QUASISYMMETRIC VARIETIES AND TEMPERLEY-LIEB ALGEBRAS

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Abstract.

1. Quasisymmetric vanishing polynomials

In this section we define polynomials in n variables that vanishes on the set QSV_n and such that the homogeneous top degree is quasisymmetric. More precisely, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be any fixed composition of $d > 0$. Let

$$M_\alpha(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

be the quasisymmetric monomial indexed by α. We will define a polynomial $P_\alpha(x_1, x_2, \ldots, x_n)$ such that $P_\alpha = M_\alpha +$ lower degree terms, and $P_\alpha(\sigma) = 0$ for all $\sigma \in QSV_n$.

$1 = f_1 < f_2 < \cdots < f_\ell < f_{\ell+1} = n+1$ and $\beta_i = \alpha_{f_i} + \alpha_{f_i+1} + \cdots + \alpha_{f_{i+1}-1}$. When α refine β, we write $\alpha \le \beta$.

Definition 1.1. labeldef:vanishP For any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, the polynomial $P_\alpha(x_1, x_2, \ldots, x_n)$ is defined as

$$P_\alpha = \sum_{\ell=1}^{k} (-1)^{k-\ell} \sum_{\substack{1=f_1<f_2<\cdots<f_{\ell+1}=k+1 \\ 1 \le i_1 < i_2 < \cdots < i_\ell \le n}} \prod_{j=1}^{\ell} \Big((x_{i_j}^{\alpha_{f_j}} - i_j^{\alpha_{f_j}}) i_j^{\alpha_{f_j+1} + \cdots + \alpha_{f_{j+1}-1}} \Big)$$

The top degree of P_α is in the sum when $\ell = k$, we must have $f_i = i$ in this case. Choosing the variable x_{i_j} in all binomials $(x_{i_j}^{\alpha_{f_j}} - i_j^{\alpha_{f_j}})$, we get M_α.

For a fix σ, we will partition the sums in P_α according to the cycle type of σ and the non-crossing structure of these cycle will play a major role in showing the following theorem.

Theorem 1.2. *For any α and any $\sigma \in QSV_n$ we have $P_\alpha(\sigma) = 0$.*

Proof. Let $\sigma = C_1 C_2 \cdots C_r$ the decomposition of σ into disjoint cycle. We include the fix points a 1-cycles. Given a set on indices $1 \le i_1 < i_2 < \cdots < i_\ell \le n$ we say that the cycle

1991 *Mathematics Subject Classification.* 05E05, ???

Key words and phrases. Quasisymmetric Polynomials,

This work is supported in part by York Research Chair and NSERC. This paper originated in a working session at the Algebraic Combinatorics Seminar at Fields Institute.

support of $S = \{i_1, i_2, \dots, i_\ell\}$ is $C(S) = \{j : S \cap C_j \neq \emptyset\}$. We have $P_\alpha = \sum_{T\subseteq[r]} P_{\alpha,T}$, where

$$P_{\alpha,T} = \sum_{\ell=|T|}^{k} (-1)^{k-\ell} \sum_{\substack{1=f_1<f_2<\cdots<f_{\ell+1}=k+1\\ 1\le i_1<i_2<\cdots<i_\ell\le n\\ C(\{i_1,i_2,\dots,i_\ell\})=T}} \prod_{j=1}^{\ell} \Big((x_{i_j}^{\alpha_{f_j}} - i_j^{\alpha_{f_j}}) i_j^{\alpha_{f_j+1}+\cdots+\alpha_{f_{j+1}-1}} \Big). \tag{1.1}$$

We show that $P_{\alpha,T}(\sigma) = 0$ for all T. If $T = \emptyset$ there is nothing to show as $P_{\alpha,\emptyset} = 0$. We first consider the case when $|T| = 1$, and then use the non-crossing structure of the cycles to reduce the case $|T| > 1$ to $|T'| = 1$.

Case $|\mathbf{T}| = \mathbf{1}$: Let $T = \{t\}$ and $C_t = (a_m \dots a_2\, a_1)$, where $a_1 < a_2 < \cdots < a_m$. Remark that on such cycle, the variable $x_{a_i} = a_{i-1}$ with the convention that $a_0 = a_m$. Expanding all the product in the definition of $P_{\alpha,T}$ and evaluating at σ, we obtain

$$P_{\alpha,\{t\}}(\sigma) = \sum_{1\le j_1\le j_2\le\cdots\le j_k\le m} \sum_{\substack{\epsilon_1,\epsilon_2,\dots,\epsilon_k\\ \epsilon_i\in\{0,1\}\\ \epsilon_i=1 \text{ if } j_{i-1}=j_i}} (-1)^{\sum \epsilon_i}\ z_{j_1,\epsilon_1}^{\alpha_1} z_{j_2,\epsilon_2}^{\alpha_2} \cdots z_{j_k,\epsilon_k}^{\alpha_k}, \tag{1.2}$$

where $z_{j_i,\epsilon_i} = a_{j_i-1}$ if $\epsilon_i = 0$, otherwise $z_{j_i,\epsilon_i} = a_{j_i}$. To show that $P_{\alpha,\{t\}}(\sigma)$, we construct a sign reversing involution on the evaluation of the terms in Equation (1.2).

Given $1 \le j_1 \le j_2 \le \cdots \le j_k \le m$ and $\epsilon_1, \epsilon_2, \dots, \epsilon_k$, then the term

$$z_{j_1,\epsilon_1}^{\alpha_1} z_{j_2,\epsilon_2}^{\alpha_2} \cdots z_{j_k,\epsilon_k}^{\alpha_k} = a_{s_1}^{\alpha_1} a_{s_2}^{\alpha_2} \cdots a_{s_k}^{\alpha_k},$$

where $s_i \in \{j_i, j_i - 1\}$ depending on ϵ_i and $0 \le s_1 \le s_2 \le \cdots \le s_k \le m$. First find the largest, rightmost $s_i \notin \{0, m\}$.

If $\epsilon_i = 1$, then let $\epsilon'_i = 0$ and $j'_i = j_i + 1$, fixing all other values $\epsilon'_p = \epsilon_p$ and $j'_p = j_p$ for $p \neq i$.

In this case remark that $z_{j_i,\epsilon_i} = a_{j_i} = z_{j'_i,\epsilon'_i}$ and $z_{j_p,\epsilon_p} = z_{j'_p,\epsilon'_p}$ for $p \neq i$. Hence

$$(-1)^{\sum \epsilon'_i}\ z_{j'_1,\epsilon'_1}^{\alpha_1} z_{j'_2,\epsilon'_2}^{\alpha_2} \cdots z_{j'_k,\epsilon'_k}^{\alpha_k} = -(-1)^{\sum \epsilon_i}\ z_{j_1,\epsilon_1}^{\alpha_1} z_{j_2,\epsilon_2}^{\alpha_2} \cdots z_{j_k,\epsilon_k}^{\alpha_k}$$

To show that $z_{j'_1,\epsilon'_1}^{\alpha_1} z_{j'_2,\epsilon'_2}^{\alpha_2} \cdots z_{j'_k,\epsilon'_k}^{\alpha_k}$ is a term of the sum Equation (1.2) we need to show that $j'_1 \le \cdots \le j'_i \le j'_{i+1} \le \cdots \le j'_k$ and since $\epsilon'_i = 0$, we need $j'_{i-1} < j'_i$. For the last inequality, we have $j'_{i-1} = j_{i-1} \le j_i < j_i + 1 = j'_i$. For the other inequality among the j's, we only need to show that $j'_i = j_i + 1 \le j_{i+1} = j'_{i+1}$. Here we recall that i is chosen so that s_i is the rightmost values such that $0 < s_i < m$. This implies that either $i = k$ and there is no j_{i+1} or $s_{i+1} = m \in \{j_{i+1}, j_{i+1} - 1\}$ and $j_{i+1} \ge m > s_i = j_i$. The last equality follows from $\epsilon_i = 1$. We thus have that all such terms cancelled each other in Equation (1.2).

The case where $\epsilon_i = 0$ is very similar and is the reverse of the operation above. The choice of the rightmost $s_i \notin \{0, n\}$ will be the same in both cases, showing that we indeed have a sign reversing involution. All terms such that the evaluation $a_{s_1}^{\alpha_1} a_{s_2}^{\alpha_2} \cdots a_{s_k}^{\alpha_k}$ contains some $0 < s_i < m$ will cancel. The only two terms that survive the cancelation are

$$(-1)^{n-1} a_0^{\alpha_1} a_m^{\alpha_2} \cdots a_m^{\alpha_k} + (-1)^n a_m^{\alpha_1} a_m^{\alpha_2} \cdots a_m^{\alpha_k}$$

that can only be obtained when $j_1 = 1$ and $j_2 = \cdots = j_k = m$ with $\epsilon_1 = 0$ and $\epsilon_2 = \cdots = \epsilon_k = 1$, for the first term; and when $j'_1 = \cdots = j'_k = m$ with $\epsilon'_1 = \cdots = \epsilon'_k = 1$ for the second term. Since $a_0 = a_m$:

$$P_{\alpha,\{t\}}(\sigma) = (-1)^{n-1} a_0^{\alpha_1} a_m^{\alpha_2} \cdots a_m^{\alpha_k} + (-1)^n a_m^{\alpha_1} a_m^{\alpha_2} \cdots a_m^{\alpha_k} = 0.$$

Case $|\mathbf{T}| > 1$: When we have more than one cycle involved, let $T = \{t_1, t_2, \ldots, t_r\}$ and we assume (without lost of generality) that $C = C_{t_1}$ is a cycle that do not contain (in the non-crossing sense) any nested cycles among C_{t_j} for $j > 1$. The fact that the cycles of σ are non-crossing guaranties the existence of such C for any given T. We now partition the terms of Equation (1.1) according to the intersection of $i_1 < i_2 < \cdots < i_\ell$ with the C_{t_j} for $j > 1$ and the corresponding possible choices of f_j's. We show that the portion of the terms intersecting C is a vanishing polynomials as in the case $|T| = 1$. Let $c = \min(C)$ and $d = \max(C)$. Assume we have $C(\{i_1, i_2, \ldots, i_\ell\}) = T$ and let

$$CQ_j(\{i_1, i_2, \ldots, i_\ell\}) = \{i_1, i_2, \ldots, i_\ell\} \cap C_{t_j} \neq \emptyset\,.$$

From our choice of $C = C_{t_1}$, we have $CQ_j(\{i_1, i_2, \ldots, i_\ell\}) = \{i_1, i_2, \ldots, i_\ell\} \cap \{i : c \le i \le d\}$. Outside the range $[c, d] = \{i : c \le i \le d\}$, we fix all the other parameters involved in the terms $P_{\alpha,T}$ in Equation (1.1). Fix $Q = (Q_2, \ldots, Q_r)$ where $\emptyset \neq Q_j \subset C_{t_j}$ such that

$$\bigcup_{j=2}^{r} Q_j = \{\underline{i}_1, \underline{i}_2, \ldots, \underline{i}_p\} \cup \{\bar{i}_1, \bar{i}_2, \ldots, \bar{i}_q\}$$

where $p + q < k$ and

$$\underline{i}_1 < \underline{i}_2 < \cdots < \underline{i}_p < c \le d < \bar{i}_1 < \bar{i}_2 < \cdots < \bar{i}_q\,.$$

We also fix $F = \{\underline{f}_1, \underline{f}_2, \ldots, \underline{f}_p, \underline{f}_{p+1}, \overline{f}_1, \overline{f}_2, \ldots, \overline{f}_q, \overline{f}_{q+1}\}$. where

$$1 = \underline{f}_1 < \underline{f}_2 < \cdots < \underline{f}_p < \underline{f}_{p+1} < \overline{f}_1 < \overline{f}_2 < \cdots < \overline{f}_q < \overline{f}_{q+1} = k+1\,.$$

For any term of the sum in Equation (1.1), we have a unique corresponding Q and F. In particular, we have $P_{\alpha,T} = \sum_{Q} P_{\alpha,T,Q,F}$ where

$$\begin{aligned}
P_{\alpha,T,Q,F} = &\prod_{j=1}^{p} \Big((x_{\underline{i}_j}^{\alpha_{\underline{f}_j}} - \underline{i}_j^{\alpha_{\underline{f}_j}}) \underline{i}_j^{\alpha_{\underline{f}_j+1} + \cdots + \alpha_{\underline{f}_{j+1}-1}} \Big) \times \\
&\sum_{\ell=1}^{k-p-q} (-1)^{k-p-q-\ell} \sum_{\substack{\underline{f}_{p+1} = f_1 < f_2 < \cdots < f_{\ell-p-q+1} = \overline{f}_1 \\ c \le i_1 < i_2 < \cdots < i_{\ell-p-q} \le d \\ C(\{i_1, i_2, \ldots, i_{\ell-p-q}\}) = \{t_1\}}} \prod_{j=1}^{\ell} \Big((x_{i_j}^{\alpha_{f_j}} - i_j^{\alpha_{f_j}}) i_j^{\alpha_{f_j+1} + \cdots + \alpha_{f_{j+1}-1}} \Big) \times \\
&\prod_{j=1}^{q} \Big((x_{\bar{i}_j}^{\alpha_{\overline{f}_j}} - \bar{i}_j^{\alpha_{\overline{f}_j}}) \bar{i}_j^{\alpha_{\overline{f}_j+1} + \cdots + \alpha_{\overline{f}_{j+1}-1}} \Big)\,.
\end{aligned}$$

When we evaluate $P_{\alpha,T,Q,F}(\sigma)$, then centred term above is $P_{(\alpha_{\underline{f}_{p+1}},\alpha_{\underline{f}_{p+1}+1},\ldots,\alpha_{\overline{f}_1}),\{t_1\}}(C_{t_1}) = 0$ using the result for $|\{t_1\}| = 1$ in first part of the proof. This complete the proof. □

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