# A BAYESIAN APPROACH TO TRACKING WIDEBAND TARGETS USING SENSOR ARRAYS AND PARTICLE FILTERS

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## ABSTRACT

This paper presents a Bayesian approach for tracking the directions-of-arrival (DOAs) of multiple *wideband* moving targets using a linear, passive sensor array, for use in smart antenna wireless applications, for example. A novel form of posterior distribution describing the parameters of the received signal is developed. The nuisance parameters (source amplitudes and variances) of this distribution are integrated out to form a more computationally efficient estimator. The desired parameters are then tracked using *particle filtering* techniques. Simulation results demonstrate the effectiveness and robustness of this method.

## 1. INTRODUCTION

Over the past few years, the use of smart antennas has emerged as a valuable tool in mobile communications. The application of smart antennas can be enhanced if the directions of arrival of the desired receivers can be easily tracked. Many techniques have been developed for tracking the DOAs of *narrowband* signal sources observed with a sensor array. These include beamforming, signal-subspace techniques and maximum likelihood methods, as well as MCMC techniques, specifically *particle filters*, [1] [2] which are modern Bayesian methods based on numerically approximating the posterior distribution of interest. Of these mentioned techniques, the *wideband* tracking problem using arrays of sensors has not received much attention to date. Nevertheless, solutions to this problem are of interest for wireless communications (particularly in the CDMA scenario), military, naval, air traffic control operations, and the 911-problem.

In this paper, we propose a novel approach that leverages recent advances in sequential MCMC techniques, allied with the persistent increase in computer power, to solve the wideband tracking problem. Here, we formulate a novel model for the wideband received signal vector, which then leads to a posterior distribution in the target angles. Tracking the moving targets is then implemented in an efficient manner through the use of particle filters. In addition, the signal waveforms from multiple emitters can be separated and restored sequentially.

## 2. STATE SPACE MODEL

The sequential sampling approach we adopt admits a first order state-space hidden Markov model. The states $[\boldsymbol{\tau}(t); \boldsymbol{a}(n)]$ evolve according to:

$$\begin{aligned} \boldsymbol{\tau}(t) &= \boldsymbol{\tau}(t-1) + \sigma_v \boldsymbol{v}(t), && (1) \\ \boldsymbol{a}(n) &\sim \mathcal{N}\left(\mathbf{0}, \delta^2 \sigma_w^2 \boldsymbol{I}_K\right), && (2) \end{aligned}$$

where $\boldsymbol{\tau}(t) \in \mathcal{R}^{K\times 1}$ are the inter-sensor delays of the received signal that vary as a function of the source DOAs, and $\boldsymbol{a}(n) \in \mathcal{R}^{K\times 1}$ are the source amplitudes respectively of $K$ (assumed known) plane waves impinging on an array of $M$ sensors, $\boldsymbol{v}(t)$ is an $iid$ Gaussian variable with zero mean and unit variance, $\sigma_v^2$ and $\sigma_w^2$ are the process and noise variances respectively, and $\delta^2$ is a hyper-parameter, corresponding to an estimate of the SNR. According to [3], a snapshot at time $t$ can be expressed as

$$\begin{aligned} \boldsymbol{y}(t) &= \sum_{k=0}^{K-1} \boldsymbol{s}_k(t - \tau_k(t)) + \sigma_w \boldsymbol{w}(t), && (3) \\ &\approx \sum_{k=0}^{K-1} \tilde{\boldsymbol{H}}(\tau_k(t)) \boldsymbol{s}_k(n) + \sigma_w \boldsymbol{w}(t), && (4) \end{aligned}$$

where[1] $\boldsymbol{w}(t)$ is an $iid$ Gaussian variable with zero mean and unit variance which is uncorrelated with the signal, $\tilde{\boldsymbol{H}}(\tau_k(t)) \in \mathcal{R}^{M\times L}$ is an $L$th–order *interpolation matrix* for the sources [3], defined as

$$\tilde{\boldsymbol{H}}(\tau_k(t)) = \begin{cases} \boldsymbol{H}(\tau_k(t)), & \text{if } \tau_k(t) \geq 0 \\ \boldsymbol{E}_M \boldsymbol{H}(\tau_k(t)), & \text{if } \tau_k(t) < 0 \end{cases}, \qquad (5)$$

where each row of $\boldsymbol{H}(\tau_k(t))$ represents $L$ coefficients computed from an appropriately selected interpolation function [2],

[1] Note that for notational convenience, from this point onwards we replace the approximation with an equality.

[2] For example, in the case of the uniform linear array, the interpolation function can be a windowed *sinc*$(\cdot)$ function.

and $\boldsymbol{E}_M$ is an exchange matrix [4]. The quantities $\boldsymbol{s}_k(t)$, the $k$th signal, and $\boldsymbol{s}_k(n)$, the corresponding discrete–time version, are defined respectively as

$$\boldsymbol{s}_k(t-\tau_k) = \\ [s_k(t), s_k(t-\tau_k(t)), \ldots, s_k(t-(M-1)\tau_k(t))]^T , \tag{6}$$

$$\boldsymbol{s}_k(n) = [s_k(n), s_k(n-1), \ldots, s_k(n-(L-1)]^T . \tag{7}$$

The interpolation matrix $\tilde{\boldsymbol{H}}(\tau_k(t))$ in (4) interpolates the discrete–time sequences $\boldsymbol{s}_k(n)$ to give the desired sequences $\boldsymbol{s}_k(t - m\tau_k(t)), m = 0, \ldots, M-1, k = 0, \ldots, K-1$, which correspond to the signals from the $k$th source at the $m$th array element at time $t$.

We now re–order (4) into a more convenient form as follows. By defining $\tilde{\boldsymbol{H}}_l(\boldsymbol{\tau}(t)) \in \mathcal{R}^{M\times K}$ as

$$\tilde{\boldsymbol{H}}_l(\boldsymbol{\tau}(t)) = \left[\tilde{\boldsymbol{H}}_l(\tau_0(t)), \tilde{\boldsymbol{H}}_l(\tau_1(t)), \ldots, \tilde{\boldsymbol{H}}_l(\tau_{K-1}(t))\right], \tag{8}$$

which collects the $l$th columns of $\tilde{\boldsymbol{H}}(\tau_k(t))$ in (5) for $K$ sources, we can then express (4) in the form

$$\boldsymbol{y}(t) = \sum_{l=0}^{L-1} \tilde{\boldsymbol{H}}_l(\boldsymbol{\tau}(t))\boldsymbol{a}(n-l) + \sigma_w \boldsymbol{w}(t), \tag{9}$$

where $\boldsymbol{a}(n) \triangleq [s_0(n), s_1(n), \ldots, s_{K-1}(n)]^T$. We define a vector $\boldsymbol{z}(n)$ (which is a function of only *past* source values) as follows

$$\boldsymbol{z}(t) \triangleq \boldsymbol{y}(t) - \sum_{l=1}^{L-1} \tilde{\boldsymbol{H}}_l(\boldsymbol{\tau}(t))\boldsymbol{a}(n-l), \tag{10}$$

and therefore according to (9) we have

$$\boldsymbol{z}(t) = \tilde{\boldsymbol{H}}_0(\boldsymbol{\tau}(t))\boldsymbol{a}(n) + \sigma_w \boldsymbol{w}(t), \tag{11}$$

which represents the desired form of the wideband model. The unknown and time–varying TOAs, $\boldsymbol{\tau}(t)$, are to be sequentially estimated based on the observations $\boldsymbol{y}(t)$.

We define $\boldsymbol{\theta}_{1:t} \triangleq \left(\{\boldsymbol{\tau}\}_{1:t}, \{\boldsymbol{a}\}_{1:t}, \sigma_v^2, \sigma_w^2\right)$, as a parameter vector, where the notation $(.)_{1:t}$ indicates all the elements from time 1 to time $t$. Hence, the joint distribution of all the parameters is $\pi(\boldsymbol{\theta}_{1:t}) \triangleq p(\boldsymbol{\theta}_{1:t}|\boldsymbol{z}_{1:t})$, which can then be expanded using appropriately selected prior distributions of the parameters, according to Bayes' theorem as

$$\pi(\boldsymbol{\theta}_{1:t}) \propto p\left(\boldsymbol{z}_{1:t}|\boldsymbol{\tau}_{1:t}, \boldsymbol{a}_{1:t}, \sigma_v^2, \sigma_w^2\right) p\left(\boldsymbol{\tau}_{1:t}|\sigma_v^2\right) \\ \times p\left(\boldsymbol{a}_{1:t}|\boldsymbol{\tau}_{1:t}, \sigma_w^2\right) p(\sigma_w^2) p(\sigma_v^2), \tag{12}$$

where $p(z_{1:t}|\cdot)$ is the likelihood term, and the remaining distributions constitute the joint prior distribution for the parameters $\boldsymbol{\theta}$.

Assuming that the observations, given the states, are $iid$ and that the state conditional update likelihood is also $iid$, we now assign distributions for each of the terms in (12) as

$$p(\boldsymbol{z}_{1:t}|\boldsymbol{\theta}_{1:t}) = \prod_{l=1}^{t} \mathcal{N}\left(\tilde{\boldsymbol{H}}_0(\boldsymbol{\tau}_l)\boldsymbol{a}(n), \sigma_w^2 \boldsymbol{I}_M\right), \tag{13}$$

$$p\left(\boldsymbol{\tau}_{1:t}|\sigma_v^2\right) = \prod_{l=1}^{t} \mathcal{N}\left(\boldsymbol{\tau}_{l-1}, \sigma_v^2 \boldsymbol{I}_K\right), \tag{14}$$

$$p\left(\boldsymbol{a}_{1:t}|\boldsymbol{\tau}_{1:t}, \sigma_w^2\right) = \prod_{l=1}^{t} \mathcal{N}\left(\boldsymbol{0}, \delta^2\sigma_w^2 \left[\tilde{\boldsymbol{H}}_0^T(\boldsymbol{\tau}_l)\tilde{\boldsymbol{H}}_0(\boldsymbol{\tau}_l)\right]^{-1}\right), \tag{15}$$

where $\mathcal{N}(\boldsymbol{\mu}|\boldsymbol{\Sigma})$ is the multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$. The prior distribution for the source amplitude vector $\boldsymbol{a}$ is chosen as in [3] [5] [6]. The prior distribution on the variances $\sigma_v^2$ and $\sigma_w^2$ are both assumed to follow inverse Gamma distributions, i.e., $p(\sigma_w^2) \sim \mathcal{IG}\left(\frac{\nu_0}{2}, \frac{\gamma_0}{2}\right)$ and $p(\sigma_v^2) \sim \mathcal{IG}\left(\frac{\nu_1}{2}, \frac{\gamma_1}{2}\right)$. The above priors are noninformative when the hyperparameters $\nu$ and $\gamma$ are set to zero. Combining these prior densities according to (12), and then simplifying yields [3] [6]

$$\pi(\boldsymbol{\theta}_{1:t}) \propto \\ \prod_{l=1}^{t} \frac{1}{(2\pi\sigma_w^2)^{M/2}} \exp\left\{\frac{-1}{2\sigma_w^2}\boldsymbol{z}_l^T \boldsymbol{P}_0^{\perp}(\boldsymbol{\tau}_l)\boldsymbol{z}_l\right\} \times \\ \prod_{l=1}^{t} \frac{1}{(2\pi\sigma_v^2)^{k/2}} \exp\left\{\frac{-1}{2\sigma_v^2}(\boldsymbol{\tau}_l - \boldsymbol{\tau}_{l-1})^T(\boldsymbol{\tau}_l - \boldsymbol{\tau}_{l-1})\right\} \times \\ \prod_{l=1}^{t} \frac{|\tilde{\boldsymbol{H}}_0^T(\boldsymbol{\tau}_l)\tilde{\boldsymbol{H}}_0(\boldsymbol{\tau}_l)|}{(2\pi\delta^2\sigma_w^2)^{k/2}} \\ \exp\left\{\frac{-1}{2\delta^2\sigma_w^2}(\boldsymbol{a}_l - \boldsymbol{m}_a(l))^T \boldsymbol{\Sigma}_0^{-1}(\boldsymbol{\tau}_l)(\boldsymbol{a}_l - \boldsymbol{m}_a(l))\right\} \\ \times \sigma_w^{2^{-(\frac{\nu_0}{2}+1)}} \exp\left\{\frac{-\gamma_0}{2\sigma_w^2}\right\} \times \sigma_v^{2^{-(\frac{\nu_1}{2}+1)}} \exp\left\{\frac{-\gamma_1}{2\sigma_v^2}\right\}, \tag{16}$$

where

$$\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{\tau}_l) = \left(1+\delta^{-2}\right)\tilde{\boldsymbol{H}}_0^T(\boldsymbol{\tau}_l)\tilde{\boldsymbol{H}}_0(\boldsymbol{\tau}_l), \tag{17}$$

$$\boldsymbol{m}_a(l) = \boldsymbol{\Sigma}_0(\boldsymbol{\tau}_l)\tilde{\boldsymbol{H}}_0^T(\boldsymbol{\tau}_l)\boldsymbol{z}_l, , \tag{18}$$

$$\boldsymbol{P}_0^{\perp}(\boldsymbol{\tau}_l) = \boldsymbol{I} - \frac{\tilde{\boldsymbol{H}}_0(\boldsymbol{\tau}_l)\left[\tilde{\boldsymbol{H}}_0^T(\boldsymbol{\tau}_l)\tilde{\boldsymbol{H}}_0(\boldsymbol{\tau}_l)\right]^{-1}\tilde{\boldsymbol{H}}_0^T(\boldsymbol{\tau}_l)}{(1+\delta^{-2})}. \tag{19}$$

From (16) and (18), a *maximum a posteriori* (MAP) esitmate of the amplitudes $\boldsymbol{a}(n)$, given all other parameters, is readily available as

$$\hat{\boldsymbol{a}}(n) \triangleq \boldsymbol{m}_a(n). \tag{20}$$

In this problem, the only parameters of interest are the $\boldsymbol{\tau}_{1:t}$, and the remaining ones can be considered as nuisance parameters and analytically integrated out. This integration on (16) yields a posterior distribution of the form

$$\boldsymbol{\pi}(\boldsymbol{\tau}_{1:t}) \propto \prod_{l=1}^{t} \frac{1}{\sigma_w^{2M}(1+\delta^2)^{k_l}} \exp\left[\frac{-1}{\sigma_w^2}\boldsymbol{z}_l^T \boldsymbol{P}_0^{\perp}(\boldsymbol{\tau}_l)\boldsymbol{z}_l\right]$$
$$\times \prod_{l=1}^{t} \frac{1}{\sigma_v^{2^{(k_l/2)}}(2\pi)^{(k_l/2)}} \exp\left[\frac{-1}{2\sigma_v^2}(\boldsymbol{\tau}_l-\boldsymbol{\tau}_{l-1})^T(\boldsymbol{\tau}_l-\boldsymbol{\tau}_{l-1})\right]$$
$$\times \kappa(\sigma_v^2,\sigma_w^2) \tag{21}$$

where $\kappa(\sigma_v^2,\sigma_w^2)$ is a function only of $\sigma_v^2$ and $\sigma_w^2$. Eq. (21) is used to estimate the $\boldsymbol{\tau}$'s using the particle filtering approach discussed below. The $\sigma_v^2$ and $\sigma_w^2$ can be estimated according to a MAP procedure outlined in [6], and the amplitudes $\boldsymbol{a}$(n) can be obtained according to (20).

## 3. SEQUENTIAL IMPORTANCE SAMPLING (SIS)

The objective of the SIS [2] [6] is to generate a numerical approximation of the desired posterior distribution $\pi(\boldsymbol{\tau}_{1:t})$, in the form of a histogram, by drawing a large number $N$ of samples, from an *importance function*, $q(\boldsymbol{\tau}_{1:t})$, whose support includes that of $\pi(\boldsymbol{\tau}_{1:t})$. It can be shown [2] that the desired posterior distribution function $\pi(\boldsymbol{\tau}_{1:t})$ can be approximated using a set of importance weights $w^{(i)}(t)$, $i = 1, ..., N$, with the following recursive update equation

$$\tilde{w}^{(i)}(t) = \tilde{w}_{t-1}^{(i)} \times \frac{p\left(\boldsymbol{z}_t|\boldsymbol{\tau}_t^{(i)}\right) p\left(\boldsymbol{\tau_t^{(i)}}|\boldsymbol{\tau_{t-1}^{(i)}}\right)}{q\left(\boldsymbol{\tau_t^{(i)}}|\boldsymbol{\tau_{1:t-1}^{(i)}},\boldsymbol{z_{1:t}}\right)}. \tag{22}$$

At each observation time $t$, particles are generated from an *optimal* importance function [2] [1], which satisfies a recurrence requirement in (22) and minimizes the variance of the weights generated by the recursion.

Let $L(\boldsymbol{\tau}_t) = L_z(\boldsymbol{\tau}_t)+L_\tau(\boldsymbol{\tau}_t)$, where $L_z(\boldsymbol{\tau}_t) = \log p\left(\boldsymbol{z}_t|\boldsymbol{\tau}_t\right)$ and $L_\tau(\boldsymbol{\tau}_t) = \log p\left(\boldsymbol{\tau}_t|\boldsymbol{\tau}_{t-1}\right)$. One choice of such optimal importance function can be given as follows [7]

$$q\left(\boldsymbol{\tau}_t^{(i)}|\boldsymbol{\tau}_{t-1}^{(i)},\boldsymbol{z}_t\right) \sim \mathcal{N}\left(\boldsymbol{m}_t,\boldsymbol{\Sigma}_t\right), \tag{23}$$

where $\boldsymbol{\Sigma}_t = -\left(\nabla^2 L(\boldsymbol{\tau}_t)\right)^{-1}$ and $\boldsymbol{m}_t = \boldsymbol{\tau}_{t-1}+\boldsymbol{\Sigma}_t\nabla L(\boldsymbol{\tau}_t)$, and $\nabla L(\boldsymbol{\tau}_t) \in \mathcal{R}^{K\times 1}$ and $\nabla^2 L(\boldsymbol{\tau}_t) \in \mathcal{R}^{K\times K}$ are the gradient and the Hessian matrix of $L(\boldsymbol{\tau}_t)$, respectively.

## 4. SIMULATION RESULTS

The proposed algorithm is now applied to a wideband scenario to demonstrate the capability for tracking the $\boldsymbol{\tau}(t)$ and restoring signal waveforms, for $K = 2$ sources, where the other parameters are listed in Table 4.1. A uniform linear array with $M = 8$ sensors is sued, and the adjacent sensors of the array are spaced by $\lambda/2$ at the highest frequency of interest. We denote the normalized sampling frequency by $F_s = 1.0$ Hz. The time-varying $\boldsymbol{\tau}(t)$ corresponding to the DOA trajectories are generated by a random walk according to (1) with $\sigma_v^2$ specified in Table 4.1, whereas the signal amplitudes are generated as Gaussian processes that are zero mean with variance $\delta^2\sigma_w^2$, and bandlimited to normalized frequency $[0.1, 0.4]$ Hz, i.e., the bandwidth is 0.3 Hz. The SNR is 13 dB, and the hyper-parameter $\delta^2 = 20$ is assumed known and constant.

| Parameter | Value |
|---|---|
| $L$ | 8 |
| $F_s$ (Hz) | 1.0 |
| $\sigma_w^2$ | 0.005 |
| $\sigma_v^2$ (sec$^2$) | $(30000F_s)^{-1}$ |

**Table 4.1**. Parameters for the experiments.

| Parameter | Mean-squared error (dB) |
|---|---|
| $\boldsymbol{\tau}(t)F_s$ | -34.54, -40.27 |
| $\boldsymbol{a}(n)$ | -18.44, -18.46 |

**Table 4.2**. Comparison between the true and estimated parameters.

A total of 10 independent trials, each consisting of 1,000 observations and using $N = 500$ particles, is used in the simulation. Figures 1-2 show the results for randomly selected trials. The proposed algorithm randomly initializes all unknown parameters. According to Figure 1, it is clear that the times-of-arrival (TOAs) $\boldsymbol{\tau}(t)$ are well tracked by their estimates throughout the entire tracking process. Moreover, Figure 2 shows that the signal amplitudes are well separated and restored by the algorithm. Note that only samples for $n \in [300, 500]$ are displayed for clear comparison. Table 4.2 lists the mean-squared errors of the estimates of $\boldsymbol{\tau}(t)$ and $\boldsymbol{a}(n)$ over the 10 trials.

In addition, the proposed algorithm was applied to 50 different scenarios each of 500 observations, for different values of SNR, in order to evaluate the performance of the algorithm, as shown in Figure 3.

## 5. CONCLUSIONS

A novel approach for wideband array signal processing is proposed. The proposed method would be useful for smart antenna applications where CDMA is used. A Bayesian approach is adopted, where a posterior density function which has the nuisance parameters integrated out is formulated.

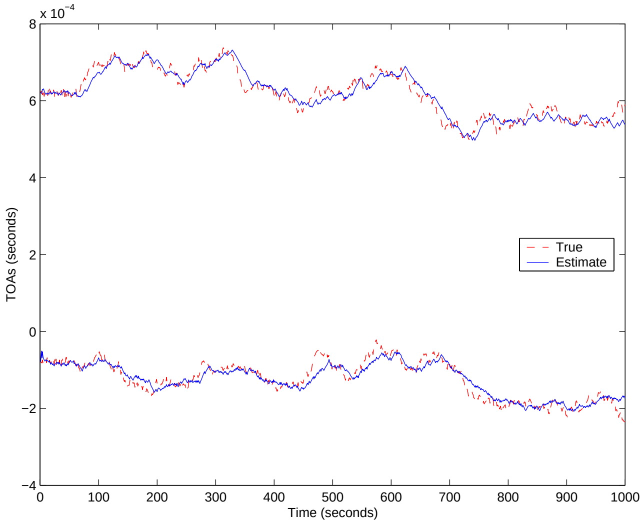


**Fig. 1**. Sequential estimates of the times-of-arrival, $\boldsymbol{\tau}(t)$.

Using sequential MCMC techniques, the TOAs of wideband sources can be tracked, and the amplitudes are separated and restored. Simulation results support the effectiveness of the method, and demonstrate reliable tracking of the times of arrival of wideband sources in a white noise environment with a uniform linear array of sensors.

## 6. REFERENCES

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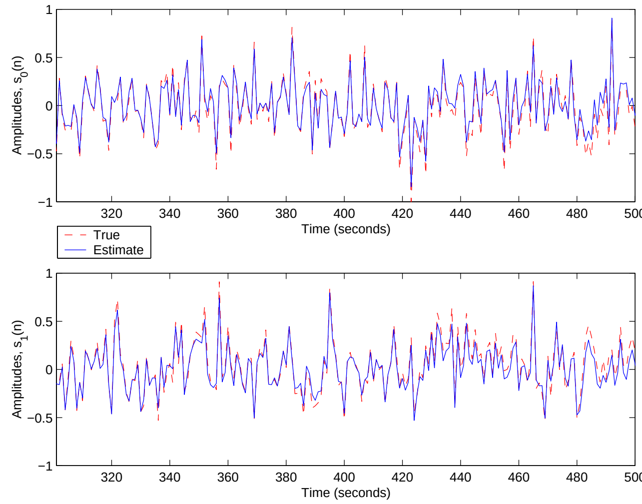


**Fig. 2**. Sequential estimates of the signal amplitudes, $\boldsymbol{a}(n)$.

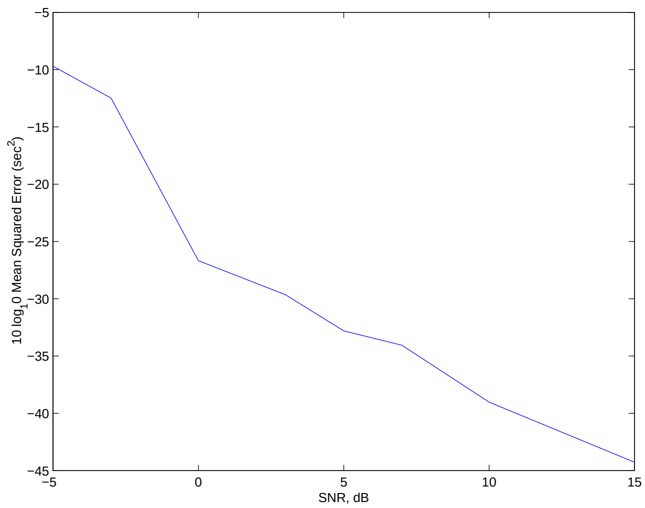


**Fig. 3**. Variances of the estimates vs. SNR.

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