APPLICATION OF THE REVERSIBLE JUMP MARKOV CHAIN MONTE CARLO METHOD TO REAL-LIFE PROPAGATION MEASUREMENTS

Jean-René Larocque and James P. Reilly

Communications Research Laboratory, McMaster University
Hamilton, Ontario, Canada
larocqjr@mcmaster.ca

ABSTRACT

This paper shows the application of the Reversible Jump Markov Chain Monte Carlo (RJMCMC)[1] method to a joint detection-estimation angle-of-arrival (AOA) problem, with simulations and processing of real-life propagation measurements. A minimum number of assumptions are made as the number of sources impinging the array, their AOA and amplitude, as well as the noise variance, are considered random variables. The algorithm jointly estimates all these parameters by sampling the posterior distribution that is made up of the union of disjoint subspaces of different dimensions.

We will apply the RJMCMC technique to simulations and to real data to jointly detect and estimate the parameters and the noise characteristics. We also present briefly an approach, using space projection, which avoids the manipulation of the nuisance parameters.

1. INTRODUCTION

The narrow-band estimation of angles of arrival for an antenna array is a well known problem. However, the most common algorithms make major assumptions in order to get good performance: known number of sources, known noise covariance matrix, etc.

We demonstrate in this paper how it is possible to jointly detect and estimate the number of sources and their angles of arrival with a minimum number of assumptions by applying the RJMCMC to real-life propagation measurements collected at 1.92 GHz. The number of sources impinging on the array, their AOA and amplitudes, as well as the noise variance, are considered random variables. The amplitude or other nuisance parameters can be integrated out numerically. Even though it would be possible to estimate the amplitudes, the analytical integration of these nuisance parameters is quite straight forward, so, we choose to do so to make the algorithm more efficient [2], but we estimate the noise variance.

The received signal, for simulations, is composed of N independent snapshots of the output of a circular antenna array of radius $0.102m$, made of $M = 8$ elements spaced uniformly. There are K sources impinging. This is the actual configuration of the system used for the measurements, as described later in the paper. The objective is to estimate the number of sources and their AOA by the estimating the maximum of the posterior distribution (aka MAP estimate):

$$\{\hat{k}, \hat{\phi_k}\} = \arg\max_{k,\phi_k} p(k, \phi_k|\mathbf{y}) \tag{1}$$

The joint detection-estimation process can be quite involved as the function to be optimized becomes quickly non-linear in the parameters. However, using sampling techniques (MCMC), estimation of the angles of arrival can easily be computed, as the maximum of the histogram of the samples. The advantages of the MCMC method are that the algorithm will converge to the global maximum with probability one and that it allows for easy numerical integration of the nuisance parameters. Since the samples are available, it is also easy to compute confidence intervals for the estimates. The objective is to construct a Markov Chain that has precisely that posterior distribution as its limiting distribution. Therefore, a sample from the chain will be a sample from the posterior distribution of interest.

2. DEFINITION OF THE PROBABILITY DISTRIBUTION FUNCTIONS

In this section, we will present the development of the algorithm and of the probability distribution functions for two noise models: 1) white noise with unknown variance, and, 2) coloured noise with unknown covariance matrix.

2.1. Algorithm

In the Metropolis-Hasting (M-H) sampling scheme, the samples from the posterior distribution are proposed by a candidate function $q(\cdot)$ and are accepted based on the acceptance function $r(\phi_{k+1}, (k+1), \phi_k, k)$, defined as:

$$\frac{p(\phi_{k+1}, (k+1)|\mathbf{y})}{p(\phi_k, k|\mathbf{y})} \frac{q(\phi_k, k|\phi_{k+1}, (k+1))}{q(\phi_{k+1}, (k+1)|\phi_k, k)} \tag{2}$$

At each iteration, the algorithm chooses randomly one of three moves: update, birth or death of a source. The randomly selected move then proposes a new state for the Markov chain, a state that will be accepted or refused based on the acceptance function. The acceptance functions for the three moves will be defined in the following section. After a burn-in period, the samples produced by the algorithm are distributed according to (12), the posterior distribution. For more details on the M-H algorithm, we refer the reader to [1].

2.2. First case: White noise with unknown variance

The output of the array at the sampling time n is:

$$\mathbf{y}_n = \mathbf{S}(\phi_k)\mathbf{a} + \mathbf{w}_n \quad n = 1, 2, \ldots, N \tag{3}$$

with $\phi_k \in \mathscr{R}^{k\times 1}$ being the vector of the angles of arrival, the subscript k indicating the dimension, and $\mathbf{S}(\phi_k) \in \mathscr{C}^{M\times k}$ is the steering matrix of the circular antenna array. In the usual way, the likelihood function of the parameters over the N i.i.d. snapshots is:

$$p(\mathbf{y}|\phi_k, \mathbf{a}, \sigma_w^2, k) = \frac{e^{-\frac{1}{\sigma_w^2}\sum_{n=1}^{N}(\mathbf{y}_n - \mathbf{S}(\phi_k)\mathbf{a})^H(\mathbf{y}_n - \mathbf{S}(\phi_k)\mathbf{a})}}{\pi^{MN}\sigma_w^{2MN}} \tag{4}$$

We use the the following priors for the parameters:

$$p(\mathbf{a}|\phi_k, k, \sigma_w^2) = \prod_{n=1}^{N} N(\mathbf{0}, \sigma_w^2\Sigma_a) \tag{5}$$

$$p(\phi_k|k, \sigma_w^2) = U[-\pi, \pi]^k \tag{6}$$

$$p(k|\sigma_w^2) = \Lambda^k e^{-\Lambda}/k! \tag{7}$$

$$p(\sigma_w^2) = IG(\nu_o, \gamma_o) \tag{8}$$

with $\Sigma_a^{-1} = \frac{S(\phi_k)^H S(\phi_k)}{d^2}$ where d^2 is the expected SNR [2] and Λ the expected number of sources. Thus, using Bayes' rule, the expression for the posterior distribution, after simplifications, is as follows:

$$\begin{aligned}
p(k,\phi_k, \mathbf{a}, \sigma_w^2|\mathbf{y}) &\propto e^{\frac{-1}{\sigma_w^2}\sum_{n=1}^{N}\mathbf{y}_n'\mathbf{S}(\phi_k)M'\mathbf{S}^H(\phi_k)\mathbf{y}_n} \\
&\times e^{\frac{-1}{\sigma_w^2}\sum_{n=1}^{N}(\mathbf{a} - M\mathbf{S}^H(\phi_k)\mathbf{y}_n)'M^{-1}(\mathbf{a} - M\mathbf{S}^H(\phi_k)\mathbf{y}_n)} \\
&\times \frac{1}{(2\pi)^{\frac{kN}{2}}\sigma_w^{kN}|\Sigma_a|^{N/2}(2\pi)^k k!} \\
&\times \Lambda^k(\sigma_w^2)^{-\nu_0-1-NM} e^{\frac{1}{\sigma_w^2}(\gamma_0 + \sum_{n=1}^{N}\mathbf{y}_n^H P_\perp \mathbf{y}_n)}
\end{aligned} \tag{9}$$

with [2]

$$P_\perp = \mathbf{I} - \mathbf{S}(\phi_k)M'\mathbf{S}'(\phi_k) \tag{10}$$

$$M^{-1} = \mathbf{S}^H(\phi_k)\mathbf{S}(\phi_k) + \Sigma_a^{-1} \tag{11}$$

The nuisance parameters $\mathbf{a}$ and σ_w^2 can be easily integrated out analytically, one at the time, as the second part of (9) is the normal distribution of the parameter $\mathbf{a}$ and the last block is the Inverted Gamma distribution of σ_w^2. To estimate the noise variance, we can easily sample the Inverted Gamma distribution. The posterior distribution of interest then simplifies to:

$$\begin{aligned}
p(k, \phi_k|\mathbf{y}) &\propto (\gamma_0 + \sum_{n=1}^{N}\mathbf{y}_n^H P_\perp \mathbf{y}_n)^{-\nu_o - MN} \\
&\times \frac{\Lambda^k}{(2\pi)^k k!(1+d^2)^{kN/2}}
\end{aligned} \tag{12}$$

For the simulations, we chose an uninformative prior for σ_w^2 by using $\nu_0 = 0$ and $\gamma_0 = 0$.The only parameters that need to be defined in order to implement the sampling scheme are the expected signal to noise ratio (d^2) and the expected number of sources (Λ), which can easily be estimated before hand [2].

2.2.1. Definition of the moves

When the update move is selected, the algorithm samples the same size posterior distribution, just as a regular Metropolis-Hasting sampling scheme. Using the uniform distribution over $[-\pi, \pi]^k$ as the proposal distribution for the angles of arrival $\phi_k^\star$, the acceptance function reduces to:

$$r_{update}(\phi_k^\star, \phi_k) = \left(\frac{\sum_{n=1}^{N}\mathbf{y}_n^H P_\perp^\star \mathbf{y}_n}{\sum_{n=1}^{N}\mathbf{y}_n^H P_\perp \mathbf{y}_n}\right)^{-MN} \tag{13}$$

Similarly, when the algorithm chooses to explore the posterior with a larger or smaller dimension through a birth or death move for the addition or the suppression of an estimated source, the proposal distributions are:

$$q(\phi_{k+1}, (k+1)|\phi_k, k) \propto \frac{\Lambda^{k+1}}{(k+1)!}\frac{1}{2\pi} \tag{14}$$

$$q(\phi_k, k|\phi_{k+1}, (k+1)) \propto \frac{\Lambda^k}{k!}\frac{1}{(k+1)} \tag{15}$$

Thus, the acceptance function of the birth move is:

$$r_{birth} = \left(\frac{\sum_{n=1}^{N}\mathbf{y}_n^H P_\perp^\star \mathbf{y}_n}{\sum_{n=1}^{N}\mathbf{y}_n^H P_\perp \mathbf{y}_n}\right)^{-MN}(k+1)^{-1}(1+d^2)^{-N/2} \tag{16}$$

and the acceptance ratio of the death move is simply $1/r_{birth}$.

2.3. Second case: Description of the sampling scheme for coloured noise

In this section, we present the development of a reversible jump MCMC scheme for the coloured noise case:

$$\mathbf{w}_n \sim N(\mathbf{0}, \Sigma_w) \tag{17}$$

This is an attempt to remove any assumptions on the noise characteristics as well as an attempt to develop an algorithm that is not a function of the amplitudes. The projector P onto the noise subspace, knowing the number of sources, is:

$$\mathbf{P}(\phi_k|k) = \mathbf{I} - \mathbf{S}(\phi_k)[\mathbf{S}'(\phi_k)\mathbf{S}(\phi_k)]^{-1}\mathbf{S}'(\phi_k) \tag{18}$$

As described in more detail in [3], the projection matrix $\mathbf{P}(\phi_k|k)$ can be written as:

$$\mathbf{P}(\phi_k|k) = \mathbf{U}_v'(\phi_k|k)\mathbf{U}_v(\phi_k|k), \tag{19}$$

where $\mathbf{U}_v(\phi_k|k)$ is a $M\times(M-k)$ matrix forming an orthogonal basis of the range subspace of $\mathbf{P}(\phi_k|k)$. Let the projected data $\mathbf{z}_n$ be:

$$\mathbf{z}_n = \mathbf{U}_v'(\phi_k|k)\mathbf{y}_n \tag{20}$$

As the coloured noise is Gaussian and independent from snapshot to snapshot, the projected data will also be Gaussian such that the likelihood function will be:

$$\begin{aligned}
p(\mathbf{z}|\phi_k, \mathbf{W}^{-1}, k) &= \pi^{-N(M-k)}|\mathbf{W}^{-1}|^N e^{-\sum_{n=1}^{N}\mathbf{z}_n^H\mathbf{W}\mathbf{z}_n} \\
&= \frac{\pi^{kN}}{\pi^{MN}}|\mathbf{W}^{-1}|^N e^{-tr(\mathbf{W}^{-1}(N\hat{\mathbf{W}}))}
\end{aligned} \tag{21}$$

where the projected expected covariance matrix $\hat{\mathbf{W}}$ is defined as:

$$\hat{\mathbf{W}} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{z}_n\mathbf{z}_n' \tag{22}$$

Once more, using Bayes' theorem:

$$\begin{aligned} p(\phi_k, \mathbf{W}^{-1}, k|\mathbf{z}) \propto\, & p(\mathbf{z}|\phi_k, \mathbf{W}^{-1}, k) \\ & \times p(\mathbf{W}^{-1}|\phi_k, k)p(\phi_k|k)p(k) \end{aligned} \tag{23}$$

The conditional prior on the projected covariance matrix $\mathbf{W}^{-1}$ is the Jeffrey's prior, $|\mathbf{W}^{-1}|^{-(M-k)}$. The conditional prior on the vector ϕ_k is again the uniform distribution of dimension k and the prior for the model order is a Poisson distribution with parameter Λ. The complete posterior distribution can be written as:

$$\begin{aligned} p(\phi_k, \mathbf{W}^{-1}, k|\mathbf{z}) \propto & |\mathbf{W}^{-1}|^{N-M+k} e^{-tr(\mathbf{W}^{-1}(N\hat{\mathbf{W}}))} \\ & \times \pi^{kN-MN}\frac{\Lambda^k}{(2\pi)^k k!} \end{aligned} \tag{24}$$

The first term can be recognised as the complex Wishart distribution and the nuisance parameter $\mathbf{W}^{-1}$ can be integrated out analytically, as we will not estimate it, to produce a more efficient sampling scheme. The posterior distribution can be simplified to:

$$\begin{aligned} p(\phi_k, k|\mathbf{z}) \propto & \pi^{\frac{k}{2}(k-2M+2N+1)}\frac{|N\hat{\mathbf{W}}|^{-N}\Lambda^k}{(2\pi)^k k!} \\ & \times \prod_{i=1}^{M-k}, (N-i+1) \end{aligned} \tag{25}$$

with

$$\hat{\mathbf{W}} = \mathbf{U}_v(\phi_k|k)'\mathbf{R}_{yy}\mathbf{U}_v(\phi_k|k) \tag{26}$$

and

$$\mathbf{R}_{yy} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{y}_n\mathbf{y}_n' \tag{27}$$

This posterior distribution is now sampled by the RJMCMC algorithm, using the following acceptance functions:

$$r_{update} = \frac{|\mathbf{U}_v(\phi_k^\star|k)'R_{xx}\mathbf{U}_v(\phi_k^\star|k)|^{-N}}{|\mathbf{U}_v(\phi_k|k)'R_{xx}\mathbf{U}_v(\phi_k|k)|^{-N}} \tag{28}$$

$$\begin{aligned} r_{birth} = & \frac{\pi^{k+N-M+1}}{(k+1),\,(N-M+k+1)} \\ & \times \frac{|N\mathbf{U}_v(\phi_k^\star|k)'R_{xx}\mathbf{U}_v(\phi_k^\star|k)|^{-N}}{|N\mathbf{U}_v(\phi_k|k)'R_{xx}\mathbf{U}_v(\phi_k|k)|^{-N}} \end{aligned} \tag{29}$$

Again, the acceptance function of the death move is the inverse of the acceptance function of the birth move.

3. SIMULATION RESULTS

In this section, we apply the two algorithms of section 2 to simulated data to show their performance. Again, no assumption on the number of sources was made, except that we fix the maximum value of the estimate to k_{max}, for computational purposes only. The marginal posterior distribution of the estimated number of sources and the marginal posterior distribution of the estimated angles of arrival are presented in figures 1 and 2 for 75000 iterations (with a burn-in period of 7000) of the Reversible Jump Sampler with 40 observations of the array output, when the SNR is 6 dB and two sources are coming from -5° and 45°. The estimated SNR from the marginal distribution is 4.95dB.

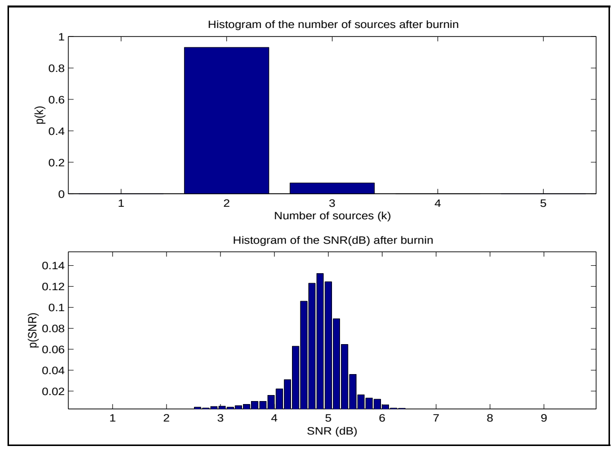

Figure 1: Noise variance Unknown: Histogram of the estimated number of sources (top); Histogram of the estimated SNR (bottom)

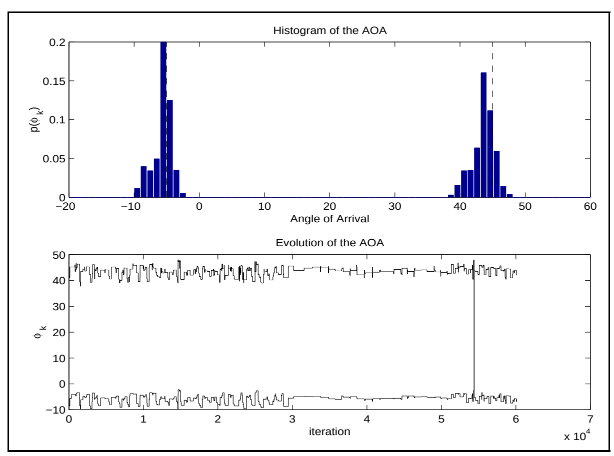

Figure 2: Noise variance Unknown: Histogram of the estimated AOA (top); Instantaneous estimate (bottom).

The slight deviation from the true values would dissapear by increasig the number of iterations of the M-H algorithm.

Figure 3 shows the marginal posterior distribution of the estimated angles-of-arrival resulting from the application of the RJMCMC in the case of coloured noise, when the number of sources is assumed known. Unfortunately, at the time of publication, results of joint detection and estimation were not available.

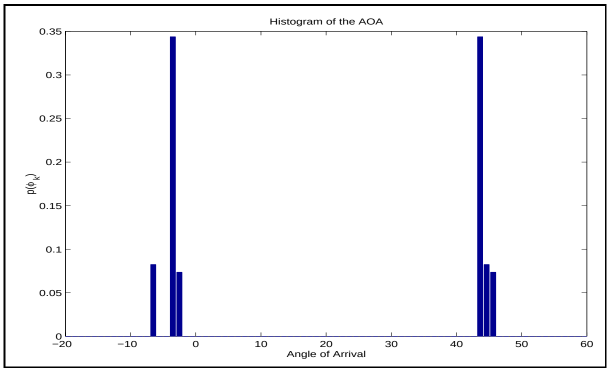

Figure 3: Coloured noise: Histogram of the AOA, with k estimated beforehand.

4. APPLICATION TO REAL-LIFE PROBLEM

In this section, we apply the Reversible Jump MCMC sampling scheme to the real-life outdoor propagation measurements.

The receiving base station is a circular antenna array made of 8 mono-pole antennas. The transmitted signal is a 512 PN sequence at 5 MHz. The received signal of each element is I-Q demodulated, converted to baseband, sampled at 10 MHz, and then stored for further processing. The measurements were conducted on the McMaster University campus, with the receiving base station at different locations and different heights in a pico-cell scenario that offered rich multipath characteristics with severe fading. To demonstrate the robustness of the discussed methods, we apply the method developed in section 2.2 to a typical data set data collected on campus. We collected 60 snapshots, from a position where the multipath characteristics have been frequently observed to be 2 rays well separated in angle. Two rays impinge the array with approximated angles of arrival of $30°$ and $-80°$. The expected signal to noise ratio is at least 20dB, but the noise is highly correlated as there is a lot of interference. This data set is adequate to test the robustness of the method.

Figure 4 shows a typical results for 75000 iterations of the Reversible Jump Sampler with 60 observations of the array output of the first leg of the ground scenario. The final objective is to develop a robust method to determine the position of a user, in angle and range. This would provide a technique to extract a propagation model. We also have an extensive data bank of wide-band measurements that will be used to validate the algorithms.

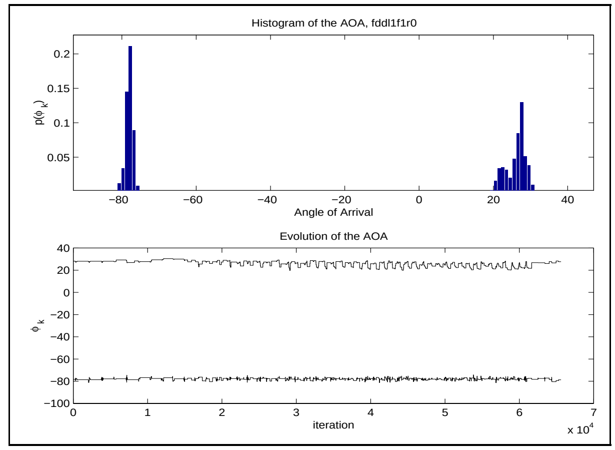

Figure 4: Measurements: Histogram of the AOA (top); Instantaneous estimate (bottom) (after the burn-in period)

5. DISCUSSION AND CONCLUSION

The choice of prior for the model order has no impact on the final convergence of the algorithm but might influence the rate of convergence, as the parameter Λ influences the choice of moves, but not the acceptance probability.

As it can be difficult or unnecessary to estimate some parameters from the measurements, the integration of the nuisance parameters was performed to give a more efficient algorithm. In one case, however, to demonstrate the effectiveness of the method, we have estimated the noise variance from the posterior distribution.

The RJMCMC offers a powerful tool to jointly estimate and detect the number of sources and their angles of arrival. It provides the global maximum with probabilty one and allows for easy integration of the nuisance parameters. Using the samples, one can easily compute confidence intervals. The application of the method to real-life outdoor propagation measurements confirms the effectiveness of the method. The next step will be the development of the projection approach for coloured noise for wide-band signals, as it requires fewer iterations, for joint detection-estimation giving us a powerfull tool for the development of a propagation model.

REFERENCES

[1] P. Green, "Reversible jump Markov Chain Monte Carlo computation and Bayesian model determination," *Biometrika*, vol. 82, no. 4, pp. 711–732, 1995.

[2] C. Andrieu and A. Doucet, "Joint Bayesian model selection and estimation of noisy sinusoids via reversible jump MCMC," *IEEE Transaction on Signal Processing*, vol. 47, p. 2667, Oct. 1999.

[3] M. Wong, J. P. Reilly, Q. Wu, and S. Qiao, "Estimation of the directions of arrival of signals in unknow collelated noise, part 1: The MAP approach and its implementation," *IEEE Transaction of Signal Processing*, vol. 40, pp. 2007–2017, August 1992.