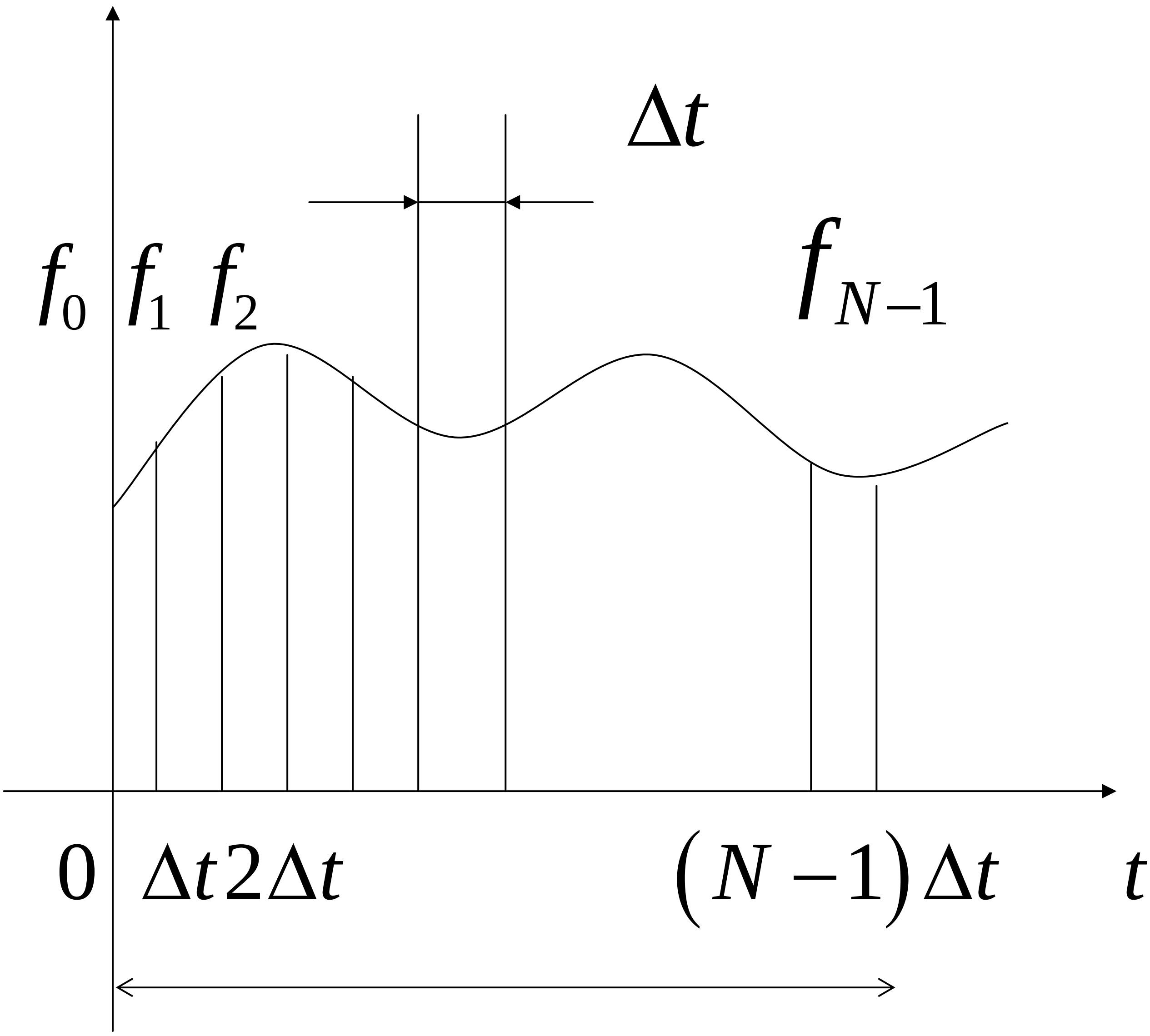
２．３　離散フーリエ変換（Ｄｉｓｃｒｅｔｅ　Ｆｏｕｒｉｅｒ　Ｔｒａｎｓｆｏｒｍ，　ＤＦＴ）

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt}$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\omega_0 kt} dt$$

$$= \frac{1}{T}\int_{0}^{T} f(t) e^{-j\omega_0 kt} dt$$

$$\omega_0 = 2\pi / T$$



$$N\Delta t = T$$

$$\left\{ f_0, f_1, f_2 \bullet\bullet\bullet f_{N-1} \right\}$$

$$\Delta\omega = 2\pi / N$$

$$\Delta\omega = 2\pi \Big/ \left(\frac{T}{\Delta t} \right) = \frac{2\pi}{T} \cdot \Delta t$$

$$e_k = \left(1, e^{jk\Delta\omega}, e^{jk\cdot 2\Delta\omega}, e^{jk\cdot 3\Delta\omega}, \right.$$

$$\left. \cdot\ \cdot\ \cdot, e^{jk(N-1)\Delta\omega} \right)$$

$$f = (f_0, f_1, \cdot\ \cdot\ \cdot\ \cdot, f_{N-1})$$

［定理］

$$\{ e_0, e_1, e_2, \cdots, e_{N-1} \}$$

は正規直交系である.

[証明]

$$\langle e_m, e_n \rangle$$

$$= \left[1, e^{jm\Delta\omega}, e^{jm2\Delta\omega}, \right.$$

$$\left. \bullet\bullet\bullet, e^{jm(N-1)\Delta\omega} \right]$$

$$\overline{\begin{bmatrix} 1 \\ e^{jm\Delta\omega} \\ e^{jm\cdot 2\Delta\omega} \\ \cdot\ \cdot\ \cdot \\ e^{jm(N-1)\Delta\omega} \end{bmatrix}} \times \frac{1}{N}$$

$$= \left(1 + e^{jm\Delta\omega} e^{\overline{jn\Delta\omega}}\right.$$

$$+ e^{jm2\Delta\omega} e^{\overline{jn2\Delta\omega}} + \bullet\bullet\bullet$$

$$\left. + e^{jm(N-1)\Delta\omega} e^{\overline{jn(N-1)\Delta\omega}}\right) \times \frac{1}{N}$$

$$\left(\begin{array}{l} \langle f(t), g(t) \rangle \\ = \frac{1}{b-a} \int_a^b f(t) \overline{g(t)} dt \end{array}\right)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{jm \cdot \frac{2\pi}{N} k} e^{-jn \frac{2\pi}{N} k}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{j(m-n) \frac{2\pi}{N} k}$$

$$\sum_{k=0}^{N-1} ar^k = \frac{a\left(1 - r^N\right)}{1 - r}$$

$$= \frac{1}{N}\frac{1-e^{j(m-n)\frac{2\pi}{N}\cdot N}}{1-e^{j(m-n)2\pi/N}} = 0 \quad (m \neq n)$$

$m = n$の時は

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{0} = \frac{1}{N}\bullet N = 1$$

$$\therefore \langle e_m, e_n \rangle = \delta_{mn}$$

$$f = \sum_{k=0}^{N-1} c_k e_k$$

$$\langle f, e_m \rangle = \left\langle \sum_{k=0}^{N-1} c_k e_k, e_m \right\rangle = c_m$$

$$\therefore c_k = \langle f, e_k \rangle = [f_0, f_1, \bullet\bullet, f_{N-1}] \begin{bmatrix} \overline{1} \\ \overline{e^{jk\cdot\frac{2\pi}{N}}} \\ \overline{e^{jk\cdot\frac{2\pi}{N}\cdot 2}} \\ \bullet\bullet\bullet \\ \overline{e^{jk\cdot\frac{2\pi}{N}\cdot(N-1)}} \end{bmatrix} \times \frac{1}{N}$$

$$= \frac{1}{N}\left(f_0 \cdot 1 + f_1 e^{-jk\cdot\frac{2\pi}{N}} + f_2 e^{-jk\cdot\frac{2\pi}{N}\cdot 2}\right.$$

$$\left. + \bullet\bullet\bullet + f_{N-1} e^{-jk\cdot\frac{2\pi}{N}(N-1)}\right.$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} f_m e^{-jk\left(\frac{2\pi}{N}\right)\cdot m}$$

DFT

$$f = \sum_{k=0}^{N-1} c_k e_k \quad ,$$

$$e_k = \left[1, e^{jk \cdot \frac{2\pi}{N}}, e^{jk \cdot \frac{2\pi}{N} 2}, \right.$$

$$\left. \bullet\bullet\bullet, e^{jk \frac{2\pi}{N} (N-1)} \right]$$

$$c_k = \frac{1}{N} \sum_{m=0}^{N-1} f_m e^{-jk\left(\frac{2\pi}{N}\right) \bullet m}$$

離散フーリエ逆変換（Ｉｎｖｅｒｓｅ Ｄｉｓｃｒｅｔｅ Ｆｏｕｒｉｅｒ Ｔｒａｎｓｆｏｒｍ ＩＤＦＴ）

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \bullet\bullet\bullet \\ f_{N-1} \end{bmatrix} = \sum_{k=0}^{N-1} c_k \begin{bmatrix} 1 \\ e^{jk\cdot\frac{2\pi}{N}\cdot 1} \\ e^{jk\cdot\frac{2\pi}{N}\cdot 2} \\ \bullet\bullet\bullet \\ e^{jk\left(\frac{2\pi}{N}\right)(N-1)} \end{bmatrix}$$

$$= c_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ e^{j \cdot \frac{2\pi}{N} \cdot 1} \\ e^{j \cdot \frac{2\pi}{N} \cdot 2} \\ \cdots \\ e^{j\left(\frac{2\pi}{N}\right)(N-1)} \end{bmatrix}$$

$$+c_2\begin{bmatrix} 1 \\ e^{j\cdot 2\cdot\frac{2\pi}{N}\cdot 1} \\ e^{j\cdot 2\cdot\frac{2\pi}{N}\cdot 2} \\ \bullet\bullet\bullet \\ e^{j2\left(\frac{2\pi}{N}\right)(N-1)} \end{bmatrix}+\bullet\bullet$$

$$+c_{N-1}\begin{bmatrix} 1 \\ e^{j\cdot(N-1)\frac{2\pi}{N}\cdot 1} \\ e^{j\cdot(N-1)\frac{2\pi}{N}\cdot 2} \\ \cdots \\ e^{j\cdot(N-1)\left(\frac{2\pi}{N}\right)(N-1)} \end{bmatrix}$$

$$f_m = c_0 + c_1 e^{j\cdot 1\left(\frac{2\pi}{N}\right)m}$$

$$+ c_2 e^{j2\cdot\left(\frac{2\pi}{N}\right)m} +$$

$$\bullet\bullet\bullet + c_{N-1} e^{j(N-1)\left(\frac{2\pi}{N}\right)m}$$

$$= \sum_{k=0}^{N-1} c_k e^{jk\left(\frac{2\pi}{N}\right)m}$$

IDFT

$$\boxed{f_m = \sum_{k=0}^{N-1} c_k e^{jk\left(\frac{2\pi}{N}\right)m}}$$