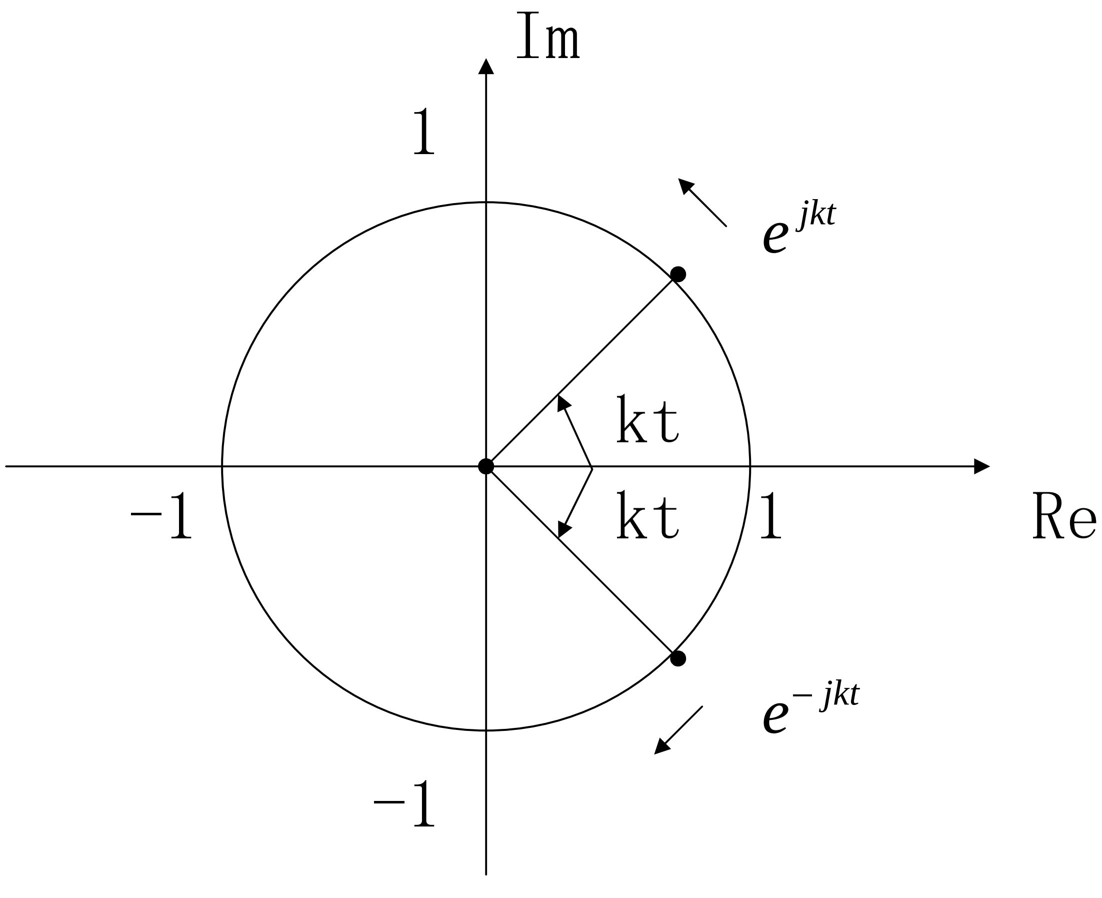
1.4 複素フーリエ級数展開

直交関数系 $\left\{e^{jkt}; k = 0, \pm 1, \pm 2, \cdots\right\}$
を用いて関数 $f(t)$ をフーリエ
展開する。

ここで、 $e^{jt} = \cos t + j \sin t$

$$
\begin{aligned}
e^{-jt} &= \cos(-t) + j \sin(-t) \\
&= \cos t - j \sin t \\
&= \overline{e^{jt}}
\end{aligned}
$$



複素関数 $f(t), \mathrm{g}(t)$ の内積を

$$\langle f(t), g(t) \rangle = \frac{1}{b-a}\int_a^b f(t)\cdot\overline{g(t)}dt$$

と定義する

$$\langle f(t), f(t) \rangle = \frac{1}{b-a}\int_a^b f(t)\cdot\overline{f(t)}dt$$

$$= \frac{1}{b-a}\int_a^b \left| f(t) \right|^2 dt$$

$$= \left\| f(t) \right\|^2$$ （ノルム）

$\|\bullet\|$：ノルム

$$\left\langle e^{jmt}, e^{jnt} \right\rangle$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jmt} \cdot e^{\overline{jnt}} dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jmt} \cdot e^{-jnt} dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(m-n)t} dt$$

$(\mathrm{i})\, m \neq n$の時

$$\left\langle e^{jmt}, e^{jnt} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(m-n)t} dt$$

$$= \frac{1}{2\pi} \left[\frac{1}{j(m-n)} e^{j(m-n)t} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(m-n)} \frac{e^{j(m-n)\pi} - e^{-j(m-n)\pi}}{2j}$$

$$= \frac{1}{\pi(m-n)} \cdot$$

$$\left\{ \frac{\cos(m-n)\pi + j\sin(m-n)\pi}{2j} \right.$$
$$\left. \frac{-\cos(m-n)\pi + j\sin(m-n)\pi}{2j} \right\}$$

$$=\frac{1}{\pi(m-n)}\sin(m-n)\pi=0$$

$(\text{ii})\,m=n$の時

$$\left\langle e^{jmt},e^{jnt}\right\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{0}dt=\frac{1}{2\pi}[t]_{-\pi}^{\pi}$$

$$=\frac{1}{2\pi}(\pi+\pi)=1$$

$$\therefore\left\langle e^{jmt},e^{jnt}\right\rangle=\delta_{mn}$$

複素フーリエ級数展開：

$$f(t) = \sum_{k=-\infty}^{\infty} cke^{jkt}$$

$$\left\langle f(t), e^{jmt} \right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} c_k \cdot e^{jkt} e^{-jmt} dt$$

$$= \sum_{k=\infty}^{\infty} c_k \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jkt} \cdot e^{-jmt} dt$$

$$= \sum_{k=-\infty}^{\infty} c_k \cdot \delta_{km}$$

$$= c_m$$

$$\therefore c_m = \left\langle f(t), e^{jmt} \right\rangle$$

複素フーリエ級数展開

$$f\left(t\right)=\sum_{k=-\infty}^{\infty}c_k e^{jkt} \quad ,$$

$$c_k=\left\langle f\left(t\right),e^{jkt}\right\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}f\left(t\right)e^{-jkt}dt$$

$$f\left(t\right)=\sum_{k=-\infty}^{\infty}c_k e^{j\omega_0 kt}, \quad \omega_0=2\pi/T$$

$$e^{jkt}\Big|_{t=\pi}=e^{jk\pi}$$

$$e^{j\omega_0 kt}\Big|_{t=\frac{T}{2}}=e^{j\frac{2\pi}{T}k\frac{T}{2}}=e^{jk\pi}$$

$$c_k=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f\left(t\right)e^{-j\omega_0 kt}dt$$

実フーリエ級数展開

$$f(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos\omega_0 kt+\sum_{k=1}^{\infty}b_k\sin\omega_0 kt$$

$$a_k=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\cos\omega_0 ktdt$$

$$b_k=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin\omega_0 ktdt$$

$$\omega_0=2\pi/T$$

11/8 宿題

関数系{1,sin$_{\omega 0}$ kt,cosω0 kt}(k=1,2,3,…)

-T/2≦t≦T/2 , ω0=2π/T

は直行関係であることを示せ

複素フーリエ級数

$$f\left(t\right)=\sum_{k=-\infty}^{\infty}c_k e^{j\omega_0 kt} \quad ,\omega_0=2\pi/T$$

$$c_k=\frac{1}{T}\int_{-T/2}^{T/2}f\left(t\right)e^{-j\omega_0 kt}dt$$

$$c_k=\frac{1}{T}\int_{-T/2}^{T/2}f\left(t\right)\left(\cos\omega_0 kt-j\sin\omega_0 kt\right)dt$$

$$=\frac{1}{T}\int_{-T/2}^{T/2}f\left(t\right)\cos\omega_0 ktdt$$

$$-j\frac{1}{T}\int_{-T/2}^{T/2}f\left(t\right)\sin\omega_0 ktdt$$

$(\mathrm{i})\, k=0$の時

$$c_k=\frac{1}{T}\int_{-T/2}^{T/2} f(t)\cdot 1dt=\frac{a_0}{2}$$

$(\mathrm{ii})\, k>0$の時

$$c_k=\frac{1}{T}\int_{-T/2}^{T/2} f(t)\cos\omega_0 kt dt$$

$$-j\frac{1}{T}\int_{-T/2}^{T/2} f(t)\sin\omega_0 kt dt$$

$$=\frac{a_k}{2}-j\frac{b_k}{2}=\frac{1}{2}(a_k-jb_k)$$

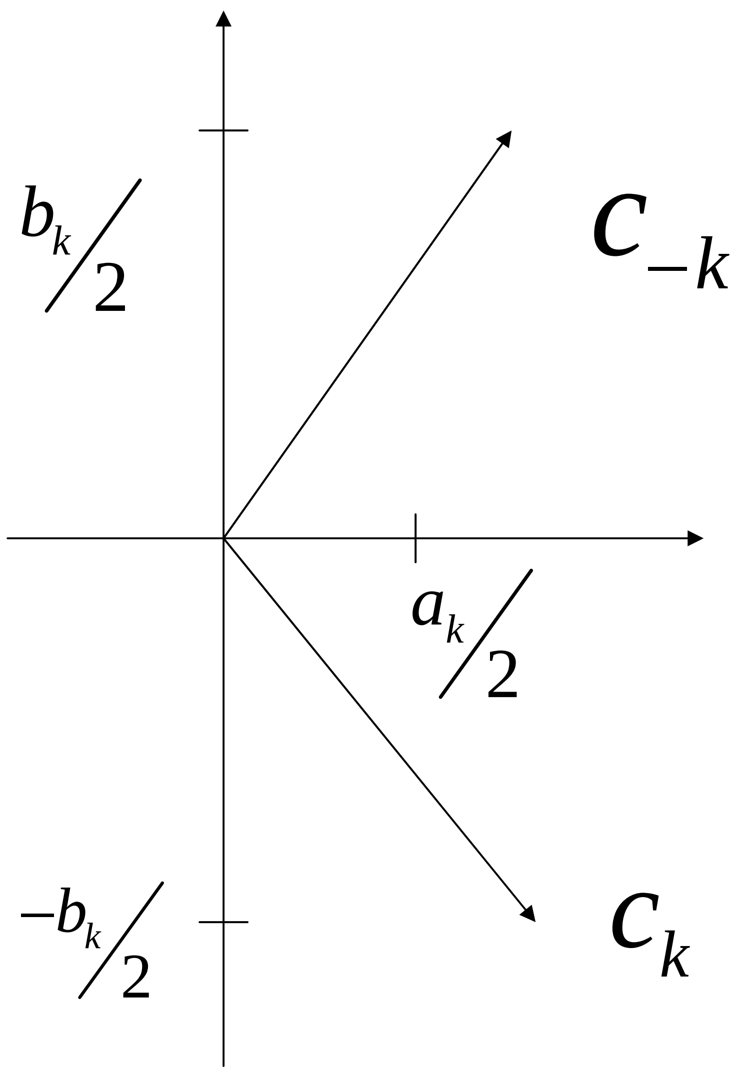
$$
\begin{aligned}
c_{-k} &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\cos\omega_0(-k)t\,dt \\
&\quad - j\frac{1}{T}\int_{-T/2}^{T/2} f(t)\sin\omega_0(-k)t\,dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\cos\omega_0 kt\,dt \\
&\quad + j\frac{1}{T}\int_{-T/2}^{T/2} f(t)\sin\omega_0 kt\,dt \\
&= \frac{a_k}{2} + j\frac{b_k}{2}
\end{aligned}
$$

$$\therefore \begin{cases} c_0 = \dfrac{a_0}{2} \\ c_k = \dfrac{1}{2}\left(a_k - jb_k\right) \\ c_{-k} = \dfrac{1}{2}\left(a_k + jb_k\right) \end{cases}$$

$$\left(k > 0\right)$$

$$c_{-k} = \overline{c_k}\ ,\ \left|c_k\right| = \left|c_{-k}\right|$$

$$\angle c_{-k} = -\angle c_k$$



$b_k/2$
c_{-k}
$a_k/2$
$-b_k/2$
c_k

$$\left|c_k\right| = \sqrt{\left(\frac{a_k}{2}\right)^2 + \left(\frac{b_k}{2}\right)^2} = \frac{\sqrt{a_k{}^2 + b_k{}^2}}{2}$$

（振幅スペクトル）

$$\angle c_k = -\tan^{-1}\frac{b_k}{a_k}$$

（位相スペクトル）

$$\left|c_k\right|^2 = \frac{a_k{}^2 + b_k{}^2}{4}$$

(パワースペクトル）

$$f\left(t\right)=\sum_{k=-\infty}^{\infty} c_k e^{j\omega_o kt}, < c_k = \varphi_k$$

$$c_k = \left|c_k\right| e^{j\varphi k}$$

$$f\left(t\right)=c_o+\sum_{k=1}^{\infty} c_k e^{j\omega_0 kt}+\sum_{k=1}^{\infty} c_{-k} e^{-j\omega_0 kt}$$

$$=c_0+\sum_{k=1}^{\infty} c_k e^{j\omega_0 kt}+\sum_{k=1}^{\infty} \overline{c}_k e^{-j\omega_0 kt}$$

$$=c_0+\sum_{k=1}^{\infty} \left|c_k\right| e^{j\varphi k} e^{j\omega_0 kt}$$

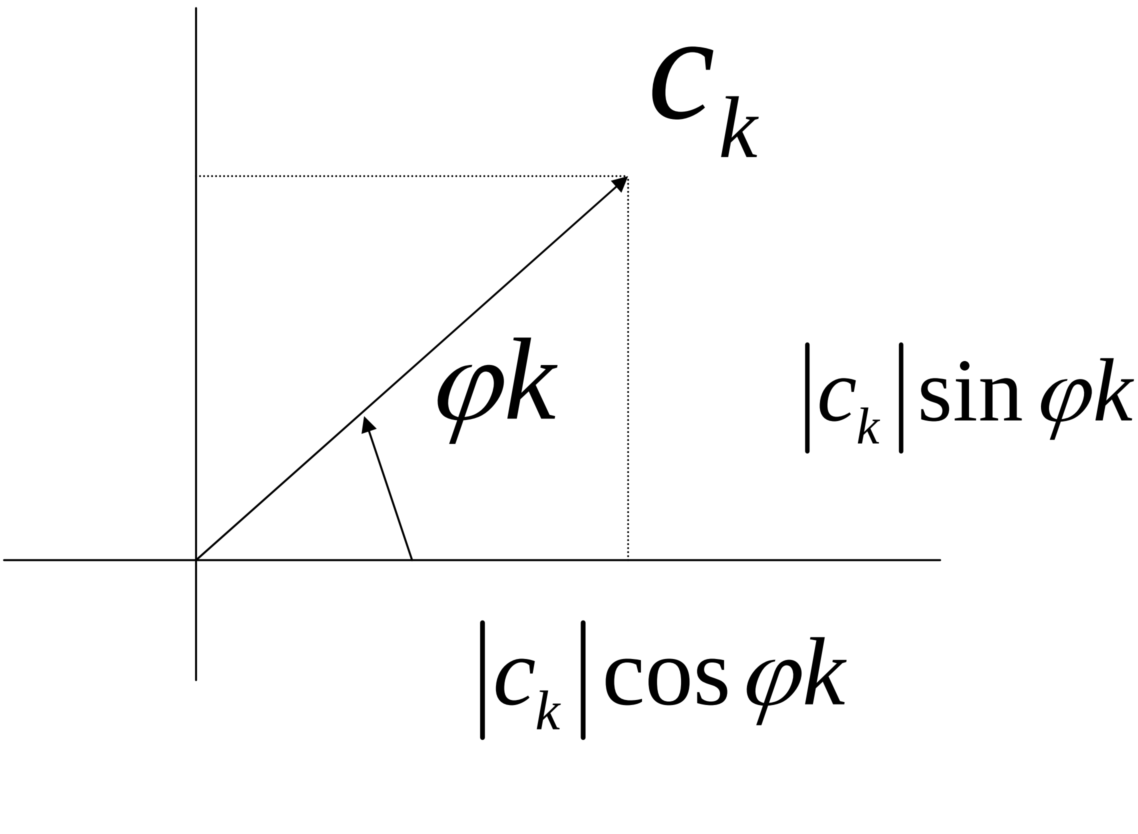
$$+\sum_{k=1}^{\infty} \left|c_k\right| e^{-j\varphi k} e^{-j\omega_0 kt}$$

$$=c_0+\sum_{k=1}^{\infty} \left|c_k\right| \left\{ \begin{array}{l} e^{j\left(\omega_0 kt+\varphi_k\right)} \\ \quad + e^{-j\left(\omega_o kt+\varphi_k\right)} \end{array} \right\}$$

$$
= c_0 + \sum_{k=1}^{\infty} |c_k| \left\{ \begin{aligned} &\cos(\omega_0 kt + \varphi_k) \\ &+ j\sin(\omega_0 kt + \varphi_k) \\ &+ \cos(\omega_0 kt + \varphi_k) \\ &- j\sin(\omega_0 kt + \varphi_k) \end{aligned} \right\}
$$

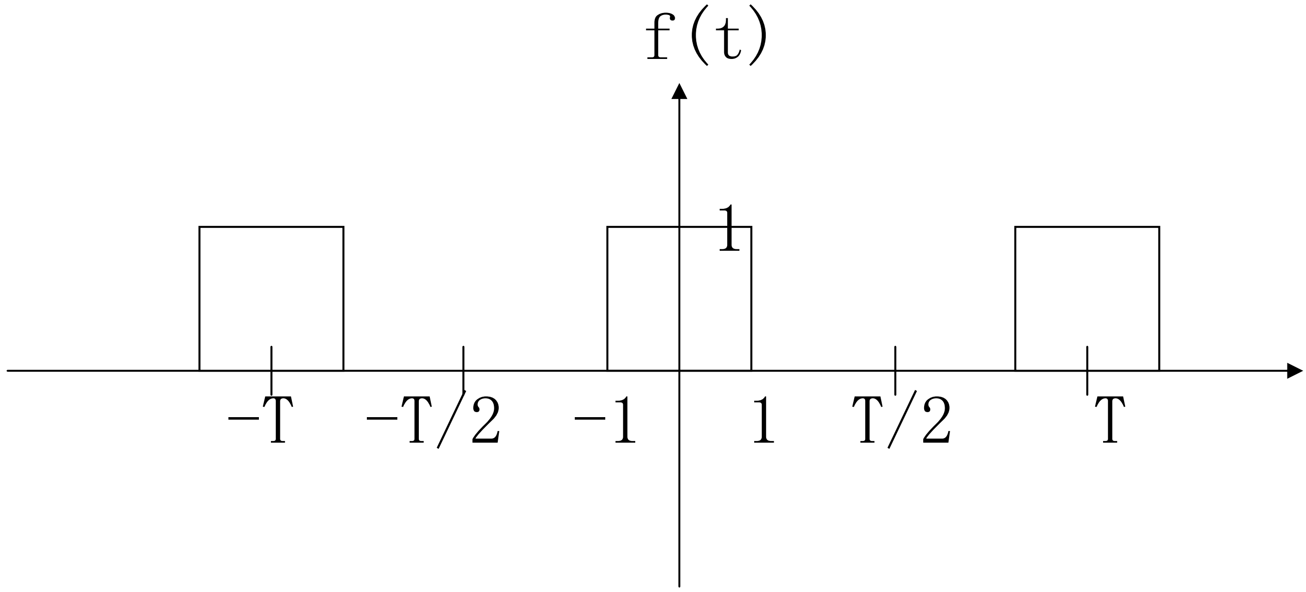
$$
= c_0 + \sum_{k=1}^{\infty} |c_k| \cdot 2 \cdot \cos(\omega_0 kt + \varphi_k)
$$

(実数)



$$
\begin{aligned}
c_k &= |c_k| \cos\varphi_k + j|c_k|\sin\varphi_k \\
&= |c_k|\left(\cos\varphi_k + j\sin\varphi_k\right) \\
&= |c_k| e^{j\varphi_k}
\end{aligned}
$$

（例）



$$f\left(t\right)=\sum_{k=-\infty}^{\infty}c_k e^{j\omega_0 kt} \qquad \omega_0 = 2\pi/T$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f\left(t\right)e^{-j\omega_0 kt}dt$$

$$= \frac{1}{T}\int_{-1}^{1} 1\cdot e^{-j\omega_0 kt}dt$$

$$= \frac{1}{T}\frac{1}{-j\omega_0 k}\left[e^{-j\omega_0 kt}\right]_{-1}^{1} \qquad \left(k \neq 0\right)$$

$$= \frac{1}{T}\frac{1}{-j\omega_0 k}\left(e^{-j\omega_0 k} - e^{j\omega_0 k}\right)$$

$$= \frac{1}{T}\frac{1}{-j\omega_0 k}\begin{Bmatrix} \left(\cos\omega_0 k - j\sin\omega_0 k\right) \\ -\left(\cos\omega_0 k + j\sin\omega_0 k\right) \end{Bmatrix}$$

$$= \frac{1}{T}\frac{2}{\omega_0 k}\sin\omega_0 k$$

$$= \frac{2}{T}\frac{\sin\omega_0 k}{\omega_0 k}$$

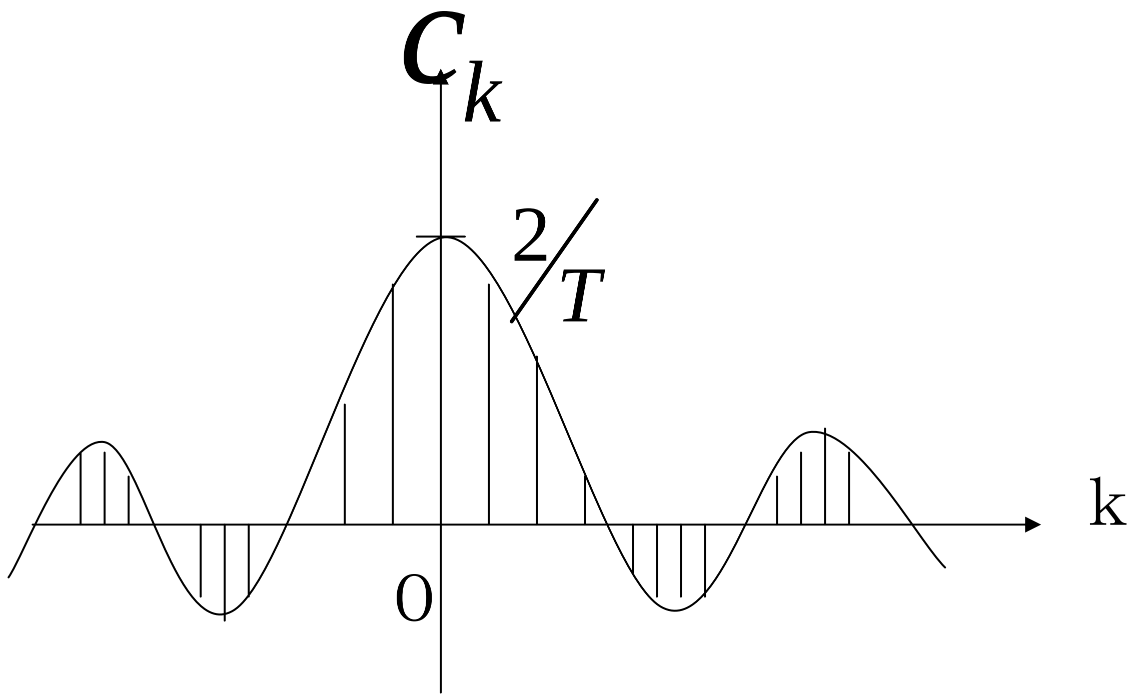
$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} f\left(t\right)\cdot 1dt$$

$$= \frac{1}{T}\int_{-1}^{1} 1dt = \frac{1}{T}\left[t\right]_{-1}^{1} = \frac{2}{T}$$

$$\omega_0 k = \pi$$

$$\frac{2\pi}{T} \cdot k = \pi$$

$$k = \frac{T}{2}$$



パーシバルの定理

$$f(t)=\sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt}$$

$$\left\| f(t) \right\|^2 = \sum_{k=-\infty}^{\infty} \left| c_k \right|^2 \qquad \left\| \bullet \right\| : ノルム$$

$$\left\| f(t) \right\|^2 = \left\langle f(t), f(t) \right\rangle$$

$$= \left\langle \sum_{m=-\infty}^{\infty} c_m e^{j\omega_0 mt}, \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 nt} \right\rangle$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left\langle c_m e^{j\omega_0 mt}, c_n e^{j\omega_0 nt} \right\rangle$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m \bar{c}_n \left\langle e^{j\omega_0 mt}, e^{j\omega_0 nt} \right\rangle$$

$$\left(\begin{aligned}\because \left\langle af(t), bg(t)\right\rangle &= \frac{1}{T}\int_{-T/2}^{T/2} af(t)\cdot\overline{bg(t)}\,dt \\ &= a\cdot\bar{b}\frac{1}{T}\int_{-T/2}^{T/2} f(t)\cdot\overline{g(t)}\,dt \\ &= a\cdot\bar{b}\left\langle f(t), g(t)\right\rangle\end{aligned}\right)$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_m \bar{c}_n \delta_{mn}$$

$$= \sum_{m=-\infty}^{\infty} c_m \bar{c}_m = \sum_{m=-\infty}^{\infty} \left|c_m\right|^2$$

$(\because)$

$$\left\langle e^{j\omega_0 mt}, e^{j\omega_0 nt} \right\rangle \qquad \omega_0 = 2\pi/T$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} e^{j\omega_0 mt}\,\overline{e^{j\omega_0 nt}}\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} e^{j\omega_0 (m-n)t}\,dt \qquad (m \neq n)$$

$$= \frac{1}{T}\frac{1}{j\omega_0(m-n)}\left[e^{j\omega_0(m-n)t}\right]_{-T/2}^{T/2}$$

$$= \frac{1}{T}\frac{1}{j\omega_0(m-n)}$$

$$\left\{e^{j\omega_0(m-n)\cdot T/2} - e^{-j\omega_0(m-n)\cdot T/2}\right\}$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$

$$=\frac{1}{T}\frac{1}{j\omega_0(m-n)}2j\sin\left(\omega_0(m-n){}^{T}\!/_{2}\right)$$

$$=\frac{1}{T}\frac{2}{\omega_0(m-n)}\sin\frac{2\pi}{T}(m-n)\cdot{}^{T}\!/_{2}$$

$$=\frac{1}{T}\frac{2}{\omega_0(m-n)}\sin(m-n)\pi=0$$

$m=n$　の時に

$$\left\langle e^{j\omega_0 mt},e^{j\omega_0 mt}\right\rangle=\frac{1}{T}\int_{-T/2}^{T/2}1dt$$

$$=\frac{1}{T}\left({}^{T}\!/_{2}-\left(-{}^{T}\!/_{2}\right)\right)=1$$

$$\therefore\left\langle e^{j\omega_0 mt},e^{j\omega_0 nt}\right\rangle=\delta_{mn}$$