3　ラプラス変換(Laplace Transform)

3.1 ラプラス変換の定義

$f(t) : t \geq 0$で定義された時間関数

（$t < 0$では$f(t) = 0$とする）

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

を $f(t)$のラプラス変換という. s は実数.

$$F(s) = L\left[f(t) \right], \quad f(t) \leftrightarrow F(s)$$

と書く場合もある.

3.2 ラプラス変換の計算例

（１）

$f(t) = e^{-at}$ のラプラス変換

（解）

$$F(s) = \int_0^\infty e^{-at} \cdot e^{-st} dt$$

$$= \int_0^\infty e^{-(a+s)t} dt \qquad (a+s>0)$$

$$= \left[\frac{1}{-(a+s)} \cdot e^{-(a+s)t} \right]_0^\infty$$

$$= 0 + \frac{1}{a+s} e^0$$

$$= \frac{1}{s+a}$$

$$\therefore\ e^{-at} \leftrightarrow \frac{1}{s+a}$$

公式 $: e^{a} \cdot e^{b} = e^{a+b}$

$$e^{0} = 1\ ,\ e^{-\infty} = 0$$

$$\frac{de^{at}}{dt} = ae^{at}$$

$$\int e^{at}dt = \frac{e^{at}}{a} + c$$

（２）

$f\left(t\right)=a$のラプラス変換

(解)

$$F\left(s\right)=\int_0^{\infty} ae^{-st}dt=a\int_0^{\infty} e^{-st}dt$$

$$=a\left[\frac{1}{-s}e^{-st}\right]_0^{\infty}$$

$$=a\left\{\left(\frac{1}{-s}e^{-\infty}\right)-\left(\frac{1}{-s}e^{0}\right)\right\}$$

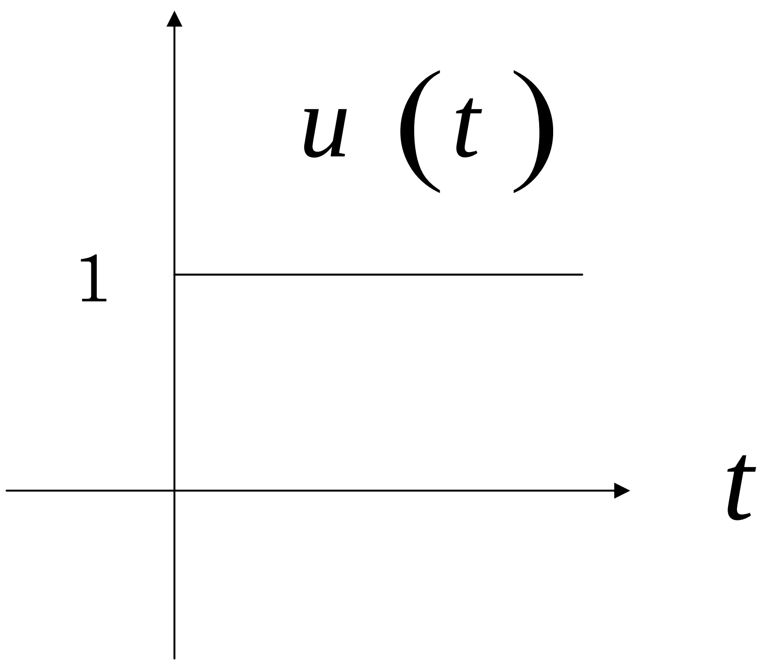
$$=a\cdot\frac{1}{s}$$

$$\therefore a \leftrightarrow \frac{a}{s}$$

$$f(t) = u(t) = \begin{cases} 1 \ (t \geq 0) \\ 0 \ (t < 0) \end{cases}$$

（単位ステップ関数）

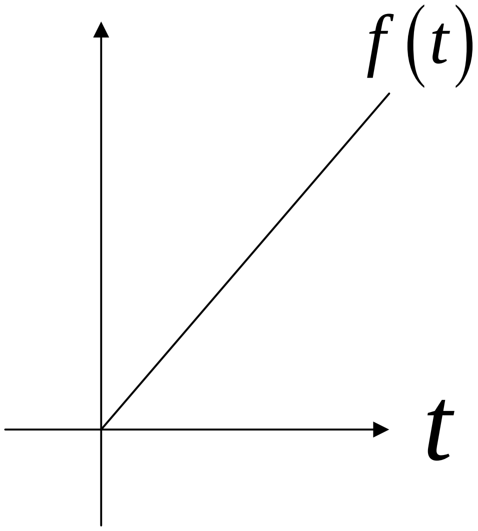
$$u(t) \leftrightarrow \frac{1}{s}$$



(3)

$$f(t)=t$$

$$F(s)=\int_0^\infty t\cdot e^{-st}dt$$



単位ランプ関数

公式：

$$\int_0^t f(t)g'(t)dt$$

$$=\left[f(t)g(t)\right]_0^t-\int_0^t f'(t)g(t)dt$$

（部分積分の公式）

$$F(s) = \int_0^\infty t \cdot e^{-st} dt$$

$$= \left[t \cdot \frac{1}{-s} e^{-st} \right]_0^\infty - \int_0^\infty \frac{1}{-s} e^{-st} dt$$

$$= \frac{1}{s} \int_0^\infty e^{-st} dt$$

$$= \frac{1}{s} \left[\frac{1}{-s} e^{-st} \right]_0^\infty$$

$$= \frac{1}{s} \cdot \frac{1}{s}$$

$$= \frac{1}{s^2}$$

$$\therefore t \leftrightarrow \frac{1}{s^2} \qquad \left(t^n \leftrightarrow \frac{n!}{s^{n+1}}\right)$$

(4)

$$f(t) = \sin \omega t$$

$$F(s) = \int_0^{\infty} \sin \omega t \cdot e^{-st} dt$$

$$= \left[\sin \omega t \cdot \frac{1}{-s} e^{-st} \right]_0^{\infty} - \int_0^{\infty} \omega \cos \omega t \cdot \frac{1}{-s} e^{-st} dt$$

$$= \frac{\omega}{s} \int_0^{\infty} \cos \omega t \cdot e^{-st} dt$$

$$= \frac{\omega}{s} \cdot \left\{ \left[\cos \omega t \cdot \frac{1}{-s} e^{-st} \right]_0^{\infty} - \int_0^{\infty} -\omega \sin \omega t \cdot \frac{1}{-s} e^{-st} dt \right\}$$

$$= \frac{\omega}{s} \cdot \left\{ \frac{1}{s} - \frac{\omega}{s} \int_0^{\infty} \sin \omega t \cdot e^{-st} dt \right\}$$

$$= \frac{\omega}{s} \left\{ \frac{1}{s} - \frac{\omega}{s} \cdot F(s) \right\}$$

$$= \frac{\omega - \omega^2 F(s)}{s^2}$$

$$\therefore F(s) = \frac{\omega - \omega^2 F(s)}{s^2}$$

$$s^2 F(s) + \omega^2 F(s) = \omega$$

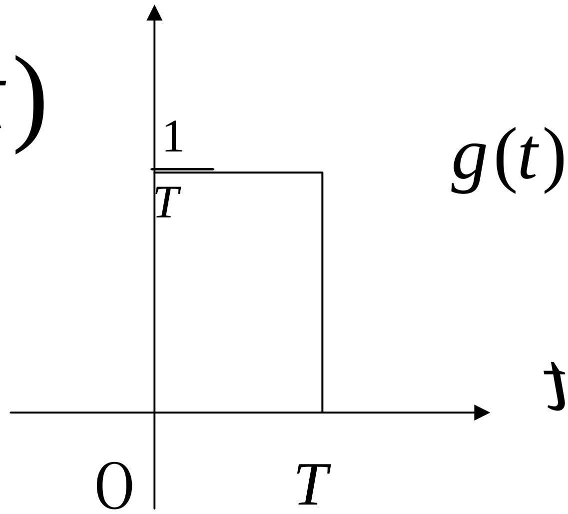
$$\therefore F(s) = \frac{\omega}{s^2 + \omega^2}$$

$$\therefore \sin \omega t \leftrightarrow \frac{\omega}{s^2 + \omega^2}$$

（5）

$$f\left(t\right)=\delta(t)$$ (デルタ関数)

$$\delta(t)=\lim_{T\to 0} g(t)$$



$$\int_{-\infty}^{\infty} g(t)dt=1$$

$$F(s)=\int_0^{\infty} f(t)e^{-st}dt=\int_0^{T}\frac{1}{T}e^{-st}dt$$

$$=\frac{1}{T}\left[\frac{1}{-s}e^{-st}\right]_0^T=\frac{1}{-st}\left(e^{-sT}-1\right)$$

$$=\frac{1-e^{-sT}}{st}$$

$$L\left[\delta(t)\right]=L\left[\lim_{T\to 0}g(t)\right]$$

$$=\lim_{T\to 0}L\left[g(t)\right]$$

$$=\lim_{T\to 0}\frac{1-e^{-sT}}{sT}$$

$$e^{x}=1+\frac{x}{1!}+\frac{x^{2}}{2!}+\cdots\frac{x^{n}}{n!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{x^{n}}{n!}$$

$$\lim_{T\to 0}\frac{1-e^{-st}}{sT}=\lim_{T\to 0}\frac{1-\left(1-\frac{sT}{1!}+\frac{(sT)^{2}}{2!}-\frac{(sT)^{3}}{3!}+\cdots\right)}{sT}$$

$$=\lim_{T\to 0}\left\{1-\frac{sT}{2!}+\frac{(sT)^{2}}{3!}-\cdots\right\}$$

$$=1$$

$$\therefore \delta(t)\leftrightarrow 1$$

演習

(1) $e^{at} \leftrightarrow \frac{1}{s-a}$

(2) $\cos\omega t \leftrightarrow \frac{s}{s^2+\omega^2}$

3.3 ラプラス変換の公式

（１）線形性

$$L[c_1 f_1(t)+c_2 f_2(t)]=c_1 L[f_1(t)]+c_2 L[f_2(t)]$$
$$=c_1 F_1(s)+c_2 F_2(s)$$

［証明］

$$L[c_1 f_1(t)+c_2 f_2(t)]=\int_0^{\infty}(c_1 f_1(t)+c_2 f_2(t))e^{-st}dt$$
$$=c_1\int_0^{\infty} f_1(t)e^{-st}dt+c_2\int_0^{\infty} f_2(t)e^{-st}dt$$
$$=c_1 F_1(s)+c_2 F_2(s)$$

$$\therefore c_1 f_1(t)+c_2 f_2(t)\leftrightarrow c_1 F_1(s)+c_2 F_2(s)$$

例題：

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$L\left[e^{j\omega t}\right] = L\left[\cos\omega t + j\sin\omega t\right]$$

$$= L[\cos\omega t] + jL[\sin\omega t]$$

$$L[e^{j\omega t}] = \frac{1}{s - j\omega} = \frac{s + j\omega}{s^2 + \omega^2}$$

$$= \frac{s}{s^2 + \omega^2} + j\frac{\omega}{s^2 + \omega^2}$$

$$\therefore \cos \omega t \leftrightarrow \frac{s}{s^2+\omega^2}$$

$$\sin \omega t \leftrightarrow \frac{\omega}{s^2+\omega^2}$$

（２）微分

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

［証明］

$$L\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt} \cdot e^{-st} dt$$

$$= \left[f(t) \cdot e^{-st} \right]_0^\infty - \int_0^\infty f(t)(-se^{-st})dt$$

$$= -f(0) + s\int_0^\infty f(t)e^{-st}dt$$

$$= -f(0) + sF(s)$$

$$\therefore \frac{df(t)}{dt} \leftrightarrow sF(s) - f(0)$$

$$\frac{d^2 f(t)}{dt^2} \leftrightarrow s\left\{sF(s) - f(0)\right\} - f'(0)$$

$$= s^2 F(s) - sf(0) - f'(0)$$

一般に

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) -$$

$$\cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

例題：

$$f(t) = \sin \omega t$$

$$\frac{df(t)}{dt} = \omega \cos \omega t$$

$$F(s) = L[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

$$= s \cdot \frac{\omega}{s^2 + \omega^2} - 0$$

$$\therefore L[\omega \cos \omega t] = \frac{s\omega}{s^2 + \omega^2}$$

$$\therefore \cos \omega t \leftrightarrow \frac{s}{s^2 + \omega^2}$$

（３）積分

$$L\left[\int_0^t f(t)dt\right]=\frac{F(s)}{s}$$

〔証明〕

$$L[\int_0^t f(t)dt]=\int_0^\infty(\int_0^t f(u)du)e^{-st}dt$$

$$=\left[\int_0^t f(u)du\cdot\left(-\frac{1}{s}e^{-st}\right)\right]_0^\infty+\int_0^\infty\frac{1}{s}e^{-st}\cdot f(t)dt$$

ここで、$\lim_{t\to\infty}e^{-st}\cdot\int_0^t f(u)du=0$ が証明できるので、

$$L\left[\int_0^t f(t)dt\right]=\frac{1}{s}\int_0^\infty f(t)e^{-st}dt=\frac{F(s)}{s}$$

一般に

$$L\left[\int_0^t \cdots \int_0^t f(t)(dt)^n\right] = \frac{F(s)}{s^n}$$

（４）移動定理

$$L\left[e^{-at}f(t)\right] = F(s+a)$$

〔証明〕

$$L\left[e^{-at}f(t)\right] = \int_0^\infty e^{-at}f(t)\cdot e^{-st}dt$$

$$= \int_0^\infty f(t)e^{-(s+a)t}dt$$

$$= F(s+a)$$

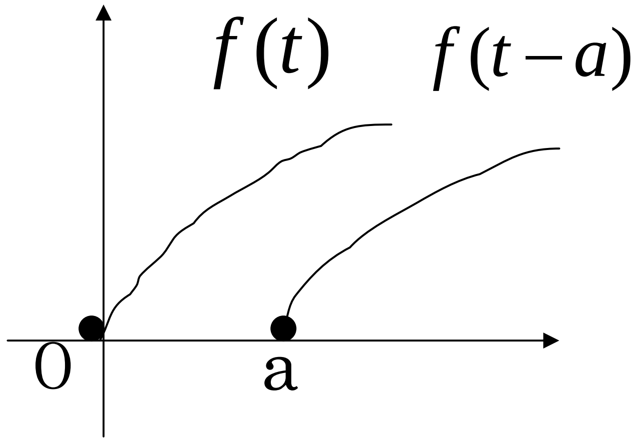
$$\left(F(s) = \int_0^\infty f(t)e^{-st}dt\right)$$

例題：

$$\sin \omega t \leftrightarrow \frac{\omega}{s^2 + \omega^2}$$

$$e^{-at} \sin \omega t \leftrightarrow \frac{\omega}{(s+a)^2 + \omega^2}$$

（５） $L\left[f(t-a)u(t-a)\right]=e^{-as}F(s)$



〔証明〕

$$L\left[f(t-a)u(t-a)\right]$$

$$=\int_0^{\infty} f(t-a)u(t-a)e^{-st}dt$$

$$=\int_a^{\infty} f(t-a)e^{-st}\,dt$$

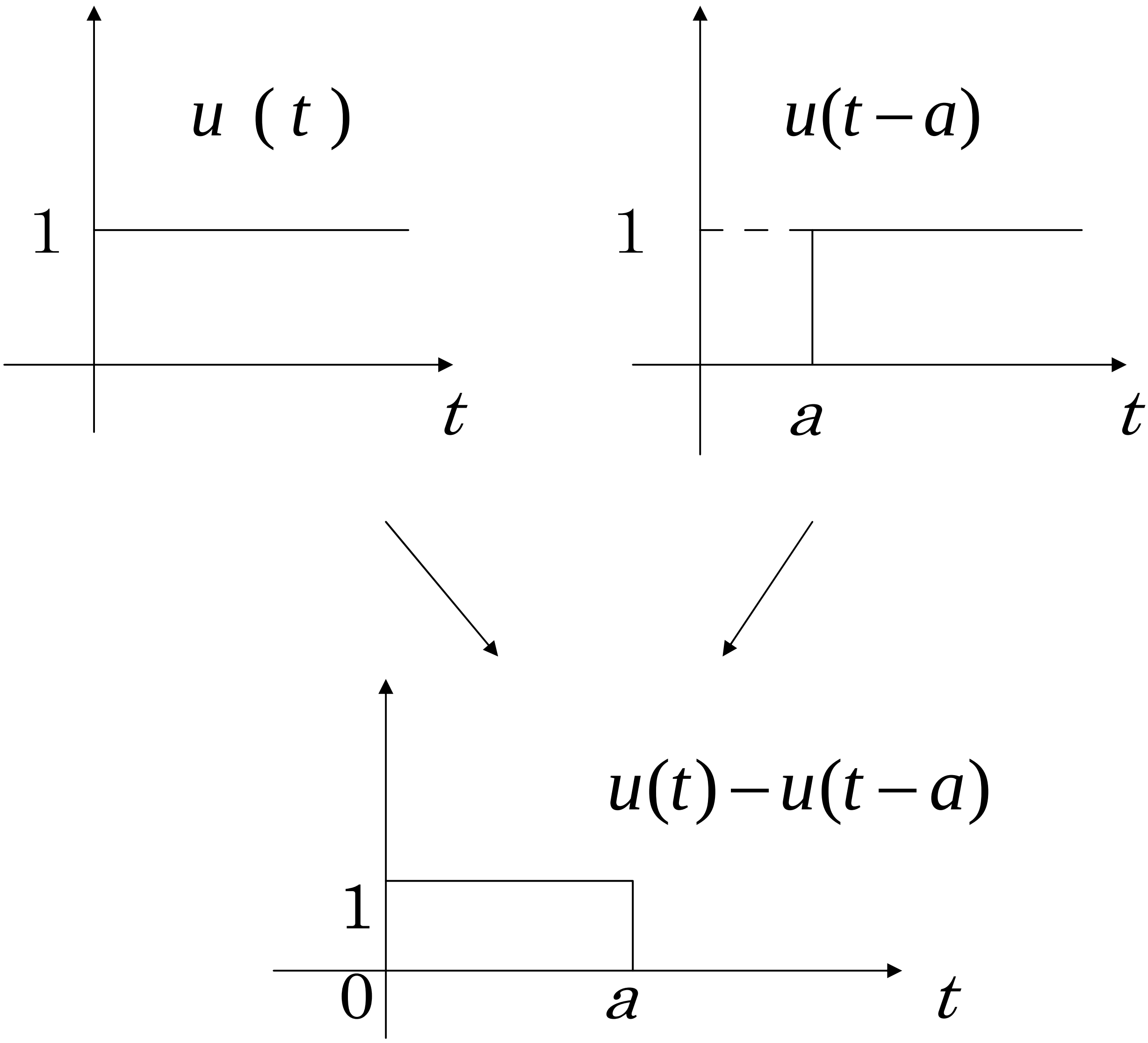
$t-a=u$ とすると

$$\int_0^{\infty} f(u)e^{-s(u+a)}du$$

$$= e^{-as}\int_0^{\infty} f(u)e^{-su}du$$

$$= e^{-as}F(s)$$

例題：



$$
\begin{aligned}
L\left[u(t)-u(t-a)\right] &= L\left[u(t)\right]-L\left[u(t-a)\right] \\
&= \frac{1}{s}-e^{-as}\cdot\frac{1}{s} \\
&= \frac{1-e^{-as}}{s}
\end{aligned}
$$

3. 4 逆ラプラス変換

$$F(s) = L\left[f(t)\right],\quad f(t) = L^{-1}\left[F(s)\right]$$

〔例題１〕

$$L\left[e^{-3t}\right] = \frac{1}{s+3}$$

$$L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$$

〔例題２〕

$L^{-1}\left[\dfrac{2}{s^2+3s+2}\right]$を求めよ.

(解)

$$F(s)=\frac{2}{s^2+3s+2}$$

$$=\frac{2}{(s+1)(s+2)}$$

$$=\frac{k_1}{s+1}+\frac{k_2}{s+2}$$

$$\frac{k_1}{s+1}+\frac{k_2}{s+2}=\frac{k_1(s+2)+k_2(s+1)}{(s+1)(s+2)}$$

$$=\frac{(k_1+k_2)s+2k_1+k_2}{(s+1)(s+2)}$$

$$\begin{cases} k_1+k_2=0 \\ 2k_1+k_2=2 \end{cases} \longrightarrow \begin{cases} k_1=2 \\ k_2=-2 \end{cases}$$

$$\therefore F(s)=\frac{2}{s+1}-\frac{2}{s+2}$$

$$\therefore L^{-1}\left[F(s)\right]=2e^{-t}-2e^{-2t}$$

〔例題３〕

$$F(s)=\frac{1}{s^{2}(s+1)^{2}}$$ の逆変換を求めよ.

(解)

$$F(s)=\frac{1}{s^2(s+1)^2}=\frac{k_0}{s}+\frac{k_1}{s^2}+\frac{k_2}{s+1}+\frac{k_3}{(s+1)^2}$$

$$F(s)=\frac{k_0 s(s+1)^2+k_1(s+1)^2+k_2 s^2(s+1)+k_3 s^2}{s^2(s+1)^2}$$

係数比較により

$$k_0=-2,\quad k_1=1,\quad k_2=2,\quad k_3=1$$

$$\therefore F(s)=-\frac{2}{s}+\frac{1}{s^2}+\frac{2}{s+1}+\frac{1}{(s+1)^2}$$

$$\therefore L^{-1}\left[F(s)\right]=-2+t+2e^{-t}+e^{-t}\cdot t$$

〔例題４〕

$F(s)=\dfrac{5s+3}{(s-1)(s^2+2s+5)}$ の逆変換を求めよ．

(解)

$$F(s)=\frac{5s+3}{(s-1)(s^2+2s+5)}$$

$$=\frac{A}{s-1}+\frac{Bs+C}{s^2+2s+5}$$

$$F(s)=\frac{A(s^2+2s+5)+(Bs+C)(s-1)}{(s-1)(s^2+2s+5)}$$

$$=\frac{A(s^2+2s+5)+Bs^2-Bs+Cs-C}{(s-1)(s^2+2s+5)}$$

$$=\frac{(A+B)s^2+(2A+C-B)s+5A-C}{(s-1)(s^2+2s+5)}$$

$\therefore$

$$\begin{cases} A+B=0 \\ 2A+C-B=5 \\ 5A-C=3 \end{cases} \longrightarrow \begin{cases} A=1 \\ B=-1 \\ C=2 \end{cases}$$

$$\begin{aligned} \therefore F(s) &= \frac{1}{s-1}+\frac{-s+2}{s^2+2s+5} \\ &= \frac{1}{s-1}+\frac{-(s+1)+3}{(s+1)^2+4} \\ &= \frac{1}{s-1}-\frac{s+1}{(s+1)^2+4}+\frac{3}{(s+1)^2+4} \end{aligned}$$

$$L^{-1}\frac{1}{s-1}=e^t$$

$$L^{-1}\frac{s+1}{(s+1)^2+4}=e^{-t}\cos 2t$$

$$\left[\, e^{-at}f(t)\leftrightarrow F(s+a) \,\right]$$

$$\left(\begin{aligned} &\cos\omega t\leftrightarrow\frac{s}{s^2+\omega^2}\\ &e^{-at}\cos\omega t\leftrightarrow\frac{s+a}{(s+a)^2+\omega^2}\end{aligned}\right)$$

$$L^{-1}\frac{3}{(s+1)^2+4}=L^{-1}\frac{3}{2}\cdot\frac{2}{(s+1)^2+4}$$

$$=\frac{3}{2}e^{-t}\cdot\sin 2t$$

$$\left(\begin{array}{l}\sin\omega t\leftrightarrow\dfrac{\omega}{s^2+\omega^2}\\[2ex] e^{-at}\sin\omega t\leftrightarrow\dfrac{\omega}{(s+a)^2+\omega^2}\end{array}\right)$$

$$\therefore\ f(t)=e^{t}-e^{-t}\cos 2t+\frac{3}{2}e^{-t}\sin 2t$$

3. 5 最終値、初期値の定理

〔最終値の定理〕

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s), \quad F(s) = L\left[f(t) \right]$$

〔証明〕

$$L\frac{df(t)}{dt} = \int_0^\infty f'(t)e^{-st}dt$$

$$= sF(s) - f(0)$$

$$\lim_{s\to 0}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to 0}\left\{ sF(s) - f(0) \right\}$$

$$左辺 = \int_0^\infty f'(t) \lim_{s \to 0} e^{-st} dt$$

$$= \int_0^\infty f'(t) dt$$

$$= \lim_{t \to \infty} \int_0^t f'(u) du$$

$$= \lim_{t \to \infty} \{ f(t) - f(0) \}$$

$$= \lim_{t \to \infty} f(t) - f(0)$$

$$右辺 = \lim_{s \to 0} sF(s) - f(0)$$

$$\therefore \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

〔初期値の定理〕

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s),\ F(s) = L\left[f(t)\right]$$

〔証明〕

$$L\frac{df(t)}{dt} = \int_0^\infty f'(t)e^{-st}dt$$

$$= sF(s) - f(0)$$

$$\lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to\infty}\left\{sF(s) - f(0)\right\}$$

$$\text{左辺} = \int_0^\infty f'(t)\lim_{s\to\infty}\cdot e^{-st}dt$$

$$= \int_0^\infty f'(t)\cdot 0dt$$

$$= 0$$

$$右边 = \lim_{s\to\infty} sF(s) - f(0)$$

$$= \lim_{s\to\infty} sF(s) - \lim_{t\to 0} f(t)$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$