of Corollary 3.5, $\lambda \preceq \mu$ is equivalent to the inequalities

$$|\lambda^+ \cap [k-1]| - |\lambda^- \cap [k]| \leqslant |\mu^+ \cap [k-1]| - |\mu^- \cap [k]|$$

and

$$|\lambda^+ \cap [k]| - |\lambda^- \cap [k]| \leqslant |\mu^+ \cap [k]| - |\mu^- \cap [k]|$$

for all $1 \leqslant k \leqslant n$. We will establish this equivalent formulation of our claim using the characterization of each side given in Lemma 4.6. From the assumption that $w \leqslant Q_\mu$, the tableau criterion (Proposition 2.3) states that the tableau $\mathcal{T}(w)$ is entry-wise less than or equal to $\mathcal{T}(Q_\mu)$. Thus, for each $1 \leqslant k \leqslant n$,

$$|\{w_i \mid 1 \leqslant i \leqslant k \text{ and } w_i > k\}| \leqslant |\{Q_\mu(i) \mid 1 \leqslant i \leqslant k \text{ and } Q_\mu(i) > k\}|,$$

since each entry of $\mathcal{T}_k(w)$ is bounded above by the corresponding entry of $\mathcal{T}_k(Q_\mu)$, and likewise

$$|\{w_i \mid n \geqslant i > k \text{ and } w_i < k\}| \leqslant |\{Q_\mu(i) \mid n \geqslant i > k \text{ and } Q_\mu(i) < k\}|,$$

since each entry of $\mathcal{T}_k(Q_\mu)$ is bounded below by the corresponding entry of $\mathcal{T}_k(w)$. This completes the proof. $\square$

We now prove Theorem 4.2. Recall the elements T_λ defined in Section 4.2.1.

Proof of Theorem 4.2. Define a relation on excedance classes:

$$\mathcal{C}_\lambda \leqslant \mathcal{C}_\mu \qquad \text{if and only if} \qquad v \leqslant w \text{ for some } v \in \mathcal{C}_\lambda \text{ and } w \in \mathcal{C}_\mu.$$

As Lemma 4.5 states that $Q_\lambda \in \mathcal{C}_\lambda$ for each noncrossing partition λ, Proposition 3.3 implies that $\mathcal{C}_\lambda \leqslant \mathcal{C}_\mu$ whenever $\lambda \preceq \mu$. It is therefore sufficient to show the converse: $\mathcal{C}_\lambda \leqslant \mathcal{C}_\mu$ only if $\lambda \preceq \mu$. To this end, suppose that $\mathcal{C}_\lambda \leqslant \mathcal{C}_\mu$, so that $v \leqslant w$ for some $v \in \mathcal{C}_\lambda$ and $w \in \mathcal{C}_\mu$. With Proposition 4.7, this implies that $v \leqslant Q_\lambda$. Then by Proposition 4.10, we have that $\lambda \preceq \mu$, completing the proof. $\square$

5 Bases for the Temperley–Lieb Algebra $\mathsf{TL}_n(2)$

The Temperley–Lieb algebra $\mathsf{TL}_n(2)$ is the $\mathbb{C}$-algebra generated by elements $e_1, \ldots, e_{n-1}$ subject to the following relations for each $1 \leqslant i, j \leqslant n$

$$
\begin{aligned}
e_i^2 &= 2e_i \\
e_i e_j &= e_j e_i \quad \text{if } |i - j| > 1 \\
e_i e_j e_i &= e_i \quad\ \ \text{if } |i - j| = 1.
\end{aligned}
$$

There is a subjective algebra morphism from the symmetric group algebra $\mathbb{C}S_n$ to $\mathsf{TL}_n(2)$ given by

$$
\begin{aligned}
\phi : \mathbb{C}S_n &\longrightarrow \mathsf{TL}_n(2) \\
s_i &\longmapsto 1 - e_i.
\end{aligned}
$$

It is well-known that the images of all 321-avoiding permutations under ϕ forms a basis for $\mathsf{TL}_n(2)$. Another basis, due to Zinno [10] can be obtained via the map ϕ using the combinatorics of noncrossing partitions described in Remark 3.2. This basis is precisely the set $\{\phi(w^{-1}) \mid w \in \mathrm{QSV}_n\}$. Since the kernel

$$
\ker(\phi) = \langle (13) - (123) - (132) + (12) + (23) - e \rangle
$$

is invariant under the $\mathbb{C}$-linear map $w \mapsto w^{-1}$, the following result is an immediate consequence of [10, Theorem 2].

Theorem 5.1. *For all $n \geqslant 0$, the set $\phi(\mathrm{QSV}_n)$ is a basis for $\mathsf{TL}_n(2)$.*

Remark 5.2. Theorem 6.1 also follows from the results of Gobet and Williams in [6] as a special case of a much stronger result which gives a basis for $\mathsf{TL}_n(2)$ for each Coxeter element of S_n. However, [6] does not provide an explicit statement of this result, so it is easier to deduce the theorem as a consequence of Zinno's work directly.

6 Try 2: Bases for the Temperley–Lieb Algebra $\mathsf{TL}_n(2)$

The Temperley–Lieb algebra $\mathsf{TL}_n(2)$ is the $\mathbb{C}$-algebra generated by elements $e_1, \ldots, e_{n-1}$, subject to relations described in Section 6.1. There is a surjective algebra morphism from the symmetric group algebra $\mathbb{C}S_n$ to $\mathsf{TL}_n(2)$ given by

$$
\begin{aligned}
\phi : \mathbb{C}S_n &\longrightarrow \mathsf{TL}_n(2) \\
s_i &\longmapsto 1 - e_i.
\end{aligned}
$$

It is well-known that the images of all 321-avoiding permutations under ϕ forms a basis for $\mathsf{TL}_n(2)$. Additional bases of permutation under ϕ have been described by Zinno [10] and later Gobet and Williams in [6], who use the general combinatorics of noncrossing partitions in the sense of Remark 3.2. In this section, we prove a similar result using excedance classes.

Theorem 6.1. *Let $n \geqslant 0$ and for each $\lambda \in \mathrm{NCP}_n$, fix an element $w_\lambda \in \mathcal{C}_\lambda$. Then the set $\{\phi(w_\lambda) \mid \lambda \in \mathrm{NCP}_n\}$ is a basis of $\mathsf{TL}_n(2)$.*

As a corollary, we re-prove the following result, which is not novel and is implicit in the aforementioned work of Gobet and Williams [6] and Zinno [10].

Corollary 6.2 ([6] and [10]). *The set $\phi(\mathrm{QSV}_n)$ is a basis of $\mathsf{TL}_n(2)$.*

In general, however, many bases obtained via Theorem 6.1 are novel. A minimal example can be obtained with $n = 4$: the set

$$\{4312, 4231, 4213, 3142, 1432, 4123, 3214, 3124, 2143, 1323, 2134, 1324, 1243, 1234\}$$

meets the criteria of Theorem 6.1, and accordingly maps to a basis of $\mathsf{TL}_n(2)$ under ϕ. This set is neither QSV_4 not the set of 321-avoiding permutations ($4312 \notin \mathrm{QSV}_4$ and is not 321-avoiding). Moreover, the set above is not described in [6, 10]: each subset of S_4 in these sources which is not $\phi(\mathrm{QSV}_n)$ contains more than one element from certain excedance classes and none from others.

6.1 A presentation of the Temperley–Lieb algebra

The Temperley–Lieb algebra $\mathsf{TL}_n(2)$ has a presentation with the generators $e_1, \ldots, e_{n-1}$ subject to the the following relations for each $1 \leqslant i, j \leqslant n$

$$
\begin{aligned}
e_i^2 &= 2e_i \\
e_i e_j &= e_j e_i \quad \text{if } |i - j| > 1 \\
e_i e_j e_i &= e_i \quad\ \ \text{if } |i - j| = 1.
\end{aligned}
$$

The kernel of the map $\phi : \mathbb{C}S_n \longrightarrow \mathsf{TL}_n(2)$ is the two-sided ideal

$$\ker(\phi) = \langle (13) - (123) - (132) + (12) + (23) - e \rangle.$$

Taking the quotient of $\mathbb{C}S_n$ by the ideal $\ker(\phi)$ is equivalent to applying a certain $\mathbb{C}$-linear relation on the natural basis of $\mathbb{C}S_n$. Recall that a 321-pattern in a permutation $w \in S_n$ is a triple $i < j < k$ for which $w_i > w_j > w_k$. Given such a pattern, we can write

$$w = \mathbf{a}w_i\mathbf{b}w_j\mathbf{c}w_k\mathbf{d},$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are (possibly empty) subwords of w. The quotient $\mathbb{C}S_n / \ker(\phi)$ is then described by the relations

$$w \equiv \mathbf{a}w_j\mathbf{b}w_i\mathbf{c}w_k\mathbf{d} + \mathbf{a}w_i\mathbf{b}w_k\mathbf{c}w_j\mathbf{d} - \mathbf{a}w_j\mathbf{b}w_k\mathbf{c}w_i\mathbf{d} - \mathbf{a}w_k\mathbf{b}w_i\mathbf{c}w_j\mathbf{d} + \mathbf{a}w_k\mathbf{b}w_j\mathbf{c}w_i\mathbf{d} \quad (6.3)$$

for each 321-pattern in each permutation $w \in S_n$.

Lemma 6.4. *Let $w \in S_n$ be a permutation with a 321-pattern in positions $i < j < k$. Then $w \geqslant w'$ for each w' in the set*

$$\{\mathbf{a}w_j\mathbf{b}w_i\mathbf{c}w_k\mathbf{d},\ \mathbf{a}w_i\mathbf{b}w_k\mathbf{c}w_j\mathbf{d},\ \mathbf{a}w_j\mathbf{b}w_k\mathbf{c}w_i\mathbf{d},\ \mathbf{a}w_k\mathbf{b}w_i\mathbf{c}w_j\mathbf{d},\ \mathbf{a}w_k\mathbf{b}w_j\mathbf{c}w_i\mathbf{d}\}$$

Proof. □

6.2 Proof of Theorem 6.1

This section proves Theorem 6.1, which follows from the triangularity established in the next result. Recall the 321-avoiding permutation T_λ defined in Section **??** for each noncrossing partition $\lambda \in \mathrm{NCP}_n$.

Proposition 6.5. *For $n \geqslant 0$, take $w \in S_n$ and let $\lambda \in \mathrm{NCP}_n$ be the unique noncrossing partition for which $w \in \mathcal{C}_\lambda$. Then*

$$w \equiv T_\lambda + \sum_{\mu \prec \lambda} a_\mu^w T_\mu \qquad (\text{mod } \ker \phi)$$

for some coefficients $a_\mu^w \in \mathbb{Z}$.

A proof of the Proposition follows the next Lemma.

Lemma 6.6. *Suppose that $w \in S_n$ has a 321-pattern. Then w has a 321-pattern $i < j < k$ with $i \in E_{pos}(w)$ and $k \notin E_{pos}(w)$, and moreover exactly one element of the set*

$$\{\mathbf{a}w_j\mathbf{b}w_i\mathbf{c}w_k\mathbf{d}, \ \mathbf{a}w_i\mathbf{b}w_k\mathbf{c}w_j\mathbf{d}, \ \mathbf{a}w_j\mathbf{b}w_k\mathbf{c}w_i\mathbf{d}, \ \mathbf{a}w_k\mathbf{b}w_i\mathbf{c}w_j\mathbf{d}, \ \mathbf{a}w_k\mathbf{b}w_j\mathbf{c}w_i\mathbf{d}\}$$

belongs to the same excedance class as w.

Proof. We begin with the first claim. By assumption, there exists an 321-pattern $i_0 < j_0 < k_0$ in w. It must be the case that either $i_0 \in E_{pos}(w)$ or $k_0 \notin E_{pos}(w)$, as the first implies that $w_{k_0} < w_{j_0} < w_{i_0} < i_0 < j_0 < k_0$, and the second that $i_0 < j_0 < k_0 \leqslant w_{k_0} < w_{j_0} < w_{i_0}$. If $i_0 \in E_{pos}(w)$, the resulting inequality implies that there exists an element $i < i_0$ for which $w_i \geqslant i_0$; as a consequence we obtain a 321-pattern $i < j_0 < k_0$ in w with $i \in E_{pos}(w)$ and $k_0 \notin E_{pos}(w)$. Following a similar argument, if $k_0 \in E_{pos}(w)$, then there exists an element $k > k_0$ for which $w_k \leqslant k_0$; we then have a 321-pattern $i_0 < j_0 < k$ for which $k \notin E_{pos}(w)$ and $i_0 \in E_{pos}(w)$.

For the second argument, we write $i < j < k$ for the 321-pattern described in the Lemma; there are two cases, in which $j \in E_{pos}(w)$ or not. We will consider the first case, as the second follows from a similar argument. In this case, it is straightforward to see that each element w' in the set

$$\{\mathbf{a}w_i\mathbf{b}w_k\mathbf{c}w_j\mathbf{d}, \ \mathbf{a}w_j\mathbf{b}w_k\mathbf{c}w_i\mathbf{d}, \ \mathbf{a}w_k\mathbf{b}w_i\mathbf{c}w_j\mathbf{d}, \ \mathbf{a}w_k\mathbf{b}w_j\mathbf{c}w_i\mathbf{d}\}$$

will have a difference excedance class: either k is an excedance position of w', in which case $E_{pos}(w') \neq E_{pos}(w)$, or or one of w_i or w_j is a not excedance value for w', in which case $E_{val}(w') \neq E_{val}(w)$. What remains is to show that with $w'' = \mathbf{a}w_j\mathbf{b}w_i\mathbf{c}w_k\mathbf{d}$,

$$E_{pos}(w'') = E_{pos}(w) \qquad \text{and} \qquad E_{val}(w'') = E_{val}(w).$$

For all $s \in [n] - \{i, j\}$, we have $w_i'' = w_i$, so the above claim reduces to $i, j \in E_{pos}(w')$ and $w_i, w_j \in E_{val}(w'')$. To show this, we observe that $i < j \leqslant w_j < w_i$, and so $w_i'' = w_j > i$ and $w_j'' = w_i > j$. $\qquad \square$

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Proof of Proposition 6.5. We proceed by induction on the Bruhat order of S_n. If w is 321-avoiding, then $w = T_\lambda$ and the claim clearly holds. Assuming that w is not 321-avoiding, we have $T_\lambda \leqslant w$, and so we assume for the sake of induction that for each $v \leqslant w$ the claim holds. As w has a 321-pattern, we apply Lemma 6.6, which along with Equation (6.3) gives an expression

$$w \equiv w'' + \sum_{\mu \prec \lambda} \sum_{w' \in \mathcal{C}_\mu} b_{w'} w',$$

where w'' is an element of $\mathcal{C}_\lambda$ with $w'' \leqslant w$, each coefficient $b_{w'} \in \{1, 0, -1\}$, and all but five of the $b_{w'}$ are zero. Using Proposition 4.10, each w' in the sum above also precedes w in the Burhat order, so the inductive hypothesis applies, giving

$$w \equiv T_\lambda + \sum_{\mu \prec \lambda} a_\mu^{w'} T_\mu + \sum_{\mu \prec \lambda} \sum_{w' \in \mathcal{C}_\mu} \left(b_{w'} T_\mu + \sum_{\nu \prec \mu} b_{w'} a_\nu^{w''} T_\nu \right).$$

This completes the proof. $\qquad\square$

7   Quasisymmetric vanishing polynomials

In this section we define polynomials in n variables that vanishes on the set QSV_n and such that the homogeneous top degree is quasisymmetric. More precisely, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be any fixed composition of $d > 0$. Let

$$M_\alpha(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

be the quasisymmetric monomial indexed by α. We will define a polynomial $P_\alpha(x_1, x_2, \ldots, x_n)$ such that $P_\alpha = M_\alpha+$ lower degree terms, and $P_\alpha(\sigma) = 0$ for all $\sigma \in QSV_n$.

$1 = f_1 < f_2 < \cdots < f_\ell < f_{\ell+1} = n+1$ and $\beta_i = \alpha_{f_i} + \alpha_{f_i+1} + \cdots + \alpha_{f_{i+1}-1}$. When α refine β, we write $\alpha \leqslant \beta$.

Definition 7.1. For any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, the polynomial $P_\alpha(x_1, x_2, \ldots, x_n)$ is defined as

$$P_\alpha = \sum_{\ell=1}^{k} (-1)^{k-\ell} \sum_{\substack{1=f_1<f_2<\cdots<f_{\ell+1}=k+1 \\ 1 \leqslant i_1 < i_2 < \cdots < i_\ell \leqslant n}} \prod_{j=1}^{\ell} \left((x_{i_j}^{\alpha_{f_j}} - i_j^{\alpha_{f_j}}) i_j^{\alpha_{f_j+1} + \cdots + \alpha_{f_{j+1}-1}} \right)$$

The top degree of P_α is in the sum when $\ell = k$, we must have $f_i = i$ in this case. Choosing the variable x_{i_j} in all binomials $(x_{i_j}^{\alpha_{f_j}} - i_j^{\alpha_{f_j}})$, we get M_α.

For a fix σ, we will partition the sums in P_α according to the cycle type of σ and the non-crossing structure of these cycle will play a major role in showing the following theorem.

Theorem 7.2. *For any α and any $\sigma \in QSV_n$ we have $P_\alpha(\sigma) = 0$.*

Proof. Let $\sigma = C_1 C_2 \cdots C_r$ the decomposition of σ into disjoint cycle. We include the fix points a 1-cycles. Given a set on indices $1 \leqslant i_1 < i_2 < \cdots < i_\ell \leqslant n$ we say that the cycle

support of $S = \{i_1, i_2, \ldots, i_\ell\}$ is $C(S) = \{j : S \cap C_j \neq \varnothing\}$. We have $P_\alpha = \displaystyle\sum_{T \subseteq [r]} P_{\alpha,T}$, where

$$P_{\alpha,T} = \sum_{\ell=|T|}^{k} (-1)^{k-\ell} \sum_{\substack{1=f_1<f_2<\cdots<f_{\ell+1}=k+1 \\ 1 \leqslant i_1 < i_2 < \cdots < i_\ell \leqslant n \\ C(\{i_1,i_2,\ldots,i_\ell\})=T}} \prod_{j=1}^{\ell} \left((x_{i_j}^{\alpha_{f_j}} - i_j^{\alpha_{f_j}}) i_j^{\alpha_{f_j+1}+\cdots+\alpha_{f_{j+1}-1}} \right). \tag{7.3}$$

We show that $P_{\alpha,T}(\sigma) = 0$ for all T. If $T = \varnothing$ there is nothing to show as $P_{\alpha,\varnothing} = 0$. We first consider the case when $|T| = 1$, and then use the non-crossing structure of the cycles to reduce the case $|T| > 1$ to $|T'| = 1$.

Case $|\mathbf{T}| = \mathbf{1}$: Let $T = \{t\}$ and $C_t = (a_m \ldots a_2\, a_1)$, where $a_1 < a_2 < \cdots < a_m$. Remark that on such cycle, the variable $x_{a_i} = a_{i-1}$ with the convention that $a_0 = a_m$. Expanding all the product in the definition of $P_{\alpha,T}$ and evaluating at σ, we obtain

$$P_{\alpha,\{t\}}(\sigma) = \sum_{1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k \leqslant m} \sum_{\substack{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k \\ \varepsilon_i \in \{0,1\} \\ \varepsilon_i=1 \text{ if } j_{i-1}=j_i}} (-1)^{\sum \varepsilon_i}\, z_{j_1,\varepsilon_1}^{\alpha_1} z_{j_2,\varepsilon_2}^{\alpha_2} \cdots z_{j_k,\varepsilon_k}^{\alpha_k}, \tag{7.4}$$

where $z_{j_i,\varepsilon_i} = a_{j_i-1}$ if $\varepsilon_i = 0$, otherwise $z_{j_i,\varepsilon_i} = a_{j_i}$. To show that $P_{\alpha,\{t\}}(\sigma)$, we construct a sign reversing involution on the evaluation of the terms in Equation (7.4).

Given $1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k \leqslant m$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$, then the term

$$z_{j_1,\varepsilon_1}^{\alpha_1} z_{j_2,\varepsilon_2}^{\alpha_2} \cdots z_{j_k,\varepsilon_k}^{\alpha_k} = a_{s_1}^{\alpha_1} a_{s_2}^{\alpha_2} \cdots a_{s_k}^{\alpha_k},$$

where $s_i \in \{j_i, j_i - 1\}$ depending on ε_i and $0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_k \leqslant m$. First find the largest, rightmost $s_i \notin \{0, m\}$.

If $\varepsilon_i = 1$, then let $\varepsilon_i' = 0$ and $j_i' = j_i + 1$, fixing all other values $\varepsilon_p' = \varepsilon_p$ and $j_p' = j_p$ for $p \neq i$.

In this case remark that $z_{j_i,\varepsilon_i} = a_{j_i} = z_{j_i',\varepsilon_i'}$ and $z_{j_p,\varepsilon_p} = z_{j_p',\varepsilon_p'}$ for $p \neq i$. Hence

$$(-1)^{\sum \varepsilon_i'}\, z_{j_1',\varepsilon_1'}^{\alpha_1} z_{j_2',\varepsilon_2'}^{\alpha_2} \cdots z_{j_k',\varepsilon_k'}^{\alpha_k} = -(-1)^{\sum \varepsilon_i}\, z_{j_1,\varepsilon_1}^{\alpha_1} z_{j_2,\varepsilon_2}^{\alpha_2} \cdots z_{j_k,\varepsilon_k}^{\alpha_k}$$

To show that $z_{j_1',\varepsilon_1'}^{\alpha_1} z_{j_2',\varepsilon_2'}^{\alpha_2} \cdots z_{j_k',\varepsilon_k'}^{\alpha_k}$ is a term of the sum Equation (7.4) we need to show that $j_1' \leqslant \cdots \leqslant j_i' \leqslant j_{i+1}' \leqslant \cdots \leqslant j_k'$ and since $\varepsilon_i' = 0$, we need $j_{i-1}' < j_i'$. For the last inequality, we have $j_{i-1}' = j_{i-1} \leqslant j_i < j_i + 1 = j_i'$. For the other inequality among the j's, we only need to show that $j_i' = j_i + 1 \leqslant j_{i+1} = j_{i+1}'$. Here we recall that i is chosen so that s_i is the rightmost values such that $0 < s_i < m$. This implies that either $i = k$ and there is no j_{i+1} or $s_{i+1} = m \in \{j_{i+1}, j_{i+1} - 1\}$ and $j_{i+1} \geqslant m > s_i = j_i$. The last equality follows from $\varepsilon_i = 1$. We thus have that all such terms cancelled each other in Equation (7.4).

The case where $\varepsilon_i = 0$ is very similar and is the reverse of the operation above. The choice of the rightmost $s_i \notin \{0, n\}$ will be the same in both cases, showing that we indeed have a sign reversing involution. All terms such that the evaluation $a_{s_1}^{\alpha_1} a_{s_2}^{\alpha_2} \cdots a_{s_k}^{\alpha_k}$ contains some $0 < s_i < m$ will cancel. The only two terms that survive the cancelation are

$$(-1)^{n-1} a_0^{\alpha_1} a_m^{\alpha_2} \cdots a_m^{\alpha_k} + (-1)^n a_m^{\alpha_1} a_m^{\alpha_2} \cdots a_m^{\alpha_k}$$

that can only be obtained when $j_1 = 1$ and $j_2 = \cdots = j_k = m$ with $\varepsilon_1 = 0$ and $\varepsilon_2 = \cdots = \varepsilon_k = 1$, for the first term; and when $j'_1 = \cdots = j'_k = m$ with $\varepsilon'_1 = \cdots = \varepsilon'_k = 1$ for the second term. Since $a_0 = a_m$:

$$P_{\alpha,\{t\}}(\sigma) = (-1)^{n-1}a_0^{\alpha_1}a_m^{\alpha_2}\cdots a_m^{\alpha_k} + (-1)^n a_m^{\alpha_1}a_m^{\alpha_2}\cdots a_m^{\alpha_k} = 0.$$

Case $|\mathbf{T}| > 1$: When we have more than one cycle involved, let $T = \{t_1, t_2, \ldots, t_r\}$ and we assume (without lost of generality) that $C = C_{t_1}$ is a cycle that do not contain (in the non-crossing sense) any nested cycles among C_{t_j} for $j > 1$. The fact that the cycles of σ are non-crossing guaranties the existence of such C for any given T. We now partition the terms of Equation (7.3) according to the intersection of $i_1 < i_2 < \cdots < i_\ell$ with the C_{t_j} for $j > 1$ and the corresponding possible choices of f_j's. We show that the portion of the terms intersecting C is a vanishing polynomials as in the case $|T| = 1$. Let $c = \min(C)$ and $d = \max(C)$. Assume we have $C(\{i_1, i_2, \ldots, i_\ell\}) = T$ and let

$$CQ_j(\{i_1, i_2, \ldots, i_\ell\}) = \{i_1, i_2, \ldots, i_\ell\} \cap C_{t_j} \neq \varnothing.$$

From our choice of $C = C_{t_1}$, we have $CQ_j(\{i_1, i_2, \ldots, i_\ell\}) = \{i_1, i_2, \ldots, i_\ell\} \cap \{i : c \leqslant i \leqslant d\}$. Outside the range $[c, d] = \{i : c \leqslant i \leqslant d\}$, we fix all the other parameters involved in the terms $P_{\alpha,T}$ in Equation (7.3). Fix $Q = (Q_2, \ldots, Q_r)$ where $\varnothing \neq Q_j \subset C_{t_j}$ such that

$$\bigcup_{j=2}^r Q_j = \{\underline{i}_1, \underline{i}_2, \ldots, \underline{i}_p\} \cup \{\bar{i}_1, \bar{i}_2, \ldots, \bar{i}_q\}$$

where $p + q < k$ and

$$\underline{i}_1 < \underline{i}_2 < \cdots < \underline{i}_p < c \leqslant d < \bar{i}_1 < \bar{i}_2 < \cdots < \bar{i}_q.$$

We also fix $F = \{\underline{f}_1, \underline{f}_2, \ldots, \underline{f}_p, \underline{f}_{p+1}, \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_q, \bar{f}_{q+1}\}$. where

$$1 = \underline{f}_1 < \underline{f}_2 < \cdots < \underline{f}_p < \underline{f}_{p+1} < \bar{f}_1 < \bar{f}_2 < \cdots < \bar{f}_q < \bar{f}_{q+1} = k+1.$$

For any term of the sum in Equation (7.3), we have a unique corresponding Q and F. In particular, we have $P_{\alpha,T} = \sum_Q P_{\alpha,T,Q,F}$ where

$$P_{\alpha,T,Q,F} = \prod_{j=1}^p \left((x_{\underline{i}_j}^{\alpha_{\underline{f}_j}} - \underline{i}_j^{\alpha_{\underline{f}_j}}) \underline{i}_j^{\alpha_{\underline{f}_j+1} + \cdots + \alpha_{\underline{f}_{j+1}-1}} \right) \times$$

$$\sum_{\ell=1}^{k-p-q} (-1)^{k-p-q-\ell} \sum_{\substack{\underline{f}_{p+1}=f_1<f_2<\cdots<f_{\ell-p-q+1}=\bar{f}_1 \\ c\leqslant i_1<i_2<\cdots<i_{\ell-p-q}\leqslant d \\ C(\{i_1,i_2,\ldots,i_{\ell-p-q}\})=\{t_1\}}} \prod_{j=1}^\ell \left((x_{i_j}^{\alpha_{f_j}} - i_j^{\alpha_{f_j}}) i_j^{\alpha_{f_j+1}+\cdots+\alpha_{f_{j+1}-1}} \right) \times$$

$$\prod_{j=1}^q \left((x_{\bar{i}_j}^{\alpha_{\bar{f}_j}} - \bar{i}_j^{\alpha_{\bar{f}_j}}) \bar{i}_j^{\alpha_{\bar{f}_j+1}+\cdots+\alpha_{\bar{f}_{j+1}-1}} \right).$$

When we evaluate $P_{\alpha,T,Q,F}(\sigma)$, then centred term above is $P_{(\alpha_{\underline{f}_{p+1}}, \alpha_{\underline{f}_{p+1}+1}, \ldots, \alpha_{\bar{f}_1}), \{t_1\}}(C_{t_1}) = 0$ using the result for $|\{t_1\}| = 1$ in first part of the proof. This complete the proof. $\qquad\square$

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