THE ITERATIVE DECONVOLUTION OF LINEARLY BLURRED IMAGES USING NON-PARAMETRIC STABILIZING FUNCTIONS

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ABSTRACT

An iterative solution to the problem of image deconvolution is presented. The previous image estimate is pre-filtered using a stabilizing function that is updated based on current error and noise estimates. Noise propagation from one iteration to the next is reduced by the use of a second, regularizing operator resulting in a hybrid iteration technique. Further, error terms are developed that shed new light on the error propagation properties of this method by quantifying the extent of noise and regularization error propagation. Optimal non-parametric stabilizing and regularization functions are then derived based on this error analysis.

1. INTRODUCTION

The degradation system for certain imaging processes can be expressed as two-dimensional linear convolutive blurring with additive white Gaussian noise:

$$r(i,j) = \sum_{k=1}^{N}\sum_{l=1}^{N} h(i-k, j-l)s(k,l) + n(i,j), \tag{1}$$

where $r(i,j)$ is a pixel value in an $N \times N$ recorded image[1], $h(i,j)$ is the unit impulse response of the point-spread function (PSF) for the image recording system, $s(i,j)$ is the underlying or desired signal, and $n(i,j)$ is additive white Gaussian noise. This can also be expressed in a matrix-vector formulation as:

$$\mathbf{r} = \mathbf{Hs} + \mathbf{n} \tag{2}$$

where $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{n}$ are columnized or lexiconographically ordered versions of the received, desired, and noise images respectively and $\mathbf{H}$ is an $N^2 \times N^2$ block-Toeplitz convolutive blurr operator. This system can be directly solved by finding an estimate desired signal $\hat{\mathbf{s}}$ that satisfies a least squares fit to the data by minimizing the norm:

$$||\mathbf{r} - \mathbf{H}\hat{\mathbf{s}}||_2^2 \tag{3}$$

This leads to the generalized inverse filter:

$$\begin{aligned} (\mathbf{H}^T\mathbf{H})\hat{\mathbf{s}} &= \mathbf{H}^T\mathbf{r} \\ \hat{\mathbf{s}} &= (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{r} \end{aligned} \tag{4}$$

Assuming that $\mathbf{H}$ is block circulant, and also that the blurred image $r(i,j)$ is sufficiently zero padded such that the results of linear and circular convolution are the same, it is well known that $\mathbf{H}$ can be diagonalized using the discrete Fourier transform[1]:

$$\mathbf{H} = \mathbf{WDW}^H \tag{5}$$

where $\mathbf{D}$ is a complex valued matrix with diagonal elements consisting of the 2-dimensional discrete Fourier transform (DFT) coefficients of the the PSF $h(i,j)$, $\mathbf{W}$ is a matrix of eigenvectors which are the complex exponential basis functions for the DFT, and the superscript H denotes the Hermitian transpose. This approach avoids the computational burden of inverting an $N^2 \times N^2$ system, reducing it to a set of N^2 scalar problems. Using this diagonalization approach, an image estimate can be expressed as[2]:

$$\hat{\mathcal{S}}(u,v) = \frac{\mathcal{H}^*(u,v)\mathcal{R}(u,v)}{|\mathcal{H}(u,v)|^2} \tag{6}$$

where $\hat{\mathcal{S}}$,$\mathcal{R}$ and $\mathcal{H}$ are the 2-dimensional discrete Fourier transforms of $\hat{s}(i,j)$, $r(i,j)$, and $h(i,j)$ respectively and * denotes complex conjugation.

The main issues to overcome in this solution, which are evident on inspection of the denominator term of the spatial frequency domain formulation of (6), are:

- high frequency additive noise amplification, particularly if the PSF is inherently low pass with small or zero DFT coefficients in the higher frequency bands.
- retrieval of lost information. If the columnspace of the convolutive operator $\mathbf{H}$ does not span the entire signal space, $\hat{\mathbf{s}}$ a many-to-one transformation of $\mathbf{r}$.

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[1] For simplicity in presentation, square images are used, however these same ideas can be used for rectangular images without loss of generality.

[2] Note that for following discussions, the DFT coefficient indices (u,v) will be omitted. Any scripted variables (e.g. $\mathcal{H}$ or $\mathcal{R}$) will be considered as a complex valued DFT coefficent

2. DIRECT DECONVOLUTION USING AN AUXILIARY IMAGE

One approach is to assume that two versions[3] of the received signal are available, $r_1(i,j)$ and $r_2(i,j)$. Then the degradation process of equation (2) can be reformulated in matrix-vector form as:

$$\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \end{bmatrix} \begin{bmatrix} \mathbf{s} \end{bmatrix} + \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \end{bmatrix} \tag{7}$$

Using the same diagonalization approach as before, we can again seek to minimize the objective function corresponding to (3) resulting in a spatial frequency domain formulation of:

$$\hat{\mathcal{S}} = \frac{\mathcal{H}_1^*\mathcal{R}_1 + \mathcal{H}_2^*\mathcal{R}_2}{|\mathcal{H}_1|^2 + |\mathcal{H}_2|^2} \tag{8}$$

This solution is well-posed if $\mathcal{H}_1$ and $\mathcal{H}_2$ do not share common zeros, which will be true if the columnspace of the matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \end{bmatrix}$$

is full rank. To undertake this approach, two versions of the image must be available. Since an auxiliary image is often not available in practical applications, an iterative approach is proposed where the image $\mathbf{r}_2$ is replaced with a filtered version of the image estimate $\hat{\mathbf{s}}_{k-1}$ at the previous iteration. Specifically, at iteration k we set $\mathbf{r}_2(k) = \mathbf{H}_s\hat{\mathbf{s}}_{k-1}$. The "pseudo"-PSF $\mathbf{H}_s$ is chosen to be a stability operator in a manner yet to be described. The DFT domain expression for the kth estimate of the image is then:

$$\hat{\mathcal{S}}_k = \frac{\mathcal{H}^*\mathcal{R} + \mathcal{H}_s^*\mathcal{H}_s\hat{\mathcal{S}}_{k-1}}{|\mathcal{H}|^2 + |\mathcal{H}_s|^2} \tag{9}$$

2.1. Error Analysis

To gain some insights into the properties of this preliminary algorithm, expressions for the error terms in equation (9) can be developed with estimates for $\mathcal{S}$ and $\mathcal{H}$ at the kth iteration defined as:

$$\hat{\mathcal{S}}_k = \mathcal{S} + \mathcal{E}_{\hat{\mathcal{S}}_k}$$

$$\hat{\mathcal{H}}_k = \mathcal{H} + \mathcal{E}_{\hat{\mathcal{H}}_k}$$

Here we also look at the stability operator $\mathcal{H}_s$, renaming the expression

$$|\mathcal{H}_s|^2 = \mathcal{H}_s^*\mathcal{H}_s = \gamma \tag{10}$$

An error expression can then be obtained by making the above substitutions, then subtracting $\mathcal{S}$ from (9).

$$\mathcal{E}_{\hat{\mathcal{S}}_k} = \frac{\hat{\mathcal{H}}_k^*\mathcal{N} - \hat{\mathcal{H}}_k^*\mathcal{E}_{\hat{\mathcal{H}}_k}\mathcal{S} + \gamma_k\mathcal{E}_{\hat{\mathcal{S}}_{k-1}}}{\left|\hat{\mathcal{H}}_k\right|^2 + \gamma_k} \tag{11}$$

where $\mathcal{N}$ are the DFT coefficients of the additive observation noise $\mathbf{n}$. The total error in the estimated image can therefore be broken down into three components; error due to additive observation noise, error due to the mis-estimation of the blur operator $\hat{\mathbf{h}}_k(i,j)$, and finally the error due to the mis-estimation in the unconstrained estimate $\hat{\mathbf{s}}_{k-1}(i,j)$. Noting that $\mathcal{E}_{\hat{\mathcal{S}}_{k-1}}$ can be expressed recursively with respect to the other two types of error in previous iterations, equation (11) can thus be re-written,

$$\mathcal{E}_{\hat{\mathcal{S}}_k} = \mathcal{N}\Sigma_{\mathcal{N}}^{(k)} - \mathcal{S}\Sigma_{\mathcal{H}}^{(k)} \tag{12}$$

where

$$\begin{aligned} \Sigma_{\mathcal{N}}^{(k)} &= \frac{\mathcal{H}_k^*}{\mathcal{Z}_k} + \sum_{i=1}^{k}\left[\prod_{j=i}^{k}\frac{\gamma_j}{\mathcal{Z}_j}\right]\frac{\mathcal{H}_{i-1}^*}{\mathcal{Z}_{i-1}} \\ &= \frac{\hat{\mathcal{H}}_k^* + \gamma_k\Sigma_{\mathcal{N}}^{(k-1)}}{\mathcal{Z}_k} \end{aligned} \tag{13}$$

$$\begin{aligned} \Sigma_{\mathcal{H}}^{(k)} &= \frac{\hat{\mathcal{H}}_k^*\mathcal{E}_{\hat{\mathcal{H}}_k}}{\mathcal{Z}_k} + \sum_{i=1}^{k}\left[\prod_{j=i}^{k}\frac{\gamma_j}{\mathcal{Z}_j}\right]\frac{\hat{\mathcal{H}}_{i-1}^*\mathcal{E}_{\hat{\mathcal{H}}_{i-1}}}{\mathcal{Z}_{i-1}} \\ &= \frac{\hat{\mathcal{H}}_k^*\mathcal{E}_{\hat{\mathcal{H}}_k} + \gamma_k\Sigma_{\mathcal{H}}^{(k-1)}}{\mathcal{Z}_k} \end{aligned} \tag{14}$$

and $\mathcal{Z}_i = \left|\hat{\mathcal{H}}_i\right|^2 + \gamma_i$. From (12) we can see that the three original error propagation terms are controlled by the additive terms $\Sigma_{\mathcal{N}}^{(k)}$ and $\Sigma_{\mathcal{H}}^{(k)}$.

3. HYBRID FORMULATION

Close inspection of (9), however, reveals the conflict introduced with this formulation. On the one hand, to reduce error due to noise amplification, the value of γ must be larger when $\left|\hat{\mathcal{H}}_k\right|^2$ is small. But from the second term in (13) we know that these are the frequency coefficients where noise propagation error is the largest. This type of error introduces "ringing" into the final estimated image, particularly with linear motion blurs[2][3]. If γ is to be used as a stability operator then, it should reinforce $\hat{\mathcal{S}}_{k-1}$ where confidence in those coefficients is high, notably where the error in $\hat{\mathcal{S}}_{k-1}$ is known to be small. Therefore, the stability operator γ should not be used for regularization. Pursuant to this discussion, we seek to reduce the effect noise propagation error by introducing a regularization operator ρ to augment the stability operator γ, which changes (9) to

$$\hat{\mathcal{S}}_k = \frac{\hat{\mathcal{H}}_k^*\mathcal{R} + \gamma_k\hat{\mathcal{S}}_{k-1}}{\left|\hat{\mathcal{H}}_k\right|^2 + \gamma_k + \rho_k} \tag{15}$$

Now the error formulation has one additional term.

$$\mathcal{E}_{\hat{\mathcal{S}}_k} = \mathcal{N}\Sigma_{\mathcal{N}}^{(k)} - \mathcal{S}\Sigma_{\mathcal{H}}^{(k)} - \mathcal{S}\Sigma_{\rho}^{(k)} \tag{16}$$

where

$$\begin{aligned} \Sigma_{\rho}^{(k)} &= \frac{\rho_k}{\mathcal{Z}_k} + \sum_{i=1}^{k}\left[\prod_{j=i}^{k}\frac{\gamma_j}{\mathcal{Z}_j}\right]\frac{\rho_{i-1}}{\mathcal{Z}_{i-1}} \\ &= \frac{\rho_k + \gamma_k\Sigma_{\rho}^{(k-1)}}{\mathcal{Z}_k} \end{aligned} \tag{17}$$

and

$$\mathcal{Z}_i = \left|\hat{\mathcal{H}}_i\right|^2 + \gamma_i + \rho_i$$

Note also that $\mathcal{Z}_i$ is changed for (13) and (14) as well.

[3] Two images of the same source corresponding to distinct PSF's

3.1. Optimal Stability and Regularization Operators

To choose γ_k and ρ_k, we start by minimizing the squared norm of the estimation error, which by Parseval's theorem is the same a minimizing $\mathcal{E}^*_{\hat{\mathcal{S}}_k}\mathcal{E}_{\hat{\mathcal{S}}_k}$ or $\left|\mathcal{E}_{\hat{\mathcal{S}}_k}\right|^2$. Note that here we minimize the error with respect to each frequency domain coefficient separately as opposed to other methods that choose a static regularization operator and minimize the expectation of this error using a regularization parameter[4]. The partial derivative of $\left|\mathcal{E}_{\hat{\mathcal{S}}_k}\right|^2$ with respect to γ_k is

$$\frac{\partial \left|\mathcal{E}_{\hat{\mathcal{S}}_k}\right|^2}{\partial \gamma_k} = \frac{\gamma_k\left(\mathcal{E}_{\hat{\mathcal{S}}_{k-1}}\mathcal{A}^* + \mathcal{E}^*_{\hat{\mathcal{S}}_{k-1}}\mathcal{A}\right) - \left(\mathcal{A}^*\mathcal{B} + \mathcal{A}\mathcal{B}^*\right)}{\left(\left|\hat{\mathcal{H}}_k\right|^2 + \gamma_k + \rho_k\right)^3} \tag{18}$$

where

$$\begin{aligned}\mathcal{A} &= \mathcal{S}\left(\rho_k + \hat{\mathcal{H}}^*_k\mathcal{E}_{\hat{\mathcal{H}}_k}\right) + \mathcal{E}_{\hat{\mathcal{S}}_{k-1}}\left(\left|\hat{\mathcal{H}}_k\right|^2 + \rho_k\right) - \hat{\mathcal{H}}^*_k\mathcal{N} \\ &= \rho_k\hat{\mathcal{S}}_{k-1} + \left|\hat{\mathcal{H}}_k\right|^2\hat{\mathcal{S}}_{k-1} - \hat{\mathcal{H}}^*_k\mathcal{R} + \hat{\mathcal{H}}^*_k\mathcal{E}_{\hat{\mathcal{H}}_k}\mathcal{S}\end{aligned}$$

and

$$\mathcal{B} = \mathcal{S}\left(\rho_k + \hat{\mathcal{H}}^*_k\mathcal{E}_{\hat{\mathcal{H}}_k}\right) - \hat{\mathcal{H}}^*_k\mathcal{N}$$

Setting this partial derivative to zero and solving for γ_k yields:

$$\gamma_k = \frac{\mathcal{A}\mathcal{B}^* + \mathcal{A}^*\mathcal{B}}{\mathcal{E}_{\hat{\mathcal{S}}_{k-1}}\mathcal{A}^* + \mathcal{E}^*_{\hat{\mathcal{S}}_{k-1}}\mathcal{A}} \tag{19}$$

Repeating the same process with respect to ρ_k yields:

$$\frac{\partial \left|\mathcal{E}_{\hat{\mathcal{S}}_k}\right|^2}{\partial \rho_k} = \frac{\mathcal{X}\left(\mathcal{Y} - \rho_k\mathcal{S}\right)^* + \mathcal{X}^*\left(\mathcal{Y} - \rho_k\mathcal{S}\right)}{\left(\left|\hat{\mathcal{H}}_k\right|^2 + \gamma_k + \rho_k\right)^3} \tag{20}$$

where

$$\begin{aligned}\mathcal{X} &= \mathcal{S}\left(\left|\hat{\mathcal{H}}_k\right|^2 + \gamma_k - \hat{\mathcal{H}}^*_k\mathcal{E}_{\hat{\mathcal{H}}_k}\right) + \gamma_k\mathcal{E}_{\hat{\mathcal{S}}_{k-1}} + \hat{\mathcal{H}}^*_k\mathcal{N} \\ &= \hat{\mathcal{H}}^*_k\mathcal{R} + \gamma_k\hat{\mathcal{S}}_{k-1} - \hat{\mathcal{H}}^*_k\mathcal{E}_{\hat{\mathcal{H}}_k}\mathcal{S}\end{aligned}$$

and

$$\mathcal{Y} = \gamma_k\mathcal{E}_{\hat{\mathcal{S}}_{k-1}} + \hat{\mathcal{H}}^*_k\mathcal{N} - \hat{\mathcal{H}}^*_k\mathcal{E}_{\hat{\mathcal{H}}_k}\mathcal{S}$$

Hence the value for ρ_k that minimizes $\left|\mathcal{E}_{\hat{\mathcal{S}}_k}\right|^2$ is:

$$\rho_k = \frac{\mathcal{X}\mathcal{Y}^* + \mathcal{X}^*\mathcal{Y}}{\mathcal{S}\mathcal{X}^* + \mathcal{S}^*\mathcal{X}} \tag{21}$$

Note that these expressions for both γ_k and ρ_k yield values that are real due to the conjugate structure of (19) and (21). This does not however mean that values for γ_k and ρ_k are positive. Hence, at each iteration γ_k and ρ_k are constrained such that any negative values as calculated per (19) or (21) are set to zero.

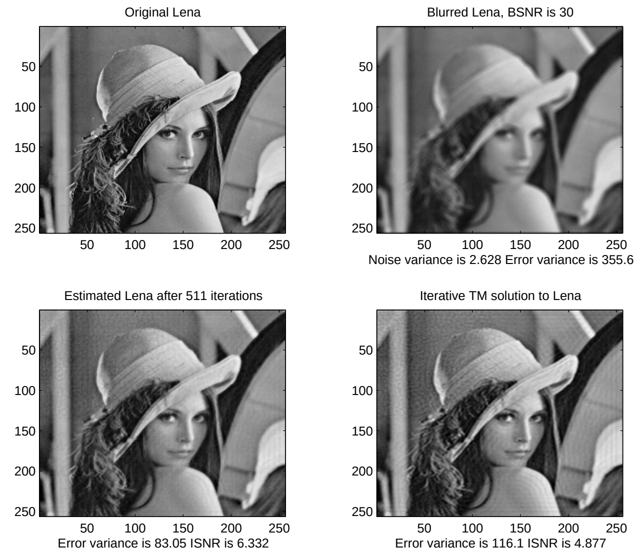

Fig. 1. Deconvolution results: BSNR = 30 dB

4. EXPERIMENTAL RESULTS

Figures 1 and 2 show experimental results using the iteration outlined by (15) using (19) and (21). Images were degraded using a 7 by 7 uniform out-of-focus blur with additive noise.

$$h(i,j) = \begin{cases} \frac{1}{\pi R} & \text{if } \sqrt{i^2 + j^2} \leq R \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

where the parameter R controls the severity of the blur. Key result indicators are described using the metrics:

$$ISNR = 10\log_{10}\left(\frac{\sum_{i,j}\left(s(i,j) - r(i,j)\right)^2}{\sum_{i,j}\left(s(i,j) - \hat{s}(i,j)\right)^2}\right)$$

$$BSNR = 10\log_{10}\left(\frac{\text{var}(\mathbf{Hs})}{\sigma_n^2}\right)$$

A maximum likelihood estimate of the noise variance σ_n^2 [4] was used to "whiten" the estimated noise spectrum $\mathcal{N}$ before calculating γ_k and ρ_k. Also, for these values the previous estimate $\hat{\mathcal{S}}_{k-1}$ was used in place of $\mathcal{S}$ at each iteration. The point spread function $\hat{\mathcal{H}}_k$ is assumed to be known a priori, therefore values for $\mathcal{E}_{\hat{\mathcal{H}}_k}$ were set to zero for all k. The stability operator γ was initialized to zeros, while the regularization operator ρ was initialized:

$$\rho_0 = \frac{1}{4}\left(1 - |\mathcal{H}|^2\right)$$

It is seen from the figures that the improvement as measured by ISNR offered by the proposed de-blurring process is significant in comparison to the iterative Tikhonov-Miller solution which is used as a benchmark. This is particularly evident at low BSNR, where in figure 2 a 65.1 percent improvement in the ISNR of this proposed method over the iterative Tikhonov-Miller solution is shown. These results are also comparable to that obtained by other methods [5][6][7].

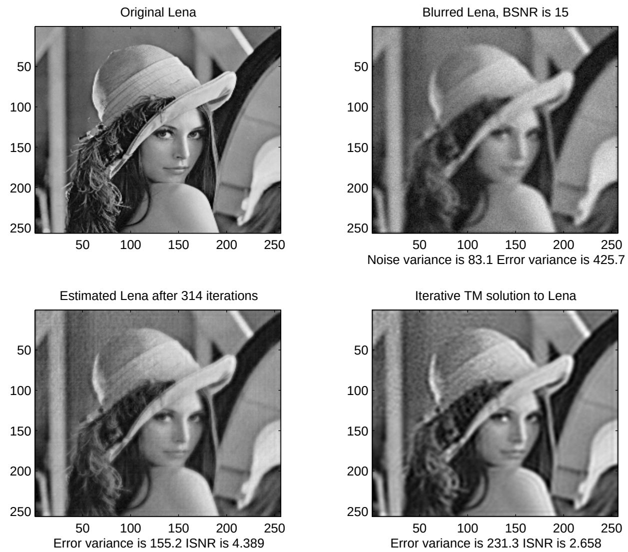

Fig. 2. Deconvolution results: BSNR = 15 dB

5. CONCLUSIONS

From the experimental results presented in the previous section, is is readily seen that the image estimation given by (15) outperforms the iterative Tikhonov Miller method. It should also be noted here that the benchmark given by the iterative Tikhonov-Miller algorithm is conducted under *ideal* condiditons, that is to say with regularization parameter α_{TM} set to the normally unknown value of:

$$\alpha_{TM} = \frac{\sum_{u,v} |\mathcal{N}(u,v)|^2}{\sum_{u,v} |\mathcal{C}(u,v)\mathcal{S}(u,v)|^2} \tag{23}$$

where $\mathcal{C}$ is the Laplacian operator, $\mathcal{N}$ is the noise, and $\mathcal{S}$ is the real or underlying image.

It is important to note that all formulations given are adaptable for use in the blind deconvoluton case, as each blur operator $\hat{\mathcal{H}}_k$ is assumed to be valid for the current iteration only. Hence, future work will include using this method in a *blind* iterative scheme similar to [8]. The inclusion of non-linear constraints as outlined in [9], [10] and [11] are also possible.

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